How to Renormalize the Schrödinger Equation III

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Example: QED in Atoms — Ps, H, He . . .

8. QED Bound States

Top 10 Reasons
to Work on
Something Else

- Bound state = nonperturbative + multi-scale.

$K \sim mv^2$, $P \sim mv$, m where $v \approx \alpha$

Eg) How big is  ?

$$= \alpha^3 m f(K/m, P/m)$$

$$= \alpha^3 m \left(\# + \# \alpha + \# \alpha^2 + \cdots \right)$$

$(= \alpha^5 m (\# + \cdots)$ in Coulomb gauge!$)$

- QED renormalization $\Rightarrow$ perturbation theory. (Feynman, Schwinger, Tomonaga . . .)

$\Rightarrow$ Bound state must be analyzed perturbatively:

Eg) $H = H_0 + \delta H_{\text{QED}}$ with $H_0 = \frac{p^2}{2m} - \frac{\alpha}{r}$

$$\Rightarrow E_0 = \frac{-\alpha^2 m}{2} \quad \psi_0 = \frac{e^{-r\alpha m}}{\sqrt{\pi}} \quad \delta H_{\text{QED}} = \text{Feyn. Diags.}$$

$+$ Rayleigh-Schrödinger Perturbation Th.

$\Rightarrow$ Big problem for He, Li . . . (no exact ψ_0, E_0!).

- α expansion of E's, Γ's not so convergent.

Eg)

$$\Gamma_{\text{O-Ps}} = \Gamma\ \left(1 - 3\alpha + 4\alpha^2 - 20\alpha^3 + \cdots\right)$$

$$\frac{3}{2\pi}\ln^2\alpha + 0.7\ln\alpha$$

$$\alpha^4\ln^3\alpha$$

$\rightarrow$ Due to multi-scales:
$\ln^2(K/m)\ldots$

QED renormalization

$+$

gauge non-invariance

$+$

relativity

$+$

nonperturbative bound state

$\vdots$

$\Rightarrow$ Too Hard!

Don’t solve bound state problem directly in relativistic QFT.

Effective Theory: Nonrelativistic QED System

- $p_e \sim m_e v \ll m_e$
  - $\Rightarrow$ Pair production highly suppressed.
  - $\Rightarrow$ e number doesn't fluctuate.
  - $\Rightarrow$ Don't need QFT for e.

- $p_\gamma \sim p_e \Rightarrow E_\gamma$ much larger:

$$(E_e \sim p_e^2/m_e \sim m_e v^2) \ll (E_\gamma \sim p_\gamma \sim m_e v)$$

$$(\Rightarrow \quad \text{photons} \approx \text{instantaneous})$$

$\Rightarrow$ Remove photons from theory with energy cutoff

$$\boxed{E_e \ll \Lambda_E < E_\gamma}$$

$\Rightarrow$ Don't need QFT for γ.

- γ interactions $\rightarrow$ renormalization correction terms.
- Cutoff in E (not p)
  - $\Rightarrow$ Corrections local in t but nonlocal in $\mathbf{r}$.
  - $\Rightarrow$ γ interaction $\rightarrow$ instantaneous potentials $V(\mathbf{r})$, $\sigma \cdot pV'(\mathbf{r})\ldots$.

$\Rightarrow$ QED $\rightarrow$ nonrelativistic Schrödinger theory:

$$H = \sum_i \frac{p_i^2}{2m_i} + V(\mathbf{r}_j, \mathbf{p}_j \ldots) + \delta H(\text{v. soft } \gamma\text{s})$$

$p_\gamma \sim mv^2$ — couple weakly

$\Rightarrow$ E-dependent δV (eg, Lamb shift)

Eg) Nonperturbative Positronium

Calculate O-Ps decay rate in two steps.

1) Define Hamiltonian ($\equiv$ QED) with finite cutoff Λ built in.

$$H^{(\Lambda)} = \frac{\mathbf{p}^2}{m} - \frac{\mathbf{p}^4}{4m^3} + V + iW$$

Decay → iW

Define V, W by matching $T = V + iW + V(E - H_0)^{-1}V + \cdots$ to QED Feynman diagrams for $e\overline{e}$ scattering, order-by-order in α and p/Λ.

