How to Renormalize the Schrödinger Equation

Peter Lepage

Cornell University

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Renormalization Review

Renormalization

Probe of wavelength λ $(= h/p)$ insensitive to structure at distances $\ll \lambda$.

λ large $\Rightarrow$

Can replace true theory (complex, unknown?) by a simpler theory (infinitely many choices!).

Eg)

 $\mathbf{J} \rightarrow$  $+$  $+ \cdots$

E_1 M_1

Effective Field Theory

Large-λ theory.

- Low-energy approximation.
- Systematically improvable.
- UV cutoff: $p < \Lambda$.

$\Lambda \approx$ threshold for next level of structure

Pedagogical Example

1. Synthetic Data

Low-energy data → effective theory.

Invent physical problem: Coulombic atom + short-range potential:

$$H = \frac{\mathbf{p}^2}{2m} + V(\mathbf{r})$$

where

$$V(\mathbf{r}) = -\frac{\alpha}{r} + V_s(\mathbf{r})$$

Arbitrary short-range potential.

- Make one up.
- Details irrelevant (secret).

Infrared (large-distance) behavior specified by:

$$m = 1 \qquad \alpha = 1 \quad (\Rightarrow \text{strongly coupled}).$$

Generate precise low-energy "data" by solving numerically.

- Bound state binding energies:

level	binding energy	level	binding energy
$1S$	1.28711542	$6S$	0.0155492598
$2S$	0.183325753	...	
$3S$	0.0703755485	$10S$	0.00534541931
$4S$	0.0371495726	$20S$	0.00129205010
$5S$	0.0229268241		

- Phase shifts computed for $r = 50$ (Coulomb tail).

energy	phase shift	energy	phase shift
10^{-10}	−0.000421343353	.03	1.232867297
10^{-5}	−0.133227246	.07	−0.579619620
.001	−1.319383451	.1	−1.156444634
.003	0.900186195	.3	−0.106466466
.007	−0.146570028	.7	−1.457426179
.01	−0.654835316	1	1.160634967

- $\langle 1s|\mathbf{p}^4|1s\rangle = 69.0\ldots$

Challenge

Given the large-r behavior, design a simple theory that reproduces the low-energy data to arbitrarily high precision.

2. Traditional Approach

Model $V_s(r)$ by $\delta^3(r)$:

$$H_{\text{eff}} = \frac{p^2}{2m} + V_{\text{eff}}(r)$$

$$V_{\text{eff}} = -\frac{\alpha}{r} + c\,\delta^3(r)$$

Large r known.

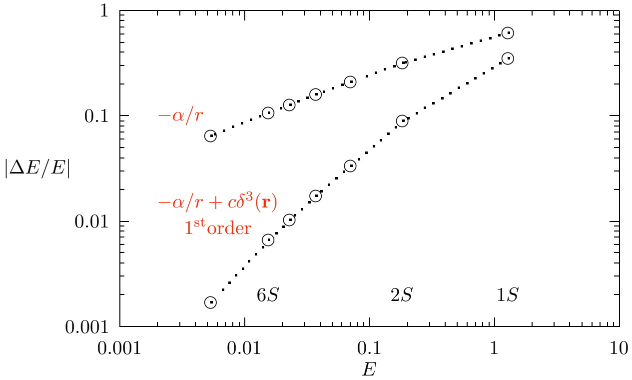
Treat as perturbation.

First order perturbation theory ⇒

$$\begin{aligned} E_{nl}^{\text{app}} &= E_{nl}^{\text{coul}} + c\left|\psi_{nl}^{\text{coul}}(0)\right|^2 \\ &= -\frac{1}{2n^2} + c\,\frac{\delta_{l,0}}{\sqrt{\pi}\,n^3} \end{aligned}$$

One parameter theory: choose c to match most infrared data.

- Match $E_{20S} \Rightarrow c = -0.5963$.
- Most IR ⇒ most accurate.
- Same c for all levels.



