How to Renormalize the Schrödinger Equation II

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Example: N-N Effective Theory

What is a Hadron?

- True theory of nuclear force is QCD.
- But e/m form factor

$$F_p(Q^2) \approx \left(\frac{1}{1+(Q/750\,\text{MeV})^2}\right)^2$$

$$\rightarrow \quad 1 \qquad \text{for } Q \ll 750\,\text{MeV}.$$

⇒ Nucleons appear point-like to probes with $p < 500\,\text{MeV}$; quarks can't be resolved.

⇒ Systematically approximate low-energy nucleon physics with nonrelativistic effective theory of point-like nucleons.

Eg) Proton NRQED with

$$\delta\mathscr{L} = \frac{\psi^\dagger e\nabla\cdot\mathbf{E}\psi}{\Lambda^2} + \cdots \quad \text{where } \Lambda \approx (750/\sqrt{2})\,\text{MeV} \approx 500\,\text{MeV}.$$

Eg) $^1S_0$ Scattering

$$V(r) = -\alpha_\pi v_\Lambda(r) + c\,\frac{\delta^3_{1/\Lambda}(\mathbf{r})}{\Lambda^2} - d\,\frac{\nabla^2\delta^3_{1/\Lambda}(\mathbf{r})}{\Lambda^4},$$

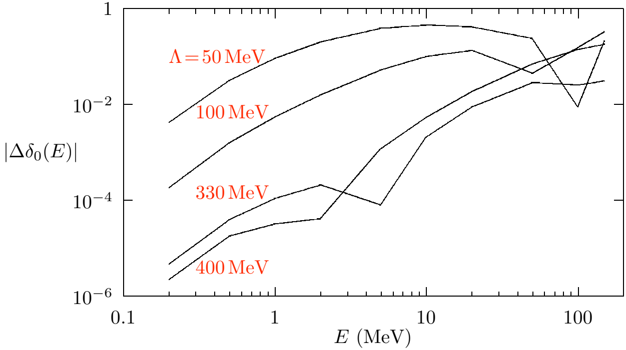
Long-range potential, due to π exchange:

$$\rightarrow \frac{\alpha_\pi}{r}\,\mathrm{e}^{-m_\pi r} \quad \text{for large } r.$$

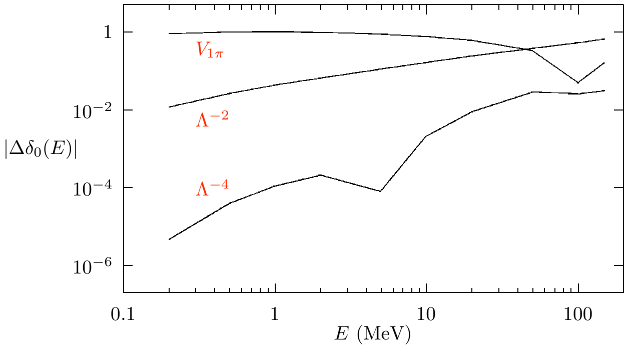
Experiment $\Rightarrow$ $\alpha_\pi \approx 0.07$ — small because of chiral symmetry.

Contact interactions — dominant!

