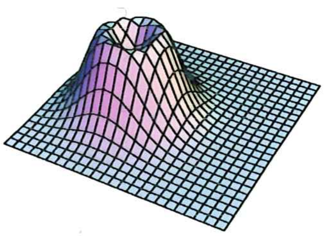
Topological objects in QCD

Schladming, February 2007

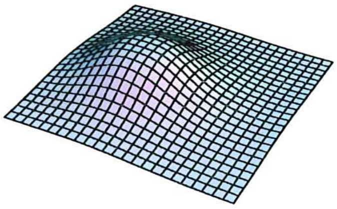
Falk Bruckmann, Universität Regensburg

Appetizer:

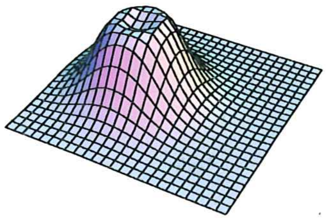
pure $SU(2)$ configuration on a $16^3 \cdot 4$ lattice, (long) overimproved cooling:



action density



Polyakov loop



zero mode profile

$\Rightarrow$ stable topological objects with various interesting (nonlinear) properties

Outline:

1. The kink
2. Magnetic monopoles and instantons
3. Analytical aspects, calorons
4. Continuum models and lattice results

with Pierre van Baal, Daniel Nogradi, Ernst-Michael Ilgenfritz, Boris Martemjanov,

Michael Müller-Preußker, Christof Gattringer, Andreas Schäfer, Stefan Solbrig

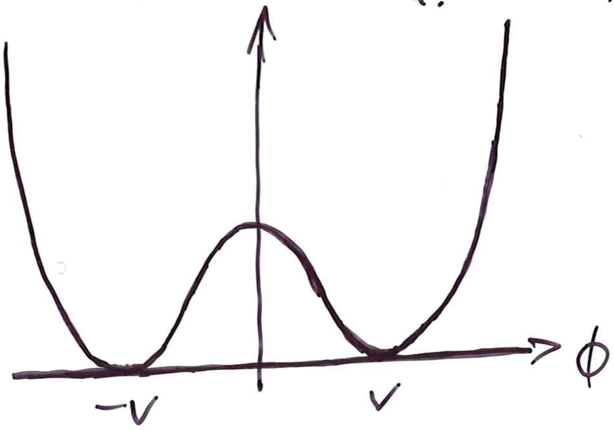
The kink

- **model**: a real scalar field ϕ in 1+1-dim. Minkowski space

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - V(\phi) \qquad g_{\mu\nu} = \mathrm{diag}(+1,-1)$$

V has several minima of same height

e.g. mexican hat: $V(\phi) = \frac{\lambda}{4!}(\phi^2 - v^2)^2$



or Sine-Gordon:
$V(\phi) \simeq 1 - \cos\phi$

two stable vacua: $V''(\phi = \pm v) = \frac{\lambda}{3}v^2 \equiv m^2$

m is the mass of perturbative excitations

plus tunnelling as a typical nonperturbative effect

there exists a static solution of the Euclidean equations of motion with finite action, connecting the vacua

Eucl.: $\mathcal{L}_E(x_1, x_2) = -\mathcal{L}_M(x_0 = i x_2, x_1) \qquad x_1 \equiv x$

$$= \cancel{\frac{1}{2}(\partial_{x_2}\phi)^2} + \frac{1}{2}(\partial_x\phi)^2 + V(\phi) \qquad \text{Hamiltonian}$$

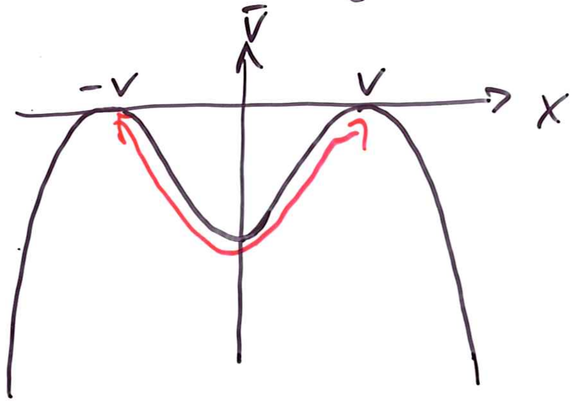
static: $\int_0^T dx_2 \equiv T$

finite action: $V(\phi(x = \pm\infty)) = 0 \Longrightarrow \phi(x = \pm\infty) \in \{\pm v\}$

- particle mechanics analogy:

$$\phi(x) \longrightarrow x(t)$$

$$L(x(t)) = \frac{1}{2}\dot{x}^2 - \bar{V}(x) \qquad \bar{V} = -V \text{ inverted potential}$$



boundary conditions:

$$x(t = \pm\infty) \in \{\pm v\}$$

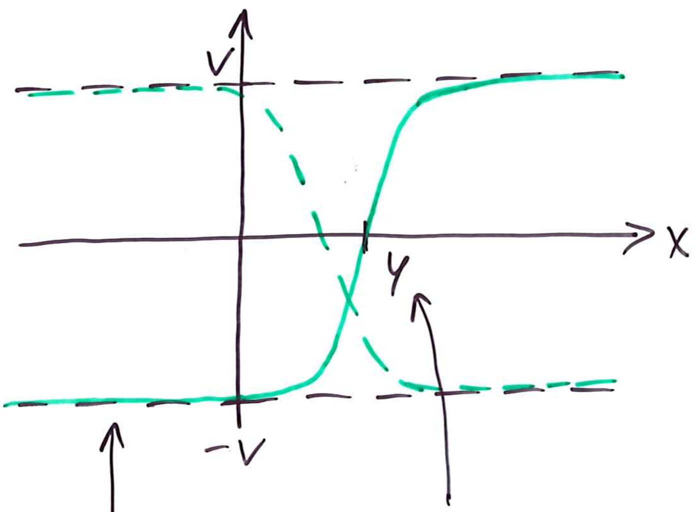
trivial solutions resting at the vacua and one rolling from one to the other

use energy conservation: $\frac{1}{2}\dot{x}^2 + \bar{V} = \bar{E} = 0 \implies \dot{x} = \pm\sqrt{2(-\bar{V})}$