1
0.1
0.01
0.001
|ΔE/E|
−α/r
−α/r + cδ³(r)
1st order
6S
2S
1S
0.001
0.01
0.1
1
10
E

More Accuracy?

1) Interaction strong $\Rightarrow$ 2$^{\text{nd}}$ order perturbation theory, but

$$\sum_{m \neq n} \frac{\langle n | c\, \delta^3(\mathbf{r}) | m \rangle \; \langle m | c\, \delta^3(\mathbf{r}) | n \rangle}{E_n - E_m} \quad = \quad \infty$$

$\sum$ over $\mathbf{k} \to \infty$ states diverges $\Rightarrow$ UV divergence.

2) Finite-range corrections to short-range $V_s \Rightarrow$

$$V_s(\mathbf{r}) \xrightarrow{\text{F.T.}} v_s(q^2) \approx v_s(0) + q^2 v_s'(0) + \cdots$$

Suggests

$$V_s(\mathbf{r}) \to V_{\text{eff}} \equiv c\,\delta^3(\mathbf{r}) + d\,\nabla^2\delta^3(\mathbf{r})$$

except

$$\langle n|\nabla^2\delta^3(\mathbf{r})|n\rangle = \infty$$

$\mathbf{k} \to \infty$ UV divergence!

Why the infinities?

$\mathbf{k} \to \infty$ behavior of V_{eff} is very bad (and wrong).

Conventional wisdom $\Rightarrow$ give up after 1^{st} order; must use real potential to go further.

3. Effective Theory

Low-energy theory insensitive to short distance details
⇒ redesign short distance so not singular & "accurate."

1) Preserve large-r behavior.

2) Introduce UV cutoff to prevent infinities, and exclude high-momentum states about which ignorant.

3) Add local operators to H_{eff} to mimic effects of states excluded by cutoff.

UV Cutoff

$$\frac{1}{r} \xrightarrow{\text{F.T.}} \frac{4\pi}{q^2}$$

$$\xrightarrow{\text{cutoff}} \frac{4\pi}{q^2}\, e^{-q^2 a^2/2} \qquad (\text{cutoff} \Rightarrow q < 1/a = \Lambda)$$

$$\xrightarrow{\text{F.T.}} \frac{\text{erf}(r/\sqrt{2}a)}{r} \qquad \left(\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt\right)$$

$\Rightarrow$ Analytic at $r = 0$, and $1/r$ at large r.

Cutoff $\Rightarrow$ errors of $\mathcal{O}((pa)^n)$. Remove errors order-by-order using local correction terms (mimics excluded $k > 1/a$ physics):

$$
\begin{aligned}
V_{\rm eff}(\mathbf{r}) \;=\; & -\frac{\alpha}{r}\,\mathrm{erf}(r/\sqrt{2}a) \\
& + c\,a^2\,\delta_a^3(\mathbf{r}) \qquad \leftarrow \text{removes } \mathcal{O}(pa)^2 \text{ errors} \\
& + d_1\,a^4\,\nabla^2\delta_a^3(\mathbf{r}) + d_2\,a^4\,\nabla\cdot\delta_a^3(\mathbf{r})\nabla \qquad \leftarrow \text{removes } \mathcal{O}(pa)^4 \\
& + \cdots \\
& + g\,a^{n+2}\,\nabla^n\delta_a^3(\mathbf{r}) \qquad \leftarrow \; \delta_a^3(\mathbf{r}) \equiv \frac{\mathrm{e}^{-r^2/2a^2}}{(2\pi)^{3/2}\,a^3} \\
& + \cdots
\end{aligned}
$$

Procedure

Focus on S states through $\mathcal{O}((pa)^4) \Rightarrow$ need only c and d_1 terms.

1) Choose an $a <$ important long-range distance scales (eg, atom size) — want pa small. (Choose $a = 1$ for now.)

2) Tune c, d_1 to fit IR data — 1 piece of data/coupling. (Use most IR data to minimize $(pa)^6$ errors.)

$$\begin{array}{ccc} c & & \delta_0(10^{-5}) \\ d_1 & \longleftrightarrow & \delta_0(10^{-10}) \end{array}$$

3) Generate everything else using same c, d_1.

