Introduction to Effective Field Theory

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Key Words Effective field theory; Low-energy approximation; Power-counting

Abstract This review summarizes Effective Field Theory techniques, which are the modern theoretical tools for exploiting the existence of hierarchies of scale in a physical problem. The general theoretical framework is described, and explicitly evaluated for a simple model. Power-counting results are illustrated for a few cases of practical interest, and several applications to Quantum Electrodynamics are described.

CONTENTS

1 Introduction

It is a basic fact of life that Nature comes to us in many scales. Galaxies, planets, aardvarks, molecules, atoms and nuclei are very different sizes, and are held together with very different binding energies. Happily enough, it is another fact of life that we don't need to understand what is going on at all scales at once in order to figure out how Nature works at a particular scale. Like good musicians, good physicists know which scales are relevant for which compositions.

The mathematical framework which we use to describe nature — quantum field theory — itself shares this basic feature of Nature: it automatically limits the role which smaller distance scales can play in the description of larger objects. This property has many practical applications, since a systematic identification of how scales enter into calculations provides an important tool for analyzing systems which have two very different scales, $m \ll M$. In these systems it is usually profitable to expand quantities in the powers of the small parameter, m/M, and the earlier this is done in a calculation, the more it is simplified.

This review intends to provide a practical introduction to the technique of Effective Field Theory, which is the main modern tool for exploiting the simplifi-

cations which arise for systems which exhibit a large hierarchy of scales (1,2,3,4). The goal is to provide an overview of the theoretical framework, but with an emphasis on practical applications and concrete examples. The intended audience is assumed to be knowledgable in the basic techniques of quantum field theory, including its path-integral formulation.

Although it is not the main focus, one of the more satisfying threads which I hope you'll find running through this review is the picture which emerges of the physics underlying the technique of renormalization. Renormalization is a practice which used to be widely regarded as distasteful, and so was largely done in the privacy of one's own home. That has all changed. As used in effective field theories renormalizing is not only respectable, it is often the smart thing to do when extracting the dependence of physical quantities on large logarithms of scale ratios, $\sim \log(M/m)$.

Another attractive conceptual spinoff of effective field theory techniques is the understanding they provide of the physical interpretation of *nonrenormalizable* theories, like Einstein's General Theory of Relativity. Although much has been made about the incompatibility of gravity and quantum mechanics, we shall find the quantization of nonrenormalizable theories can make perfect sense provided they are only applied to low-energy predictions.

1.1 A toy model

In order make the discussion as concrete as possible, consider a system involving two spinless particles, l and H, with one — l — being very light compared with the other — H. Taking the classical action for the system to be the most general

renormalizable one consistent with the discrete symmetry $l \to -l$ gives[1]

$$S_c[l,H] = -\int d^4x \, \left[\frac{1}{2}\left(\partial_\mu l \, \partial^\mu l + \partial_\mu H \, \partial^\mu H\right) + V(l,H)\right], \tag{1}$$

where the interaction potential is

$$V(l,H) = \frac{1}{2}\, m^2 l^2 + \frac{1}{2}\, M^2 H^2 + \frac{g_l}{4!}\, l^4 + \frac{g_h}{4!}\, H^4 + \frac{g_{lh}}{4} l^2 H^2 + \frac{\tilde{m}}{2}\, l^2 H + \frac{\tilde{g}_h M}{3!}\, H^3. \tag{2}$$

M and m denote the two particle masses, and we imagine the three dimensionful quantities — m, $\tilde{m}$ and M — to satisfy $M \gg m, \tilde{m}$, in order to ensure a large hierarchy of scales.

Now imagine computing a low-energy physical process in this model, which we take for simplicity to be two-body $l-l$ scattering at centre-of-mass energies much smaller than the heavy-particle mass: $E_{\rm cm} \ll M$. The Feynman graphs which give rise to this scattering at tree level are given in Fig. (1).

The S-matrix element which follows from these graphs may be written:

$$S(p_1,p_2;p_3,p_4) = i(2\pi)^4 \delta^4(p_1+p_2-p_3-p_4)\, \mathcal{A}(p_1,p_2,p_3,p_4), \tag{3}$$

where

$$\begin{aligned} \mathcal{A}(p_1,p_2,p_3,p_4) &= -g_l + \tilde{m}^2 \left[\frac{1}{(p_1-p_3)^2+M^2} + \frac{1}{(p_1-p_4)^2+M^2} \right. \\ &\qquad \left. + \frac{1}{(p_1+p_2)^2+M^2}\right] \\ &\approx -g_l + \frac{3\,\tilde{m}^2}{M^2} + \frac{4\,\tilde{m}^2 m^2}{M^4} + \mathcal{O}\left(M^{-6}\right) . \end{aligned} \tag{4}$$

This last (approximate) equality assumes momenta and energies to be much smaller than M. The final line also uses the identity $(p_1-p_4)^2 + (p_1-p_3)^2 + (p_1+p_2)^2 = -4m^2$, which follows from the mass-shell condition $p^2 = -m^2$.

There are two points to be emphasized about this last expression. First, this scattering amplitude (and all others) simplifies considerably in the approximation

[1] We use units $\hbar = c = k_B = 1$ and adopt the 'mostly plus' metric signature.

that powers of m/M, $\tilde{m}/M$ and $E_{\rm cm}/M$ may be neglected. Second, the result up to $O(M^{-4})$ could be obtained for a theory involving only l particles interacting through the following 'effective' potential:

$$V_{\rm eff}^{(4)} = \frac{1}{4!}\left(g_l - \frac{3\,\tilde{m}^2}{M^2} - \frac{4\,\tilde{m}^2 m^2}{M^4}\right)\, l^4. \tag{5}$$

Even more interesting, the $O(M^{-4})$ contribution to *all* other observables involving low-energy l scattering are completely captured if the following terms are added to the above potential:

$$V_{\rm eff}^{(6)} = \frac{\tilde{m}^2}{M^4}\left(\frac{g_{lh}}{16} - \frac{g_l}{6}\right)\, l^6. \tag{6}$$

There are several reasons why it is very useful to establish at the outset that the low-energy interactions amongst light particles are described, in the large-M limit, by some sort of effective interactions like eqs. (5) and (6). Most prosaically, it is obviously much easier to calculate more complicated processes starting from eqs. (5) and (6) than by computing the full result using eqs. (1) and (2), and only *then* expanding in powers of $1/M$. For more difficult calculations — such as the cross section for the reaction $6\,l \to 12\,l$, for instance — such ease of calculation can be the difference which decides whether a computation is feasible or not. Furthermore, knowledge of the types of effective interactions which are possible at a given order in $1/M$ might guide us to identify which low-energy observables are most (or least) sensitive to the properties of the heavy particles.

Fortunately, it appears to be a basic property of quantum field theories that so long as a large hierarchy of masses exists, a low-energy description in terms of a collection of effective interactions is indeed possible. The point of this review is to sketch why this is so, what may be said about its properties, and how to compute it (if possible) from a full theory of both light and heavy particles.

2 General formulation

A great variety of observables could be used as the vehicle for illustrating effective field theory techniques, but since these can be computed from knowledge of the various correlation functions of the theory, it is convenient to phrase the discussion in terms of the generating functional for these. In particular, we focus attention on the generator, Γ, of *one-particle-irreducible* (1PI) correlation functions. Although this quantity is often called the 'effective action,' particularly in older references, we reserve this name for another quantity of more direct interest which we come to later.

2.1 The 1PI and 1LPI actions

We start by reviewing the standard definition for the generating functional. Consider a theory whose fields are generically denoted by ϕ. Our interest in this theory is in the correlation functions of these fields, since other physical quantities can generically be constructed from these. These correlations may be obtained by studying the response of the theory to the application of an external field, $J(x)$, which couples to $\phi(x)$.

For instance, a path-integral definition of the correlation function would be

$$\langle \phi(x_1) \dots \phi(x_k) \rangle_J \equiv e^{-iW[J]} \int \mathcal{D}\phi \, [\phi(x_1) \dots \phi(x_k)] \, \exp\left\{ i \int d^4x \, \Big[\mathcal{L} + J\phi \Big] \right\}, \tag{7}$$

where $\mathcal{L}$ denotes the lagrangian density which describes the system's dynamics, and the quantity $W[J]$ is defined by:

$$\exp\Big\{ iW[J] \Big\} = \int \mathcal{D}\phi \, \exp\left\{ i \int d^4x \, \Big[\mathcal{L}[\phi] + J\phi \Big] \right\}. \tag{8}$$

$W[J]$ generates the *connected* correlations of the operator ϕ, in the sense that

$$\langle \phi(x_1) \cdots \phi(x_k) \rangle_{c,J} = (-i)^k \frac{\delta^k W}{\delta J(x_1) \cdots \delta J(x_k)}. \tag{9}$$

This can be taken to define the connected part, but it also agrees with the usual graphical sense of connectedness. When this average is evaluated at $J = 0$ it coincides with the covariant time-ordered — more properly, T^*-ordered — vacuum expectation value of ϕ.

2.1.1 The 1PI generator: One-particle reducible graphs are defined as those connected graphs which can be broken into two disconnected parts simply by cutting a single internal line. One-particle *irreducible* (1PI) graphs are those connected graphs which are not one-particle reducible.

A non-graphical formulation of 1-particle reducibility of this sort can be had by performing a Legendre transformation on the functional $W[J]$ (5). With this choice, if the mean field, φ, is defined by

$$\varphi(J) \equiv \frac{\delta W}{\delta J} = \langle \phi(x) \rangle_J, \tag{10}$$

then the Legendre transform of $W[J]$ is defined to be the functional $\Gamma[\varphi]$, where

$$\Gamma[\varphi] \equiv W[J(\varphi)] - \int d^4x \; \varphi \, J. \tag{11}$$

Here we imagine $J(\varphi)$ to be the external current that is required to obtain the expectation value $\langle \phi \rangle_J = \varphi$, and which may be found, in principle, by inverting[2] eq. (10). If $\Gamma[\varphi]$ is known, $J(\varphi)$ can be found by directly differentiating the defining equation for $\Gamma[\varphi]$, which gives:

$$\frac{\delta \Gamma[\varphi]}{\delta \varphi} + J = 0. \tag{12}$$

[2]For simplicity, the inversion of eq. (10) is assumed to be possible.

This last equation implies another important property for $\Gamma[\varphi]$: its stationary point specifies the expectation value of the original operator, $\langle\phi(x)\rangle_{J=0}$. This can be seen from eqs. (10) and (12) above. For time-independent field configurations, u, this argument can be sharpened to show that $\Gamma[\varphi]$ is the minimum expectation value of the system's Hamiltonian, given that the expectation of the field $\phi(x)$ is constrained to equal $\varphi(x)$ (6).

A graphical representation for $\Gamma[\varphi]$ can be obtained by setting up a path-integral representation for it. An expression for $\Gamma[\varphi]$ as a path integral is found by combining the definitions of eqs. (8) and (11):

$$\begin{aligned} \exp\Big\{i\Gamma[\varphi]\Big\} &= \int \mathcal{D}\phi \, \exp\left\{ i \int d^4x \, \Big[\mathcal{L}(\phi) + J\,(\phi - \varphi)\Big]\right\} \\ &= \int \mathcal{D}\hat{\phi} \, \exp\left\{ i \int d^4x \, \Big[\mathcal{L}(\varphi + \hat{\phi}) + J\hat{\phi}\Big]\right\}. \end{aligned} \tag{13}$$

At first sight this last equation is a doubtful starting point for calculations since it gives only an implicit expression for $\Gamma[\varphi]$. Eq. (13) is only implicit because the current, J, appearing on its right-hand side must itself be given as a function of φ using eq. (12): $J = -(\delta\Gamma/\delta\varphi)$, which again depends on $\Gamma[\varphi]$. The implicit nature of eq. (13) turns out not to be an obstacle for computing $\Gamma[u]$, however. To see this recall that a saddle-point evaluation of eq. (8) gives $W[J]$ as the sum of all connected graphs that are constructed using vertices and propagators built from the classical lagrangian, $\mathcal{L}$, and having the currents, J, as external lines. But $\Gamma[\varphi]$ just differs from $W[J]$ by subtracting $\int d^4x \, J\varphi$, and evaluating the result at the specific configuration $J[\varphi] = -(\delta\Gamma/\delta\varphi)$. This merely lops off all of the 1-particle reducible graphs, ensuring that $\Gamma[\varphi]$ is given by summing 1-particle irreducible graphs.

This gives the following result for $\Gamma[\varphi]$ semiclassically:

$$\Gamma[\varphi] = S[\varphi] + \Gamma_1[\varphi] + (\text{Feynman Graphs}). \tag{14}$$

Here, the leading (tree-level) term is the classical action: $S = \int \mathcal{L}(\varphi)\, d^4x$. The next-to-leading result is the usual one-loop functional determinant,

$$\Gamma_1 = \mp \frac{1}{2}\, \log\det \left[\delta^2 S/\delta\phi(x)\delta\phi(y)\right]_{\phi=\varphi}, \tag{15}$$

of the quadratic part of the action expanded about the configuration $\phi = \varphi$. The sign, $\mp$, is $-$ for bosonic fields and $+$ for fermionic ones.

