

Phases of Dense Quarks at Large N_c

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Abstract

In the limit of a large number of colors, N_c , we suggest that gauge theories can exhibit several distinct phases at nonzero temperature and quark density. Two are familiar: a cold, dilute phase of confined hadrons, where the pressure is ~ 1 , and a hot phase of deconfined quarks and gluons, with pressure $\sim N_c^2$. When the quark chemical potential $\mu \sim 1$, the deconfining transition temperature, T_d , is independent of μ . For $T < T_d$, as μ increases above the mass threshold, baryons quickly form a dense phase where the pressure is $\sim N_c$. As illustrated by a Skyrme crystal, chiral symmetry can be both spontaneously broken, and then restored, in the dense phase. While the pressure is $\sim N_c$, like that of (non-ideal) quarks, the dense phase is still confined, with interactions near the Fermi surface those of baryons, and not of quarks. Thus in the chirally symmetric region, baryons near the Fermi surface are parity doubled. We suggest possible implications for the phase diagram of QCD.

1 Introduction

Many of the observed properties of QCD can be understood, at least qualitatively, by generalizing from three to a large number of colors, $N_c \rightarrow \infty$. [1,2,3]. For example, consider the phase transition at a nonzero temperature, T [4]. At low temperature confinement implies that all states are color singlets, such as mesons and glueballs, with a pressure $\sim N_c^0 \sim 1$. At high temperature gluons in the adjoint representation deconfine, contributing $\sim N_c^2$ to the pressure. Thus one can define the deconfining transition simply by the point where the term $\sim N_c^2$ in the pressure turns on. The transition temperature for deconfinement, T_d , is expected to be of order one at large N_c , on the order of a typical QCD mass scale, such as the renormalization mass parameter, $\Lambda_{QCD} \sim 200$ MeV. Arguments suggest that deconfinement is a strongly first order transition, with a latent heat $\sim N_c^2$. Since the free energy of N_f flavors of deconfined quarks is $\sim N_c N_f$ in the limit of large N_c , and small N_f , deconfinement probably drives chiral symmetry restoration at T_d . Several of these features have been confirmed by numerical simulations on the lattice [5].

In this paper we consider the phase diagram in the plane of temperature and quark chemical potential, μ [6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24]. It is usually assumed that for all T and μ , there is a single transition, at which both deconfinement and chiral symmetry restoration occurs, shaped something like a semi-circle. If the transition is crossover for $\mu = 0$ and $T \neq 0$ [17], there could be a chiral critical end point in the plane of T and μ [13]. From numerical simulations on the lattice [17,18] within errors the two transitions always appear to coincide, at least for $T \neq 0$ and small μ .

At large N_c , we find a very different phase diagram in the T - μ plane, in which the deconfining and chiral transitions split from one another. For $\mu \sim 1$, the deconfining transition temperature is independent of μ . At low temperatures and densities, there is the usual confined phase of hadrons, with chiral symmetry breaking. The confined phase is baryon free, as a Fermi sea of baryons first forms at a value near the lightest baryon mass. Within a narrow window in μ , $\sim 1/N_c^2$, there is then a rapid transition to a dense phase, with a pressure $\sim N_c$. The properties of dense phase(s) are illustrated by a Skyrme crystal [25,26]. Although the total pressure is $\sim N_c$, like that of quarks, the dense phases are confined, with interactions near the Fermi surface dominated by baryons. We suggest that the dense phase undergoes a chiral transition for $\mu \sim 1$. In the chirally symmetric phase, the baryons are parity doubled [27], consistent with the constraint of anomalies at nonzero density [28].

Admittedly, all of our arguments are merely qualitative. Even so, we think it worth pursuing them, because they are so different from naive expectation. Of course our analysis could simply be an artifact of the large N_c expansion, and of limited relevance to QCD, where $N_c = 3$. At the end, we suggest what our analysis might imply about the phase diagram of QCD.

2 Review of Large N_c

If g is the gauge coupling, the 't Hooft limit is to take $N_c \rightarrow \infty$, holding $g^2 N_c$ fixed [1,2]. This selects all planar diagrams of gluons. Holding N_f fixed as $N_c \rightarrow \infty$, quark loops are suppressed, and the only states which survive have a definite number of quarks and anti-quarks.

Mesons are composed of one quark anti-quark pair, and are free at infinite N_c : cubic interactions vanish $\sim 1/\sqrt{N_c}$, quartic interactions, $\sim 1/N_c$, *etc.* Glueballs are pure glue states, with no quarks or anti-quarks; their cubic interaction vanish $\sim 1/N_c$, and so on. Except for Goldstone bosons, the lightest bosons have masses $\sim \Lambda_{QCD}$.

In contrast, baryons are rather nontrivial. To form a color singlet, they have N_c quarks. Assuming that each quark has an energy of order Λ_{QCD} , the mass of a baryon $M_B \sim N_c \Lambda_{QCD}$ [2,3]. For instance, a gluon exchanged between any two quarks contributes $\sim g^2 N_c^2 \sim N_c$, and contributes to an average Hartree potential. Care must be taken with diagrams to higher order: one includes diagrams which modify the average potential, but not iterations

thereof. Thus two gluons exchanged between two different pairs of quarks is $\sim N_c^2$, but represents an iteration of the average potential. A diagram which modifies the average potential is given by three quarks which emit three gluons, and which then interact through a three gluon coupling, fig. (41) of [2]; this is $\sim g^4 N_c^3 \sim N_c$.

The scattering of two baryons begins with the exchange of two quarks between the baryons, with a gluon emitted between them, fig. (35) of [2]. Since each quark can have a different color, this diagram is $\sim g^2 N_c^2 \sim N_c$. It is also possible to exchange quarks of the same color without gluon emission; this is also $\sim N_c$, fig. (36) of [2]. As for the average potential in one baryon, the two body scattering amplitude remains $\sim N_c$ to higher order in g^2 . To see this, it is necessary to pick out contributions which are two baryon irreducible, from those which represent iterations of the two baryon potential.

Continuing, the scattering between three baryons is $\sim N_c$: if a quark in each baryon emits a single gluon, which interact through a three gluon interaction, the amplitude is $\sim g^4 N_c^3 \sim N_c$. Again, multiple gluon exchange does not produce higher powers of N_c , once one picks out interactions which are three baryon irreducible.

In general, the scattering of \mathcal{M} baryons is of order $\sim N_c$. Thus unlike mesons or glueballs, whose interactions vanish at infinite N_c , baryons interact strongly, with couplings of strength $\sim N_c$.