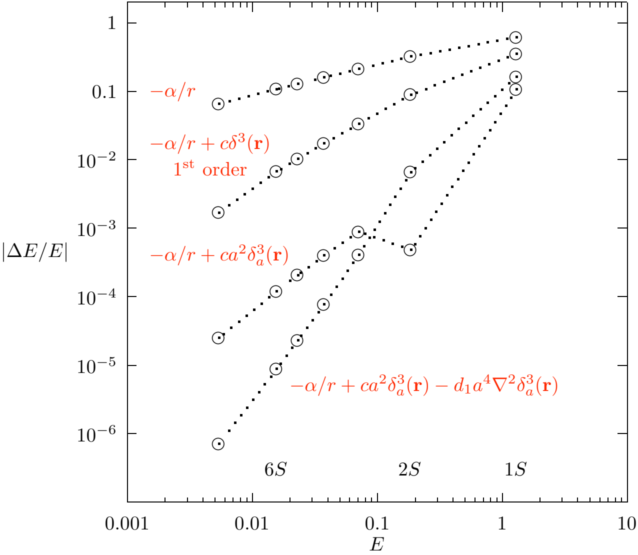
Note:

$a \neq 0$

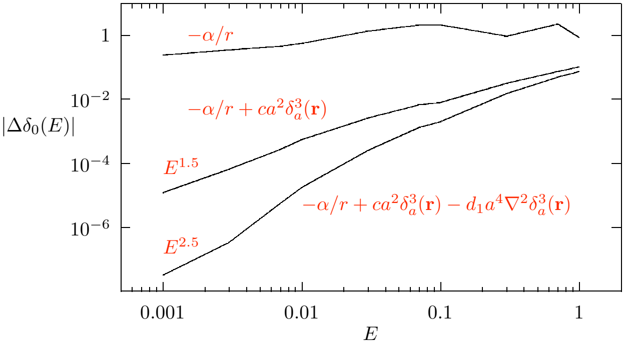
$\Rightarrow$ $V_{\text{eff}}(\mathbf{r})$ simple, nonsingular (analytic!) at $r = 0$.

$\Rightarrow$ Trivial to solve for any c, d_1, a (eg, numerically).

$\rightarrow$ No infinities, because of cutoff.



1
0.1
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
$|\Delta E/E|$
$-\alpha/r$
$-\alpha/r + c\delta^3(\mathbf{r})$
1^{st} order
$-\alpha/r + ca^2\delta^3_a(\mathbf{r})$
$-\alpha/r + ca^2\delta^3_a(\mathbf{r}) - d_1a^4\nabla^2\delta^3_a(\mathbf{r})$
$6S$
$2S$
$1S$
0.001
0.01
0.1
1
10
E



$-\alpha/r$
$-\alpha/r + ca^2\delta_a^3(\mathbf{r})$
$E^{1.5}$
$-\alpha/r + ca^2\delta_a^3(\mathbf{r}) - d_1 a^4\nabla^2\delta_a^3(\mathbf{r})$
$E^{2.5}$
$|\Delta\delta_0(E)|$
1
10^{-2}
10^{-4}
10^{-6}
0.001
0.01
0.1
1
E

Note:

- 2 phase shifts $\Rightarrow$ all these results!
- Errors smaller for smaller Es — $(pa)^n$.
- Adding $\nabla^2 \delta_a$ term $\Rightarrow$ error curve slope steeper by one power of $(pa)^2 \propto E$.
- Corrections stop working for $pa \approx 1$. (Here $a = 1$.)
- 1% error in E_{2S} even though $a \approx r_{2S}/4$;
  E_{10S} accurate to 6 digits.