Finally, the 'Feynman Graphs' contribution denotes the sum of all possible graphs which (*i*) involve two or more loops; (*ii*) have no external lines (with φ-dependent internal lines constructed by inverting $\delta^2 S/\delta\phi(x)\delta\phi(y)$ evaluated at $\phi = \varphi$); and (*iii*) are one-particle irreducible (1PI).

2.1.2 Light-particle correlations: We now specialize this general framework to the specific model described above containing the two scalar fields, working in the limit where one is much more massive than the other. If we couple an external current, j, to l and another, J, to H, then the generator of 1PI correlations in this model, $\Gamma[\ell, h]$, depends on two scalar variables, $\ell = \langle l \rangle_{jJ}$ and $h = \langle H \rangle_{jJ}$. The correlation functions it generates can be used to construct general scattering amplitudes for both l and H particles by using standard techniques.

Our interest is in determining the dependence on the heavy mass, M of low-energy observables, where 'low energy' here means those observables for which all of the particles involved have centre-of-mass (CM) momenta and energies which satisfy $p, E \ll M$. If only l particles are initially present in any scattering process at such low energies, then no heavy particles can ever appear in the final state since there is not enough energy available for their production. As a result,

the heavy-field correlations can be ignored, and it suffices to consider only the generator of 1PI correlations exclusively for the light fields in the problem.

Since H-correlators are of no interest to us we are free to set $J = 0$ when evaluating $\Gamma[\ell, h]$, since we never need differentiate with respect to J. As the previous sections show, the condition $J = 0$ is equivalent to evaluating $\Gamma[\ell, h]$ with its argument, h, evaluated at the stationary point, $h = \overline{h}(\ell)$, where $\delta\Gamma/\delta h$ vanishes.

Furthermore, if only low-energy observables are of interest we may also restrict the external current j to be slowly varying in space, so that its Fourier transform only has support on low-momentum modes for which $p, E \ll M$. As above for J and h, the vanishing of these modes of j correspond to using the condition $\delta\Gamma/\delta\ell = 0$ to eliminate the high-frequency components of ℓ in terms of the low-frequency components. The restriction of $\Gamma[\ell, h]$ with these two conditions, denoted $\gamma[\ell]$, in principle contains all of the information required to compute any low-energy observable.

How is $\gamma[\ell]$ computed? Recall that $\Gamma[\ell, h]$, obtained by performing a Legendre transformation on both j and J, is obtained by summing over all 1PI graphs in the full theory. But $\gamma[\ell]$ differs from $\Gamma[\ell, h]$ only by setting both J and the short-wavelength components of j to zero (rather than to the configurations $J = -\delta\Gamma/\delta h$ and $j = -\delta\Gamma/\delta\ell$). Since it is the currents which are responsible for cancelling out the one-particle reducible graphs in $\Gamma[\ell, h]$, we see that setting currents to vanish simply means that this cancellation does not take place.

We see from this that $\gamma[\ell]$ is given by the sum of one-*light*-particle-irreducible (1LPI) graphs: *i.e.* the graphs which contribute to $\gamma[\ell]$ are one-particle irreducible only with respect to cutting low-momentum, low-frequency light-particle, l, lines,

but are one-particle reducible with respect to cutting high-momentum l lines or H lines having any momentum.

To make these manipulations concrete, the remainder of this section derives explicit expressions for both $\Gamma[\ell, h]$ and $\gamma[\ell]$ in the toy model, working to the tree and one-loop approximations.

2.1.3 TREE-LEVEL CALCULATION: Suppose both $\Gamma[\ell, h]$ and $\gamma[\ell]$ are computed approximately within a semiclassical loop expansion: $\Gamma = \Gamma^t + \Gamma_{1-\text{loop}} + \cdots$ and $\gamma = \gamma^t + \gamma^{1-\text{loop}} + \cdots$. Explicit formulae for both are particularly simple at tree level since the tree approximation, Γ^t, to Γ is simply given by evaluating the classical action at $l = \ell$ and $H = h$: $\Gamma^t[\ell, h] = S[\ell, h]$. The tree approximation for $\gamma^t[\ell]$ is therefore obtained by solving the classical equations of motion for h — and the high-frequency part of ℓ) as a function of (the low-frequency part of) ℓ — and then substituting this result, $\overline{h}_t(\ell)$, back into the classical action.

We next compute the resulting expression for $\gamma^t[\ell]$ in the limit that M is much larger than all of the other scales of interest. The classical equation of motion for h which is obtained from eqs. (1) and (2) is

$$\Box h - M^2 h - \frac{1}{2}\, g_{lh} \ell^2 h - \frac{g_h}{3!}\, h^3 - \frac{\tilde{m}}{2}\, \ell^2 - \frac{\tilde{g}_h M}{2}\, h^2 = 0, \tag{16}$$

so the solution, $\overline{h}_t(\ell)$ can be formally written:

$$\overline{h}_t(\ell) = \left(\Box - M^2 - \frac{g_{lh}}{2}\, \ell^2\right)^{-1} \left(\frac{\tilde{m}}{2}\, \ell^2 + \frac{g_h}{3!}\, \overline{h}^3 + \frac{\tilde{g}_h M}{2}\, \overline{h}^2\right). \tag{17}$$

This may be solved perturbatively in powers of g_h, $\tilde{g}_h$ and $1/M$, with the leading contribution obtained by taking $\overline{h} = 0$ on the right-hand side. This leads to the explicit solution

$$\begin{aligned} \overline{h}_t(\ell) &= \left[-\frac{1}{M^2} - \frac{1}{M^4}\left(\Box - \frac{g_{lh}}{2}\, \ell^2\right) + \cdots\right] \left(\frac{\tilde{m}}{2}\, \ell^2\right) + \mathcal{O}(g_h, \tilde{g}_h) \\ &= -\frac{\tilde{m}}{2M^2}\, \ell^2 + \frac{g_{lh}\tilde{m}}{4M^4}\, \ell^4 - \frac{\tilde{m}}{2M^4} \Box\left(\ell^2\right) + \mathcal{O}\left(\frac{1}{M^5}\right), \end{aligned} \tag{18}$$

where the leading contributions involving nonzero g_h and $\tilde{g}_h$ contribute at $\mathcal{O}(M^{-5})$.

Substituting this last result back into the classical action gives the tree level expression for the generating functional, $\gamma[\ell]$, as an expansion in powers of $1/M$. We find the result $\gamma^t[\ell] = \gamma_0^t[\ell] + \gamma_2^t[\ell]/M^2 + \gamma_4^t[\ell]/M^4 + \cdots$, with:

$$\begin{aligned} \gamma_0^t[\ell] &= \int d^4x \left(-\frac{1}{2}\,\partial_\mu \ell\, \partial^\mu \ell - \frac{m^2}{2}\,\ell^2 - \frac{g_l}{4!}\,\ell^4 \right), \\ \frac{\gamma_2^t[\ell]}{M^2} &= \int d^4x \left(\frac{\tilde{m}^2}{8M^2}\,\ell^4 \right), \\ \frac{\gamma_4^t[\ell]}{M^4} &= \int d^4x \left(-\frac{\tilde{m}^2}{2M^4}\,\ell^2\,\partial_\mu \ell\, \partial^\mu \ell - \frac{g_{lh}\,\tilde{m}^2}{16M^4}\,\ell^6 \right). \end{aligned} \tag{19}$$

These expressions for $\gamma^t[\ell]$ also have a simple representation in terms of Feynman graphs. All of the M-dependent terms may be obtained by summing connected tree diagrams having only ℓ particles as external lines and only h particles for internal lines, and then Taylor expanding all of the h propagators in powers of $1/M$. The graphs of this type which contribute up to and including $O(1/M^6)$ are given explicitly in Fig. (2).

This calculation brings out several noteworthy points:

• *Decoupling:* All of the M dependence in $\gamma^t[\ell]$ vanishes as $M \to \infty$, reflecting (at tree level) the general result that particles decouple from low-energy physics in the limit that their mass becomes large.

• *Truncation:* The only part of $\gamma^t[\ell]$ which survives as $M \to \infty$ consists of those terms in the classical action which are independent of the heavy field, h. That is, at tree level $\gamma_0^t[\ell]$ is obtained from the classical action, $S[\ell, h]$, simply by truncating it to $h = 0$: $\gamma_0^t[\ell] = S[\ell, 0]$. Notice that this relies on two assumptions: the absence of $M^3 H$ terms in the potential, and the condition $\tilde{m} \ll M$. (Even though $\overline{h}^t(\ell)$ would still vanish like $1/M$ if $\tilde{m}$ were $O(M)$, this is not enough to ensure the vanishing of terms like $M^2(\overline{h}^t)^2$. This latter condition often breaks

down in supersymmetric theories (7).)

• *Locality:* Although the exact expression for $\overline{h}_t(\ell)$, and hence for $\gamma^t[\ell]$, involves nonlocal quantities such as $G(x,x') = \langle x| \left(-\Box + M^2\right)^{-1} |x'\rangle$, the entire result becomes local once these are expanded in powers of $1/M$ because of the locality of propagators in this limit:

$$G(x,x') = \int \frac{d^4p}{(2\pi)^4} \left[\frac{e^{ip(x-x')}}{p^2+M^2}\right] = \left[\frac{1}{M^2} + \frac{\Box}{M^4} + \cdots\right] \delta^4(x-x'). \qquad (20)$$

This locality is ultimately traceable to the uncertainty principle. The key observation is that the M-dependent interactions in $\gamma[\ell]$ express the effects of virtual heavy particles whose energies $E_h \geq M$, are much higher than those of any of the light particles whose scatterings are under consideration in the low-energy limit. Indeed, it is precisely the high energy of these particles which precludes their being inadvertently produced as real final-state particles in any low-energy scattering process, and so guarantees that they only mediate transitions amongst the light-particle states.

But the virtual contribution of heavy particles to low-energy processes can nevertheless occur within perturbation theory because the uncertainty principle permits the violation of energy conservation required for their production, provided it takes place only over short enough times, $\Delta t \lesssim 1/\Delta E_h \lesssim 1/M$. As a consequence, from the low-energy perspective the influence of heavy particles appears to be instantaneous (*i.e.* local in time). The uncertainty principle similarly relates the momentum required to produce heavy virtual particles with the distances over which they can travel, thereby making their influence also *local* in space.[3]

[3] We assume a relativistic dispersion relation, $E^2 = p^2 + m^2$, so states having large momentum also have high energy, and must therefore be excluded from the low-energy theory.

The influence at low energies, E, of very massive particles, $M \gg E$, can therefore generically be reproduced to a fixed order in E/M by local interactions involving only the particles which are present at low energies. It is precisely this locality which makes possible the construction of a low-energy effective theory which accurately describes these virtual effects.

•*Redundant interactions:* At face value eq. (19) is not precisely the same as the effective potential, $V^{(4)} + V^{(6)}$, encountered in eqs. (5) and (6) of the introduction. This apparent difference is illusory, however, because the difference can be removed simply by performing a field redefinition.

To see how this works, suppose we have an action, $S[\phi^i]$, given as a series in some small parameter ε:

$$S[\phi^i] = S_0[\phi^i] + \varepsilon\ S_1[\phi^i] + \varepsilon^2\ S_2[\phi^i] + \cdots\,, \tag{21}$$

where ε could be a small loop-counting parameter or a small energy ratio, E/M. Suppose also that somewhere amongst the interactions which appear in the $O(\varepsilon^n)$ contribution, $S_n[\phi^i]$, there is a term, $S_n^R[\phi^i]$, which vanishes when the fields, ϕ^i are chosen to satisfy the equations of motion for the lowest-order action, $S_0[\phi^i]$. In equations:

$$S_n^R[\phi] = \int d^4x\ f^i(x)\,\frac{\delta S_0}{\delta\phi^i(x)}, \tag{22}$$

where the coefficients $f^i(x)$ are ultra-local functions of the fields and their derivatives at the spacetime point x.

The claim is that, *to order* ε^n, any such interaction, $S_n^R[\phi^i]$, can be removed by performing a field redefinition without altering any of the other terms at lower or equal order, and so can have no physical consequences. The required field

redefinition is:

$$\phi^i(x) \to \tilde{\phi}^i(x) = \phi^i(x) - \varepsilon^n \; f^i(x), \tag{23}$$

since under this redefinition the $O(\varepsilon^n)$ terms in the action vary into:

$$S[\phi^i] \to S[\tilde{\phi}^i] = S[\phi^i] - \varepsilon^n \int d^4x \; f^i(x) \, \frac{\delta S_0}{\delta \phi^i(x)} + O(\varepsilon^{n+1}). \tag{24}$$

Clearly, to $O(\varepsilon^n)$ the sole effect of this redefinition is simply to cancel $S_n^R[\phi^i]$.

Applying this reasoning to $\gamma[\ell]$, above, notice that integrating by parts gives

$$\int d^4x \; \ell^2 \, \partial_\mu \ell \, \partial^\mu \ell = -\frac{1}{3} \int d^4x \; \ell^3 \, \Box \ell, \tag{25}$$

and so using the lowest-order equations of motion for ℓ derived from $\gamma_0^t[\ell]$ – *i.e.* $\Box \ell = m^2 \ell + g_l \, \ell^3/3!$ – implies

$$\int d^4x \; \ell^2 \, \partial_\mu \ell \, \partial^\mu \ell = -\frac{1}{3} \int d^4x \; \left(m^2 \, \ell^4 + \frac{g_l}{3!} \, \ell^6 \right). \tag{26}$$

Using this in eq. (19) reproduces the potential of eqs. (5) and (6).