Of course baryon interactions can also be viewed as arising from the exchange of color singlet mesons. The coupling between a meson and a baryon $\sim \sqrt{N_c}$, and so single meson exchange gives a two baryon interaction $\sim N_c$, as above. While it appears that multiple meson exchange will lead to higher powers of N_c , this does not occur, due to an extended $SU(2N_f)$ symmetry [3]. These cancellations are subtle, and surely have analogies in nuclear matter. For our purposes, however, all we require is that the scattering of \mathcal{M} baryons is always of order $\sim N_c$.

The $SU(2N_f)$ symmetry implies the low energy spectrum of baryons is highly degenerate. The lowest mass baryons form multiplets of isospin, I , and spin, J [3]. These multiplets have $I = J$, from $1/2$ to $N_c/2$ for odd N_c . (For QCD, there is one such state, the Δ .) The splitting in energy between the states in these multiplets is of order $M_B \sim M(1 + \kappa J^2/N_c^2)$, where κ is a constant. These are the lightest states: there are other excited baryons with masses $\sim \Lambda_{QCD}$ above the lightest.

At zero temperature, there is no Fermi sea until the chemical potential exceeds the mass of the fermion. Let M be the mass of the lightest baryon, $M \sim N_c$ [2]. It is natural to define a ‘‘constituent’’ quark mass,

$$m_q = M/N_c ,$$

which is of order one at large N_c . Thus at $T = 0$, there is no Fermi sea until the baryon chemical potential $\mu_B > M$; for the quark chemical potential, this is $\mu > m_q$.

To illustrate how quarks enter at large N_c , consider the gluon self energy at nonzero T and μ . To lowest order in g^2 , at zero momentum this is gauge independent, equal to the square of the Debye mass. For N_f massless flavors, its trace equals

$$\Pi^{\mu\mu}(0) = g^2 \left(\left(N_c + \frac{N_f}{2} \right) \frac{T^2}{3} + \frac{N_f \mu^2}{2\pi^2} \right), \quad (1)$$

Taking $N_c \rightarrow \infty$, holding $g^2 N_c$ fixed, we see that the gluon contribution, $\sim g^2 N_c T^2 \sim T^2$, survives. This is the first in an infinite series of planar, gluon diagrams at infinite N_c . In contrast, whether for $T \neq 0$ and $\mu \neq 0$, the quark contribution is only $\sim g^2$, and so suppressed by $\sim 1/N_c$.

This is true order by order in perturbation theory, both in vacuum and for all T and $\mu \sim 1$: holding N_f fixed as $N_c \rightarrow \infty$, the effects of quarks loops are suppressed by $\sim 1/N_c$ [1,2]. This is simply because there are $\sim N_c^2$ gluons in the adjoint representation, but only $\sim N_c$ quarks in the fundamental representation. Since the quark contribution, relative to that of gluons, is $\sim N_f/N_c$, it is essential to hold N_f fixed as $N_c \rightarrow \infty$; *i.e.*, to take of limit of large N_c , but small N_f .

In this limit, we can immediately make some broad conclusions about the phase diagram in the $T - \mu$ plane. At $\mu = 0$, one expects that the deconfining transition temperature $T_d \sim \Lambda_{QCD}$ [4], which appears to be confirmed by numerical simulations on the lattice [5]. Since quarks don't affect the gluons, the deconfining transition temperature is then *independent* of μ , $T_d(\mu) = T_d(0)$ for values of $\mu \sim 1$. This is illustrated in fig. (1): in the plane of T and μ , the phase boundary for deconfinement is a straight line. The theory is in a deconfined phase when $T > T_d$, and in a confined phase for $T < T_d$.

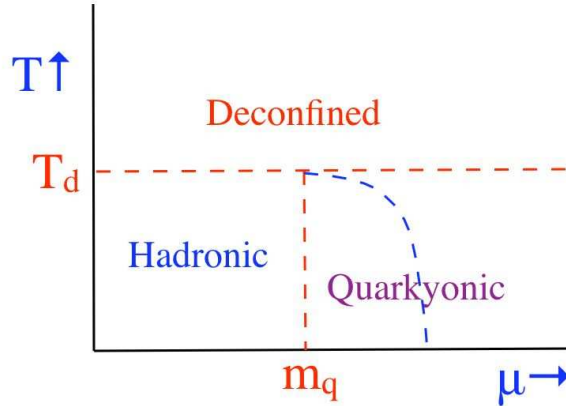


Fig. 1. Phase diagram at infinite N_c in the plane of temperature and quark chemical potential. The blue line in the quarkyonic phase indicates a guess for the position of the chiral phase transition.

In fact, consider the “box” in the lower, left hand corner of the $T - \mu$

plane, where $T < T_d$ and $\mu < m_q$, fig. (1). For particles of finite mass, as long as $T \neq 0$, one expects some population of fermions in the thermal ensemble. At large N_c , however, the baryons all have a mass $\sim N_c$; thus if $\mu < m_q$, as long as $T < T_d$, the relative abundance of baryons is $\sim \exp(-\kappa N_c)$, with κ a number of order one, and so the baryon abundance is exponentially small at large N_c . (There is a small window in which baryons can be excited, when $\mu - m_q \sim 1/N_c$, where it costs ~ 1 , and not $\sim N_c$, to excite baryons.)

This box, $T < T_d$ and $\mu < m_q$, is the usual, confined phase of hadrons. At nonzero temperature, the pressure ~ 1 is due exclusively to mesons and glueballs, with only exponentially small contributions from baryons. Even for $\mu > m_q$, the only baryons are those in the Fermi sea, or excitations thereof. Simply because they are too heavy, virtual baryon anti-baryon pairs never contribute at large N_c . This is not true if μ grows like a power of N_c , but we generally do not consider this regime, except following eq. (9).

Henceforth we concentrate on cool, dense quarks: remaining in the confined phase, $T < T_d$, and moving out in μ from m_q . This is the “quarkyonic” phase in fig. (1). We explain this terminology later, but stress that as it occurs for $T < T_d$ (and $\mu \sim 1$), that it *is* confined. We note that Cleymans and Redlich showed that in QCD, phenomenologically the boundary for chemical equilibration begins at $\mu \approx m_q = M_N/3 = 313$ MeV when $T = 0$ [14], reminiscent of fig. (1).