- the (anti) kink/soliton solution: $\partial_x \phi = \pm\sqrt{2V} = \pm\sqrt{\frac{\lambda}{12}}(\phi^2 - v^2)$

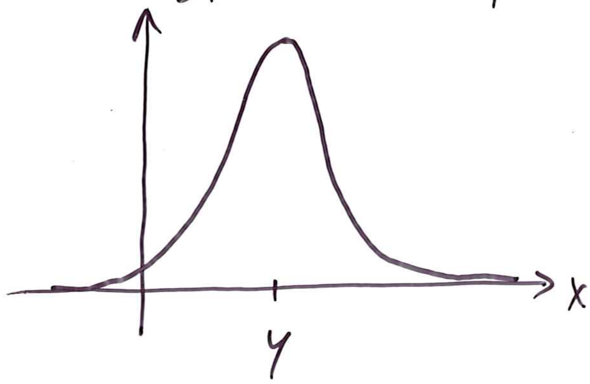
$$\phi(x) = \pm v \tanh(m(x-y))$$

nonlinear



exp. tail with decay constant $\frac{1}{m}$

energy/action density



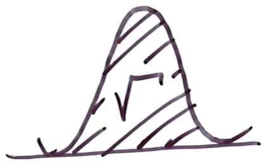
localised

- total energy of the kink:

$$E = \int_{-\infty}^{\infty} dx \left[\tfrac{1}{2} (\partial_x \phi)^2 + V \right] = \int_{-\infty}^{\infty} dx \, 2V \qquad \partial_x \phi = \pm \sqrt{2V}$$

$$= \int_{x=-\infty}^{x=+\infty} dx \, \frac{d\phi}{dx} \sqrt{2V}$$

$$= \int_{-v}^{v} d\phi \, \sqrt{2V}$$



tunnelling amplitude (WKB)

$$T = \exp(-E_{\text{kink}})$$

$$= W \Big|_{-v}^{v} \quad \text{where} \quad \frac{dW}{d\phi} = \sqrt{2V(\phi)} \qquad W = \sqrt{\frac{\lambda}{12}} \left(v^2 - \frac{\phi^2}{3} \right) \phi$$

$$\underline{\underline{E = \frac{m^3}{\lambda}}}$$

perturbative limit $\lambda \to 0$: very massive

- Bogomolnyi bound:

$$S = \int dx \, dx_2 \left\{ \tfrac{1}{2} (\partial_{x_2} \phi)^2 + \tfrac{1}{2} (\partial_x \phi)^2 + V(\phi) \right\}$$

$$= \int dx \, dx_2 \left\{ \tfrac{1}{2} (\partial_{x_2} \phi)^2 + \tfrac{1}{2} \left[\partial_x \phi \mp \sqrt{2V} \right]^2 \pm \partial_x \phi \sqrt{2V} \right\}$$

$$S \geq \left| \int dx \, dx_2 \, \partial_x \phi \sqrt{2V} \right|$$

equality iff both squares vanish $\rightleftarrows$ (anti)kink

$$= \left| T \int_{-\infty}^{\infty} dx \, \partial_x W(\phi(x)) \right|$$

boundary term indep. of the shape of $\phi(x)$

$$\underline{\underline{S \geq T \frac{m^3}{\lambda} |q|}}$$

with the topological quantum number

$$q = \frac{1}{2v} \left[\phi(x=+\infty) - \phi(x=-\infty) \right] = \begin{cases} 0 & \text{trivial vacua} \\ 1 & \text{kink} \\ -1 & \text{antikink} \end{cases}$$

applies to every configuration (not just class. sol.ns) with finite action as an integer, q cannot be deformed continuously

[would require $\phi(x=\pm\infty) \neq \pm v$, $V(x=\pm\infty)\neq 0$, $S\to\infty$]

the space of finite action solutions splits into sectors labelled by the topological quantum number q and separated by infinite barrieres; $S \geq \text{const}\,|q|$ where the equality holds for class. solutions

[sine-Gordon: $q\in\mathbb{Z}$]

- topological current: $J^\mu = \frac{1}{2v}\epsilon^{\mu\nu}\partial_\nu\phi$

  $\partial_\mu J^\mu = 0$ without using equations of motion, not a Noether current