2.1.4 One-loop calculation: More can be learned by examining some of the subdominant terms in the loop expansion for $\gamma[\ell]$.

At face value keeping one-loop corrections changes $\gamma[\ell]$ in two different ways: through the one-loop corrections to the functional form of $\Gamma[\ell, h]$; and through the corrections that these imply for the stationary point, $\overline{h}(\ell)$. In practice, however, the correction to $\overline{h}(\ell)$ does not contribute to $\gamma[\ell]$ at one loop. To see this use the expansions $\Gamma = \Gamma^t + \Gamma^{1-\text{loop}} + \cdots$ and $\overline{h} = \overline{h}^t + \overline{h}^{1-\text{loop}} + \cdots$ into the definition $\gamma[\ell]$. To one-loop order this gives

$$\begin{aligned} \gamma[\ell] &= \Gamma^t[\ell, \overline{h}_t + \overline{h}^{1-\text{loop}}] + \Gamma^{1-\text{loop}}[\ell, \overline{h}^t] + \cdots \\ &= \left[\Gamma^t[\ell, \overline{h}^t] + \int d^4x \; \left(\frac{\delta \Gamma^t}{\delta h} \right)_{\overline{h}^t} \overline{h}^{1-\text{loop}} \right] + \Gamma^{1-\text{loop}}[\ell, \overline{h}^t] + \cdots \\ &= \gamma^t[\ell] + \Gamma^{1-\text{loop}}[\ell, \overline{h}^t] + \cdots, \end{aligned} \tag{27}$$

where the second equality follows by expanding Γ^t about $h = \overline{h}^t$, keeping only terms of tree and one-loop order, and the last equality uses the fact that $\delta\Gamma^t/\delta h = 0$ when evaluated at $\overline{h}^t$, together with the tree-level result, $\gamma^t[\ell] = \Gamma^t[\ell, \overline{h}^t(\ell)]$. Eq. (27) states that the one-loop approximation to γ is obtained by evaluating $\Gamma^{1-\text{loop}}$ at the *tree-level* configuration $\overline{h}^t$: *i.e.* $\gamma^{1-\text{loop}}[\ell] = \Gamma^{1-\text{loop}}[\ell, \overline{h}^t(\ell)]$.

To proceed we require the one-loop contribution, $\Gamma_{1-\text{loop}}$, which is given by:

$$\Gamma_{1-\text{loop}}[\ell, h] = \frac{i}{2} \log\det \begin{bmatrix} (-\Box + V_{\ell\ell})/\mu^2 & V_{\ell h}/\mu^2 \\ V_{\ell h}/\mu^2 & (-\Box + V_{hh})/\mu^2 \end{bmatrix}, \tag{28}$$

where μ is an arbitrary scale required on dimensional grounds, and the matrix of second derivatives of the scalar potential is given by

$$\begin{aligned} \mathbf{V} &= \begin{pmatrix} V_{\ell\ell} & V_{\ell h} \\ V_{\ell h} & V_{hh} \end{pmatrix} \\ &= \begin{pmatrix} m^2 + \frac{g_l}{2}\,\ell^2 + \frac{g_{lh}}{2}\,h^2 + \tilde{m}h & g_{lh}\ell h + \tilde{m}\ell \\ g_{lh}\ell h + \tilde{m}\ell & M^2 + \tilde{g}_h\, Mh + \frac{g_h}{2}\,h^2 + \frac{g_{lh}}{2}\,\ell^2 \end{pmatrix}. \end{aligned} \tag{29}$$

Evaluating the functional determinant in the usual way gives an expression which diverges in the ultraviolet. For the two-scalar theory under consideration these divergences only appear (at one loop) in that part of $\Gamma[\ell, h]$ which does not depend on derivatives of ℓ or h (*i.e.* the scalar 'effective potential'). If the divergent terms are written $\Gamma_{\text{div}} = -\int d^4x\, V_{\text{div}}$, then V_{div} is:

$$V_{\text{div}} = \frac{1}{32\pi^2}\left[C + (V_{\ell\ell} + V_{hh})\Lambda^2 - \frac{1}{2}\,(V_{\ell\ell}^2 + V_{hh}^2 + 2V_{\ell h}^2)\; L\right] \tag{30}$$

where C is a field-independent, divergent constant, and $L = \log\left(\Lambda^2/\mu^2\right)$. Λ is an ultraviolet cutoff which has been used to regulate the theory, and which is assumed to be sufficiently large compared to all other scales in the problem that all inverse powers of Λ can be neglected.

Alternatively, using dimensional regularization instead of an ultraviolet cutoff leads to the same expressions as above, with two changes: (i) all of the terms

proportional to C or to Λ^2 are set to zero, and (ii) the logarithmic divergence is replaced by $L \to 2/(4-n) = 1/\varepsilon$, where $n = 4-2\varepsilon$ is the dimension of spacetime.

All of the divergences can be absorbed into renormalizations of the parameters of the lagrangian, by defining the following renormalized couplings:

$$\begin{aligned}
A_R &= A_0 + \frac{1}{32\pi^2}\left[C + (M^2+m^2)\Lambda^2 - \frac{1}{2}\,(M^4+m^4)\,L\right], \\
B_R &= B_0 + \frac{1}{32\pi^2}\left[(\tilde{m}+\tilde{g}_h M)\Lambda^2 - (m^2\tilde{m}+\tilde{g}_h M^3)\,L\right], \\
m_R^2 &= m^2 + \frac{1}{32\pi^2}\left[(g_l+g_{lh})\Lambda^2 - (g_l m^2 + g_{lh}M^2 + 2\tilde{m}^2)\,L\right], \qquad (31) \\
M_R^2 &= M^2 + \frac{1}{32\pi^2}\left[(g_h+g_{lh})\Lambda^2 - \left(g_{lh}m^2 + (g_h+\tilde{g}_h^2)M^2 + \tilde{m}^2\right)L\right], \\
\tilde{m}_R &= \tilde{m} - \frac{1}{32\pi^2}\,(g_l\tilde{m} + g_{lh}\tilde{g}_h M + 4g_{lh}\tilde{m})\,L, \\
(\tilde{g}_h)_R &= \tilde{g}_h - \frac{3}{32\pi^2}\left(g_{lh}\frac{\tilde{m}}{M} + g_h\tilde{g}_h\right)L \\
(g_l)_R &= g_l - \frac{3}{32\pi^2}\,(g_l^2+g_{lh}^2)\,L \\
(g_h)_R &= g_h - \frac{3}{32\pi^2}\,(g_h^2+g_{lh}^2)\,L \\
(g_{lh})_R &= g_{lh} - \frac{1}{32\pi^2}\,(g_l+g_h+4g_{lh})g_{lh}\,L.
\end{aligned}$$

In these expressions A_0 and B_0 are the coefficients of the terms $A+Bh$ which do not appear in the classical potential, but which are not forbidden by any symmetries. By not including such terms in the classical potential we are implicitly choosing A_0 and B_0 to satisfy the renormalization condition $A_R = B_R = 0$, which can be ensured by appropriately shifting the renormalized fields. Similarly, the assumed heirarchy of masses between the light field, ℓ, and the heavy field, h, assumes the *renormalized* quantities satisfy $m_R, \tilde{m}_R \ll M_R$. Notice that these conditions are 'unnatural' in that they require that large $O(M)$ renormalization corrections must be cancelled by the bare quantities A_0, B_0, m and $\tilde{m}$, in addition to the cancellation of any divergent parts. In what follows we assume that this renormalization has been performed, and that all parameters which appear

in expressions are the renormalized quantities — even though the subscript 'R' is not explicitly written.

Once this renormalization has been performed the remaining expression is finite and its dependence on the heavy mass scale, M, can be identified. As for the tree-level analysis, it is convenient to organize $\gamma^{1-\text{loop}}[\ell]$ into an expansion in powers of $1/M$. A new feature which arises at one loop is that the dominant term in this expansion now varies as $M^4 \log M$ rather than as M^0 when $M \to \infty$. We therefore write: $\gamma^{1-\text{loop}} = \gamma_{-4}^{1-\text{loop}} M^4 + \gamma_{-2}^{1-\text{loop}} M^2 + \gamma_{-1}^{1-\text{loop}} M + \gamma_0^{1-\text{loop}} + \gamma_2^{1-\text{loop}}/M^2 + \cdots$.

It turns out that those terms in $\gamma^{1-\text{loop}}[\ell]$ which involve derivatives of ℓ first appear at order $1/M^2$ in this expansion.[4] To identify all of the terms which are larger than $O(1/M^2)$ it therefore suffices to work only with the effective scalar potential. After performing the renormalizations indicated in eqs. (31), the one-loop result for the scalar potential becomes:

$$\gamma_{\text{pot}}^{1-\text{loop}}[\ell] = -\frac{1}{64\pi^2} \int d^4x \left[V_+^2 \log\left(\frac{V_+}{\mu^2}\right) + V_-^2 \log\left(\frac{V_-}{\mu^2}\right) \right], \tag{32}$$

where $V_\pm$ denotes the two eigenvalues of the matrix $\mathbf{V}$, which are given explicitly by:

$$V_\pm = \frac{1}{2} \left[(V_{\ell\ell} + V_{hh}) \pm \sqrt{(V_{\ell\ell} - V_{hh})^2 + 4V_{\ell h}^2} \right]. \tag{33}$$

Our interest is in a potential for which $V_{hh} \gg V_{\ell\ell}, V_{\ell h}$, and so these expressions can be simplified to:

$$\begin{aligned} V_+ &\approx V_{hh} + \frac{V_{\ell h}^2}{V_{hh}} + \frac{V_{\ell\ell} V_{\ell h}^2}{V_{hh}^2} + O\left(\frac{1}{V_{hh}^3}\right), \\ V_- &\approx V_{\ell\ell} - \frac{V_{\ell h}^2}{V_{hh}} - \frac{V_{\ell\ell} V_{\ell h}^2}{V_{hh}^2} + O\left(\frac{1}{V_{hh}^3}\right), \end{aligned} \tag{34}$$

[4] This result is a special feature of the two-scalar system at one loop. At two loops, or at one loop for other systems, kinetic terms like $\partial_\mu \ell \, \partial^\mu \ell$ can appear proportional to logarithms of M.

and:

$$\gamma_{\rm pot}^{1-{\rm loop}}[\ell] = -\frac{1}{64\pi^2}\int d^4x\left\{V_{hh}^2\,\log\left(\frac{V_{hh}}{\mu^2}\right)+V_{\ell\ell}^2\,\log\left(\frac{V_{\ell\ell}}{\mu^2}\right)\right. \tag{35}$$
$$\left.+V_{\ell h}^2\left[1+2\log\left(\frac{V_{hh}}{\mu^2}\right)\right]+\frac{2V_{\ell\ell}V_{\ell h}^2}{V_{hh}}\,\log\left(\frac{V_{hh}}{V_{\ell\ell}}\right)+O\left(\frac{1}{V_{hh}^2}\right)\right\}.$$

Using the explicit expressions given above for the scalar potential, and evaluating the result at $h=\overline{h}^t(\ell)$, the first few terms of the $1/M$ expansion at one loop are given by:

$$\begin{aligned}
M^4\gamma_{-4}^{1-{\rm loop}} &= -\frac{M^4}{64\pi^2}\,\log\left(\frac{M^2}{\mu^2}\right)\int d^4x,\\
M^2\gamma_{-2}^{1-{\rm loop}}[\ell] &= -\frac{M^2}{64\pi^2}\,\left[\log\left(\frac{M^2}{\mu^2}\right)+\frac{1}{2}\right]\int d^4x\; g_{lh}\;\ell^2,\\
M\gamma_{-1}^{1-{\rm loop}}[\ell] &= +\frac{M}{64\pi^2}\,\left[\log\left(\frac{M^2}{\mu^2}\right)+\frac{1}{2}\right]\int d^4x\;\tilde{g}_h\tilde{m}\;\ell^2,\\
\gamma_0^{1-{\rm loop}} &= -\frac{1}{64\pi^2}\int d^4x\;\left[\left(\frac{g_{lh}^2}{4}\;\ell^4+2\tilde{m}^2\ell^2\right)\log\left(\frac{M^2}{\mu^2}\right)\right.\\
&\qquad\left.+\frac{3g_{lh}^2}{8}\;\ell^4+\tilde{m}^2\ell^2+\left(m^2+\frac{g_l^2}{2}\ell^2\right)^2\log\left(\frac{m^2+\frac{g_l^2}{2}\;\ell^2}{\mu^2}\right)\right].
\end{aligned} \tag{36}$$

We see that this shares two of the crucial properties of the tree-level result:

•*Decoupling:* Notice that — superficially — the effects of the heavy particle no longer appear to vanish as $M\to\infty$. However, all of the terms which grow as M grows have the same form as does the classical lagrangian, and so they can all be absorbed into finite renormalizations of A_R, m_R^2 and $(g_l)_R$. That is, if we define the new quantities:

$$\begin{aligned}
A_R' &= A_R+\frac{M^4}{64\pi^2}\,\log\left(\frac{M^2}{\mu^2}\right)\\
(m_R^2)' &= m_R^2+\frac{1}{64\pi^2}\left\{M^2\left(g_{lh}-\tilde{g}_h\frac{\tilde{m}}{M}\right)\left[1+2\log\left(\frac{M^2}{\mu^2}\right)\right]+4\tilde{m}^2\log\left(\frac{M^2}{\mu^2}\right)\right\}\\
(g_l)_R' &= (g_l)_R+\frac{3\,g_{lh}^2}{32\pi^2}\,\log\left(\frac{M^2}{\mu^2}\right),
\end{aligned} \tag{37}$$

then the one-loop contribution, $\gamma^{1-\text{loop}}[\ell]$, to the 1LPI generator becomes:

$$\gamma^{1-\text{loop}}[\ell] = -\frac{1}{64\pi^2}\int d^4x \left[\frac{3g_{lh}^2}{8}\,\ell^4 + \tilde{m}^2\ell^2 + \left(m^2 + \frac{g_l^2}{2}\ell^2\right)^2 \log\left(\frac{m^2 + \frac{1}{2}\,g_l^2\,\ell^2}{\mu^2}\right)\right] + O\left(\frac{1}{M}\right), \tag{38}$$

where m^2 represents $(m_R^2)'$, and so on.