3 Narrow Window of Dilute Baryons

We start by working at zero temperature, very close to the point where a Fermi sea of baryons forms. The Fermi momentum for baryons, k_F , is $k_F^2 + M^2 = \mu_B^2$. If k_F is (arbitrarily) small, we have an ideal gas of baryons. For such a gas, the baryon density is $n(k_F) \sim k_F^3$, the energy is $\epsilon \sim k_F^2/2M$, and the pressure is

$$P_{\text{ideal baryons}} \sim n(k_F) \frac{k_F^2}{M} \sim \frac{1}{N_c} \frac{k_F^5}{\Lambda_{QCD}}. \quad (2)$$

For such a dilute gas of baryons, the pressure at $T = 0$ is very small, $\sim 1/N_c$.

Now consider increasing k_F further, so that the additional resonances of the baryon condense. There are several effects which enter. The first is to include the resonances with $I = J$, representing the generalization of the Δ , *etc.*, at large N_c . Each species contributes to the pressure as $(k_F^2 - (\kappa_I I^2 + \kappa_J J^2) \Lambda_{QCD}^2)^{5/2}/M$. Summing over spin and isospin gives a total contribution of order

$$\delta P_{\text{resonances}} \sim \frac{1}{M} \frac{k_F^8}{\Lambda_{QCD}^3} \sim \frac{1}{N_c} \frac{k_F^8}{\Lambda_{QCD}^4}. \quad (3)$$

Thus while the sum over resonances changes the dependence upon k_F , it still contributes to the pressure $\sim 1/N_c$.

In contrast, once the nucleon Fermi momentum increases, the effect of interactions quickly dominates. The amplitude for the four point interaction

between baryons includes a term $\sim N_c(\psi^\dagger\psi)^2/\Lambda_{QCD}^2$. The coupling for this interaction has dimensions of inverse mass squared, which we assume is typical of Λ_{QCD} . At low densities this interaction contributes of order density squared, or

$$\delta P_{\text{two body int.'s}} \sim N_c \frac{n(k_F)^2}{\Lambda_{QCD}^2} \sim N_c \frac{k_F^6}{\Lambda_{QCD}^2}. \quad (4)$$

Likewise, six point nucleon interactions contribute as density cubed, or $\sim N_c k_F^9/\Lambda_{QCD}^5$, etc.

Now clearly one cannot trust this series when the nucleon Fermi momentum is of order Λ_{QCD} . What is interesting is when the series breaks down. Consider balancing (2) and (4): the two terms are comparable when $k_F^5/N_c \sim N_c k_F^6$, or

$$k_F \sim \frac{1}{N_c^2} \Lambda_{QCD}. \quad (5)$$

Thus at *very* low nucleon Fermi momentum, $k_F \sim 1/N_c^2$, two body nucleon interactions are as important as the kinetic terms. Contributions from resonances, (3), are suppressed by one factor of the density, $k_F^3 \sim 1/N_c^6$, as are three body interactions.

As k_F increases beyond this point, contributions from the Fermi sea, even including the increasing number of states, are irrelevant. Instead, when $k_F \sim \Lambda_{QCD}$, the pressure is completely dominated by baryon *interactions*. Since the interactions between \mathcal{M} baryons are $\sim N_c$, all baryon interactions contribute equally, to give a dense phase in which the pressure is $\sim N_c$.

It is important to stress that at large N_c , the window in which baryons are dilute is *very* narrow. We expect to enter a dense phase, with pressure $\sim N_c$, when the baryon Fermi momentum $k_F \sim 1$. In terms of the quark chemical potential, though,

$$\mu - m_q = \frac{\mu_B - M}{N_c} = \frac{k_F^2}{2MN_c} \sim \frac{1}{N_c^2} k_F^2. \quad (6)$$

That is, for μ one enters a dense regime within $1/N_c^2$ of the mass threshold. This is why in fig. (1), we have indicated that the quarkyonic phase begins right at m_q : the window in which one has dilute baryons is only $\sim 1/N_c^2$ in width.

This discussion is somewhat naive. If the potential between two nucleons is attractive at large N_c — as it is in QCD — then at arbitrarily low densities free nucleons collapse to form bound nuclear matter. (This is analogous of the tendency of nuclear matter in QCD to go to the most stable state, which is one of iron nuclei.) The mass threshold for baryons is then not at $\mu_B = M_B$, but at $\mu_B = M_B - \delta E$, where δE is the binding energy of nuclear matter at large N_c [29]. One expects $\delta E = N_c \delta e$, with $\delta e \sim \Lambda_{QCD}$. Then in fig. (1), the quarkyonic phase begins not at m_q , but within $\sim 1/N_c^2$ of $\mu = m_q - \delta e$.

In QCD, the nuclear binding energy is anomalously small, $\delta e = \delta E/3 \sim 5$ MeV, versus $m_q = 313$ MeV. We do not know if δe is generically small in

the limit of large N_c . If so, it is surely related to the $SU(2N_f)$ symmetry [3]. In QCD, however, this may be an “accident” of $2 + 1$ light flavors.

In QCD, a gas of nucleons doesn’t directly collapse to nuclear matter, but instead exhibits a liquid-gas transition. This is because of two effects, the kinetic energy of individual nucleons and Coulomb repulsion. Neither is important at large N_c . As seen above, effects of kinetic energy are automatically $\sim 1/N_c$. Likewise, taking the baryon electric charge to be of order one at large N_c (as is necessary for electromagnetism to remain weakly coupled), then Coulomb repulsion is ~ 1 , and negligible relative to the nuclear potential, $\sim N_c$.

4 Dense Baryons as a Skyrme Crystal

In the previous section we saw that when the nucleon Fermi momentum $k_F \sim \Lambda_{QCD}$, that one goes into a dense phase, dominated by baryon-baryon interactions. To understand what might happen in this regime, in this section we review the properties of Skyrme crystals [25,26]. Our principal interest is as an example of a confined theory which nevertheless has a chirally symmetric phase.

The usual Skyrme model is a sum of two terms,

$$L = f_\pi^2 \text{tr}[V_\mu]^2 + \kappa \text{tr}[V_\mu, V_\nu]^2 \quad , \quad V_\mu = U^\dagger \partial_\mu U \quad , \quad U = \exp(i\pi/f_\pi) \quad , \quad (7)$$

where f_π is the pion decay constant, and κ a coupling constant, with $f_\pi^2 \sim \kappa \sim N_c$. We limit ourselves here to the Skyrme model for two flavors, where π is the pion field. There are many other terms besides those in (7); terms from the anomaly contribute through the Wess-Zumino-Witten (WZW) Lagrangian. The two terms above must be viewed as the leading terms in a derivative expansion. Terms with higher numbers of derivatives have coupling constants with dimensions of inverse mass squared. The mass dimension of these other terms is presumably set by (inverse) powers of Λ_{QCD} .