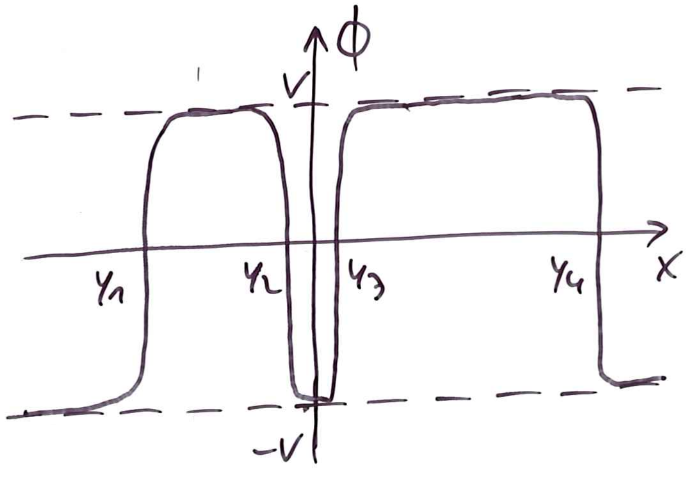
  $$\int_{-\infty}^{\infty} dx\, J^0 = \frac{1}{2v}\int_{-\infty}^{\infty} dx\, \partial_x\phi = q$$

- ϕ as a mapping:

  $$\phi\big|_{x=\pm\infty} : \underbrace{\partial\mathbb{R} \cong Z_2}_{\text{boundary of space}} \longrightarrow \underbrace{\{\pm v\} = Z_2}_{\text{vacuum manifold}}$$

  q characterises $\phi\big|_{x=\pm\infty}$: whether the image is fully covered and "in which direction"

- multi-solitons:



chains of kink and antikinks are approximate solutions when diluted: $\Delta y \gg \frac{1}{m}$

• application: semiclassical calculation of path integrals

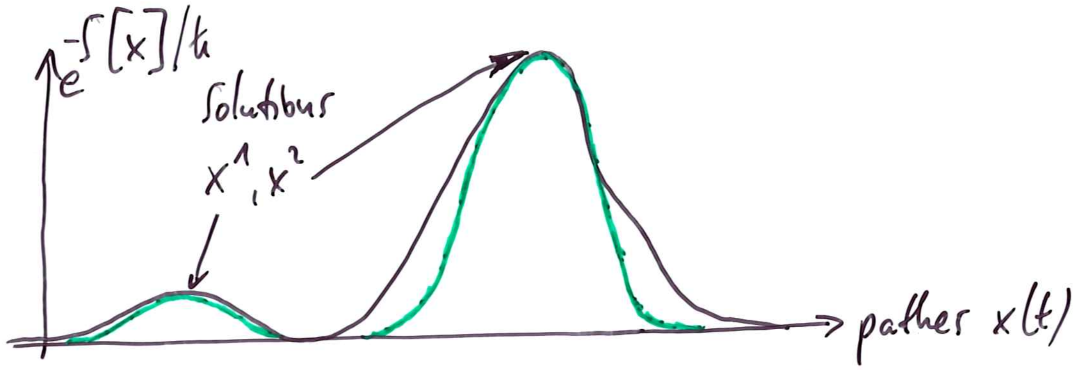
[Euclidean, particle mechanics]

time evolution:

$$\langle x=\pm v | e^{-Ht/\hbar} | x=v \rangle$$

$$= N \int_{x(0)=\pm v}^{x(t)=v} \mathcal{D}x \cdot e^{-S[x]/\hbar}$$ sum over all pathes $x(t)$...

semiclassically: second order around solutions plus Gaussian integration



$$= \# \int \mathcal{D}x \sum_A e^{-S[x^A]/\hbar - 0 - \frac{1}{2}\int dt\, [x(t)-x^A(t)] \frac{\delta^2 \mathcal{L}}{\delta x^2}(x^A) [x(t)-x^A(t)] /\hbar}$$

eigenmodes

decompose: $x(t) - x^A(t) = \sum_n c_n x_n(t)$

$$\int \mathcal{D}x = \int \prod_n dc_n \cdot 1$$ unitary: $L_2 \to l_2$

$$\int dc_n\, e^{-\frac{1}{2}\lambda_n c_n^2/\hbar} \sim \frac{1}{\sqrt{\lambda_n}}$$

$$= \# \sum_A e^{-S[x_A]/\hbar} \frac{1}{\sqrt{\det[-\partial_t^2 + V''(x^A(t))]}}$$

but S is independent of the parameters of the solution

kink: location $\Longrightarrow$ zero mode of $-\partial_t^2 + V''(x^A(t))$

split off these "flat directions":

$$x(t) - x^A(t) = c_0 x_0(t) + \dots$$

$$Dx = dc_0 \cdot \dots = dy \cdot \text{Jacobian} \cdot \dots$$

$$\langle x = \pm v | e^{-Ht/\hbar} | x = v \rangle$$

$$= \# \sum_A \int dy_A \cdot \text{Jacobian} \cdot e^{-S[x^A]/\hbar} \frac{1}{\sqrt{\det'[-\partial_t^2 + V''(x^A(t))]}}$$

dilute gas of multisolitons:

A = number n of solitons and antisolitons $= \begin{cases} \text{even} \\ \text{odd} \end{cases}$

$$S[x^A] \simeq n S_{kink} \qquad y_A = \{y_1 \dots y_n\}$$

$$= \# \sum_{n \,\substack{\text{even} \\ \text{odd}}} \frac{t^n}{n!} \sqrt{S_{kink}}^{\,n} \cdot e^{-n S_{kink}/\hbar} \cdot K^n e^{-mt/2}$$

$$= \# e^{-mt/2} \left(e^{K' e^{-S_{kink}/\hbar} t} \pm e^{-K' e^{-S_{kink}/\hbar} t} \right)$$

$$= \Psi_0^*(\pm v) \Psi_0(v) e^{-E_0 t/\hbar} + \Psi_1^*(\pm v) \Psi_1(v) e^{-E_1 t/\hbar} + \dots$$

$$\underline{\underline{E_{0,1} = \frac{\hbar m}{2} \mp \hbar K' e^{-S_{kink}/\hbar}}}$$

perturbative: ground states of harmonic oscillators at each vacuum

splitting by tunnelling $S_{kink} \sim \frac{1}{\lambda}$: $e^{-\frac{1}{\lambda \hbar}}$ not seen in perturbation theory.