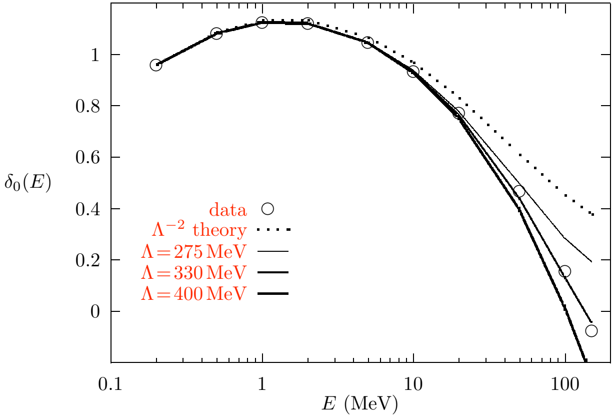
Lambda Dependence



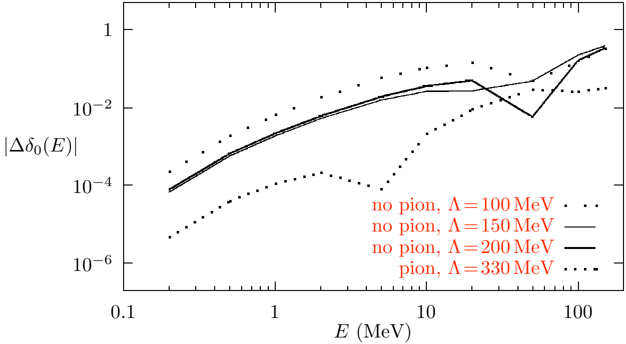
$1/\Lambda^2$ vs $1/\Lambda^4$ Theories



$^1S_0$ Phase Shift



With and Without Pions

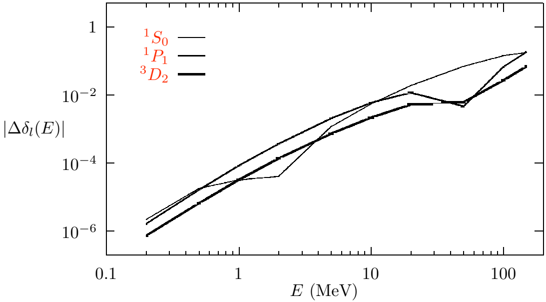


S, P and D Wave Phase Shifts

$$S: \qquad V_{\pi,S} + \frac{\delta^3_{1/\Lambda}}{\Lambda^2} + \frac{\nabla^2 \delta^3_{1/\Lambda}}{\Lambda^4} + \mathcal{O}(1/\Lambda^6)$$

$$P: \qquad V_{\pi,P} + \frac{\nabla \delta^3_{1/\Lambda} \nabla}{\Lambda^4} + \mathcal{O}(1/\Lambda^6)$$

$$D: \qquad V_{\pi,D} + \mathcal{O}(1/\Lambda^6)$$



Perturbative Analysis

5. Perturbative Analysis

QED, QCD . . . $\Rightarrow$ complex, nonperturbative low-p physics
+ simple, perturbative high-p physics.

Nonperturbative solution (eg, numerical) hard $\Rightarrow$ don't want to use it for the much simpler high-p sector.

Strategy

Introduce cutoff that separates high from low momenta:

$$H \to H_{\text{eff}}(a, c(a), d(a) \ldots)$$

Solve in two steps:

1) Compute couplings $c(a)$, $d(a) \ldots$ using perturbation theory (easy!) since c, $d \ldots$ contain only $p > 1/a$ physics.

Solves $p > 1/a$ part.

2) Analyze H_{eff} nonperturbatively — much easier since only $p < 1/a$ modes left (ie, fewer degrees of freedom).

Solves $p < 1/a$ part.

A Simple Example

Exact theory:

$$H = \frac{\mathbf{p}^2}{2m} - \frac{\alpha}{r} \qquad \text{with} \quad \begin{cases} \alpha = 0.01 \\ m = 100. \end{cases}$$

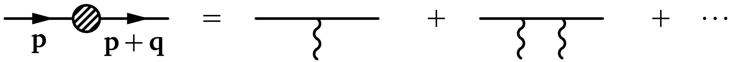
Effective theory:

$$H_{\text{eff}} = \frac{\mathbf{p}^2}{2m} - \frac{\alpha}{r}\,\text{erf}(r/\sqrt{2}\,a) - 2\pi\alpha\, c\, a^2\, \delta_a^3(\mathbf{r}).$$

Challenge: Compute c using perturbation theory.

