# Instanton

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An **instanton** or **pseudoparticle** is a notion appearing in theoretical and mathematical physics. It is a classical solution to equations of motion with a finite, non-zero action, either in quantum mechanics or in quantum field theory. More precisely, it is a solution to the equations of motion of the classical field theory on a Euclidean spacetime. In such a theory, solutions to the equations of motion may be thought of as critical points of the action. The critical points of the action may be local maxima of the action, local minima, or saddle points. Instantons are important in quantum field theory because (a) they appear in the path integral as the leading quantum corrections to the classical behavior of a system, and (b) they can be used to study the tunneling behavior in various systems such as a Yang-Mills theory.

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## Instantons in Quantum Mechanics

An **instanton** can be used to calculate a certain transition probability for a quantum mechanical particle to tunnel through a region of potential energy. Maybe, the easiest example for a system with an **instanton** effect is the particle in a double-well potential. In contrast to a particle in classical mechanics, there is a non-vanishing probability for it to cross a region of potential energy higher than its own energy. One way to calculate this probability is by means of the semi-classical WKB approximation, which requires the value of $\hbar$ to be small. The Schrödinger equation for the particle reads

$$\frac{d^2\psi}{dx^2} = \frac{2m(V(x)-E)}{\hbar^2}\psi.$$

If the potential were constant, the solution would (up to proportionality) be a plane wave,

$$\psi = exp(-ikx)$$

with

$$k = \frac{\sqrt{2m(E-V)}}{\hbar}.$$

This means, that if the energy of the particle is smaller than the potential energy, one obtains an exponentially decreasing function. The associated probability for the particle to tunnel results to be

$$e^{-\frac{i}{\hbar}\int_a^b \sqrt{2m(V(x)-E)}dx},$$

where $a$ and $b$ are beginning and endpoint of the tunneling trajectory.

Alternatively, the use of path integrals allows an **instanton** interpretation and the same result can be obtained with this approach. In path integral formulation, the transition amplitude can be expressed as

$$K(a,b;t) = \langle x=a|e^{\frac{i\mathbf{H}t}{\hbar}}|x=b\rangle = \int d[x(t)]e^{\frac{iS[x(t)]}{\hbar}}.$$

Following the process of Wick rotation (analytic continuation) to Euclidean spacetime ($t \longrightarrow i\tau$), one gets

$$K_E(a,b;t) = \langle x=a|e^{\frac{-\mathbf{H}\tau}{\hbar}}|x=b\rangle = \int d[x(\tau)]e^{\frac{iS_E[x(\tau)]}{\hbar}},$$

with the Euclidean action

$$S_E = \int_{\tau_a}^{\tau_b}\left(\frac{1}{2}m\left(\frac{dx}{d\tau}\right)^2 + V(x)\right)d\tau.$$

For the potential, this means, that it gets rotated by 180 degrees, thus standing on its head, exhibiting two "hills" of maximal energy. Results obtained from the mathematically well-defined Euclidean path integral may be Wick-rotated back and give the same physical results as would be obtained by appropriate treatment of the (potentially divergent) Minkowskian path integral. As can be seen from this example, calculating the transition probability for the particle to tunnel through a classically forbidden region ($V(x)$) with the Minkowskian path integral corresponds to calculating the transition probability to tunnel through a classically allowed region (with potential $-V(X)$) in the Euclidean path integral (pictorially speaking - in the Euclidean picture - this transition corresponds to a particle rolling from one hill of a double-well potential standing on its head to the other hill). This classical solution of the Euclidean equations of motion is often named "kink solution" and is an example of an **instanton**. In this example, the two "vacua" of the double-well potential, turn into hills in the Euclideanized version of the problem. Thus, the **instanton** field solution of the (1+1)-dimensional field theory (first quantized quantum mechanical system) allows to be interpreted as a tunneling effect between the two vacuas of the physical Minkowskian system.

Note that a naive perturbation theory around one of those two vacua would never show this non-perturbative tunneling effect, dramatically changing the picture of the vacuum structure of this quantum mechanical system.

# Instantons in Quantum Field Theory

Studying Quantum Field Theory (QFT), interest in the vacuum structure of a theory may draw attention to instantons. Just as a double-well quantum mechanical system illustrates, a naive vacuum may not be the true vacuum of a field theory. Moreover, the true vacuum of a field theory may be an "overlap" of several topologically inequivalent sectors, so called "topological vacua".

A well understood and illustrative example of an **instanton** and its interpretation can be found in the context of a QFT with a non-abelian gauge group, a Yang-Mills theory. For a Yang-Mills theory these inequivalent sectors can be (in an appropriate gauge) classified by the third homotopy group of SU(2) (whose group manifold is the 3-sphere $S^3$). A certain topological vacuum (a "sector" of the true vacuum) is labelled by a topological invariant, the Pontryagin index. As the third homotopy group of $S^3$ has been found to be the set of integers,

$$\pi_3(S^3) = \mathbf{Z}$$

there are infinitely many topologically inequivalent vacua, denoted by $|N\rangle$, where $N$ is their corresponding Pontryagin index. An **instanton** is a field configuration fulfilling the classical equations of motion in Euclidean spacetime, which is interpreted as a tunneling effect between these different topological vacua. It is again labelled by a whole number, its Pontryagin index, $Q$. One can imagine an **instanton** with index $Q$ to quantify tunneling between topological vacua $|N\rangle$ and $|N+Q\rangle$. The true vacuum of the theory is labelled by an "angle" theta and is an overlap of the topological sectors:

$$|\theta\rangle = \sum_{N=-\infty}^{N=+\infty} e^{i\theta N}|N\rangle.$$

# Instantons in Yang-Mills theory

The classical Yang-Mills action on a principal bundle with structure group $G$, base $M$, connection $A$, and curvature (Yang-Mills field tensor) $F$ is

$$S_{YM} = \int_M |F|^2 \, d\mathrm{vol}_M,$$

where $d\mathrm{vol}_M$ is the volume form on $M$. If the inner product on $\mathfrak{g}$, the Lie algebra of $G$ in which $F$ takes values, is given by the Killing form on $\mathfrak{g}$, then this may be denoted as $\int_M \mathrm{Tr}(F \wedge *F)$, since

$$F \wedge *F = \langle F, F\rangle d\mathrm{vol}_M.$$

For example, in the case of the gauge group U(1), $F$ will be the electromagnetic field tensor. From the principle of stationary action, the Yang-Mills equations follow. They are

$$\mathrm{d}F = 0, \quad \mathrm{d}{*F} = 0.$$

The first of these is an identity, because $\mathrm{d}F = \mathrm{d}^2A = 0$, but the second is a second-order partial differential equation for the connection $A$. But notice how similar these equations are; they differ by a Hodge star. Thus a solution to the simpler first order (non-linear) equation

$$*F = \pm F$$

is automatically also a solution of the Yang-Mills equation. Such solutions usually exist, although their precise character depends on the dimension and topology of the base space M, the principal bundle P, and the gauge group G.

In nonabelian Yang-Mills theories, $DF = 0$ and $D * F = 0$ where D is the exterior covariant derivative. Furthermore, the Bianchi identity

$$DF = dF + A\wedge F - F\wedge A = d(dA + A\wedge A) + A\wedge(dA + A\wedge A) - (dA + A\wedge A)\wedge A = 0$$

is satisfied.

In quantum field theory, an **instanton** is a topologically nontrivial field configuration in four-dimensional Euclidean space (considered as the Wick rotation of Minkowski spacetime). Specifically, it refers to a Yang-Mills gauge field **A** which locally approaches pure gauge at spatial infinity. This means the field strength defined by **A**,

$$\mathbf{F} = d\mathbf{A} + \mathbf{A} \wedge \mathbf{A}$$

vanishes at infinity. The name *instanton* derives from the fact that these fields are localized in space and (Euclidean) time - in other words, at a specific instant.

Instantons may be easier to visualise in two dimensions than in four. In the simplest case the gauge group is U(1). In this case the field can be visualised as an arrow at each point in two-dimensional spacetime. An instanton is a configuration where, for example, the arrows point away from a central point. More complicated configurations are also possible.

The field configuration of an instanton is very different to that of the vacuum. Because of this instantons cannot be studied by using Feynman diagrams, which only include perturbative effects. Instantons are fundamentally non-perturbative.

The Yang-Mills energy is given by

$$\frac{1}{2}\int_{\mathbb{R}^4} \mathrm{Tr}[*\mathbf{F} \wedge \mathbf{F}]$$

where $*$ is the Hodge dual. If we insist that the solutions to the Yang-Mills equations have finite energy, then the curvature of the solution at infinity (taken as a limit) has to be zero. This means that the Chern-Simons invariant can be defined at the 3-space boundary. This is equivalent, via Stokes' theorem, to taking the integral

$$\int_{\mathbb{R}^4} \mathrm{Tr}[\mathbf{F} \wedge \mathbf{F}].$$

This is a homotopy invariant and it tells us which homotopy class the instanton belongs to.

Since the integral of a nonnegative integrand is always nonnegative,

$$0 \leq \frac{1}{2}\int_{\mathbb{R}^4} \mathrm{Tr}[(*\mathbf{F} + e^{-i\theta}\mathbf{F}) \wedge (\mathbf{F} + e^{i\theta} * \mathbf{F})] = \int_{\mathbb{R}^4} \mathrm{Tr}[*\mathbf{F} \wedge \mathbf{F} + 2\cos\theta \mathbf{F} \wedge \mathbf{F}]$$

for all real $\theta$. So, this means

$$\frac{1}{2}\int_{\mathbb{R}^4} \mathrm{Tr}[*\mathbf{F} \wedge \mathbf{F}] \geq \frac{1}{2}\left|\int_{\mathbb{R}^4} \mathrm{Tr}[\mathbf{F} \wedge \mathbf{F}]\right|.$$

If this bound is saturated, then the solution is a BPS state. For such states, either $*\mathbf{F} = \mathbf{F}$ or $*\mathbf{F} = -\mathbf{F}$ depending on the sign of the homotopy invariant.

Instanton effects are important in understanding the formation of condensates in the vacuum of quantum chromodynamics (QCD) and in explaining the mass of the so-called 'eta-prime particle', a Goldstone-boson which has acquired mass through the axial current anomaly of QCD. Note that there is sometimes also a corresponding soliton in a theory with one additional space dimension. Recent research on **instantons** links them to topics such as D-branes and Black holes and, of course, the vacuum structure of QCD. For example, in oriented string theories, a Dp brane is a gauge theory instanton in the world volume (p+5)-dimensional U(N) gauge theory on a stack of N D(p+4)-branes.

# References


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