- **fermions in the kink background:** [Jackiw, Rebbi]

$$\mathcal{L} = \mathcal{L}_{bosonic} - \bar{\psi}\left(\gamma^\mu \partial_\mu + g\phi\right)\psi$$ Yukawa coupling to $\phi = \phi_{kink}$

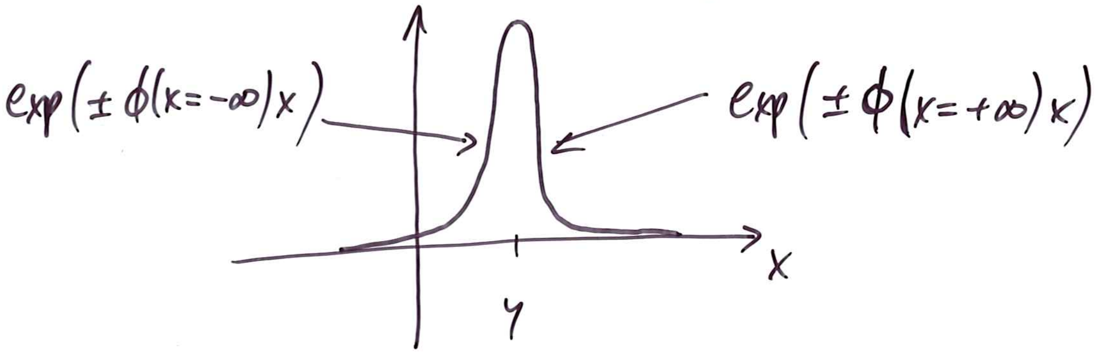
Dirac-Hamiltonian: $H_D = \gamma^2\left(\gamma^1 \partial_x + g\phi\right)$ [$\gamma^\mu \sim$ Pauli matrices]

$\{H_D, \gamma^1\} = 0$: γ^1 relates eigenstates with E and $-E$

on $E=0$: diagonalise γ^1

$$\psi_{E=0}(x) = \chi_\pm(x)\, s_\pm$$

$$\chi_\pm(x) = \exp\left(\mp \int_0^x dx'\, \phi(x')\right)$$



a normalisable zero mode exists and is exp. localised (with vev v)

for $q=1$ $\longleftarrow$ γ^1-eigenvalue $= -1$

for $q=-1$ " $+1$

none in the trivial sector $q=0$

index theorem [Bott, Seeley] for all configurations

- **Derrick's Theorem:**

let $S = \int d^d x \left[\frac{1}{2} (\partial_\mu \phi)^2 + V(\phi) \right] \equiv I_{kin} + I_{pot}$

a solution has $\delta S = 0$, but is it stable, $\delta^2 S \geq 0$?

$\Rightarrow$ a specific variation: $\phi_\lambda(x) = \phi(\lambda x)$

$$S(\lambda) = \lambda^{2-d} I_{kin}(\lambda=1) + \lambda^{-d} I_{pot}(\lambda=1)$$

$$\left.\begin{aligned} \frac{\delta S}{\delta \lambda} &= 0 \\ \frac{\delta^2 S}{\delta \lambda^2} &\geq 0 \end{aligned}\right\} (2-d) \geq 0 \qquad d \leq 2$$

Magnetic monopoles

- a gauge-Higgs system [Georgi-Glashow]

$SU(2)$ gauge + scalar fields : close to electroweak

$$\mathcal{L} = -\frac{1}{2}\,\mathrm{tr}\,F_{\mu\nu}F^{\mu\nu} + \mathrm{tr}\,(D_\mu\phi)^2 - \frac{\lambda}{8}\left(2\,\mathrm{tr}\,\phi^2 - v^2\right)^2$$

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu - ig\,[A_\mu, A_\nu] \qquad D_\mu\phi \equiv \partial_\mu\phi - ig\,[A_\mu, \phi]$$

self-interaction of gauge bosons   covariant deriv., ϕ in adjoint

all (hermitian) matrices: $\phi = \phi^a \frac{\sigma^a}{2}$, $A_\mu = A_\mu^a \frac{\sigma^a}{2}$ $a = 1,2,3$ σ^a: Pauli mat.

- vacua : $V(\phi) = 0 \Longrightarrow \phi^a\phi^a = v^2$: a whole S^2 in 3D color space

a particular realisation : $\underline{\phi^a = v\,n^a}$ $n^a n^a = 1$

symmetry breaking : Lagrangian has $SU(2)/Z_2 \cong SO(3)$ ... color rotations of ϕ^a

vacuum has $SO(2)$ ... color rotations, that leave n^a inv.