Recipe (“Perturbative Matching”)

Compute scattering amplitude



in perturbation theory (Born Series):

$$
\begin{aligned}
H &\longrightarrow T(\mathbf{p},\mathbf{q}) = \sum_n \alpha^n T^{(n)} \\
H_{\text{eff}} &\longrightarrow T_{\text{eff}}(\mathbf{p},\mathbf{q},c,a)
\end{aligned}
$$

Require

$$
\boxed{T(\mathbf{p},\mathbf{q}) = T_{\text{eff}}(\mathbf{p},\mathbf{q},c,a) \times \left(1+\mathcal{O}(pa)^4\right)}
$$

$\Rightarrow$ solve for c, in limit $pa \to 0$, order-by-order in α.

Note:

α-expansion of T (and $T_{\rm eff}$) diverges as $\mathbf{p},\mathbf{q} \to 0$ due to IR Coulomb threshold singularity.

But c has convergent expansion — IR parts of T and $T_{\rm eff}$ cancel in calculation of c since sensitive only to UV part of theory.

(Crucial for QCD applications — eg, lattice QCD.)

Time Dependent Perturbation Theory

Formally, scattering amplitude from

$$T = V + V \frac{1}{E - H_0 + i\epsilon} V + V \frac{1}{E - H_0 + i\epsilon} V \frac{1}{E - H_0 + i\epsilon} V + \cdots$$

In momentum representation,

$$T(E, \mathbf{p} \to \mathbf{p} + \mathbf{q}) = V(\mathbf{q}) + \int \frac{d^3k}{(2\pi)^3} V(\mathbf{p} + \mathbf{q}, \mathbf{k}) \frac{1}{E - \mathbf{k}^2/2m + i\epsilon} V(\mathbf{k}, \mathbf{p}) + \cdots$$

Lowest Order ("Tree Level")

$$T^{(1)}(\mathbf{q}) = -\frac{4\pi\alpha}{q^2}$$

$$T^{(1)}_{\text{eff}}(\mathbf{q}) = -\frac{4\pi\alpha}{q^2}\, e^{-q^2a^2/2}\,(1 + c\, q^2a^2/2)$$

$-\alpha\, \mathrm{erf}(\ldots)/r$ $\qquad$ $-2\pi c\alpha a^2 \delta^3_a(r)$

$$= -\frac{4\pi\alpha}{q^2}\,\left(1 + (c-1)\, q^2a^2/2 + \mathcal{O}(q^4a^4)\right)$$

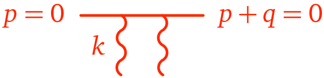
$$T^{(1)} = T^{(1)}_{\text{eff}} \quad \Rightarrow \quad c = 1$$

$$1S \text{ error } = \begin{cases} 0.04\% & c = 0 \\ 0.0004\% & c = 1 \end{cases}$$

$$\text{for } a = 0.01.$$

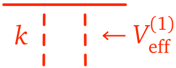
Order α^2 Matching

$$T^{(2)} = \int \frac{d^3k}{(2\pi)^3}\,\frac{-4\pi\alpha}{k^2}\,\frac{-2m}{k^2}\,\frac{-4\pi\alpha}{k^2}$$



$$T^{(2)}_{\text{eff}} = -\,2\pi\alpha^2 a^2 c^{(1)} \qquad \longleftarrow \mathcal{O}(\alpha^2) \text{ part of } \delta^3_a \text{ term}$$

$$+ \int \frac{d^3k}{(2\pi)^3}\left[\frac{-4\pi\alpha}{k^2}\,\frac{-2m}{k^2}\,\frac{-4\pi\alpha}{k^2}\right] \mathrm{e}^{-k^2a^2}\,\left(1 + k^2a^2/2\right)^2$$



Set $T^{(2)} = T^{(2)}_{\text{eff}}$ and solve for $c^{(1)}$:

$$c^{(1)} = \frac{1}{-2\pi\alpha^2 a^2} \left[\quad \text{(diagram)} \quad - \quad \text{(diagram)} \quad \right]_{p,q\to 0}$$