R. Hill and G.P. Lepage, Phys Rev D62, 111301,2000; hep-ph/000327.

$$\langle \mathbf{l}|V|\mathbf{k}\rangle = -\frac{4\pi\alpha}{|\mathbf{k}-\mathbf{l}|^2}\,\mathrm{e}^{-|\mathbf{k}-\mathbf{l}|^2/2\Lambda^2} \quad \leftarrow \mathcal{O}(\alpha^2 m)$$

$$\mathcal{O}(\alpha^4 m) \rightarrow \quad + \left[\frac{\pi\alpha}{m^2}\frac{(l^2-k^2)^2}{|\mathbf{k}-\mathbf{l}|^4} - \frac{4\pi\alpha}{2\Lambda^2} + \cdots\right]\mathrm{e}^{-|\mathbf{k}-\mathbf{l}|^2/2\Lambda^2}$$

$$\mathcal{O}(\alpha^5 m) \rightarrow \quad + \frac{8\pi\alpha}{3\pi m^2}\,\mathrm{e}^{-|\mathbf{k}-\mathbf{l}|^2/2\Lambda^2}\langle \mathbf{p}\cdot(H-E)\ln(\Lambda/(H-E))\cdot\mathbf{p}\rangle + \cdots$$

(Lamb Shift)

$$+ \frac{\alpha^2}{m^2}\left[\frac{14}{3}\ln(|\mathbf{l}-\mathbf{k}|/m) - \frac{74}{15} - \frac{16}{3}\ln 2 + D\right]\mathrm{e}^{-|\mathbf{k}-\mathbf{l}|^2/2\Lambda^2}$$

Match $e\bar{e} \rightarrow e\bar{e}$

$$D = -\sqrt{\pi}\left[\frac{-121}{36}\frac{\Lambda}{m} - 9\frac{m}{\Lambda} + \frac{5}{3}\left(\frac{m}{\Lambda}\right)^2\right] - \frac{16}{3}\ln\frac{\Lambda}{m}$$

Decay piece:

$$\langle \mathbf{l}|W|\mathbf{k}\rangle = \left[A + B\,\frac{|\mathbf{k}-\mathbf{l}|^2}{m^2}\right]\, \mathrm{e}^{-|\mathbf{k}-\mathbf{l}|^2/2\Lambda^2}$$

$A^{(0)}(1+\alpha A^{(1)}+\cdots)$ from matching Born series for T with QED.

Not real QED behavior but equivalent for low-energy e's, provided A, B correct.

$$\Rightarrow \qquad W(\mathbf{r}) \propto \left[A - B\,\frac{\nabla^2}{m^2}\right]\, \delta^3_{1/\Lambda}(\mathbf{r})$$

$$
\begin{aligned}
A^{(1)} &= a_0 + \frac{1}{\sqrt{\pi}}\left[\frac{4}{3}\left(\frac{\Lambda}{m}\right) + 3\left(\frac{\Lambda}{m}\right)^{-1}\right] + \mathcal{O}\left(\frac{\lambda}{m}\right), \\
A^{(2)} &= b_0 - 2a_1 + \frac{1}{\sqrt{\pi}}\left[\frac{4}{3}\left(\frac{\Lambda}{m}\right) + 3\left(\frac{\Lambda}{m}\right)^{-1}\right] a_0 + \frac{1}{3}\ln\frac{\Lambda}{m} \\
&\quad + \frac{1}{\pi\sqrt{\pi}}\left\{\left[-\frac{44\sqrt{6}}{81}\left(\gamma - \ln\frac{2\Lambda^2}{3m^2} - 2\right)\right]\left(\frac{\Lambda}{m}\right)^3\right. \\
&\quad \left. + \left[\frac{7}{3}\ln\frac{\Lambda}{m} + \frac{56\sqrt{6}}{27}\left(\gamma - \ln\frac{2\Lambda^2}{3m^2} - \frac{2}{7}\right) - \frac{37}{15} + \frac{1}{3}\ln 2 - \frac{7}{6}\gamma\right]\left(\frac{\Lambda}{m}\right)\right\} \\
&\quad + \left(\frac{83}{24\pi} - \frac{11\sqrt{3}}{12\pi} + \frac{11}{48}\right)\left(\frac{\Lambda}{m}\right)^2 \\
&\quad + \left(\frac{25}{2\pi} - \frac{4\sqrt{3}}{3\pi} + \frac{17}{18} - \frac{5}{6}\ln 2 - \frac{1}{3}\gamma + \frac{2}{\sqrt{\pi}}\kappa\right) \\
&\quad + \left(\frac{49}{6\pi} - \frac{3\sqrt{3}}{2\pi} - \frac{1}{4}\right)\left(\frac{\Lambda}{m}\right)^{-2} + \mathcal{O}\left(\frac{\lambda}{m}\right).
\end{aligned}
$$