A single Skyrmion is given by a solution to the field equations from (7) over all of space-time. Since the terms in the action are $\sim N_c$, the energy of a configuration is also of the same order, and represents a single baryon.

As shown by Klebanov [26], a realistic crystal is given by considering periodic solutions in a finite box. Like the energy of a single baryon, the energy of the Skymrion crystal is automatically $\sim N_c$, with one baryon per box. Solving the Skyrme equations of motion for a system with cubic symmetry is technically involved. For many crystals, however, it is known that a reasonable approximation is to chop off the corners of the cube, and to consider the theory on a sphere. This approximation was adopted by Kutschera, Pethick, and Ravenhall, and also by Manton [26]. Many properties of the crystal with cubic symmetry are especially transparent for a spherical geometry.

If R is the size of the sphere, the solution is constructed so that there is one baryon per sphere, so the baryon density is $1/(4\pi R^3/3)$. A crystal is a

bad approximation for large R , but presumably reasonable when $R \sim \sqrt{\kappa}/f_\pi$. At large N_c , this mass scale ~ 1 .

For large spheres, the chiral symmetry is broken, as the U field points in a given direction in isospin space; typically, $\pi \rightarrow 0$ at spatial infinity. As R decreases, the stationary point is distorted by the finite volume of the sphere.

As the radius of the sphere becomes small, there is a phase transition to a chirally symmetric phase. For small spheres, the stationary point is just the identity map, from from S^3 of the U 's to S^3 of space, taking $\pi^a \sim \hat{r}^a$. This has unit baryon number per spherical volume, but it is also easy to see that the integral of U , over the sphere, vanishes.

The restoration of chiral symmetry is less obvious for a crystal with cubic symmetry. It follows from the half-Skyrmion symmetry of Goldhaber and Manton, where the total chiral order parameter cancels between different regions of the crystal [26]. (For more than two flavors, the correct representation of the chirally symmetric phase is presumably a generalization of this half-Skyrmion symmetry.)

The crucial test for the restoration of chiral symmetry is that the excitation modes fall into chiral multiplets of the unbroken symmetry. This was shown for the mesonic excitations of the crystal by Forkel *et al.* [26]. Since the baryon current is topological, the baryon excitations are not evident. In a chirally symmetric phase, they must be parity doubled. In detail, this happens because the baryon is a topological current. Thus it is given by integrating over the entire box; if the configuration is chirally symmetric, so are the integrals thereof.

The Skyrme crystal does not give one insight into all properties of the system. The pressure of the system is not obvious, nor even the chemical potential of baryons. What one can do is to compute how the energy, per cell, depends upon the density. *If* one takes only the two terms in the Skyrme Lagrangian of (7), then the term with four derivatives dominates at small R , which is high density. Since this term is scale invariant, one automatically finds that the relationship between the energy density, e , and the density, n , is that for a conformally symmetric theory, $e \sim n^{4/3}$, controlled by the coupling κ .

This is an accident of keeping only two terms in the Skyrme lagrangian. Terms with six derivatives, for example, are also proportional to N_c , with dimensions of $1/\Lambda_{QCD}^2$. When the size of the crystal is $\sim 1/\Lambda_{QCD}$, however, *all* such interactions are equally important. This is analogous to the counting for baryon baryon interactions, which are always $\sim N_c$, and which are characterized by mass scales $\sim \Lambda_{QCD}$.

We stress that the Skyrme crystal is only meant as an illustration. For example, while naively one expects that a system of heavy particles forms a crystal, since the interactions are as large, also $\sim N_c$, even this may not necessarily follow.

5 Quarkyonic Matter at Large N_c

Skyrme crystals illustrate what might happen as μ increases from m_q (or μ_B from M). Alternately, one can consider working down in μ . For very large μ , $\mu \gg \Lambda_{QCD}$, one should be able to compute the total pressure by perturbation theory, in the QCD coupling g^2 :

$$P_{\text{pert.}}(\mu) \sim N_c N_f \mu^4 F_0(g^2(\mu/\Lambda_{QCD}), N_f), \quad (8)$$

The function F_0 has been computed to $\sim g^4$ [6]. At large N_c , it is given by planar diagrams with a single quark line, and an arbitrary number of gluon insertions. The pressure of ideal (massless) quarks is $\sim \mu^4$. Perturbative corrections are given by a gluon knocking a quark in the Fermi sea out of it, the subsequent rescattering of the quark(s) and gluons, *etc.* When $\mu \gg \Lambda_{QCD}$, for quarks deep in the Fermi sea, all such contributions are due to hard scattering, with the energy and momenta transferred $\sim \mu$. Since the gluon has momentum components of order μ , the coupling runs according to this scale, with the β -function of the pure glue theory, (8). Thus in perturbation theory, the pressure is a power series in $\sim 1/\log(\mu/\Lambda_{QCD})$, *etc.* times μ^4 .

To be able to compute the total pressure reliably in perturbation theory, we only need to assume that $\mu \gg \Lambda_{QCD}$. In particular, it is *not* necessary to assume that μ grows like a power of N_c . Presumably a perturbative calculation is applicable for, *e.g.*, $\mu > 10^2 \Lambda_{QCD}$. When N_c is absurdly large, such as $N_c = 10^{12}$, this is still smaller than any other scale which enters at large N_c , such as $N_c^{1/4} \Lambda_{QCD} \sim 10^3 \Lambda_{QCD}$; see the discussion following eq. (9).

Now consider pushing the perturbative computation of the pressure down to $\mu \sim \Lambda_{QCD}$. Non-perturbative contributions to the pressure enter, through terms such as $\sim \mu^2$, eq. (10). Even so, outside of the window of dilute baryons, sec. 3, baryons are dense, and we expect that the pressure remains $\sim N_c$.

This then raises the central conundrum of our work: for $T < T_d$, and $\mu \sim 1$, it appears that we can describe the system *either* as one of confined baryons, *or* as one of quarks. Admittedly, the baryons interact strongly, $\sim N_c$; likewise, and especially for $\mu \sim \Lambda_{QCD}$, the Fermi sea of quarks is far from ideal. But how can *both* pictures apply?