Clearly, after such renormalizations are performed, all of the remaining M dependence vanishes in the limit $M \to \infty$. Provided the values of renormalized couplings are in any case inferred from experiment, all of the *physical* effects of the heavy particle really are suppressed for large M, ensuring the heavy particle does decouple from physical observables.

•*Locality:* Since the one-loop action, $\gamma^{1-\text{loop}}[\ell]$, is the integral over spacetime of a quantity which is evaluated at a single spacetime point when expanded in inverse powers of M, it shares the locality of the classical result. The underlying source of this locality is again the uncertainty principle, which precludes violations of energy and momentum conservation over large distances – a result which hinges on our keeping only states that are defined by their low energy.

2.2 The Wilson action

We next reorganize the same calculation, with the goal of making the M dependence of physical results manifest from the outset. To this end, suppose we start from the path-integral expression for the 1LPI generating functional, $\gamma[\ell]$, derived as eq. (13) above,

$$\exp\Big\{i\gamma[\ell]\Big\} = \int \mathcal{D}l\;\mathcal{D}H\;\exp\left\{i\int d^4x\,\Big[\mathcal{L}(\ell + l, H) + jl\Big]\right\}, \tag{39}$$

with the external current, j, regarded as being defined by $j[\ell] = -\delta\gamma[\ell]/\delta\ell$. No similar current is coupled to the heavy field, H, in eq. (39) since our attention

is restricted to low-energy processes for which no heavy particles appear in the initial or final states.

2.2.1 THE WILSON ACTION: Now imagine schematically dividing the functional integral into its low-energy and high-energy parts, $\mathcal{D}l\,\mathcal{D}H = [\mathcal{D}l]_{\rm l.e.}[\mathcal{D}l\,\mathcal{D}H]_{\rm h.e.}$, relative to some arbitrary intermediate scale, λ. (For instance, this might be done by requiring high-energy modes to satisfy $p^2 + m^2 > \lambda^2$ in Euclidean signature.) Using this distinction between low- and high-energy modes it becomes possible to perform the functional integration over — or to 'integrate out' — the high-energy modes once and for all:

$$\exp\Big\{i\gamma[\ell]\Big\} = \int [\mathcal{D}l]_{\rm l.e.} \exp\left\{i\int d^4x\,[\mathcal{L}_W\,(\ell + l_{\rm l.e.}) + jl_{\rm l.e.}]\right\}, \qquad (40)$$

where $S_W = \int d^4x\,\mathcal{L}_W$ is called the *Wilson action*, and is defined as the result of performing the high-energy part of the functional integral:

$$\exp\Big\{iS_W\,[\ell + l_{\rm l.e.}]\Big\} = \int [\mathcal{D}l\,\mathcal{D}H]_{\rm h.e.}\ \exp\Big\{iS\,[\ell + l_{\rm l.e.}, l_{\rm h.e.}, H_{\rm h.e.}]\Big\}. \qquad (41)$$

Eqs. (40) and (41) are the central definitions from which the calculation of $\gamma[\ell]$ *à la* Wilson proceed.

There are two points about these last expressions which bear special emphasis. Notice first that $\mathcal{L}_W$ appears in eq. (40) in precisely the same way as would the classical lagrangian in a theory for which no heavy field existed. Consequently, once $\mathcal{L}_W$ is known it may be used to compute $\gamma[\ell]$ in the usual way: one must sum over all 1LPI vacuum Feynman graphs using the interactions and propagators for the light fields which are dictated by the effective lagrangian density, $\mathcal{L}_W$.

The second point — which is what makes eq. (40) so useful — is that because $\mathcal{L}_W$ is computed by integrating only over high-energy modes, the uncertainty principle guarantees that it is *local* once it is expanded in inverse powers of the

heavy scales. Consequently, to the extent that we work only to a fixed order in this expansion, we need not worry that eq. (41) will generate arbitrary non-local interactions.

2.2.2 The physics of renormalization: Eqs. (40) and (41) share another beautiful feature. Although the Wilson action depends explicitly on the scale λ which is used in its definition, this dependence *always* drops out of any physical observables. It must drop out since it only arises from our choice to perform the calculation in two steps: first integrating modes heavier than λ, and then integrating the lighter modes.

In detail, this cancellation arises because λ enters into the low-energy part of the calculation in two ways. The first way is through the explicit λ dependence of all of the couplings of the Wilson action, S_W. However, λ also enters because all of the contributions of virtual particles in the low-energy theory have their momenta cutoff at the scale λ. The λ dependence of the couplings in S_W is just what is required to cancel the λ's which enter through the cutoff.

This entire discussion of λ-cancellation induces a strong sense of *déja vu*, because it exactly parallels the traditional renormalization programme wherein the regularization dependence of divergent loop integrals are cancelled by introducing regularization-dependent interactions (or counter-terms) into the classical action, S. This similarity makes it irresistible to regard the original classical action, $S[l, H]$, as itself being the Wilson action for a yet more fundamental theory which applies at still higher energies, above the cutoff Λ. Any such Wilson action would be used to compute physical observables in precisely the same way as one traditionally uses the classical action, including the renormalization of all couplings to cancel the cutoff dependence of all observable quantities. The great benefit of

adopting this point of view is the insight it gives into the physical nature of this cancellation.

2.2.3 The dimensionally regularized Wilson action: The Wilson action defined with an explicit cutoff is somewhat cumbersome for practical calculations, for a variety of reasons. Cutoffs make it difficult to keep the gauge symmetries of a problem manifest when there are spin-one gauge bosons (like photons) in the problem. Cutoffs also complicate our goal of following how the heavy scale, M, appears in physically interesting quantities like $\gamma[\ell]$, because they muddy the dimensional arguments used to identify which interactions in S_W contribute to observables order-by-order in $1/M$.

It is much more convenient to use dimensional regularization, even though dimensional regularization seems to run counter to the entire spirit of a low-energy action by keeping momenta which are arbitrarily high. This is not a problem in practice, however, because the error we make by keeping such high-momentum modes can itself always be absorbed into an appropriate renormalization of the effective couplings. This is always possible precisely because our 'mistake' is to keep high-energy modes, whose contributions at low energies can always be represented using local effective interactions. Whatever damage we do by using dimensional regularization to define the low-energy effective action can always be undone by appropriately renormalizing our effective couplings.

We are led to the following prescription for defining a dimensionally-regularized effective action in the two-scalar toy model. First dimensionally regulate the full theory involving both fields l and H, using for convenience the mass-independent $\overline{MS}$ renormalization scheme. At one loop this amounts to renormalizing as in eqs. (31), but with all of the quartically and quadratically divergent terms set to

zero, and substituting $L \to 1/\varepsilon + k$ in the logarithmically divergent terms, where $k = \gamma - \log(4\pi)$ and $\gamma = 0.577\ 215\cdots$ is the Euler-Mascherelli constant.

Next define the effective theory to include only the light field l, also regulated using dimensional regularization. However rather than using minimal subtraction in the effective theory we instead renormalize the effective couplings by demanding that they successfully reproduce the low-energy limit of the full theory, using for this purpose any convenient set of observables. Once this *matching* calculation has been done, the resulting effective theory can be used to compute any other quantities as required. This construction is best understood using the concrete example of the two-scalar model.

2.2.4 Tree-Level Calculation: Imagine computing both γ and $\mathcal{L}_W$ within the loop expansion: $\gamma = \gamma^t + \gamma^{1-\text{loop}} + \cdots$ and $\mathcal{L}_W = \mathcal{L}_W^t + \mathcal{L}_W^{1-\text{loop}} + \cdots$. At tree level the distinction between $S_W = \int d^4x\ \mathcal{L}_W$, and the 1LPI generator, γ, completely degenerates. This is because the tree approximation to $\gamma[\ell]$ is simply obtained by evaluating the integrands of eqs. (40) and (41) at the classical saddle point. In the present case this implies that γ^t is simply given by evaluating $\mathcal{L}_W^t$ at $l = \ell$, leading to:

$$\gamma^t[\ell] = \int d^4x\ \mathcal{L}_W^t(\ell). \tag{42}$$

Similarly, evaluating the path integral expression of eq. (41) to obtain $\mathcal{L}_W$ in the tree approximation entails evaluating the classical action at the saddle point $l = \ell$ and $H = \overline{h}^t(\ell)$. Graphically, this gives the tree-level Wilson action as the sum over all tree graphs which have only heavy particles propagating in their internal lines, and only light particles for external lines. Retracing the steps taken in previous sections to compute $\gamma[\ell]$ at tree level gives an explicit expression for

$\mathcal{L}_W^t = \mathcal{L}(\ell, \overline{h}^t(\ell))$:

$$\begin{aligned}\mathcal{L}_W^t(\ell) &= -\frac{1}{2}\,\partial_\mu\ell\,\partial^\mu\ell - \frac{m^2}{2}\,\ell^2 - \left(\frac{g_l}{4!} - \frac{\tilde{m}^2}{8M^2}\right)\ell^4 \qquad (43)\\ &\quad - \left(\frac{\tilde{m}^2}{2M^4}\right)\ell^2\,\partial_\mu\ell\,\partial^\mu\ell - \left(\frac{g_{lh}\,\tilde{m}^2}{16M^4}\right)\ell^6 + \cdots,\\ &= -\frac{1}{2}\,\partial_\mu\ell\,\partial^\mu\ell - \frac{m^2}{2}\,\ell^2 - \left(\frac{g_l}{4!} - \frac{\tilde{m}^2}{8M^2} - \frac{m^2\tilde{m}^2}{6M^4}\right)\ell^4 \qquad (44)\\ &\quad - \left(\frac{g_{lh}\,\tilde{m}^2}{16M^4} - \frac{g_l\,\tilde{m}^2}{36M^4}\right)\ell^6 + \cdots,\end{aligned}$$

where the freedom to redefine fields has been used, and ellipses represent terms which are higher order in $1/M$ than are those displayed. This result also could be obtained by asking what local lagrangian involving only light fields reproduces the 2-body and 3-body scattering of the full theory to $O(1/M^4)$.

2.2.5 One-Loop Calculation: At one loop many of the nontrivial features of the Wilson action emerge for the first time. As usual we assume the renormalized dimensional parameters which ensure the hierarchy of scales in the two-scalar model satisfy $m \sim \tilde{m} \ll M$.

The most general possible Wilson action which is local and consistent with the symmetry $\ell \to -\ell$ is

$$S_W[\ell] = -\int d^4x\,\left[a_0 + \frac{a_2}{2}\,\ell^2 + \frac{a_4}{4!}\,\ell^4 + \frac{1}{2}(1+b_2)\,\partial_\mu\ell\,\partial^\mu\ell + \cdots\right], \qquad (45)$$

where ellipses denote higher-dimension interactions and the constants a_k, b_k, *etc.* to be determined by matching to the full theory, as is now described.

For simplicity we specialize to matching to $O(M^0)$, since this allows us to specialize to background configurations for which ℓ is spacetime independent — *i.e.* $\partial_\mu\ell = 0$. The only effective couplings which are relevant in this case (more about this in the next section) are a_0, a_2 and a_4, and it is convenient to determine these by requiring the one-loop result for $\gamma^{1-\text{loop}}[\ell]$, computed using S_W, agrees

with the result computed with the full theory, to $O(M^0)$.