Higgs effect [perturbative expansion around one vacuum]

gauge fields along S^2_{color} : $m_{W\text{-boson}} = vg$

scalar fields perpendicular to S^2_{color} : $m_{Higgs} = v\sqrt{\lambda}$

gauge field of remaining $U(1)$-symmetry : $m_{photon} = 0$

- solitonic solutions

$\phi^a\phi^a \longrightarrow v^2$ (finite action)

static and $A_0 = 0$: $F_{\mu\nu} \rightsquigarrow \vec{B}$

- **BPS trick:**

$$E = \int d^3x \left\{ \mathrm{tr}\,(\vec{D}\phi)^2 + \mathrm{tr}\,\vec{B}^2 + V(\phi) \right\}$$

$$= \int d^3x \left\{ \mathrm{tr}\left[\vec{D}\phi \mp \vec{B}\right]^2 + V(\phi) \mp 2\,\mathrm{tr}\,\vec{B}\vec{D}\phi \right\}$$

$$E \geq 2 \left| \int d^3x\, \mathrm{tr}\,\vec{B}\vec{D}\phi \right| = 2 \left| \underbrace{\int d^3x\, \vec{\partial}\, \mathrm{tr}(\vec{B}\phi)}_{\text{surface term}} \right| = v \left| \int_{S^2_\infty} d\vec{\sigma}\, (\vec{B}^a n^a) \right|$$

magnetic flux projected onto Abel. direction given by n

$\underline{\underline{E \geq 4\pi v \cdot |q_{mag}|}}$ ... magnetic charge

- **explicit solution** ['t Hooft–Polyakov '74]

radial ansatz: $A_i^a = \epsilon_{iaj} \dfrac{x_j}{|\vec{x}|} \cdot A(|\vec{x}|)$ mixing between

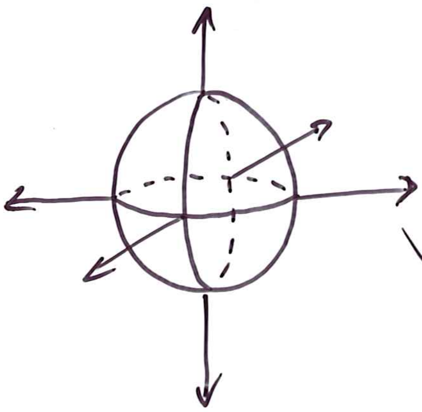
$\phi^a = \dfrac{x_a}{|\vec{x}|} \cdot \phi(|\vec{x}|)$ space and color space

analytical sol.us available for vanishing potential, $\vec{D}\phi = \pm\vec{B}$

asymptotics: $\left.\begin{array}{l} \phi(|\vec{x}|) \longrightarrow v \\ A(|\vec{x}|) \longrightarrow \dfrac{1}{g|\vec{x}|} \end{array}\right\}$ $\vec{B}^a n^a \longrightarrow \dfrac{\vec{x}}{g|\vec{x}|^3}$

Coulomb with $q_{mag} = \dfrac{1}{g}$

shape: $\phi^a \sim \dfrac{x^a}{|\vec{x}|}$



"hedgehog"

$\phi \longrightarrow v \dfrac{x^a}{|\vec{x}|}$ $\leftarrow n^a$

- as a mapping:

$$n^a(\vec{x}) : \partial \mathbb{R}^3 \cong S^2_\infty \longrightarrow S^2_{color} \cong SO(3)/SO(2) \quad \text{coset space}$$

boundary of space — vacuum manifold

characterised by the winding number $\deg(n) \in \pi_2(S^2) = \mathbb{Z}$

counts, how many times the image sphere is covered by the preimage sphere and in which direction

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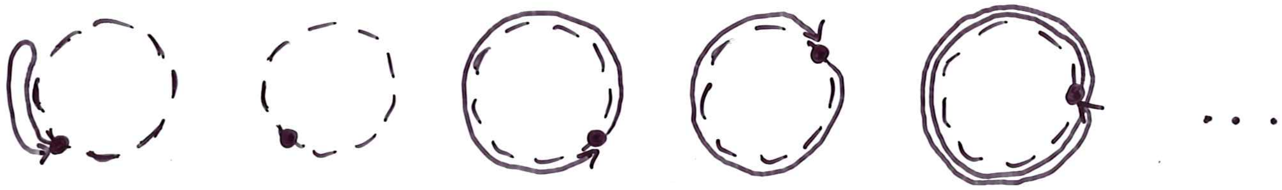
the one-minute-topologist

mappings : $S^1 \longrightarrow S^1$ $\quad \deg \in \pi_1(S^1) = \mathbb{Z}$: first homotopy group



deg=0 deg=0 deg=1 deg=−1 deg=2

$$\deg(n_{hedgehog}) = 1$$

- $\exists$ topological current J^μ such that $\int d^3x\, J^0 = \deg(n)$
- $q_{mag} = \frac{1}{g} \cdot \deg(n)$ $\quad q_{mag}$ is a topological quantum number
- $n^a(\vec{x})$ cannot be extended smoothly into the bulk

$$\Longrightarrow \phi^a(\vec{y}) = 0 \quad \text{for some } \vec{y}$$

location of the monopole = free parameter $\quad$ [here origin]

- $\exists$ fermionic zero modes [Jackiw, Rebbi]

- physical consequences:

charge quantisation: $q_{mag} \cdot g = \deg(n) \in \mathbb{Z}$

magnetic monopole quantises electric charges

mass of the monopole: $m_{mon} = v \cdot \frac{4\pi}{g} = \frac{4\pi}{g^2} \cdot m_W$ heavy

there are no magnetic monopoles in the Standard Modell

since $SU(2)_{isospin} \times U(1)_{hypercharge} \xrightarrow[\theta_{Weinberg}]{\text{nontrivially}} U(1)_{electromagn.}$

but generic in Grand Unified Theories

- can one 'abelianize' the monopole?