a Dependence

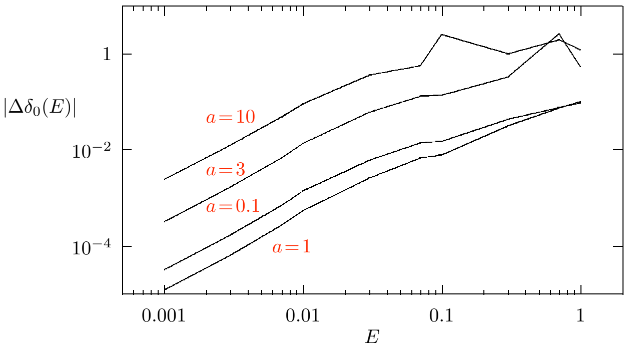
Errors $\propto (pa)^n$

$\rightarrow$ large $a \Rightarrow$ larger errors

But $a \rightarrow 0 \Rightarrow$

- infinities
- bad (unphysical) high-energy states
- no values of $c, d_1 \ldots$ fit data
- $\ldots$

Typically want $a \approx$ true scale of $V_s(\mathbf{r})$.



$|\Delta\delta_0(E)|$
1
10^{-2}
10^{-4}
$a = 10$
$a = 3$
$a = 0.1$
$a = 1$
0.001
0.01
0.1
1
E

Physics in a

Tune a to point $a = a_c$ where errors stop decreasing with decreasing a.

$\Rightarrow$ $a_c \approx$ true scale of $V_s(\mathbf{r})$.

Point where true V_s starts to compete with long-range potential.

$$\text{Errors} \propto \begin{cases} (pa)^n & a > a_c \\ (pa_c)^n & a \leq a_c \end{cases}$$

Here $a_c \approx 1$.

“Running” Couplings

- Depend upon number of corrections:

$$
\begin{aligned}
a = 1,\, d_1 = 0 \quad &\Rightarrow \quad c = 6.9 \\
a = 1 \quad &\Rightarrow \quad c = 6.4,\, d_1 = 1.6
\end{aligned}
$$

- Depend upon a (hence “running”):

$a =$		$c =$	
	10		5.8
	3		6.1
	1		6.9
	0.1		12.5
	0.01		147.9
	$\vdots$		$\vdots$

$c \approx$ const. (for $a = 10, 3, 1$)

$c \to \infty$ (cancels wrong physics at $r < a_c$) (for $a = 0.1, 0.01, \ldots$)

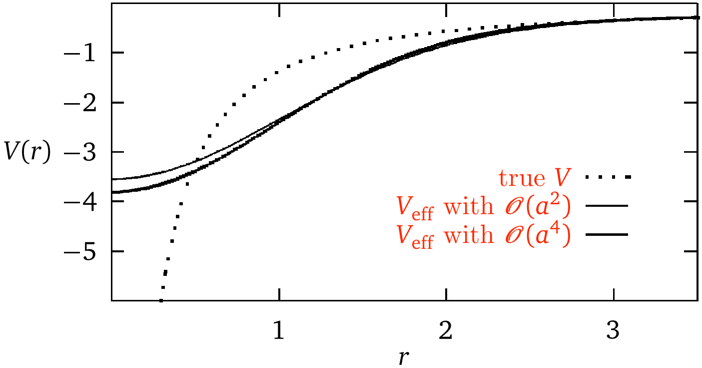
Misconceptions

1) As high-order corrections are added

$$V_{\text{eff}}(\mathbf{r}) \to \text{true } V(\mathbf{r})$$

Wrong! Infinitely many V_{eff}s: all give same low-energy results, but totally different high-energy results.

N.B. "True V" may not exist! (Eg, quantum field theory at short distances, quarks in nucleus, etc.)



V(r)
−1
−2
−3
−4
−5
1
2
3
r
true V
V_eff with 𝒪(a²)
V_eff with 𝒪(a⁴)

2) Just ordinary curve fitting — add more parameters, get better answers.

Wrong! Highly optimized curve fitting — systematically removes errors order-by-order in (pa).

Eg) Compare two 2-parameter fits:

1) vary a,c holding $d_1 = 0 \Rightarrow E_{10S}$ error $\approx 10^{-3}$;

2) vary c, d_1 holding $a = 1 \Rightarrow E_{10S}$ error $\approx 10^{-6}$.