The previous section gives the one-loop calculation in the full theory to this order as eq. (35), evaluated at $h = \overline{h}^t(\ell) = -(\tilde{m}/2M^2)\,\ell^2 + O(1/M^4)$. This gives $\gamma_{\rm pot}^{\rm 1-loop}[\ell] = -\int d^4x\, V^{\rm 1-loop}$, with $V^{\rm 1-loop}$ as given in eq. (36):

$$\begin{aligned} V^{\rm 1-loop}(\ell) &= \frac{1}{64\pi^2}\left\{ M^4 \log\left(\frac{M^2}{\mu^2}\right) \right. \\ &\quad + (g_{lh}M^2 - \tilde{g}_h\tilde{m}M + 2\tilde{m}^2)\left[\log\left(\frac{M^2}{\mu^2}\right) + \frac{1}{2}\right]\,\ell^2 \\ &\quad + \frac{g_l^2}{4}\left[\log\left(\frac{M^2}{\mu^2}\right) + \frac{3}{2}\right]\ell^4 \\ &\quad \left. + \left(m^2 + \frac{g_l}{2}\,\ell^2\right)^2\,\log\left(\frac{m^2 + \frac{1}{2}\,g_l\,\ell^2}{\mu^2}\right)\right\}, \end{aligned} \tag{46}$$

neglecting $O(1/M)$ terms. With ultraviolet divergences regularized in $n = 4 - 2\varepsilon$ dimensions, and renormalized using the $\overline{MS}$ renormalization scheme, the counterterms used to obtain this expression are those of eq. (31):

$$\begin{aligned} A_R &= A_0 - \frac{1}{64\pi^2}\,(M^4 + m^4)\,L, \\ B_R &= B_0 - \frac{1}{32\pi^2}\,(m^2\tilde{m} + \tilde{g}_h M^3)\,L, \\ m_R^2 &= m^2 - \frac{1}{32\pi^2}\,(g_l m^2 + g_{lh}M^2 + 2\tilde{m}^2)\,L, \\ M_R^2 &= M^2 - \frac{1}{32\pi^2}\left(g_{lh}m^2 + (g_h + \tilde{g}_h^2)M^2 + \tilde{m}^2\right)L, \\ \tilde{m}_R &= \tilde{m} - \frac{1}{32\pi^2}\,(g_l\tilde{m} + g_{lh}\tilde{g}_h M + 4g_{lh}\tilde{m})\,L, \\ (\tilde{g}_h)_R &= \tilde{g}_h - \frac{3}{32\pi^2}\,\left(g_{lh}\frac{\tilde{m}}{M} + g_h\tilde{g}_h\right)\,L \\ (g_l)_R &= g_l - \frac{3}{32\pi^2}\,(g_l^2 + g_{lh}^2)\,L \\ (g_h)_R &= g_h - \frac{3}{32\pi^2}\,(g_h^2 + g_{lh}^2)\,L \\ (g_{lh})_R &= g_{lh} - \frac{1}{32\pi^2}\,(g_l + g_h + 4g_{lh})g_{lh}\,L, \end{aligned} \tag{47}$$

with $L = 1/\varepsilon + k$.

Repeating the same calculation using the Wilson action, eq. (45), instead leads

to:

$$V_{\rm eff}^{1-{\rm loop}} = \frac{1}{64\pi^2}\left(\hat{a}_2 + \frac{\hat{a}_4}{2}\,\ell^2\right)^2 \log\left(\frac{\hat{a}_2 + \frac{\hat{a}_4}{2}\,\ell^2}{\mu^2}\right), \tag{48}$$

where $\hat{a}_2 = a_2/(1+b_2) \approx a_2(1-b_2+\cdots)$ and $\hat{a}_4 = a_4/(1+b_2)^2 \approx a_4(1-2b_2+\cdots)$. Here the constants a_k have been renormalized in the effective theory, being related to the bare couplings by expressions similar to eq. (47):

$$\begin{aligned}
(a_0)_R &= a_0 - \frac{a_2^2}{64\pi^2}\,L, \\
(a_2)_R &= a_2 - \frac{a_4 a_2}{32\pi^2}\,L, \\
(a_4)_R &= a_4 - \frac{3\,a_4^2}{32\pi^2}\,L.
\end{aligned} \tag{49}$$

The constants a_k are now fixed by performing the UV-finite renormalization required to make the two calculations of $V^t + V^{1-{\rm loop}}$ agree. The b_k are similarly determined by matching the two-derivative terms in $\gamma^{1-{\rm loop}}$, and so on. (For the scalar model under consideration this gives $b_2 = 0$ to the order of interest, since a single H loop first contributes to the $\partial_\mu\ell\,\partial^\mu\ell$ term at $O(1/M^2)$.) Requiring the tree contribution, $V_{\rm eff}^t = a_0 + \frac{1}{2}a_2\,\ell^2 + \frac{1}{4!}\,a_4\,\ell^4$, to capture the terms in $V^{1-{\rm loop}}$ which are missing in $V_{\rm eff}^{1-{\rm loop}}$ gives the required effective couplings in the Wilson action:

$$\begin{aligned}
(a_0)'_R &= A + \frac{1}{64\pi^2}\,M^4 \log\left(\frac{M^2}{\mu^2}\right) \\
(a_2)'_R &= m^2 + \frac{1}{64\pi^2}\,(g_{lh}M^2 - \tilde{g}_h\tilde{m}M + 2\tilde{m}^2)\left[2\log\left(\frac{M^2}{\mu^2}\right) + 1\right] \\
(a_4)'_R &= g_l + \frac{3\,g_l^2}{32\pi^2}\left[\log\left(\frac{M^2}{\mu^2}\right) + \frac{3}{2}\right].
\end{aligned} \tag{50}$$

Such finite renormalizations, arising as a heavy particle is integrated out, are called *threshold corrections*, and it is through these that the explicit powers of M get into the low-energy theory in dimensional regularization. (Notice also that the coefficients of M^2 in these expressions need not agree with those of Λ^2 in a

cutoff low-energy theory, revealing the fallacy of using quadratic divergences in an effective theory to track heavy-mass dependence (8).)

3 Power counting

As is clear from the two-scalar model, effective lagrangians typically involve potentially an infinite number of interactions corresponding to the ultimately infinite numbers of terms which can arise once the M-dependence of physical observables is expanded in powers of $1/M$. If it were necessary to deal with even a large number of these terms there would be no real utility in using them in practical calculations.

The most important part of an effective-lagrangian analysis is therefore the identification of which terms in $\mathcal{L}_{\rm eff}$ are required in order to compute observables to any given order in $1/M$, and this is accomplished using the power-counting rules of this section.

3.1 A class of effective interactions

To keep the discussion interestingly general, in this section we focus on a broad class of effective lagrangians that can be written in the following way:

$$\mathcal{L}_{\rm eff} = f^4 \sum_k \frac{c_k}{M^{d_k}} \, \mathcal{O}_k \left(\frac{\phi}{v} \right) . \tag{51}$$

In this expression, ϕ is meant to generically represent the fields of the problem, which for simplicity of presentation are taken here to be bosons and so to have the canonical dimension of mass, using fundamental units for which $\hbar = c = 1$. (The generalization to more general situations, including fermions, is straightforward.) The quantities f, v and M are all constants also having the dimensions of mass. The index 'k' runs over all of the labels of the various effective interactions which

appear in $\mathcal{L}_{\rm eff}$, and which are denoted by $\mathcal{O}_k$, and which are assumed to have dimension $(\text{mass})^{d_k}$. Since the ratio ϕ/v is dimensionless, all of this dimension is carried by derivatives, $\partial\phi$. As a result, d_k simply counts the number of derivatives which appear in the effective interaction, $\mathcal{O}_k$.

3.2 Power-counting rules

Imagine now computing Feynman graphs using these effective interactions, with the goal of tracking how the result depends on the scales f, v and M, as well as the mass scale, m, of the low-energy particles. Consider in particular a graph, $\mathcal{A}_E(q)$, involving E external lines whose four-momenta are collectively denoted by q. Suppose also that this graph has I internal lines and V_{ik} vertices. The labels i and k indicate two of the properties of the vertices: with i counting the number of lines which converge at the vertex, and k counting the power of momentum which appears in the vertex. Equivalently, i counts the number of powers of the fields, ϕ, which appear in the corresponding interaction term in the lagrangian, and k counts the number of derivatives of these fields which appear there.

3.2.1 Some useful identities: The positive integers, I, E and V_{ik}, which characterize the Feynman graph in question are not all independent since they are related by the rules for constructing graphs from lines and vertices. One such a relation is obtained by equating the two equivalent ways of counting the number of ends of internal and external lines in a graph. On one hand, since all lines end at a vertex, the number of ends is given by summing over all of the ends which appear in all of the vertices: $\sum_{ik} i\, V_{ik}$. On the other hand, there are two ends for each internal line, and one end for each external line in the graph: $2I + E$.

Equating these gives the identity which expresses the 'conservation of ends':

$$2I + E = \sum_{ik} i\, V_{ik}, \qquad \text{(Conservation of Ends)}. \tag{52}$$

A second useful identity *defines* of the number of loops, L, for each (connected) graph:

$$L = 1 + I - \sum_{ik} V_{ik}, \qquad \text{(Definition of } L\text{)}. \tag{53}$$

This definition doesn't come out of thin air, since for graphs which can be drawn on a plane it agrees with the intuitive notion of the number of loops in a graph.

3.2.2 Estimating integrals: Reading the Feynman rules from the lagrangian of eq. (51) shows that the vertices in the Feynman graph of interest contribute a factor

$$\text{(Vertex)} = \prod_{ik} \left[i(2\pi)^4 \delta^4(p) \, \left(\frac{p}{M}\right)^k \, \left(\frac{f^4}{v^i}\right) \right]^{V_{ik}}, \tag{54}$$

where p generically denotes the various momenta running through the vertex.

Similarly, each internal line contributes the additional factors:

$$\text{(Internal Line)} = \left[-i \int \frac{d^4p}{(2\pi)^4} \, \left(\frac{M^2 v^2}{f^4}\right) \, \frac{1}{p^2 + m^2} \right]^I, \tag{55}$$

where, again, p denotes the generic momentum flowing through the line. m denotes the mass of the light particles which appear in the effective theory, and it is assumed that the kinetic terms which define their propagation are those terms in $\mathcal{L}_{\text{eff}}$ involving two derivatives and two powers of the fields, ϕ.

As usual for a connected graph, all but one of the momentum-conserving delta functions in eq. (54) can be used to perform one of the momentum integrals in eq. (55). The one remaining delta function which is left after doing so depends only on the external momenta, $\delta^4(q)$, and expresses the overall conservation of four-momentum for the process. Future formulae are less cluttered if this factor

is extracted once and for all, by defining the reduced amplitude, A, by

$$\mathcal{A}_E(q) = i(2\pi)^4 \delta^4(q)\ A_E(q). \tag{56}$$

The number of four-momentum integrations which are left after having used all of the momentum-conserving delta functions is then $I - \sum_{ik} V_{ik} + 1 = L$. This last equality uses the definition, eq. (53), of the number of loops, L.

In order to track how the result depends on the scales in $\mathcal{L}_{\rm eff}$ it is convenient to estimate the results of performing the various multi-dimensional momentum integrals using dimensional analysis. Since these integrals are typically ultraviolet divergent, they must first be regulated, and this is where the use of dimensional regularization pays off. The key observation is that if a dimensionally regulated integral has dimensions, then its size is set by the light masses or external momenta which appear in the integrand. That is, dimensional analysis applied to a dimensionally-regulated integral implies

$$\int \cdots \int \left(\frac{d^n p}{(2\pi)^n}\right)^A \ \frac{p^B}{(p^2+m^2)^C} \sim \left(\frac{1}{4\pi}\right)^{2A} m^{nA+B-2C}, \tag{57}$$

with a dimensionless prefactor which depends on the dimension, n, of spacetime, and which may be singular in the limit that $n \to 4$. Here m represents the dominant scale which appears in the integrand of the momentum integrations. If the light particles appearing as external states in $A_E(q)$ should be massless, or highly relativistic, then the typical external momenta, q, are much larger than m, and m in the above expression should be replaced by q.[5] q is used as the light scale controlling the size of the momentum integrations in the formulae quoted below.

[5] Any logarithmic infrared mass singularities which may arise in this limit are ignored here, since our interest is in following *powers* of ratios of the light and heavy mass scales.

With this estimate for the size of the momentum integrations, we find the following quantity appears in the amplitude $A_E(q)$

$$\int \cdots \int \left(\frac{d^4p}{(2\pi)^4}\right)^L \frac{p^{\sum_{ik} kV_{ik}}}{(p^2+q^2)^I} \sim \left(\frac{1}{4\pi}\right)^{2L} q^{4L-2I+\sum_{ik} kV_{ik}}, \tag{58}$$

which, with liberal use of the identities (52) and (53), gives as estimate for $A_E(q)$:

$$A_E(q) \sim f^4 \left(\frac{1}{v}\right)^E \left(\frac{Mq}{4\pi f^2}\right)^{2L} \left(\frac{q}{M}\right)^{2+\sum_{ik}(k-2)V_{ik}}. \tag{59}$$

This last formula is the main result, which is used in the various applications considered later. Its utility lies in the fact that it links the contributions of the various effective interactions in the effective lagrangian, (51), with the dependence of observables on small mass ratios such as q/M. Notice in particular that more and more complicated graphs – for which L and V_{ik} become larger and larger – are only suppressed in their contributions to observables if q is much smaller than the scales M and f.

Notice also that the basic estimate, eq. (57), would have been much more difficult to do if the effective couplings were defined using a cutoff, λ, because in this case it is the cutoff which would dominate the integral, since it is then the largest external scale. But we knowing this cutoff dependence is less useful because the general arguments of the previous sections show that λ is guaranteed to drop out of any physical quantity.