We suggest the following resolution. At large $\mu \gg \Lambda_{QCD}$, for quarks far from the Fermi surface, their scattering can be reliably computed in perturbation theory. This is reasonable: at large μ , the density of quarks per hadronic volume $\sim 1/\Lambda_{QCD}^3$, is large. In such a dense medium, a quark doesn't know which baryon it belongs to; then, for the most part, it is appropriate to view the system as one of (non-ideal) quarks.

Even at large μ , however, it is essential to consider separately the scattering of particles within $\sim \Lambda_{QCD}$ of the Fermi surface. In this regime, quarks interact by exchanging gluons with momenta $\sim \Lambda_{QCD}$. At infinite N_c , where quarks cannot screen gluons, we know how quarks at $\mu \neq 0$ scatter: exactly as for $\mu = 0$. When $T < T_d$, then, the theory is in a confined phase, and so near the

Fermi surface, it is appropriate to speak not of the scattering of quarks, but of baryons.

We term this a “*quarkyonic*” phase: a quark Fermi sea, with a baryonic Fermi surface. The width of the baryonic surface is $\sim \Lambda_{QCD}$, so when $\mu \sim \Lambda_{QCD}$, it is all baryons. As μ increases, the baryons form a band, of approximately constant width, on the edge of the Fermi surface. There is no quantitative difference between a quarkyonic phase, with a wide baryon surface, and one with a narrow surface: they smoothly interpolate from one to the other. At large N_c , it is possible to differentiate the quarkyonic phase, with a pressure $\sim N_c$, from that in the hadronic phase, ~ 1 , or the deconfined phase, $\sim N_c^2$. This clear distinction is only possible at large N_c (and small N_f).

The effects of a baryonic Fermi surface show up in the pressure through terms which are powers of $\sim (\Lambda_{QCD}/\mu)^2$ times the ideal gas term, eq. (10). This is typical of a nonperturbative correction, as an inverse power of a (hard) mass scale. When $\mu \sim \Lambda_{QCD}$, this is a large correction. When $\mu \gg \Lambda_{QCD}$, numerically this is a very small contribution to the total pressure. Even so, as particles at the edge of the Fermi surface are the lightest excitations, even at large μ baryons dominate processes with low momenta. This implies that at large N_c , phenomena involving the Fermi surface, such as superconductivity and superfluidity, are properly described by baryons, and not by quarks, for *all* $\mu \sim 1$.

By considering gluonic probes, it is clear that the theory is in a confined phase for $T < T_d$ and $\mu \sim 1$. As discussed by Greensite and Halpern [4], at zero temperature the Wilson loop is insensitive to quarks at large N_c : it exhibits an area law, with a nonzero string tension. Screening due to quarks enters through corrections $\sim 1/N_c$. Adding a Fermi sea of quarks doesn’t change this, as long as $\mu \sim 1$.

At nonzero temperature, the order parameter for deconfinement is the renormalized Polyakov loop [21,22]. Quarks induce an expectation value for the Polyakov loop, but this is $\sim 1/N_c$, versus a value ~ 1 in the deconfined phase. Thus up to corrections $\sim 1/N_c$, as a function of temperature the expectation value of the renormalized Polyakov loop is independent of μ (for $\mu \sim 1$); *e.g.*, in fig. (1), $T_d(\mu)$ is a straight line.

That the theory confines can also be seen by exciting a quark, in the Fermi sea, with some external probe. If the quark is deep in the Fermi sea, knocking it out takes a probe with large momentum. When $\mu \gg \Lambda_{QCD}$, at first the resulting quark propagates like a hard quark. It, and the remaining hole in the Fermi sea, then scatter off of other quarks in the Fermi sea, knocking some out, creating other holes, and so on. Eventually, one ends up with not a single, unconfined quark, but a perturbed Fermi sea, characterized by some number of excited *baryons* and their holes. That the theory confines is clearer if the external probe carries momentum $\sim \Lambda_{QCD}$: then one immediately sees that the only particles (and holes) excited near the Fermi surface are not quarks,

but baryons.

The above discussion applies to quarks of any mass, and leads to the phase diagram of fig. (1). An example is provided by the solution of QCD in $1 + 1$ dimensions [1]. In two dimensions there is only a confined phase, $T_d = \infty$. Schon and Thies showed that at nonzero quark density, the quark propagator remains infrared divergent, and so confined, as it is in vacuum [20]. This is exactly what one expects of a quarkyonic phase: that only color singlet excitations, such as mesons and baryons, have finite energy. It would be interesting to perform more detailed calculations, such as of the free energy, and how the properties of mesons and baryons change with μ .

Returning to four dimensions, the crucial question is: where is the chiral phase transition for light quarks? For $\mu < m_q$, it surely coincides with the deconfining phase transition, and occurs at T_d . We suggest, however, that the two transitions no longer coincide when $\mu > m_q$. If we take the Skyrme crystal as a guide to the quarkyonic phase, then chiral symmetry restoration occurs not at the mass threshold, but above $\mu = m_q$. This is because there must be some significant density of baryons in the Fermi sea to drive the transition. Further, the transition occurs when $\mu \sim m_q$, on the order of the constituent quark mass, and not at some μ which is asymptotically large in a power of N_c . Schematically, this gives the blue line in fig. (1).

This can also be seen in illustrative models. As an example of a possible solution to the Schwinger-Dyson equations [23], Wagenbrunn and Glozman [27] studied a model with a confining gluon propagator, $\sim 1/(k^2)^2$ in momentum space. At infinite N_c , solutions to the Schwinger-Dyson equations are those where the quark propagator (and its vertex with gluons) change, but the gluon propagator doesn't. Computing with the quark propagator at $\mu \neq 0$, but leaving the gluon propagator unchanged, it is not difficult to see that increasing μ drives chiral symmetry restoration at some nonzero value of μ above the mass threshold.

This can also be seen from the eigenvalue distribution of the Dirac operator. In vacuum, Banks and Casher showed that chiral symmetry breaking is driven by a nonzero density eigenvalues at zero eigenvalue [28]. When $\mu \neq 0$, for $N_f \geq 3$ the eigenvalues spread out in the complex plane, and there is no simple analogous condition [15]. Even so, it is most natural then as μ increases above the mass threshold, that whatever effect the gauge fields have on the eigenvalues, that eventually it is overwhelmed by $\mu \neq 0$. An explicit example of this is chiral symmetry restoration in random matrix models [15].