Note:

- Given V, W can solve theory even if completely ignorant of QED, renormalization, Effective Field Theory.
  - $\Lambda = m$ (or $m/2$ or $2m$)
    $\Rightarrow$No divergences (V analytic at $r = 0$).
  - Renormalization built in, automatic.
  - High-order QED/relativity built in, automatic.

- Can solve nonperturbatively in V (eg, numerically).
  - $\Rightarrow$ Don't need Rayleigh-Schrödinger perturbation theory.
  - $\Rightarrow$ Move trivially to many-e analysis (He...).

- No $\ln\alpha$s in V $\Rightarrow$ perturbation theory for V is more convergent than for E_n.
  - $\Rightarrow$ Compute V in (QED) perturbation theory; solve nonperturbatively in V.
  - $\Rightarrow$ Schrödinger equation generates $\ln\alpha$s and resums them automatically.

2) Solve theory:

a) Diagonalize $H^{((\Lambda))}$ (eg, on finite basis set of Gaussians).
$\Rightarrow E_n$s and $|\psi_n\rangle$s.

b) Compute

$$\begin{aligned}\Gamma_n = & -2\langle\psi_n|W|\psi_n\rangle \\ & + \overline{A}\,\langle\psi_n|\mathrm{e}^{-r^2\Lambda^2/2}|\psi_n\rangle \\ & - \overline{B}\,\langle\psi_n|\nabla^2\mathrm{e}^{-r^2\Lambda^2/2}|\psi_n\rangle \\ & + \cdots .\end{aligned}$$

c) Publish numerical values obtained for Γ_ns.
$\Rightarrow$ ($\Gamma_{1S} = 7.039967(10)\,\mu s^{-1}$.)

QED in He

Traditional approach:

- Brute force numerical diagonalization of

$$H_0 = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} - \frac{2\alpha}{r_1} - \frac{2\alpha}{r_2} + \frac{\alpha}{|r_1 - r^2|}$$

  $\Rightarrow E_n^0$ to 14–17 digits; ψ_n^0 to ? digits.

- Add QED + relativity using Rayleigh-Schrödinger P.Th. and numerical ψ_n^0s:

$$\delta E_n = \langle \psi_n^0 | \frac{-p_1^4}{8m^3} | \psi_n^0 \rangle + \cdots$$

Problems:

- Hard! Need expert in brute-force numerical analysis and QED.

- Numerical accuracy of ψ_n^0 uncertain and well explored by highly singular QED/relativity corrections.

Solution:

- QED+relativity → potential $V_{\text{eff}}(r\ldots)$ (from Ps, H analyses).
- Replace V_{coul} in brute-force evaluation by V_{eff}.

⇒ Brute force numerics yields E_n that includes QED to 14-17 digits. (⇒ finished; no Step 2!)

Numerical Analysis Bonus

- $-\alpha/r \to \infty$ as $r \to 0$.
  - $\Rightarrow$ Cusp in $\psi(r)$ at $r = 0$.
  - $\Rightarrow$ Expansions (eg, on basis set) converge more slowly.

- Gaussian cutoff in $V_{\mathrm{eff}}(r) \;\Rightarrow\;$ analytic at $r = 0$.
  - $\Rightarrow$ $\psi(r = 0)$ analytic.
  - $\Rightarrow$ Expansions converge exponentially faster.

Eg) Lamb Shift in H, He

- $N = \#$ basis functions $< \infty$
  $\Rightarrow$ effective cutoff Λ_N.