$\equiv$ rotate Φ onto fixed color direction, say σ_3 'unitary gauge'

fails at the monopole location: $\Phi(\vec{x}=\vec{y})=0$

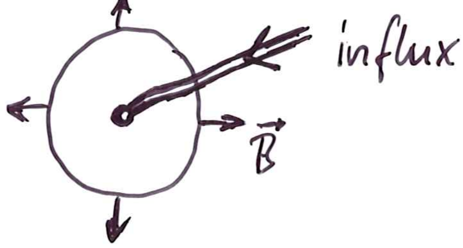
fails around it: $\deg(n_{hedgehog})=1$, but $\deg(\hat{n}^a=(0,0,1))=0$

still magnetically charged: $q_{mag} = \int_{S^2_\infty} d^2\vec{\sigma}\,(\vec{B}^a n^a)$ gauge inv.

[Dirac '31]

the gauge transformed $\vec{A}(\vec{x})$ is the one of the Dirac monopole

with location $\vec{x}=\vec{y}$ and a Dirac string from $\vec{y}$ to infinity (use fibre bundles)



plus exponentially decaying parts

'massive', fine-tuned to avoid the singularities in the full theory

superpositions difficult!

Instantons

- **Yang–Mills theory** = purely gluonic part of QCD

$$\mathcal{L} = \frac{1}{2} \operatorname{tr} F_{\mu\nu} F^{\mu\nu} = \frac{1}{2}\left(\vec{E}^{a2} + \vec{B}^{a2}\right)$$

4D, Euclidean
mostly $SU(2)$

self-interaction of gluons (special running of the coupling $\Rightarrow$ asymptotic freedom)

- **finite action**:

$r = \sqrt{x_\mu^2} \to \infty$: $F_{\mu\nu} \to 0$, $A_\mu \to$ pure gauge

[Uhlenbeck: $\mathbb{R}^4 \rightsquigarrow S^4$]

- **BPS trick**:

$$S = \int d^4x \, \frac{1}{2} \operatorname{tr} F_{\mu\nu}^2 = \int d^4x \, \frac{1}{4}\left(\operatorname{tr} F_{\mu\nu}^2 + \underbrace{\tilde{F}_{\mu\nu}^2}_{\vec{E}^2 + \vec{B}^2}\right) \qquad \tilde{F}_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F_{\rho\sigma} \quad (\vec{E} \leftrightarrow \vec{B})$$

$$= \int d^4x \left\{ \frac{1}{4} \operatorname{tr} \underbrace{\left(F_{\mu\nu} \mp \tilde{F}_{\mu\nu}\right)^2}_{(\vec{E} \mp \vec{B})^2} \pm \frac{1}{2} \operatorname{tr} \underbrace{F_{\mu\nu} \tilde{F}_{\mu\nu}}_{\vec{E}\cdot\vec{B}} \right\}$$

$$\underline{S \geq \left| \int d^4x \, \frac{1}{2} \operatorname{tr} F_{\mu\nu} \tilde{F}_{\mu\nu} \right| \equiv \frac{8\pi^2}{g^2} |Q|}$$

instanton number, top. charge
Pontryagin index, 2nd Chern class

$$Q = \int d^4x \, \partial_\mu K_\mu(A) \qquad K_\mu = \frac{g^2}{16\pi^2} \epsilon_{\mu\nu\rho\sigma}\left(A_\nu^a \partial_\rho A_\sigma^a + \frac{g}{3} \epsilon_{abc} A_\nu^a A_\rho^b A_\sigma^c\right)$$

Chern–Simons current

$$= \int_{S^3_\infty} d^3\sigma \, K_\perp\left(A_\mu = \frac{i}{g} \Omega^\dagger \partial_\mu \Omega\right) = \ldots$$

$$= \deg(\Omega)$$

$$\Omega : \partial\mathbb{R}^4 \cong S^3_\infty \longrightarrow SU(2) \cong S^3 \qquad \deg(\Omega) \in \pi_3(S^3) = \mathbb{Z}$$

$SU(N)$: —— " —— $SU(N)$ [higher dim., still] $\pi_3(SU(N)) = \mathbb{Z}$

- (anti) selfdual solutions ≡ (anti) instantons

$$F_{\mu\nu} = \pm \tilde{F}_{\mu\nu} \qquad \left(\text{Bianchi identity } D_\mu F_{\mu\nu} = 0 \longrightarrow \text{equ. of motion } D_\mu \tilde{F}_{\mu\nu} = 0\right)$$

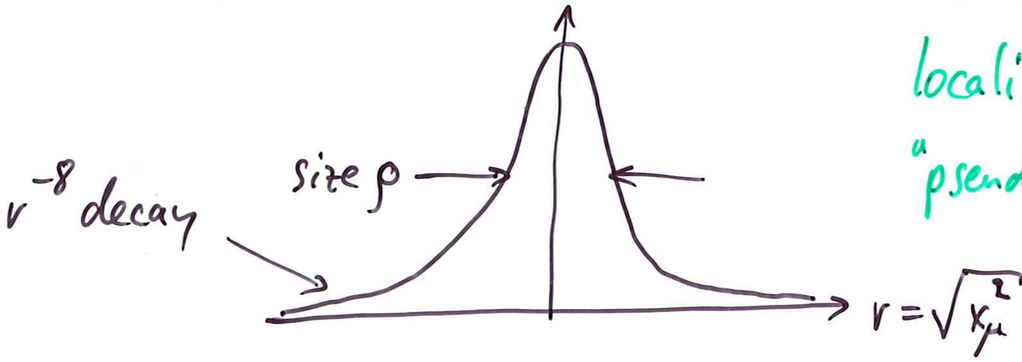
$$\uparrow \; \vec{E} = \pm \vec{B} \qquad \text{only first order in } A_\mu$$