The splitting of the deconfining and chiral transitions can be represented naturally in effective models. Mocsy, Sannino, and Tuominen [7] showed that if the coupling between the Polyakov loop and the chiral condensate has one sign, the transitions coincide; for the other, they diverge. Thus this coupling vanishes at the point where the transitions diverge.

At large N_c , because of the changes in the magnitude of the pressure, the deconfining transition is expected to be of first order. Once the chiral transi-

tion no longer coincides with deconfinement, its order is presumably controlled by the usual renormalization group analysis [7], and depends strongly upon the number of flavors.

There is one caveat which must be noted. Chiral symmetry, and its possible restoration, can be sensitive to pairing near the Fermi surface. Certainly for $\mu \gg \Lambda_{QCD}$, one expects that quarks deep in the Fermi sea are best described as chirally symmetric. It is possible, however, that the baryons near the Fermi surface may experience non-perturbative effects which cause them to pair in a chirally asymmetric manner. This effect will manifestly be small, suppressed at least by $\sim (\Lambda_{QCD}/\mu)^2$.

Up to this point, we have assumed that $\mu \sim 1$, so that gluons are blind to quarks and their Fermi sea. Especially to understand finite N_c , however, it is also necessary to consider values of μ which grow with (fractional) powers of N_c . We assume that the temperature $T \sim \Lambda_{QCD}$, like $T_d(0)$, the temperature for deconfinement at $\mu = 0$. In perturbation theory, the pressure includes terms as

$$P_{\text{pert.}}(\mu, T) \sim N_c N_f \mu^4 F_0 \quad , \quad N_c N_f \mu^2 T^2 F_1 \quad , \quad N_c^2 T^4 F_2 \quad . \quad (9)$$

In the limit of large N_c , F_0 , F_1 and F_2 are functions of the coupling constant g^2 and N_f . The coupling runs with both mass scales, μ and T . Thus when μ grows like a power of N_c times Λ_{QCD} , perturbation theory in g^2 is a good approximation. As always, this is excepting power like corrections from the region near the Fermi surface.

Consider $\mu \sim N_c^{1/4} \Lambda_{QCD}$. In this region, the quark contribution to the pressure, $\sim N_c \mu^4 F_0 \sim N_c^2 F_0$, is as large as that of deconfined gluons, $\sim N_c^2 F_2$. In this regime, the quark contribution to the pressure is independent of temperature, since $\sim N_c \mu^2 F_1 \sim N_c^{3/2} F_1$ is down by $\sim 1/\sqrt{N_c}$. It is possible that there is a temperature dependent term in the pressure, induced by baryons near the Fermi surface; *e.g.*, $\sim N_c N_f T^2 \Lambda_{QCD}^2$, which is down by $\sim 1/N_c$ to the leading term, $\sim N_c^2$.

At larger values of $\mu \sim N_c^{1/2} \Lambda_{QCD}$, quarks contribute to the Debye mass in the limit of large N_c , eq. (1). In this regime, the pressure is completely dominated by that of quarks at zero temperature, $\sim N_c \mu^4 F_0 \sim N_c^3 F_0$. In perturbation theory, the gluon contribution to the pressure remains $\sim N_c T^4 F_2 \sim N_c^2$, with the temperature dependent part of the quark pressure also $\sim N_c \mu^2 T^2 F_2 \sim N_c^2$; thus both are down by $1/N_c$, relative to the quark term at zero temperature.

When $\mu \sim N_c^{1/2} \Lambda_{QCD}$, gluons are screened by quarks, and one is in a qualitatively new regime. At zero temperature, the Wilson loop no longer exhibits an area law; at nonzero temperature, the renormalized Polyakov loop acquires an expectation value of order one. Consequently, eventually the first order phase transition for deconfinement ends. It can do so in one of two ways: the phase boundary for deconfinement can either bend over to zero temperature, or it can end in a critical end point at $T \neq 0$. Our simple

arguments cannot predict which occurs. It is reasonable that either is only possible once gluons feel the quarks; *i.e.*, when $\mu \sim N_c^{1/2} \Lambda_{QCD}$. If so, and there is a critical end point, the critical behavior is a correction in $1/N_c$ to the total pressure, which is dominated by the zero temperature term for quarks.

We remark that if N_f goes to infinity with N_c , then all terms in eq. (9) are $\sim N_c^2$. Indeed, when both N_c and N_f are large, then even in vacuum one cannot speak, rigorously, of confinement. We have no insight into this limit.

Deryagin, Grigoriev, and Rubakov showed that at large N_c , the color singlet pairing of chiral density waves dominates over the di-quark pairing of color superconductivity [12]. Their perturbative analysis is reliable for $\mu \gg \Lambda_{QCD}$, as long as the quarks do not lie within $\sim \Lambda_{QCD}$ of the Fermi surface. When they do, the pairing of baryons also contributes.

Son and Shuster [12] showed that even for extremely large values of N_c , Debye screening disfavors the pairing to chiral density waves, and quark color superconductivity dominates. In sec. (7), we adopt a similar criterion to estimate when in QCD there is a transition from a quarkyonic, to a perturbative, regime.

6 Confinement and Chiral Symmetry Breaking

At zero temperature and density, the effect of anomalies usually ensures that any confined phase is one in which chiral symmetry is broken. We now give a heuristic argument as to why this need not be true at nonzero density.

Casher, and then Casher and Banks, argued that in the vacuum, confinement automatically implies the breaking of chiral symmetry [28]. Consider a meson, in which the quark propagates to the right, with a spin along its direction of motion. To remain a meson at rest, this must mix with a quark propagating to the left, which can happen by scattering off of a gluon. Its spin, however, is now opposite to the direction of motion, so its helicity has been flipped. Since in QCD the interactions preserve chirality, which for a massless field equals helicity, this change of direction cannot occur. It can if there is a mass condensate in the vacuum, which the quark can scatter off of, and flip its helicity. Note that this argument is especially tight in the limit of large N_c , where the number of quarks in a meson is fixed.

Now consider the similar process at nonzero temperature. Then besides scattering off of a gluon, one can scatter off of a quark in a thermal distribution. However, if we consider the processes of both emission and adsorption, the total is $\tilde{n}(E_k) - \tilde{n}(E_{k'})$; E_k is the energy of the rightgoing quark, $E_{k'}$ is the energy of the leftgoing quark, and $\tilde{n}(E)$ is the Fermi-Dirac statistical distribution function.