- Amplitudes cancel for $k \ll 1/a$, but not for $k > 1/a$.
- Removes wrong $k > 1/a$ behavior (H_{eff}) and replaces it with correct behavior (H).
- Universal since p, q negligible relative to $k > 1/a$.

$$\Rightarrow \quad c = 1 + \alpha\, am \frac{5}{3\sqrt{\pi}} + \mathscr{O}(\alpha^2)$$

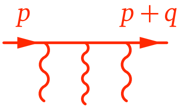
$$1S \text{ error } = \begin{cases} 4 \times 10^{-2}\,\% & c = 0 \\ 4 \times 10^{-4}\,\% & c = 1 \\ 8 \times 10^{-7}\,\% & c = 1.0094 \end{cases}$$

for $a = 0.01$.

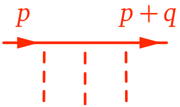
More Precision: $\mathcal{O}(\alpha^3)$

$$c = 1 + \alpha c^{(1)} + \alpha^2 c^{(2)} + \cdots$$

$$c^{(2)} = \frac{1}{-2\pi\alpha^2 a^2} \Bigg[$$



$$-$$

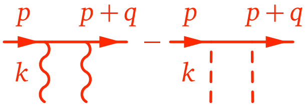


$$\Bigg|_{p,q\to 0}$$

$$= \frac{(am)^2}{2\pi}\left(\frac{5\sqrt{3}}{72} + \frac{1633}{384}\right)$$

More Precision: $\mathcal{O}(a^4)$

Taylor expand in powers of $p, q \ll 1/a, k$:



$$
\begin{aligned}
= \quad & A_0^{(S)} && \longleftarrow \delta_a^3 \\
& + A_2^{(S)}\, a^2\, \mathbf{q}^2 && \longleftarrow \nabla^2 \delta_a^3 \\
& + A_2^{(P)}\, a^2\, \mathbf{p}\cdot(\mathbf{p}+\mathbf{q}) && \longleftarrow \nabla \cdot \delta_a^3 \nabla \\
& + \mathcal{O}(a^4 q^4, a^4 p^4)
\end{aligned}
$$

Add $\delta H = (\pi\alpha/2)\, d\, a^4 \nabla^2 \delta_a^3(\mathbf{r})$ to H_{eff}:

$\Rightarrow\ T = T_{\text{eff}}\left(1 + \mathcal{O}((pa)^6)\right)$

$\Rightarrow$ Solve for $d = 1 + \alpha\, d^{(1)} + \cdots$

from $A_2^{(S)}$.

Note:

$d \neq 0 \quad \Rightarrow \quad$ shift in c:

$$c = 1 + \frac{\alpha\, am}{\sqrt{\pi}} \left[\frac{5}{3} - \frac{5}{4}\, d - \frac{3}{64}\, d^2 \right] + \cdots$$

"Operator Mixing"

Operator Mixing

Quantum fluctuations (ie, loops) generate $\alpha m\,\delta_a^3(r)/a$ piece in $\nabla^2\delta_a^3(r)$.

Naive expectation:

$$a^2\,\langle\nabla^2\delta_a^3\rangle = \mathcal{O}((pa)^2\,\langle\delta_a^3\rangle)$$

Actual calculation:

$$\begin{aligned} a^2\,\langle\nabla^2\delta_a^3\rangle &= \mathcal{O}(\alpha\, am\,\langle\delta_a^3\rangle) \\ &= \mathcal{O}((pa)^1\,\langle\delta_a^3\rangle) \end{aligned}$$

Much larger.