3.3 The Effective-Lagrangian Logic

Power-counting estimates of this sort suggest the following general logic concerning the use of effective lagrangians:

Step I Choose the accuracy (*e.g.* one part per mille) with which observables, such as $A_E(q)$, are to be computed.

Step II Determine the order in the small mass ratios q/M or m/M that must be required in order to achieve the desired accuracy.

Step III Use the power counting result, eq. (59), to find which terms in the effective lagrangian are needed in order to compute to the desired order in q/M and m/M. Eq. (59) also determines which order in the loop expansion is required for each effective interaction of interest.

Step IVa Compute the couplings of the required effective interactions using the full underlying theory. If this step should prove to be impossible, due either to ignorance of the underlying theory or to the intractability of the required calculation, then it may be replaced by the following alternative:

Step IVb If the coefficients of the required terms in the effective lagrangian cannot be computed then they may instead be regarded as unknown parameters which are to be taken from experiment. Once a sufficient number of observables are used to determine these parameters, all other observables may be unambiguously predicted using the effective theory.

A number of points cry out for comment at this point.

• Utility of Step IVb: The possibility of treating the effective lagrangian phenomenologically, as in Step IVb above, immeasurably broadens the utility of effective lagrangian techniques, since they need not be restricted to situations for which the underlying theory is both known and calculationally simple. Implicit in such a program is the underlying assumption that there is no loss of generality in working with a local field theory. This assumption has been borne out in all known examples of physical systems. It is based on the conviction that the restrictions which are implicit in working with local field theories are simply

those that follow from general physical principles, such as unitarity and cluster decomposition.

• *When to expect renormalizability:* Since eq. (59) states that only a finite number of terms in $\mathcal{L}_{\rm eff}$ contribute to any fixed order in q/M, and these terms need appear in only a finite number of loops, it follows that only a finite amount of labour is required to obtain a fixed accuracy in observables. Renormalizable theories represent the special case for which it suffices to work only to zeroth order in the ratio q/M. This can be thought of as being the reason why renormalizable theories play such an important role throughout physics.

• *How to predict using nonrenormalizable theories:* An interesting corollary of the above observations is the fact that only a finite number of renormalizations are required in the low-energy theory in order to make finite the predictions for observables to any fixed order in q/M. Thus, although an effective lagrangian is not renormalizable in the traditional sense, it nevertheless *is* predictive in the same way that a renormalizable theory is.

4 Applications

We next turn briefly to a few illustrative applications of these techniques.

4.1 Quantum Electrodynamics

The very lightest electromagnetically interacting elementary particles are the photon and the electron, and from the general arguments given above we expect that the effective field theory which describes their dominant low-energy interactions should be renormalizable, corresponding to the neglect of all inverse masses heavier than the electron. The most general such renormalizable interactions which

are possible given the electron charge assignment is

$$\mathcal{L}_{\rm QED}(A_\mu,\psi) = -\frac{1}{4}\,F_{\mu\nu}F^{\mu\nu} - \overline{\psi}(\not{D} + m_e)\psi, \tag{60}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ gives the electromagnetic field strength in terms of the electromagnetic potential, A_μ, and the electron field, ψ, is a (four-component) Dirac spinor. The covariant derivative for the electron field is defined by $D_\mu\psi = \partial_\mu\psi + ieA_\mu\psi$, where e is the electromagnetic coupling constant.

Notice that this is precisely the lagrangian of Quantum electrodynamics (QED). In this we have the roots of an explanation of *why* QED is such a successful description of electron-photon interactions.

4.1.1 Integrating Out the Electron: Many practical applications of electromagnetism involve the interaction of photons with macroscopic electric charge and current distributions at energies, E, and momenta, p, much smaller than the electron mass, m_e. As a result they fall within the purview of low-energy techniques, and so lend themselves to being described by an effective theory which is defined below the electron mass, m_e, as is now described.

For present purposes it suffices to describe the macroscopic charge distributions using an external electromagnetic current, $J^\mu_{\rm em}$, which can be considered as an approximate, mean-field, description of a collection of electrons in a real material. This approximation is extremely good for macroscopically large systems, if these systems are only probed by electromagnetic fields whose energies are very small compared to their typical electronic energies. We take the interaction term coupling the electromagnetic field to this current to be the lowest dimension interaction that is possible:

$$\mathcal{L}_J = -e\,A_\mu\,J^\mu_{\rm em}. \tag{61}$$

This coupling is only consistent with electromagnetic gauge invariance if the external current is identically conserved: $\partial_\mu J^\mu_{\rm em} = 0$, independent of any equations of motion, and falls off sufficiently quickly to ensure that there is no current flow at spatial infinity. Two practical examples of such conserved configurations would be those of (i) a static charge distribution: $J^0_{\rm em} = \rho(\mathbf{r})$, $\mathbf{J}_{\rm em} = 0$, or (ii) a static electrical current: $J^0_{\rm em} = 0$ and $\mathbf{J}_{\rm em} = \mathbf{j}(\mathbf{r})$, where $\rho(\mathbf{r})$ and $\mathbf{j}(\mathbf{r})$ are localized, time-independent distributions, satisfying $\nabla \cdot \mathbf{j} = 0$.

Our interest is in the properties of electromagnetic fields outside of such distributions, and of tracking in particular the low-energy effects of virtual electrons. The most general effective theory involving photons only which can govern the low-energy limit is:

$$\mathcal{L}_4 = \mathcal{L}_{\rm eff}(A) - e\, A_\mu\, J^\mu_{\rm em}. \tag{62}$$

where $\mathcal{L}_{\rm eff}$ may be expanded in terms of interactions having successively higher dimensions. Writing $\mathcal{L}_{\rm eff} = \mathcal{L}_4 + \mathcal{L}_6 + \mathcal{L}_8 + \cdots$ we have

$$\mathcal{L}_4 = -\frac{Z}{4}\, F_{\mu\nu}F^{\mu\nu}, \tag{63}$$

$$\mathcal{L}_6 = \frac{a}{m_e^2}\, F_{\mu\nu}\Box F^{\mu\nu} + \frac{a'}{m_e^2}\, \partial_\mu F^{\mu\nu}\partial^\lambda F_{\lambda\nu},$$

$$\mathcal{L}_8 = \frac{b}{m_e^4}\, (F_{\mu\nu}F^{\mu\nu})^2 + \frac{c}{m_e^4}\, (F_{\mu\nu}\tilde{F}^{\mu\nu})^2 + (\partial^4 F^2 \text{ terms}), \tag{64}$$

and so on. In this expression for $\mathcal{L}_8$, $\tilde{F}$ represents the 'dual' field-strength tensor, defined by $\tilde{F}_{\mu\nu} = \frac{1}{2}\,\epsilon_{\mu\nu\lambda\rho}F^{\lambda\rho}$, and possible terms involving more derivatives exist, but have not been written. A power of $1/m_e$ has been made explicit in the coefficient of each term, since we work perturbatively in the electromagnetic coupling, and m_e is the only mass scale which appears in the underlying QED lagrangian. With this power extracted, the remaining constants, Z, a, a', b, c, *etc.*, are dimensionless.

Notice that no $\partial^2 F^3$ terms appear since any terms involving an odd number of F's are forbidden by charge-conjugation invariance, for which $F_{\mu\nu} \to -F_{\mu\nu}$. This is Furry's theorem in its modern, low-energy, guise.

Notice also that we may use the freedom to redefine fields to rewrite the $1/m_e^2$ terms in terms of direct current-current 'contact' interactions. The simplest way to do so is to rewrite these interactions using the lowest-dimension equations of motion, $\partial_\mu F^{\mu\nu} = eJ^\nu_{\rm em}$ – and so also $\Box F^{\mu\nu} = e(\partial^\mu J^\nu_{\rm em} - \partial^\nu J^\mu_{\rm em})$ – following the general arguments of earlier sections. Together with an integration by parts, this allows $\mathcal{L}_6$ to be written

$$\mathcal{L}_6 = \frac{e^2}{m_e^2}\,(a' - 2a)\, J^\nu_{\rm em} J_{{\rm em}\,\nu}. \tag{65}$$

This shows in particular that these interactions are irrelevant for photon propagation and scattering.[6]

In principle, we may now proceed to determine which of these effective interactions are required when working to a fixed order in $1/m_e^2$ in any given physical observable. Having done so, we may then compute the relevant dimensionless coefficients Z, $a' - 2a$, b, c *etc.*. and thereby retrieve the low-energy limit of the full QED prediction for the observable in question.

4.1.2 The Scattering of Light by Light: Inspection of the effective lagrangian, eq. (63), shows that the simplest interaction to receive contributions to $\mathcal{O}(m_e^{-4})$ is the scattering of light by light, for which the leading contribution to the cross section at centre-of-mass energies $E_{\rm cm} \ll m_e$ can be inferred relatively easily.

The first step is to determine precisely which Feynman graphs built from the interactions in the effective lagrangian of eq. (63) contribute, order-by-order in

[6] Things are more interesting if conducting boundaries are present however, see (9).

$1/m_e$. For this we may directly use the general power-counting results of the previous chapter, since the effective lagrangian of eq. (63) is a special case of the form considered in eq. (51), with the appropriate dimensionful constants being $f = M = v = m_e$. Directly using eq. (59) for the E-point scattering amplitude, $A_{\scriptscriptstyle E}(q)$, leads in the present case to:

$$A_{\scriptscriptstyle E}(q) \sim q^2 m_e^2 \left(\frac{1}{m_e}\right)^E \left(\frac{q}{4\pi\, m_e}\right)^{2L} \left(\frac{q}{m_e}\right)^{\sum_{ik}(k-2)V_{ik}} . \tag{66}$$

We may also use some specific information for the QED lagrangian which follows from the gauge invariance of the problem. In particular, since the gauge potential, A_μ, only appears in $\mathcal{L}_{\rm eff}$ through its field strength, $F_{\mu\nu}$, all of the interactions of the effective theory must contain at least as many derivatives as they have powers of A_μ. In equations: $V_{ik} = 0$ unless $k \geq i$. In particular, since $k = 2$ therefore implies $i \leq 2$, we see that the only term in $\mathcal{L}_{\rm eff}$ having exactly two derivatives is the kinetic term, $F_{\mu\nu}F^{\mu\nu}$. Since this is purely quadratic in A_μ, it is not an interaction (rather, it is the unperturbed lagrangian), and so we may take $V_{ik} = 0$, for $k \leq 2$. A consequence of these considerations is the inequality $\sum_{ik}(k-2)V_{ik} \geq 2$, and this sum equals 2 only if $V_{ik} = 0$ for all $k > 4$, and if $V_{i4} = 1$.

For two-body photon-photon scattering we may take $E = 4$, and from eq. (66) it is clear that the minimum power of q/m_e which can appear in $A_4(q)$ corresponds to taking (*i*) $L = 0$, and (*ii*) $V_{ik} = 0$ for $k \neq 4$, and $V_{i4} = 1$ for precisely one vertex for which $k = 4$. This tells us that the only graph which is relevant for photon-photon scattering at leading order in $E_{\rm cm}/m_e$ is the one shown in Fig. (3), which uses precisely one of the two vertices in $\mathcal{L}^{(8)}$, for which $i = k = 4$, and gives a result of order q^6/m_e^8 (where $q \sim E_{\rm cm}$).

The effective lagrangian really starts saving work when sub-leading contribu-

tions are computed. For photon-photon scattering eq. (66) implies the terms suppressed by two additional powers of $E_{\rm cm}/m_e$ requires one insertion of the $k = 4, i = 2$ interaction, but since this is in $\mathcal{L}_6$ it cannot contribute, making the next-to-leading contribution down by at least $(E_{\rm cm}/m_e)^4$ relative to the leading terms.

The leading term comes from using the quartic interactions of $\mathcal{L}_8$ in eq. (63) in the Born approximation. This gives the following differential cross section for unpolarized photon scattering, in the CM frame:

$$\frac{d\sigma_{\gamma\gamma}}{d\Omega} = \frac{278}{65\pi^2}\ \left[(b+c)^2 + (b-c)^2\right]\ \left(\frac{E_{\rm cm}^6}{m_e^8}\right)\left(3+\cos^2\theta\right)^2\ \left[1+\mathcal{O}\left(\frac{E_{\rm cm}^4}{m_e^4}\right)\right]. \tag{67}$$

Here $E_{\rm cm}$ is the energy of either photon in the CM frame, $d\Omega$ is the differential element of solid angle for one of the outgoing photons, and θ is the angular position of this solid-angle element relative to the direction of (either of) the incoming photons. Notice that, to this point, we have used only the expansion in powers of $E_{\rm cm}/m_e$, and have *not* also expanded the cross section in powers of $\alpha = e^2/4\pi$.