- Lamb shift

$\psi^\dagger$  $\psi \quad \sim \quad \psi^\dagger \delta^3(r) \psi$

$$\sim \left| \int^{\Lambda_N} d^3k \, \psi(k) \right|^2 \qquad \text{where } \psi(k) \to \frac{1}{k^4} \text{ for } k \text{ large}$$

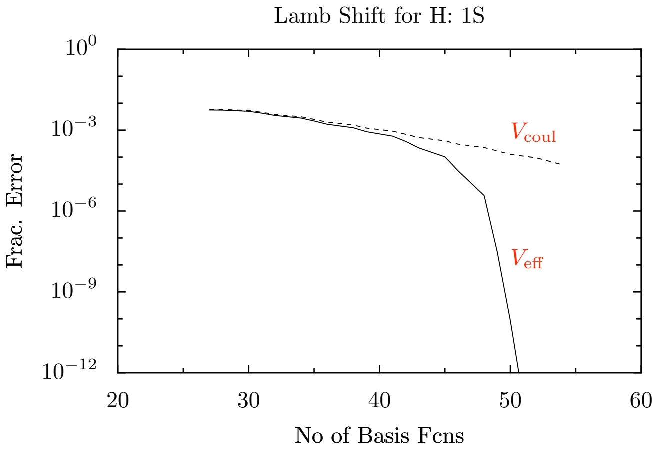
$$\Rightarrow \boxed{\text{error} \propto 1/\Lambda_N^2}$$

- $\psi_{\rm eff} \sim e^{-k^2/2\Lambda^2}$ for k large

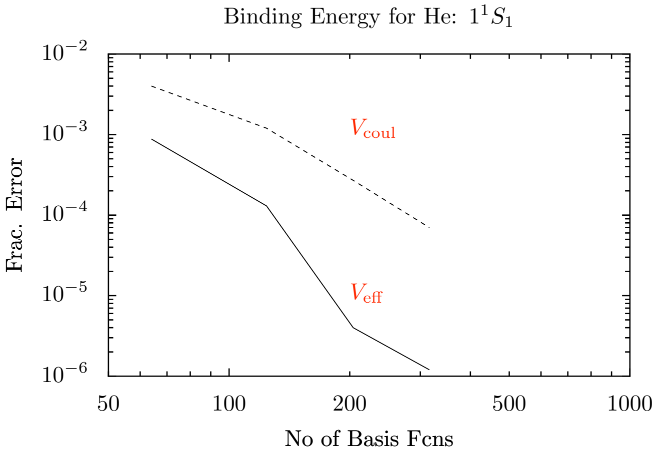
$$\Rightarrow \boxed{\text{error} \propto e^{-\Lambda_N^2/\Lambda^2}}$$

$\Rightarrow$ Errors exponentially suppressed.

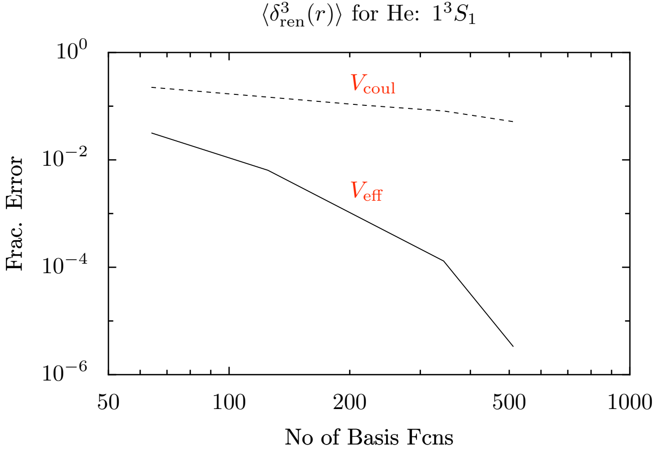
H(1S): Lamb Shift



He(1^1S): Binding Energy



He$(1^1 S)$: QED/Relativistic Corrections



Note:

- Computing $\langle\psi_{\rm coul}|\delta^3(r_1)|\psi_{\rm coul}\rangle$ for He using $\psi_{\rm eff}$.
- $\psi_{\rm eff}(r_1 \approx 0, r_2)$ totally different from $\psi_{\rm coul}(r_1 \approx 0, r_2)$.
- Renormalization $\Rightarrow$ c,d such that

$$\langle\psi_{\rm coul}|\delta^3(r_1)|\psi_{\rm coul}\rangle = \langle\psi_{\rm eff}|(c - d\nabla^2/\Lambda^2)\,\delta^3_{1/\Lambda}(r_1)|\psi_{\rm eff}\rangle$$

(N.B. c, d from H analysis — simple!)