For an isotropic distribution, as in thermal equilibrium, if the momenta are the same, then the two distribution functions cancel. Thus the process is only allowed when $k \neq k'$. Since the momenta of the right and left moving quarks are different, however, we end up with an excited meson, different from

the initial meson. That is, this process represents not a meson at rest, but scattering between a meson, and some thermally excited state, such as another meson.

In general, this is fine in a thermal distribution: what we mean by a “meson” is a sum over states anyway. However, there is a problem at large N_c : if $T \neq 0$ and $\mu = 0$, then all interactions vanish at large N_c , and this process must be suppressed by powers of $1/N_c$. Thus Casher’s argument suggests that at nonzero temperature, and $\mu = 0$, the connection between chiral symmetry and confinement remains.

This connection could be lost in the presence of a Fermi sea, however. The argument goes through as before, except now we scatter off of a quark in the Fermi sea. Physically, the quark in the test meson scatters off a quark in a baryon, which then scatters into a baryon hole. There is no inconsistency with the large N_c expansion, because the scattering amplitude is large, of order one. This suggests that it is possible to have a confined, but chirally symmetric, phase at $\mu \neq 0$.

What of the constraints from anomalies, which after all, are due to ultra-violet effects? Certainly the anomaly itself is unchanged by temperature or density. However, their implications are less obvious when T or μ are nonzero. Because of the breaking of Lorentz invariance at T and $\mu \neq 0$, Itoyama and Mueller showed that many more amplitudes arise [28]. The anomaly relates these amplitudes, but not as directly as in vacuum. For example, Pisarski, Trueman, and Tytgat showed that the Sutherland-Veltman theorem, which relates the amplitudes for $\pi^0 \rightarrow \gamma\gamma$, does not apply at nonzero temperature [28]. Thus the connection between chiral symmetry breaking, and confinement, need not remain at nonzero T or μ .

The above generalization of Casher’s argument suggests that a system at $\mu \neq 0$ is uniquely different from $\mu = 0$ and $T \neq 0$. This may arise as follows. Anomalies are saturated by excitations with arbitrarily low energies; for example, when chiral symmetry breaking occurs, by pions. To model a chirally symmetric phase, consider massive, parity doubled baryons [27]. In a thermal distribution with $\mu = 0$, massive modes are Boltzmann suppressed, and cannot be excited at low energy. At nonzero density, however, a Fermi sea of massive particles can be excited, with arbitrarily small energy, by forming a particle hole pair.

We conclude this section by noting that the Skyrme model provides a direct example of a confined theory which satisfies the anomaly conditions. There, the Wess-Zumino-Witten term automatically incorporates all effects of the anomaly, such as $\pi^0 \rightarrow \gamma\gamma$, *etc.* This happens whether the background field is chirally asymmetric, as for large R , or chirally symmetric, as for small R . In either case, fluctuations from the Wess-Zumino-Witten terms automatically incorporate all anomalous amplitudes.

7 Quarkyonic Matter in QCD

The large N_c limit we consider is only simple because N_f is held fixed as $N_c \rightarrow \infty$. Since $N_c = N_f = 3$ in QCD, then especially at $\mu \neq 0$, it is far from clear that QCD really is close to this limit of large N_c , and small N_f . For the purposes of discussion, we henceforth assume that it is.

To extend the analysis at large N_c to QCD, the principal physical effect to include is that of Debye screening. From (1), the Debye mass is $m_{Debye}^2 = (2N_f/\pi)\alpha_s(\mu)\mu^2$, where $\alpha_s = g^2/(4\pi)$. This is to be compared with the scale of confinement, Λ_{QCD} . The latter is only approximate: probably a better measure of the confinement scale is not Λ_{QCD} *per se*, but the mass of the ρ meson, ≈ 1 GeV.

The Debye mass can be computed, at zero temperature and nonzero density, to higher order in perturbation theory [6]. The really essential question is to know how the effective coupling runs. At nonzero temperature, and $\mu = 0$, Braaten and Nieto suggested in the imaginary time formalism, as energies are always multiples of $2\pi T$, perhaps the effective coupling runs in the same way [24]. This was confirmed by computations to two loop order [24]. This implies that while $T_d \sim 200$ MeV is relatively low, that the effective coupling is moderate in strength, even down to T_d [21].

There doesn't appear to be any similar factor at nonzero density; at $T = 0$, the coupling should run like $\alpha_s(\mu/(c\Lambda_{QCD}))$, where c is a number of order one. For purposes of discussion, we that assume perturbation theory is reliable for $\mu > 1$ GeV; at this scale, the Debye mass is also ~ 1 GeV.

A Fermi sea first forms when the quark chemical potential $\mu > M_N/3 \approx 313$ MeV. Large N_c suggests that dilute baryons persist only in a narrow window, $\sim \Lambda_{QCD}/N_c^2$. Then, at some scale above this, QCD becomes quarkyonic, in that the pressure rises rapidly. Notice that the increase in pressure is *not* associated with a phase transition. In terms of baryons, it appears to be due entirely to their strong interactions. Below $\mu \sim 1$ GeV, it can also be viewed as due to the strong interactions amongst highly non-ideal quarks.

Once Debye screening becomes significant above $\mu \sim 1$ GeV, gluons are shielded, and the coupling may be (relatively) moderate in strength. Because of Debye screening, at large μ scattering within Λ_{QCD} of the Fermi surface should be under control. This is unlike large N_c , where Debye screening doesn't contribute until $\mu \sim \sqrt{N_c}$. The transition from a quarkyonic regime, to one which is perturbative in quarks and gluons, is presumably smooth, as there is no order parameter to distinguish one from the other. (Assuming that the deconfining transition doesn't extend down to $T = 0$, see below.)

Needless to say, our estimates for μ in the quarkyonic phase are *extremely* crude. Understanding the lower bound on μ requires matching onto models of nuclear matter; perhaps it might help by matching onto models consistent with large N_c counting. The upper limit can be pinned down better through higher order calculations in perturbation theory at $\mu \neq 0$ and $T = 0$ [6].