The final result for QED is reproduced once the constants b and c are computed from the underlying theory, and it is at this point that we first appeal to perturbation theory in α. To lowest order in α, the relevant graph is shown in Fig. (4). This graph is ultraviolet finite so no counterterms are required, and the result is simply:

$$b = \frac{4}{7}\ c = \frac{\alpha^2}{90}, \tag{68}$$

leading to the standard result (10)

$$\frac{d\sigma_{\gamma\gamma}}{d\Omega} = \frac{139}{4\pi^2}\ \left(\frac{\alpha^2}{90}\right)^2\ \left(\frac{E_{\rm cm}^6}{m_e^8}\right)\left(3+\cos^2\theta\right)^2\ \left[1+\mathcal{O}\left(\frac{E_{\rm cm}^4}{m_e^4}\right)\right]. \tag{69}$$

4.1.3 Renormalization and Large Logs: We next return to the terms in $\mathcal{L}_{\rm eff}$ which are unsuppressed at low energies by powers of $1/m_e$, for both practical and pedagogical reasons. Pedagogy is served by using this to introduce the *Decoupling Subtraction* renormalization scheme, which is the natural generalization of minimal subtraction to the effective lagrangian framework. The practical purpose of this example is to demonstrate how classical electromagnetism gives an *exact* description of photon response at low energies, whose corrections require powers of $1/m_e$ (and *not*, for example, simply more powers of α).

The only term in the effective lagrangian of eq. (63) which is not suppressed by powers of $1/m_e$ is the term

$$\mathcal{L}_4 = -\frac{Z}{4}\, F_{\mu\nu}F^{\mu\nu} - e\, A_\mu\, J^\mu_{\rm em}. \tag{70}$$

All of the influence of the underlying physics appears here only through the dimensionless parameter, Z, whose leading contribution from virtual electrons may be explicitly computed in QED to be

$$Z = 1 - \frac{\alpha}{3\pi}\left[\frac{1}{\varepsilon} + k + \log\left(\frac{m_e^2}{\mu^2}\right)\right], \tag{71}$$

where we regulate ultraviolet divergences using dimensional regularization. k is the constant encountered in section 2.2.3, which appears universally with the divergence, $1/\varepsilon$, in dimensional regularization, and μ is the usual (arbitrary) mass scale introduced in dimensional regularization to keep the coupling constant, e, dimensionless.

The physical interpretation of Z is found by performing the rescaling $A_\mu = Z^{-\frac{1}{2}}\, A^R_\mu$, to put the photon kinetic term into canonical form. In this case we recover the effective theory

$$\mathcal{L}_4 = -\frac{1}{4}\, F_{\mu\nu}F^{\mu\nu} - e_{\rm phys}\, A_\mu\, J^\mu_{\rm em}. \tag{72}$$

where $e \equiv Z^{\frac{1}{2}}\, e_{\rm phys}$. We see that to leading order virtual electrons affect low-energy photon properties only through the *value* taken by the physical electric charge, $e_{\rm phys}$, and does not otherwise affect electromagnetic properties.

This conclusion has important practical implications concerning the accuracy of the calculations of electromagnetic properties at low energies in QED. It states that the justification of simply using Maxwell's equations to describe photon properties at low energies is the neglect of terms of order $1/m_e$, and not the neglect of powers of α. Thus, for example, even though the classical formulae for the scattering of electromagnetic waves by a given charge distribution is a result which is obtained only in tree approximation in QED, corrections to these formulae do not arise at any order in α unsuppressed by powers of $E_{\rm cm}/m_e$. To zeroth order in $1/m_e$ the sole affect of all higher-loop corrections to electromagnetic scattering is to renormalize the value of α in terms of which all observables are computed. The significance of this renormalization enters once it is possible to measure the coupling at more than one scale, since then the logarithmic running of couplings with scale causes real physical effects, in particular encoding the potential dependence of observables on some of the logarithms of large mass ratios.

To see how to extract large logarithms most efficiently, in this section we first contrasting two useful renormalization schemes. The first of these is the one defined above, in which all of Z is completely absorbed into the fields and couplings:

$$\begin{aligned} & A_\mu = Z_{\rm phys}^{-\frac{1}{2}}\, A_\mu^{\rm phys}, \qquad \text{and} \qquad e = Z_{\rm phys}^{\frac{1}{2}}\, e_{\rm phys}, \\ \text{with} \qquad & Z_{\rm phys} = Z = 1 - \frac{\alpha}{3\pi}\left[\frac{1}{\varepsilon} + k + \log\left(\frac{m_e^2}{\mu^2}\right)\right]. \end{aligned} \tag{73}$$

The subscript 'phys' emphasizes that the charge, $e_{\rm phys}$, is a physical observable

whose value can be experimentally determined, in principle. For instance, it could be measured by taking a known static charge distribution, containing a predetermined number of electrons, and then using Maxwell's equations to predict the resulting flux of electric field at large distances from these charges. Comparing this calculated flux with the measured flux gives a measurement of $e_{\rm phys}$.

The alternative scheme of choice for most practical calculations is the $\overline{MS}$ scheme, for which the renormalization is defined to subtract only the term $1/\varepsilon + k$ in Z. That is:

$$\begin{aligned} &A_\mu = Z_{\overline{MS}}^{-\frac{1}{2}}\, A_\mu^{\overline{MS}}, \qquad \text{and} \qquad e = Z_{\overline{MS}}^{\frac{1}{2}}\, e_{\overline{MS}}, \\ &\text{with} \qquad Z_{\overline{MS}} = 1 - \frac{\alpha}{3\pi}\left[\frac{1}{\varepsilon} + k\right]. \end{aligned} \tag{74}$$

In terms of this scheme the effective lagrangian becomes, to this order in α:

$$\mathcal{L}_{\rm eff} = -\frac{1}{4}\left[1 - \frac{\alpha}{3\pi}\log\left(\frac{m_e^2}{\mu^2}\right)\right]\, F_{\mu\nu}^{\overline{MS}} F_{\overline{MS}}^{\mu\nu} - e_{\overline{MS}}\, A_\mu^{\overline{MS}}\, J_{\rm em}^\mu. \tag{75}$$

The $\overline{MS}$ coupling defined in this way is not itself a physical quantity, but is simply a parameter in terms of which the effective lagrangian is expressed. The physical coupling $\alpha_{\rm phys} = e_{\rm phys}^2/4\pi$ and the $\overline{MS}$ coupling $\alpha_{\overline{MS}} = e_{\overline{MS}}^2/4\pi$ are related in the following way:

$$\alpha_{\overline{MS}} = \left(\frac{Z_{\rm phys}}{Z_{\overline{MS}}}\right)\, \alpha_{\rm phys} = \left[1 - \frac{\alpha}{3\pi}\log\left(\frac{m_e^2}{\mu^2}\right)\right]\, \alpha_{\rm phys}. \tag{76}$$

A key observation is that, since $\alpha_{\rm phys}$ is a physical quantity, it cannot depend on the arbitrary scale μ. As a result, this last equation implies a μ-dependence for $\alpha_{\overline{MS}}$. It is only for $\mu = m_e$ that the two couplings agree:

$$\alpha_{\overline{MS}}(\mu = m_e) = \alpha_{\rm phys}. \tag{77}$$

As usual, there is profit to be gained by re-expressing the μ dependence of $\alpha_{\overline{MS}}$

of eq. (76) as a differential relation:

$$\mu^2 \frac{d\alpha_{\overline{MS}}}{d\mu^2} = + \frac{\alpha^2_{\overline{MS}}}{3\pi}. \tag{78}$$

This is useful because the differential expression applies so long as $\alpha_{\overline{MS}} \ll 1$, while eq. (76) also requires $\alpha_{\overline{MS}} \log(m_e^2/\mu^2) \ll 1$. Integrating eq. (78) allows a broader inference of the μ-dependence of $\alpha_{\overline{MS}}$:

$$\frac{1}{\alpha_{\overline{MS}}(\mu)} = \frac{1}{\alpha_{\overline{MS}}(\mu_0)} - \frac{1}{3\pi} \log\left(\frac{\mu^2}{\mu_0^2}\right), \tag{79}$$

which is accurate to all orders in $(\alpha/3\pi)\log(m_e^2/\mu^2)$, so long as $(\alpha/3\pi) \ll 1$.

Eq. (79) is useful because it provides a simple way to keep track of how some large logarithms appear in physical observables. For instance, consider the cross section for the scattering of electrons (plus an indeterminate number of soft photons, having energies up to $E_{\max} = fE$) with $1 > f \gg m_e/E$). Such a quantity has a smooth limit as $m_e/E \to 0$ when it is expressed in terms of $\alpha_{\overline{MS}}(\mu)$ (11,12), so on dimensional grounds we may write

$$\sigma(E, m_e, \alpha_{\rm phys}) = \frac{1}{E^2} \left[\mathcal{F}\left(\frac{E}{\mu}, \alpha_{\overline{MS}}(\mu), f, \theta_k\right) + O\left(\frac{m_e}{E}\right)\right], \tag{80}$$

where the θ_k denote any number of dimensionless quantities (like angles) on which the observable depends, and the explicit μ-dependence of $\mathcal{F}$ must cancel the μ-dependence which appears implicitly through $\alpha_{\overline{MS}}(\mu)$. However $\mathcal{F}$ *is* singular when $m_e/E \to 0$ when it is expressed in terms of $\alpha_{\rm phys}$, because of the appearance of large logarithms. These may be included to all orders in $\alpha \log(E^2/m_e^2)$ simply by choosing $\mu = E$ in eq. (80) and using eq. (79) with eq. (77).

For the next section it is important that eq. (78) integrates so simply because the $\overline{MS}$ renormalization is a *mass-independent* scheme. That is, $d\alpha/d\mu$ depends only on α and does not depend explicitly on mass scales like m_e. ('On shell'

renormalizations, such as where e is defined in terms of the value of a scattering amplitude at a specific momentum transfer, furnish examples of schemes which are not mass-independent.)

4.1.4 Muons and the Decoupling Subtraction scheme: Consider now introducing a second scale into the problem by raising the energies of interest to those above the muon mass. In this case the underlying theory is

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} - \overline{\psi}(\not{D} + m_e)\psi - \overline{\chi}(\not{D} + M_\mu)\chi, \tag{81}$$

where χ is the Dirac spinor representing the muon, and M_μ is the muon mass. Our interest is in following how large logarithms like $\log(M_\mu/m_e)$ appear in observables.

In this effective theory the $\overline{MS}$ and physical electromagnetic couplings are related to one another by:

$$\alpha_{\overline{MS}} = \left\{1 - \frac{\alpha}{3\pi}\left[\log\left(\frac{m_e^2}{\mu^2}\right) + \log\left(\frac{M_\mu^2}{\mu^2}\right)\right]\right\}\ \alpha_{\rm phys}. \tag{82}$$

This relation replaces eq. (76) of the purely electron-photon theory. Notice, in particular, that in this theory the physical coupling, $\alpha_{\rm phys}$ is no longer simply equal to $\alpha_{\overline{MS}}(\mu = m_e)$ but is now equal to $\alpha_{\overline{MS}}(\mu = \sqrt{m_e M_\mu})$. The corresponding RG equation for the running of $\alpha_{\overline{MS}}$ therefore becomes:

$$\mu^2 \frac{d\alpha_{\overline{MS}}}{d\mu^2} = +\frac{2\alpha^2}{3\pi}, \tag{83}$$

and its solution is

$$\frac{1}{\alpha_{\overline{MS}}(\mu)} = \frac{1}{\alpha_{\overline{MS}}(\mu_0)} - \frac{2}{3\pi}\ \log\left(\frac{\mu^2}{\mu_0^2}\right). \tag{84}$$

Here we see an inconvenience of the $\overline{MS}$ renormalization scheme: the right-hand-side of eq. (83) is twice as large as its counterpart, eq. (78), in the pure electron-photon theory, simply because the mass-independence of the $\overline{MS}$ scheme

ensures that both the electron and the muon contribute equally to the running of $\alpha_{\overline{MS}}$. The problem is that this is equally true for all μ, and it even applies at scales $\mu \ll M_\mu$, where we expect the physical influence of the muon to decouple. Of course, the physical effects of the muon indeed *do* decouple at scales well below the muon mass, it is just that this decoupling is not manifest at intermediate steps in any calculation performed with the $\overline{MS}$ scheme.

The dimensionally-regularized effective lagrangian furnishes a way to circumvent this disadvantage, by keeping the decoupling of heavy particles manifest without giving up the benefits of a mass-independent renormalization scheme. The remedy is to work with minimal subtraction, but to do so only when running couplings in an energy range between charged-particle thresholds. As the energy falls below each charged-particle threshold, a new effective theory is defined by 'integrating out' this particle, with the couplings in the new low-energy theory found by matching to the coupling defined in the underlying theory above the relevant mass scale. The scheme defined by doing so through all particle thresholds is called the *decoupling subtraction* ($\overline{DS}$) renormalization scheme (13).

For instance, for the electrodynamics of electrons and muons, the coupling constant as defined in the $\overline{MS}$ and $\overline{DS}$ schemes is identical for the full theory which describes energies greater than the muon mass: $\mu > M_\mu$. For $m_e < \mu < M_\mu$ we integrate out the muon and construct an effective theory involving only photons and electrons. This effective theory consists of the usual QED lagrangian, plus an infinite number of higher-dimension effective interactions encoding the low-energy implications of virtual muons. Within this effective lagrangian the coupling constant is again defined by the coefficient Z in front of the $F_{\mu\nu}F^{\mu\nu}$ term, using minimal subtraction. But, because there is no muon within this

effective theory, only the electron contributes to its running.