All-Orders Derivation: Short-Range Interactions

9. Short-Range Interactions

All interactions short-range $\Rightarrow$ model by

$$\mathscr{L}^{(\Lambda)} = \psi^\dagger \left(i\partial_t + \frac{\nabla^2}{2m} \right) \psi$$
$$- \frac{g}{\Lambda m} \left(\psi^\dagger \psi\right)^2 - \frac{g\,d}{\Lambda^3 m} \{\psi^\dagger \nabla^2 \psi, \psi^\dagger \psi\} - \frac{g_p}{\Lambda^3 m} \psi^\dagger \nabla \psi \cdot \psi^\dagger \nabla \psi - \cdots$$
$$- \frac{h}{\Lambda^4 m} \left(\psi^\dagger \psi\right)^3 - \cdots$$

(Why not g/Λ^2?)

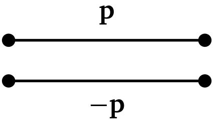
For example:

- BEC atoms
- N-N interactions for $p < 100\,\text{MeV}$.

  ...

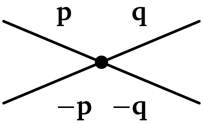
Perturbation Theory: 2-Body

Leading order in $1/\Lambda$ (c-of-m frame):



$$\frac{1}{E - \frac{p^2}{2m} - \frac{p^2}{2m} + i\epsilon} = \frac{-m}{p^2 + \gamma^2}$$

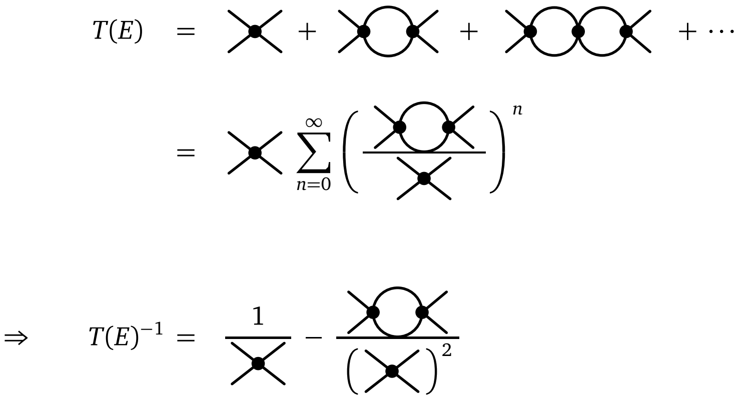
$\gamma^2 \equiv -mE - i\epsilon$



$$\frac{4\pi^2 g}{\Lambda m}\,\theta(p < \Lambda)\theta(q < \Lambda)$$

UV cutoff *separable!*

All Orders in g



$$-\frac{m}{4\pi^2}\int_0^{\Lambda} k^2 dk\, \frac{1}{k^2+\gamma^2}$$

IR vs UV

$$\int_0^\Lambda k^2 dk \, \frac{1}{k^2+\gamma^2} = \int_0^\Lambda dk \, \frac{(k^2+\gamma^2)-\gamma^2}{k^2+\gamma^2} = \Lambda - \gamma^2 \int_0^\Lambda \frac{dk}{k^2+\gamma^2}$$

$$= \Lambda - \gamma^2 \int_0^\infty \frac{dk}{k^2+\gamma^2} + \gamma^2 \int_\Lambda^\infty \frac{dk}{k^2+\gamma^2}$$

IR term $\Rightarrow$

$$-\frac{\gamma\pi}{2}$$

$\Rightarrow$ odd power of γ.

UV term: $\gamma \ll \Lambda \;\Rightarrow$

$$\gamma^2 \int_\Lambda^\infty \frac{dk}{k^2} \sum_{n=0}^{\infty} \left(\frac{-\gamma^2}{k^2} \right)^n$$

$$= \Lambda \left[\frac{\gamma^2}{\Lambda^2} - \frac{\gamma^4}{3\Lambda^4} + \cdots \right]$$

$\Rightarrow$ even powers of γ^2/Λ^2.