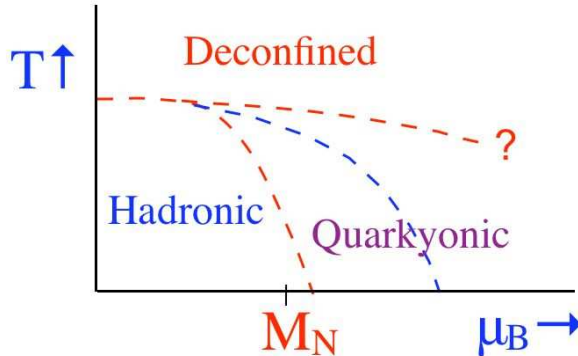


Fig. 2. Possible phase diagram for QCD in the plane of temperature and baryon chemical potential. The blue line in the quarkyonic phase indicates the chiral phase transition. There is a critical end point for deconfinement.

A possible phase diagram is drawn in fig. (2); following phenomenology [14], we plot this as a function of the temperature and the baryon chemical potential, μ_B . If large N_c is a reasonable guide to $N_c = 3$, this should look something like fig. (1), except that the sharp edges are smoothed out. For example, below the mass threshold, T_d should change little with μ ; this appears to be true from numerical simulations on the lattice [18]. Similarly, at large N_c nuclear matter rapidly goes from a dilute phase, to one which is dense and quarkyonic. We indicate this in the figure by drawing the quarkyonic phase slightly above M_N , the nucleon mass.

We expect that the chiral phase transition occurs in the quarkyonic phase, well above the mass threshold. For QCD, at present numerical simulations on the lattice indicate that for small μ , the deconfining and chiral transitions coincide, and are crossover. A chiral critical end point may exist in the plane of T and μ_B [13]. One might conjecture that if such a critical end point exists, that the deconfining and chiral transitions split from one another at that point.

Speculating in this manner, in the quarkyonic phase, the latent heat associated with the chiral transition might be relatively small. Certainly at large N_c , the large increase in pressure, $\sim N_c$, is not tied to the chiral transition. The behavior of the chiral transition is very sensitive to the number of flavors, and possible restoration of the axial $U(1)$ symmetry, though.

Consider the deconfining phase transition, after it splits from the chiral transition. At fixed μ , as T increases, one goes from a confined phase of parity doubled baryons, to one of quarks and gluons. Deconfinement could either remain crossover, or perhaps become first order again (from the splitting point?). If it does turn first order, it will then have to end in a critical end point, now for deconfinement. Alternately, a first order deconfining transition could persist down to zero temperature. We indicate this uncertainty by the

question mark in fig. (2).

How can the quarkyonic phase be studied? For the total pressure, one should use a description not in terms of baryons, but in terms of quarks. Admittedly, they are highly non-ideal quarks, but there are hints to their possible behavior from numerical simulations, on the lattice, at nonzero temperature. At $T \neq 0$ and $\mu = 0$, the pressure can be characterized by a generalized (or “fuzzy”) bag model. This is a power series in $1/T^2$ times the ideal gas term [16,17,21]. At $T = 0$, and nonzero μ , this suggests

$$P_{\text{quarkyonic}}(\mu) = f_{\text{pert}} \mu^4 - \mu_c^2 \mu^2 - B + \dots \quad (10)$$

Perturbative corrections are subsumed into f_{pert} . Nonperturbative corrections, such as due to confinement, are included in μ_c^2 and B . Because of the term $\sim \mu_c^2 \mu^2$, the constant B need not agree with the usual MIT bag term, even in sign. We note that such a parametrization arises naturally from a Skyrme crystal. In the simplest model, what is equivalent to a conformally symmetric term $\sim \mu^4$ arises from the Skyrme term, $\sim \kappa$. Power like corrections then arise from the usual sigma Lagrangian, $\mu_c^2 \sim f_\pi^2$, etc. The pressure from this generalized bag model should then match *smoothly* onto that of nuclear matter, with no phase transition between the two.

In contrast, in order to compute properties near the Fermi surface, it is necessary to consider effective theories of baryons. In a phase with chiral symmetry breaking, at low density these must match onto models of nuclear matter. In a chirally symmetric phase, the baryons are parity doubled. One possibility is to use Nambu-Jona-Lasino (NJL) models, not of quarks [10,11], but of baryons. Linear models of parity doubled baryons may also be of use [27]. Phenomenon such as superfluidity and superconductivity, and transport properties in general, are dominated by these states. For parity doubled baryons, the patterns of baryonic superfluidity and superconductivity will be significantly constrained by anomaly conditions. One might guess that the scales of baryon pairing in the quarkyonic phase is on the order of those in ordinary nuclear matter; *i.e.*, that the gaps are small, tens of MeV.

Schäfer and Wilczek [9] noted that for three light flavors, there is continuity between a nucleonic phase and one with quark color superconductivity. While chiral symmetry breaking is large in a nucleonic phase, it is also generated by color-flavor locking. This suggests that for quarkyonic matter with three flavors and three colors, that one possibility is for the massive, parity doubled baryons to form (small) gaps which spontaneously break chiral symmetry. This is the simplest way by which massive baryons, which are now only approximately parity doubled, can satisfy anomaly constraints at $\mu \neq 0$, although we suspect there are others.

One way of computing the properties of a quarkyonic phase is to use approximate solutions of Schwinger-Dyson equations [23]. These are, almost uniquely, the one approximation scheme which includes both confinement and chiral symmetry breaking. They do have features reminiscent of large N_c : at

low momentum, if chiral symmetry breaking occurs, the gluon propagator for $N_f = 3$ is numerically close to that for $N_f = 0$. At present, solutions at $\mu \neq 0$ assume a Fermi surface dominated by quarks; if quark screening is not too large at moderate μ , these models should exhibit a quarkyonic phase.

On the lattice, it is well known that while gauge theories have a sign problem at nonzero quark density when $N_c \geq 3$, that numerical simulations can be done for two colors. Recently, these were done at $T = 0$ and $\mu \neq 0$ for heavy quarks: they exhibit superfluidity at the mass threshold and deconfinement well above it [19]. These simulations could be extended to light quarks, to see if the phase diagram is anything like that of fig. (2): *e.g.*, at low temperature, is chiral symmetry restored before deconfinement?

While our analysis is crude, existing prejudice has been that the phase transitions for deconfinement and chiral symmetry are inexorably linked together. Large N_c suggests that by moving out in chemical potential, that potentially one has the chance to see the two transitions separate. This could happen at rather high temperature, near that for the deconfining transition temperature at zero density, and relatively low density, less than that for nuclear matter. Experimentally, it is possible to move out in the plane in μ , at high T , by going to “low” energies, such as at critRHIC and FAIR. Thus these facilities may explore not just a chiral critical end point [13], but quarkyonic phases, including one which is confined, yet chirally symmetric.

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