The initial conditions for the RG equation at the muon mass is obtained by matching, as in the previous sections. They are chosen to ensure that the effective theory reproduce the same predictions for all physical quantities as does the full theory, order-by-order in the low-energy expansion. If there were other charged particles in the problem, each of these could be integrated out in a similar fashion as μ falls below the corresponding particle threshold.

Quantitatively, to one loop the RG equation for the $\overline{DS}$ scheme for the theory of electrons, muons and photons becomes:

$$\begin{aligned} \mu^2 \frac{d\alpha_{\overline{DS}}}{d\mu^2} &= \frac{2\alpha^2}{3\pi}, \qquad \text{if} \qquad \mu > M_\mu; \\ &= \frac{\alpha^2}{3\pi}, \qquad \text{if} \qquad m_e < \mu < M_\mu; \\ &= 0, \qquad \text{if} \qquad \mu < m_e, \end{aligned} \tag{85}$$

with the boundary conditions that the $\alpha_{\overline{DS}}$ should be continuous at $\mu = M_\mu$ and $\alpha_{\overline{DS}}(\mu = m_e) = \alpha_{\rm phys}$ at $\mu = m_e$. Integrating then gives

$$\begin{aligned} \frac{1}{\alpha_{\overline{DS}}(E)} &= \frac{1}{\alpha_{\rm phys}} - \frac{1}{3\pi} \log\left(\frac{E^2}{M_\mu^2}\right) \qquad \text{for } m_e < E < M_\mu, \\ &= \frac{1}{\alpha_{\rm phys}} - \frac{1}{3\pi} \log\left(\frac{M_\mu^2}{m_e^2}\right) - \frac{2}{3\pi} \log\left(\frac{E^2}{M_\mu^2}\right) \quad \text{for } M_\mu < E. \end{aligned} \tag{86}$$

This last expression shows how to efficiently display the various large logarithms by running a coupling with the ease of a mass-independent scheme, but with each particle explicitly decoupling as μ drops through the corresponding particle threshold.

4.2 Power-counting examples: QCD and Gravity

We close with a sketch of the utility of eq. (59) for two important examples: the interactions amongst pions and kaons at energies well below a GeV; and

gravitational self-interactions for macroscopic systems.

4.2.1 Below the QCD scale: mesons We start with the interactions of pions and kaons at energies below a GeV. This represents a useful low-energy limit of the Standard Model because these mesons are Goldstone bosons for the spontaneous breaking of an approximate symmetry of the strong interactions (14). As such, their interactions are suppressed in this low-energy limit, as a general consequence of Goldstone's theorem (15). Because they interact so weakly we represent them with fundamental scalar fields in the effective theory which applies at energies $E \ll \Lambda \sim 1$ GeV (**?**).

The resulting scalar lagrangian has the form of eq. (51), with the constants appearing in the effective lagrangian being: $f = \sqrt{F_\pi \Lambda}$, $M = \Lambda$ and $v = F_\pi$, where $F_\pi \sim 100$ MeV defines the scale of the order parameter which describes the spontaneous breaking of the relevant approximate symmetry. In this case the powercounting estimate of eq. (59) becomes:

$$A_{\scriptscriptstyle E}(q) \sim F_\pi^2 q^2 \left(\frac{1}{F_\pi}\right)^E \left(\frac{q}{4\pi F_\pi}\right)^{2L} \left(\frac{q}{\Lambda}\right)^{\sum_{ik}(k-2)V_{ik}}, \qquad (87)$$

which is a famous result, due first to Weinberg (16). The explicit suppression of all interactions by powers of $q/\Lambda \sim q/(4\pi F_\pi)$ explicitly encodes the suppression of interactions that Goldstone's theorem requires. Since the pion mass is $m_\pi \sim 140$ MeV, this suppression is clearly only suppressed for scattering at energies near threshhold, $E_{\rm cm} \sim m_\pi$.

The dominant terms in $\mathcal{L}_{\rm eff}$ which govern the scattering at these energies corresponds to choosing the smallest possible value for which $L = 0$, and $V_{ik} \neq 0$ only if $k = 2$. Since it happens that symmetries determine the effective couplings of all such terms purely in terms of F_π (in the limit of massless quarks), a great deal can be said about such low-energy meson interactions without knowing any

of the dynamical details about their explicit wave-functions. These predictions are consequences of the symmetry-breaking pattern, and are known as 'soft pion' theorems. Comparison of these predictions, including next-to-leading corrections and nonzero quark masses, are in good agreement with observations (17).

4.2.2 General Relativity as an Effective Field Theory It is instructive to repeat this powercounting analysis for the gravitational effective theory, since this case furnishes a less familiar example. The result obtained also justifies the neglect of quantum effects in performing practical calculations with gravity on macroscopic scales. Even better: it permits the systematic calculation of the leading corrections in the semiclassical limit, should these ever be desired.

The field relevant for gravity (18) is the metric, $g_{\mu\nu}$ (whose matrix inverse is denoted $g^{\mu\nu}$). For applications on macroscopic scales we use the most general effective lagrangian consistent with general covariance:[7]

$$-\mathcal{L}_{\rm eff} = \sqrt{-g}\left[\frac{1}{2}\,M_p^2 R + c_1\,R^2 + c_2\,R_{\mu\nu}R^{\mu\nu} + c_3 R_{\mu\nu\lambda\rho}R^{\mu\nu\lambda\rho} + \frac{e_1}{m_e^2}R^3 + \cdots\right], \tag{88}$$

where $g = \det(g_{\mu\nu})$, while $R_{\rho\mu\lambda\nu}$, $R_{\mu\nu} = g^{\lambda\rho}R_{\lambda\mu\rho\nu}$ and $R = g^{\mu\nu}R_{\mu\nu}$ respectively denote the Riemann and Ricci tensors, and the Ricci scalar, each of which involves two derivatives of $g_{\mu\nu}$. The ellipses denote terms involving at least six derivatives, one term of which is displayed explicitly in eq. (88). The term linear in R is the usual Einstein-Hilbert action, with M_p denoting the usual Planck mass. The remaining effective couplings — c_k and e_k — are dimensionless, and not all of the terms written need be independent of one another. The scale m_e denotes the lightest particle (say, the electron) to have been integrated out to obtain this

[7] No cosmological term is written here since this precludes a perturbative expansion about flat space.

effective lagrangian.

Eq. (88) has the form considered earlier, with $f = \sqrt{m_e M_p}$, $\Lambda = m_e$ and $v = M_p$. Furthermore, with these choices, the dimensionless couplings of all of the interactions *except* for the Einstein term, are explicitly suppressed by the factor m_e^2/M_p^2. With these choices the central powercounting result, eq. (59), becomes (19, 20):

$$\begin{aligned} \overline{\mathcal{A}}_E(q) &\sim q^2 M_p^2 \left(\frac{1}{M_p}\right)^E \left(\frac{q}{4\pi M_p}\right)^{2L} \left(\frac{m_e^2}{M_p^2}\right)^{\sum_{i;k>2} V_{ik}} \left(\frac{q}{m_e}\right)^{\sum_{ik}(k-2)V_{ik}} \\ &\sim q^2 M_p^2 \left(\frac{1}{M_p}\right)^E \left(\frac{q}{4\pi M_p}\right)^{2L} \left(\frac{q^2}{M_p^2}\right)^{\sum_{i;k>2} V_{ik}} \left(\frac{q}{m_e}\right)^{\sum_{i;k>4}(k-4)V_{ik}} \end{aligned} \tag{89}$$

where covariance requires $V_{ik} = 0$ unless $k = 2, 4, 6...$, with $k = 2$ corresponding to the Einstein-Hilbert term, $k = 4$ the curvature-squared terms, and so on.

As before, the dominant term comes from choosing $L = 0$ and using only the interactions of the usual Einstein-Hilbert action: *i.e.* $V_{ik} = 0$ for $k > 2$. The dominant contribution to gravitational physics is therefore obtained by working to tree level with the Einstein action, which is to say that one is to compute the classical response of the gravitational field, using the full Einstein equations to compute this response.

The graphs responsible for the next-to-leading terms are also simple to determine. The minimum additional suppression by q/M_p is obtained either by working to one loop order ($L = 1$) using the Einstein action ($V_{ik} = 0$ for $k > 2$), or by working to tree level ($L = 0$) using precisely one insertion of one of the curvature-squared interactions (*i.e.* with V_{i2} arbitrary but $V_{i4} = 1$ for one interaction with $k = 4$). Both cases give an additional suppression of q^2/M_p^2 relative to the leading contribution. The one-loop contribution also carries the usual additional loop factor, $(1/4\pi)^2$.

Notice that the derivative expansion is an expansion in q/m_e as well as in q/M_p, due to the inverse powers of m_e which appear in the higher-curvature terms. Notice also that all of the m_e-dependence drops out for graphs constructed only using the Einstein and the curvature-squared terms (*i.e.* $V_{ik} = 0$ for $k > 4$), as it should since $1/m_e^2$ first enters eq. (88) at order curvature-cubed. Although the condition $q \ll m_e$ may come as something of a surprise, it is nevertheless an excellent expansion for macroscopic applications, such as in the solar system.

Acknowledgements

This review is based on a series of lectures given for the Swiss Troisième Cycle in Lausanne, and in the University of Oslo, in June 1995, whose organizers I thank for their kind invitations, and whose students I thank for their questions and comments. My research during the preparation of these lectures was funded in part by NSERC (Canada), FCAR (Québec) and the Killam Foundation.

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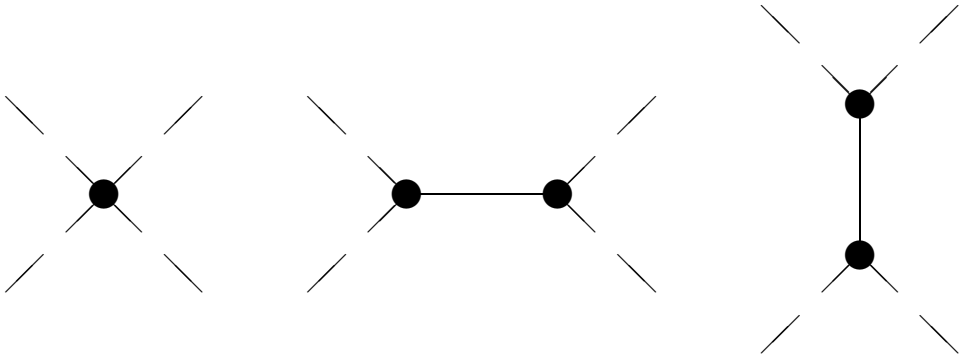

Figure 1: The Feynman graphs which contribute to two-body light-particle scattering at tree level in the Toy Model. Solid (dashed) lines represent the heavy (light) scalar.

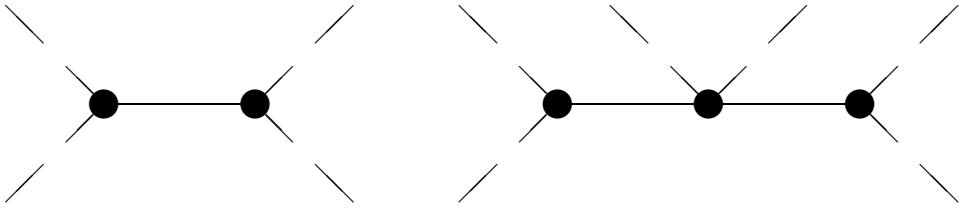

Figure 2: The Feynman graphs which contribute the corrections to the tree level generating functional, $\gamma[\ell]$, to order $1/M^4$. Solid (dashed) lines represent heavy (light) scalars.

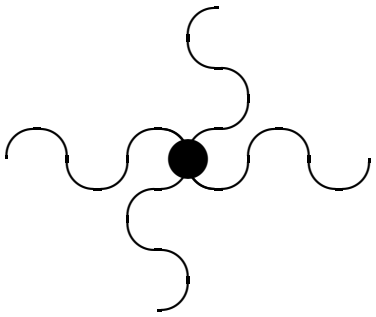

Figure 3: The Feynman graph which contributes the leading contribution to photon-photon scattering in the effective theory for low-energy QED. The vertex represents either of the two dimension-eight interactions discussed in the text.

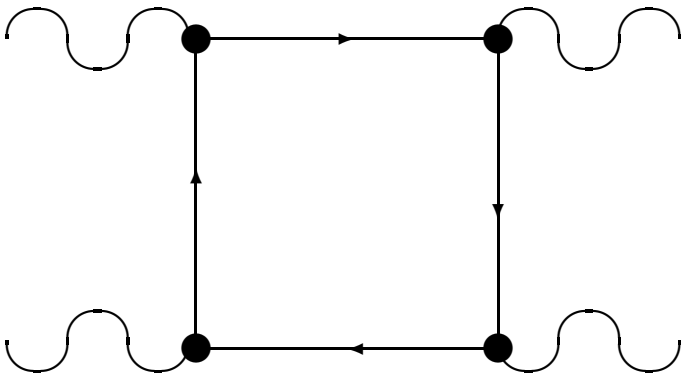

Figure 4: The leading Feynman graphs in QED which generate the effective four-photon operators in the low energy theory. Straight (wavy) lines represent electrons (photons).