Key Theorem (for later)

Prove by induction:

$$\int_0^\Lambda k^2 dk \, \frac{1}{k^2+\gamma^2} \left(\frac{k^2}{\Lambda^2}\right)^n = \Lambda \left[c_0 + c_1 \frac{\gamma^2}{\Lambda^2} + c_2 \frac{\gamma^4}{\Lambda^4} + \cdots \right]$$

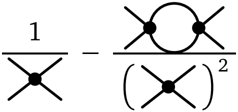
UV piece:
$\Rightarrow$ even in γ
$\Rightarrow$ analytic in E.

$$- \frac{\gamma\pi}{2} \left(\frac{-\gamma^2}{\Lambda^2}\right)^n$$

IR piece: $k^{2n} \rightarrow (-\gamma)^{2n}$
$\Rightarrow$ odd in γ
$\Rightarrow$ non-analytic.

Final result (where $\gamma = \sqrt{-mE}$):

$$T(E)^{-1} =$$



$$= \frac{\Lambda m}{4\pi^2 g} + \frac{m}{4\pi^2}\left[\Lambda\left(1 + \frac{\gamma^2}{\Lambda^2} + \cdots\right) - \frac{\gamma\pi}{2}\right]$$

$$= \frac{\Lambda m}{4\pi^2}\left(\frac{1}{g} + 1\right) - \frac{m\gamma}{8\pi} + \mathscr{O}\left(\frac{\gamma^2}{\Lambda^2}\Lambda m\right)$$

Remove cutoff dependence
by tuning $g \to g(\Lambda)$;
$\Rightarrow$ physical!

No coupling to tune away
cutoff dependence;
$\Rightarrow$ unphysical *but suppressed.*

Note:

- $T(E)^{-1} = (m/8\pi)(1/a - \gamma) + \cdots$ where

$$\frac{1}{a} = \frac{2\Lambda}{\pi}\left(\frac{1}{g} + 1\right)$$

defines physical "scattering length" a.

$\Rightarrow$ Cutoff independent physics with

$$g(\Lambda)^{-1} = \frac{\pi}{2a\Lambda} - 1$$

- $(1/g+1)>0 \;\Rightarrow\;$ single bound state with

$$\gamma_{\rm bd} \equiv \sqrt{-mE_{\rm bd}} = \frac{2\Lambda}{\pi}\left(\frac{1}{g}+1\right).$$

- Unphysical (and uninteresting) unless $\gamma_{\rm bd} \ll \Lambda$ (ie, $g \approx -1$).
  ($\Rightarrow\;\mathcal{O}(\Lambda m\,(\gamma_{\rm bd}^2/\Lambda^2))$ only suppressed by $\gamma_{\rm bd}/\Lambda$ relative to leading order.)

- "Natural" size of $\gamma_{\rm bd}$ is $\mathcal{O}(\Lambda)$; $\gamma_{\rm bd} \ll \Lambda \;\Rightarrow\;$ tuning (BEC atoms) or luck (deuteron).

All Orders in $1/\Lambda^n$ — S-Waves

Include finite-size corrections to contact term:

 $$\rightarrow \quad \frac{4\pi^2 g}{\Lambda m} f(p^2/\Lambda^2)\,\theta(p<\Lambda)\theta(q<\Lambda) f(q^2/\Lambda^2)$$

$\Rightarrow$ still separable with

$$f(p^2/\Lambda^2) \equiv 1 + \sum_{n=1}^{\infty} d_n \left(\frac{p^2}{\Lambda^2}\right)^n$$

$\Rightarrow$ tuneable couplings: $g, d_1, d_2 \ldots$

Note:

- Vertex p,q dependence:

$$f(p^2/\Lambda^2)f(q^2/\Lambda^2) = 1 + d_1\left(\frac{p^2}{\Lambda^2} + \frac{q^2}{\Lambda^2}\right) + d_1^2\left(\frac{p^2q^2}{\Lambda^4}\right) + d_2\left(\frac{p^4}{\Lambda^4} + \frac{p^4}{\Lambda^4}\right) + \cdots$$

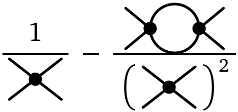
- But $p^2 = q^2 = -\gamma^2$ on energy shell $\Rightarrow$ $p^2q^2 \equiv p^4 \equiv q^4$ in correction terms for $\mathscr{L}^{(\Lambda)}$.

$$\Rightarrow \qquad (\psi^\dagger\nabla^4\psi)(\psi^\dagger\psi) \equiv (\psi^\dagger\nabla^2\psi)^2. \qquad \text{(“Redundant” operator)}$$

$\Rightarrow$ One coupling, d_n, enough to tune entire $1/\Lambda^{2n}$ correction.