Naive expectation is more relevant:

Eg) Remove $\mathscr{O}(a^4)$ from H_{eff} by adding

$$\delta H = \frac{\pi\alpha}{2} a^4 d \left[\nabla^2 \delta_a^3(r) + \frac{\alpha m}{a\sqrt{\pi}} \left(5 + \frac{3d}{16}\right) \delta_a^3(r) \right]$$

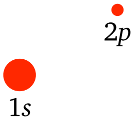
$$\equiv \frac{\pi\alpha}{2} a^4 d \left[\nabla^2 \delta_a^3(r) \right]_{\text{ren.}} \qquad \longleftarrow \text{ context specific.}$$

$$\Rightarrow \qquad a^2 \left[\nabla^2 \delta_a^3 \right]_{\text{ren.}} = \mathscr{O}((pa)^2 \, \delta_a^3) \qquad \longleftarrow \text{ naive expectation.}$$

Example: Rigorous Pseudopotentials

6. Rigorous Pseudopotentials

Interesting He state for QED is $1s2p$:



$\Rightarrow \; r_{2s} \approx 8 r_{1s}$.

$\Rightarrow$ Replace

$$2e + 1 \text{ nucleus} \longrightarrow 1e + 1 \text{ “fat nucleus”}$$

Mimic $1s$ + nucleus core state
by “pseudopotential”.

$$H_{\text{eff}} = \frac{\mathbf{p}^2}{2m} - \left[\frac{\alpha}{r}\right]^{(\Lambda)}$$ ⟵ Screened Coulomb.

$$+ \frac{c}{\Lambda^2}\,\mathrm{e}^{-r^2\Lambda^2/2}$$ ⟵ Mimics finite core radius.

$$+ \frac{d}{\Lambda^4}\,\nabla^2\mathrm{e}^{-r^2\Lambda^2/2}$$

$$+ \frac{f}{\Lambda^5}\left[\int \mathrm{e}^{-r^2\Lambda^2/2}|\psi|^2\right]\mathrm{e}^{-r^2\Lambda^2/2}$$ ⟵ Core reaction to valence e.

(Nonlinear Schröd. eq'n.)

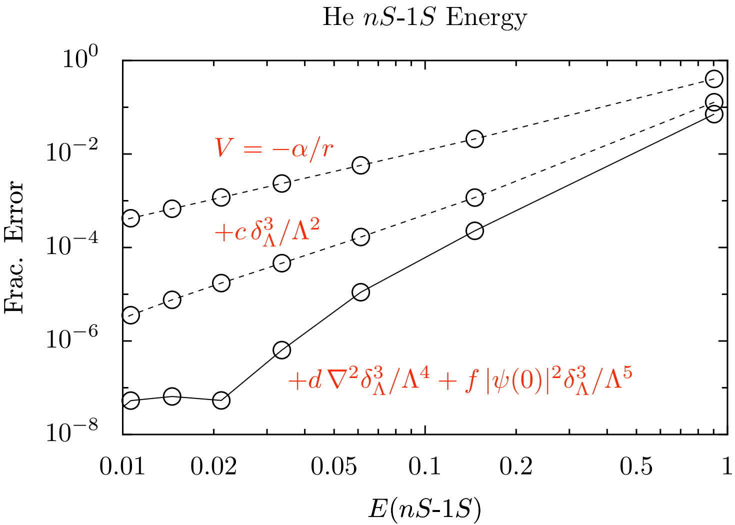
$$+ \cdots + \frac{\alpha}{r}\sum_{n=3}^{\infty}\frac{c_n}{(\Lambda r)^n} + \cdots$$ ⟵ Core pol'n (neglect).

- $c, d, f \ldots$ computed from properties of core state (easy in He), or fit using valence spectrum measurements (easy for Cs).

- Easy to generalize potential to include QED/relativity.

$\Rightarrow$ He problem becomes 1-e problem $\Rightarrow$ trivial!

He nS-$1S$ Energies



Example: μ Decay in Materials

7. Muon Decay

- Measure μ lifetime $\Gamma_\mu \Rightarrow$ μs stopped in matter.
- Significant number bind to form muonium (μe).
- Effect of binding? Effect of interactions with matter?