4. Improved Operators

Connection between true and effective theories is subtle.

- Low-energy spectra, $\delta_l(E)$s nearly identical.
- $\psi(\mathbf{r})$ and $\psi_{\text{eff}}(\mathbf{r})$ totally different at small r.

  Eg) $\langle n|\mathbf{p}^4|n\rangle$ for $n = 1S, 2S \ldots$:

level	$\langle\mathbf{p}^4\rangle$	$\langle\mathbf{p}^4\rangle_{\text{eff}}$
$1S$	69	5.9
$2S$	5.50	1.7
$3S$	1.309	0.4
$4S$	0.5070	0.17
$5S$	0.24740	0.08
$6S$	0.138784	0.05

  $\Rightarrow$ Complete disagreement even for $E_n \to 0$!

- Renorm'n theory ⇒ local corrections for $\mathbf{p}^4$, as for $H_{\rm eff}$:

$$\langle \mathbf{p}^4 \rangle_{\rm true} = Z \, \langle \mathbf{p}^4 \rangle_{\rm eff} + \frac{\gamma}{a} \, \langle \delta_a^3(\mathbf{r}) \rangle_{\rm eff} + \eta \, a \, \langle \nabla^2 \delta_a^3(\mathbf{r}) \rangle_{\rm eff} + \mathcal{O}(a^3)$$

Z, γ, η from IR data; same for all other states.
20S, 15S, 10S data ⇒ $Z = 1$, $\gamma = -96.2$, $\eta = -140.6$,

level	$\langle \mathbf{p}^4 \rangle$	$\langle \mathbf{p}^4 \rangle_{\rm eff}$	$\langle Z\mathbf{p}^4 + \gamma \delta_a^3 / a + \cdots \rangle_{\rm eff}$
1S	69	5.9	28
2S	5.50	1.7	5.34
3S	1.309	0.4	1.306
4S	0.5070	0.17	0.5068
5S	0.24740	0.08	0.24738
6S	0.138784	0.05	0.138780

- Renorm'n theory $\Rightarrow$ local corrections for $\mathbf{p}^4$, as for $H_{\rm eff}$:

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$20S$, $15S$, $10S$ data $\Rightarrow$ $Z = 1$, $\gamma = -96.2$, $\eta = -140.6$,

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True Theory		Effective Theory
$\mathbf{p}^2/2m + V(r)$	$\longleftrightarrow$	$\mathbf{p}^2/2m - \alpha\,\mathrm{erf}()/r + ca^2\delta_a^3 + \cdots$
$\mathbf{p}^4$	$\longleftrightarrow$	$Z\mathbf{p}^4 + \gamma/a\,\delta_a^3 + \cdots$
	$\vdots$	

Couplings $c, Z\ldots$ depend on a but not on state (universal!)

Operator Product Expansion (OPE)

$$\begin{aligned}\psi_{\text{true}}(r) &= \overline{\gamma}(r)\int d^3r\,\psi_{\text{eff}}\,\delta_a^3(\mathbf{r}) \\ &+ \overline{\eta}(r)\,a^2\int d^3r\,\psi_{\text{eff}}\,\nabla^2\delta_a^3(\mathbf{r}) \\ &+ \mathcal{O}(a^4) \qquad \text{for } r < a.\end{aligned}$$

Take $r = 0$ $\Rightarrow$ $\overline{\gamma}(0) = -28$ and $\overline{\eta}(0) = -3.6$,

level	$\psi(0)$	$\psi_{\text{eff}}(0)$	$\overline{\gamma}\int\psi_{\text{eff}}\delta_a^3 + \cdots$
1S	1.50	0.53	−3.4
2S	0.383	0.19	0.369
3S	0.1837	0.09	0.1830
4S	0.11353	0.06	0.11344
5S	0.079005	0.04	0.078986
6S	0.059031	0.03	0.059025

Operator Product Expansion (OPE)

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