Final result (where $\gamma = \sqrt{-mE}$):

$T(E)^{-1} =$



$$= \frac{1}{f(p^2/\Lambda^2)f(q^2/\Lambda^2)} \left\{ \frac{\Lambda m}{4\pi^2 g} + \frac{m}{4\pi^2} \int_0^\Lambda k^2 dk \frac{f^2(k^2/\Lambda^2)}{k^2+\gamma^2} \right\}$$

$$\frac{m}{4\pi^2} \left[\Lambda \left(c_0 + c_1 \frac{\gamma^2}{\Lambda^2} + \cdots \right) - \frac{m\gamma}{8\pi} f^2(-\gamma^2/\Lambda^2) \right]$$

$$\xrightarrow{q^2,k^2 \to -\gamma^2} \frac{\Lambda m}{4\pi^2} \left(\frac{1}{g} + t_0 + t_1 \frac{\gamma^2}{\Lambda^2} + t_2 \frac{\gamma^4}{\Lambda^4} + \cdots \right) - \frac{m\gamma}{8\pi}$$

Renormalized:

$$T^{-1} = \frac{\Lambda m}{4\pi^2}\left(\frac{1}{g} + 1 + \frac{2d_1}{3} + d_1^2 + \cdots\right)$$

g removes Λ dependence.
$\Rightarrow$ Physical.
N.B. Operator mixing.

$$+ \frac{\Lambda m}{4\pi^2}\left(\frac{\gamma^2}{\Lambda^2}\right)\left(d_1^2 - 2d_1 - \frac{1}{3} + \right.$$

$$\left. +2d_1\left(\frac{1}{g} + 1 + \frac{3d_1}{2} + \frac{d_1^2}{5}\right) + \cdots\right)$$

d_1 removes Λ dependence.
$\Rightarrow$ Now physical!

$$+ \frac{\Lambda m}{4\pi^2}\left(\frac{\gamma^2}{\Lambda^2}\right)^2\left(\quad\cdots\quad\right) + \cdots$$

d_n fixes $(\gamma^2/\Lambda^2)^n$ term.
$\Rightarrow$ Physical.

$$- \frac{m\gamma}{8\pi}$$

IR contribution
$\Rightarrow$ cutoff independent
$\Rightarrow$ physical.

Note:

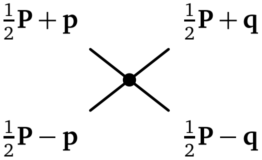
- Each d_n added and tuned $\Rightarrow$ errors reduced by γ^2/Λ^2.
  - $\Rightarrow$ If leading Λm term tuned/accidently small, unlikely $(\gamma^2/\Lambda^2)^n$ corrections have same suppression (although still relatively small).
  - $\Rightarrow$ Effective range (t_1 term) can give more reliable indicator of scale of new physics than scattering length.

- No $(\gamma^2/\Lambda^2)^n$ corrections to IR term $-m\gamma/8\pi$.
  - $\Rightarrow$ Good: no coupling constants to tune away Λ dependence.

- Exercise: Show there is still only one *physical* bound state.

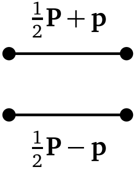
- **Solution to all orders in γ^2/Λ^2 and g!**
  - $\Rightarrow$ Universal behavior for *all* short-range potentials.
  - $\Rightarrow$ Effective range theory (Bethe, Schwinger, etc.).

Moving Frame: Galilean Invariance



$$\rightarrow \quad \frac{4\pi^2 g}{\Lambda m} f(p^2/\Lambda^2)\,\theta(p<\Lambda)\,\theta(q<\Lambda) f(q^2/\Lambda^2)$$

Gal. Inv. $\Rightarrow$ Independent of $\mathbf{P}$.



$$\rightarrow \quad \frac{1}{E-\frac{(\mathbf{P}/2+\mathbf{p})^2}{2m}-\frac{(\mathbf{P}/2-\mathbf{p})^2}{2m}} = \frac{-m}{p^2+\tilde{\gamma}^2}$$

$$\tilde{\gamma}^2 \equiv -m\left(E-\frac{P^2}{4m}\right)$$

$\Rightarrow$ $T(E)^{-1}$ same but with $\gamma \rightarrow \tilde{\gamma}$