A. Czarnecki, G.P. Lepage, and W.J. Marciano, hep-ph/9908439 and Phys. Rev.

Simple estimates

- Binding energy reduces phase space
  $\Rightarrow$ correction of $\mathcal{O}(\alpha^2 m_e/m_\mu)$
  $\Rightarrow$ Important!

- Final state interactions
  $\Rightarrow$ $\mathcal{O}(\langle V\rangle/m_\mu) = \mathcal{O}(\alpha^2 m_e/m_\mu)$.
  $\Rightarrow$ Important!

- Except these cancel (gauge invariance).
  $\Rightarrow$ Dominant contribution is $\mathcal{O}(\alpha^2 m_e^2/m_\mu^2)$.
  $\Rightarrow$ Unimportant!

NRQED for μ

- Typical $p_\mu \sim \alpha m_e$ for stopped μ in matter.
  $\Rightarrow$ Use NRQED to describe it.

- $\mu \to e\nu\overline{\nu} \to \mu \;\Rightarrow\;$ distances $\sim 1/m_\mu$
  $\Rightarrow$ Include in NRQED as local (non-unitary) terms.

$$\begin{aligned}
\mathscr{L}_{\text{nrqed}} =& \psi_\mu^\dagger \Bigg\{ iD_t + \frac{\mathbf{D}^2}{2m_\mu} + \cdots \\
& + \frac{i\Gamma_\mu}{2}\Bigg(1 + c_1 \frac{\mathbf{D}^2}{2m_\mu^2} + c_2 \frac{3\mathbf{D}^4}{8m_\mu^4} + \cdots && \leftarrow \mu \to e\nu\overline{\nu} \\
& + d_1 \frac{\psi_e^\dagger \psi_e}{m_\mu^3} + d_2 \frac{\sigma\cdot\psi_e^\dagger \sigma \psi_e}{m_\mu^3} + \cdots && \leftarrow \mu e \to \nu\overline{\nu} \\
& + f_1 \frac{e\sigma\cdot\mathbf{B}}{m_\mu^2} + f_2 \frac{e\nabla\cdot\mathbf{E}}{m_\mu^3} + \cdots \Bigg)\Bigg\}\psi_\mu .
\end{aligned}$$

Note:

- No term $i\Gamma_\mu \psi_\mu^\dagger A^0 \psi_\mu / m_\mu$ (not gauge invariant) $\Rightarrow$ No $\alpha^2 m_e/m_\mu$. (Compare $J = 0$ "photon.")

Q. Why no $i\Gamma_\mu \psi_\mu^\dagger i D_t \psi_\mu / m_\mu$?

A. In $\mathscr{L}_{\text{nrqed}}$

$$\psi_\mu^\dagger i D_t \psi_\mu \equiv -\psi_\mu^\dagger \frac{\mathbf{D}^2}{2m} \psi_\mu + \cdots$$

because "equations of motion" are

$$i D_t \psi_\mu = -\frac{\mathbf{D}^2}{2m} \psi_\mu + \cdots .$$

N.B. "Equivalent" not "equal." Prove using field transformation in path integral (change integration variables) $\Rightarrow$ "redundant operators".

- Muon decay from

$$\delta\mathscr{L}_{\text{decay}} \equiv \frac{i\Gamma_\mu}{2}\,\psi_\mu^\dagger\psi_\mu + \mathscr{O}((\alpha m_e/m_\mu)^2\,\Gamma_\mu)$$

Not renormalized; free-μ decay rate in rest frame..

Conserved current; μ number operator.

$\Rightarrow$ Decay rate for *any* state $|\mu\phi\rangle$

$$\langle\mu\phi|\delta\mathscr{L}_{\text{decay}}|\mu\phi\rangle = \frac{i\Gamma_\mu}{2} + \mathscr{O}((\alpha m_e/m_\mu)^2\,\Gamma_\mu)$$

- Here ϕ is e in muonium, conduction band in metal ... or any other single/multi-electron state in ordinary matter.

$\Rightarrow$ Decay rate of μ unaffected by all ordinary materials at ppb level.