- explicit solution for charge 1 [Belavin, Polyakov, Schwartz, Tyupkin '75]

radial ansatz: $\Omega(x) = \frac{x_4}{r} \mathbb{1}_2 + i \frac{x_a}{r} \sigma_a \qquad \deg(\Omega) = 1$

$$A_\mu^a \longrightarrow 2 \eta^a_{\mu\nu} \frac{x_\nu}{r^2} \qquad \eta^a_{\mu\nu} \in \{-1, 0, 1\} \text{ 't Hooft tensor}$$

$$A_\mu^a = 2 \eta^a_{\mu\nu} \frac{x_\nu}{r^2 + \rho^2} \qquad \mathrm{tr}\, F_{\mu\nu}^2 = \# \frac{1}{g^2} \frac{\rho^4}{(r^2+\rho^2)^4}$$



localised in 4D
"pseudoparticle"

most general:

$$A_\mu = \Omega^\dagger \left(2 \eta^a_{\mu\nu} \frac{(x-y)_\nu}{(x-y)^2 + \rho^2} \frac{\sigma_a}{2} \right) \Omega \qquad \Omega: \text{color orientation}$$

anti-instanton: $\eta \longrightarrow \bar{\eta}$

- ansatz for higher charge [Corrigan, Fairlie, 't Hooft, Wilczek]

$$A_\mu^a = \eta^a_{\mu\nu} \partial_\nu \log \left(1 + \sum_{p=1}^{Q} \frac{\rho^{(p)}}{\left(x_\mu - y_\mu^{(p)}\right)^2} \right) \qquad [\text{singular gauge}]$$

Q lumps with arbitrary locations $y_\mu^{(p)}$ and sizes $\rho^{(p)}$, but same color orientation

$5Q$ out of $8Q-3$ moduli

- massless fermions coupled to instantons

$$\mathcal{L}_\Psi = \bar{\Psi}\,(i\gamma^\mu D_\mu + im)\,\Psi$$

chiral symmetry in the massless case : $\{ i\gamma^\mu D_\mu , \gamma_5 \} = 0$

$\Rightarrow$ eigenvalues come in pairs $\pm\lambda$

on the zero modes $\lambda = 0$: diagonalise γ_5, pos. and neg. chirality

Weyl representation (Eucl.)

$$\gamma_\mu = \begin{pmatrix} 0 & \sigma_\mu \\ \bar{\sigma}_\mu & 0 \end{pmatrix} = \begin{pmatrix} 0 & (i\vec{\sigma}, \mathbb{1}_2) \\ (-i\vec{\sigma}, \mathbb{1}_2) & 0 \end{pmatrix} \qquad \gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4 = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}$$

$$\gamma_5 \begin{pmatrix} \Psi_L \\ 0 \end{pmatrix} = (+1)\begin{pmatrix} \Psi_L \\ 0 \end{pmatrix} \qquad \gamma_5 \begin{pmatrix} 0 \\ \Psi_R \end{pmatrix} = (-1)\begin{pmatrix} 0 \\ \Psi_R \end{pmatrix}$$

in an instanton background there is 1 left-handed zero mode, but no right handed one

[vice versa for antiinstantons]

$$\bar{\sigma}_\mu D_\mu \Psi_L = 0 \qquad \Psi_L \sim \frac{\rho}{((x-y)^2 + \rho^2)^{3/2}}$$ centered at inst. location and spher. symmetric

$$\sigma_\mu D_\mu \Psi_R = 0 \implies \Psi_R \text{ not normalisable}$$

$(-D_\mu^2$ is positive$)$

- Index theorem [Atiyah, Singer]

$$\text{index} \equiv n_L - n_R = Q \qquad \text{for any configuration}$$

instanton: just the minimal number of zero modes: $1 - 0 = 1$

6 higher color-rep.s of the fermion: factor on the r.h.s., e.g. 4 for adjoint

- **tunnelling picture**

Weyl (temporal) gauge: $A_0 = 0$

$$\mathcal{L}_M = \frac{1}{2}\left((\partial_0 \vec{A}^a)^2 - \vec{B}^{a2} \right)$$

$$H = \int d^3x \, \frac{1}{2}\left(\vec{\Pi}^{a2} + \vec{B}^{a2} \right) \qquad \vec{\Pi}^a = \partial_0 \vec{A}^a = \vec{E}^a$$

vacua: $E=0$ for pure gauges $\vec{A}(\vec{x}) = \frac{i}{g}\Omega^\dagger(\vec{x})\vec{\partial}\Omega(\vec{x})$

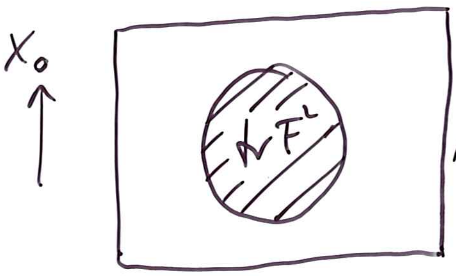
characterised by Chern–Simons number:

$$CS(\vec{A}) = \int d^3x \, K_0(\vec{A}) = \deg\left(\Omega(\vec{x}) : S^3 \to SU(2)\right) \in \mathbb{Z}$$

vacua with different CS cannot be deformed into each other within vacua

configuration space connected $\Rightarrow$ $E>0$ inbetween

the instanton as a tunnelling process:

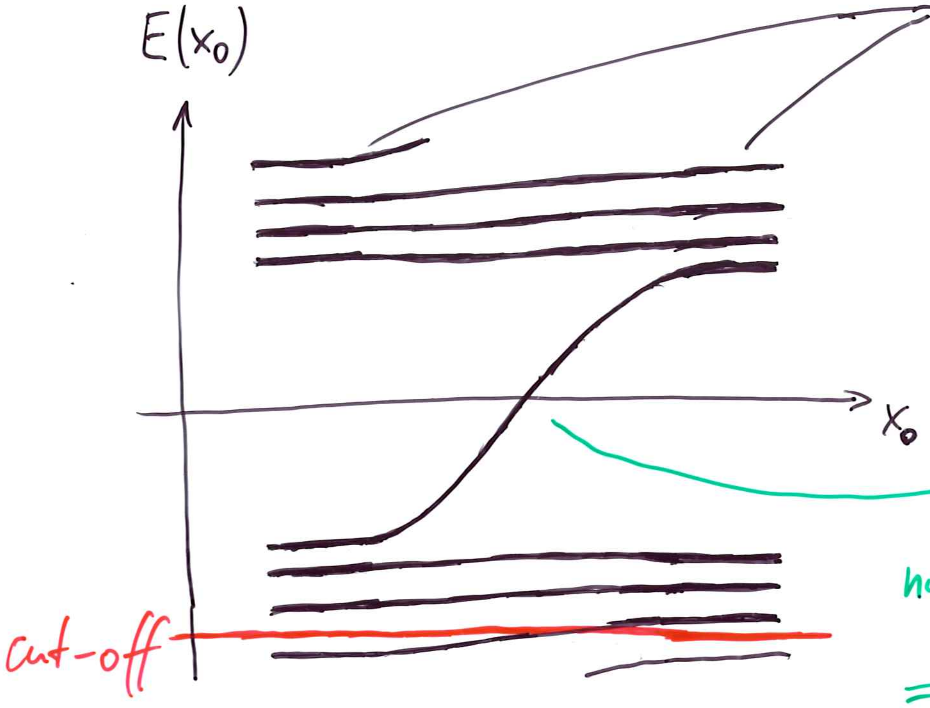


$$Q = \int d^4x \, \partial_\mu K_\mu = CS(\vec{A}_{+\infty}) - CS(\vec{A}_{-\infty}) + 0$$

tunnelling between vacua with $\Delta CS(\vec{A}) = Q$

• spectral flow and the axial anomaly

$$H_{D,x_0} = -\gamma_0 \vec{\gamma}\, \vec{D}_{x_0}$$



$\vec{A}_{-\infty}$ and $\vec{A}_{+\infty}$:
both vacua,
different $CS(\vec{A})$,
same spectrum

spectral flow ≡
number of zero crossings
= index $(\not{D}) = Q$

[Atiyah, Patodi, Singer]

adiabatic approximation:
normalisability of 4D zero mode of $\not{D}$ requires $E_{\pm\infty}$ of different sign

classically: axial current conserved

$$\partial_\mu j^5_\mu = 0 \qquad \text{(infinite hotel)}$$

quantum: renormalised axial current:

$$\partial_\mu j^5_\mu = 2\, \frac{g^2}{16\pi^2}\, \mathrm{tr}\, F_{\mu\nu} \tilde{F}_{\mu\nu}$$

cut-off the Dirac see: modes reappear after tunnelling:

$$\Delta Q^5 = Q^5_{+\infty} - Q^5_{-\infty} = 2Q$$

because in an instanton background: $\Delta Q^L = 1 \quad \Delta Q^R = -1$

one fermion flips chirality

(Some) Analytical aspects of instantons

instantons on torus-like manifolds [lattice!]

- the Nahm transform: a mapping between instantons: [Nahm; Braam, van Baal]

charge Q, $SU(N)$, on T^4 $(x_\mu \sim x_\mu + L_\mu)$

$\updownarrow$ [squares to identity, hyperKähler isometry]

charge N, $SU(Q)$, on $\tilde{T}^4$ $(z_\mu \sim z_\mu + \frac{1}{L_\mu})$

constructive, via the fermion zero modes:

$$\hat{A}_\mu^{pq}(z) \equiv \int d^4x \; \Psi_z^{(p)}(x)^\dagger \, i\partial_{z_\mu} \Psi_z^{(q)}(x)$$

$$\nabla_\mu \big(\partial_\mu 1\!\!1_N \underbrace{- iA_\mu + 2\pi i z_\mu 1\!\!1_N}\big) \Psi_z^{(p)}(x) = 0 \qquad p = 1..Q$$

$su(N)$ + trace part $\in u(N)$, same $F_{\mu\nu}$ [assume: no zero modes of wrong chirality]

$\Longrightarrow$ same charge and # zero modes

$2\pi i z_\mu 1\!\!1_N$ can be gauged away by $g_\mu = e^{2\pi i x_\mu z_\mu 1\!\!1_N}$ iff $L_\mu z_\mu = 1$ (no sum)

the new gauge field $\hat{A}_\mu(z)$:

- lives on $\tilde{T}^4$: $z_\mu \sim z_\mu + \frac{1}{L_\mu}$
- inv. under gauge transformations
- transforms like a gauge field under base change
- $|Q| \times |Q|$, hermitean $\Longrightarrow u(|Q|)$, even $su(|Q|)$
- selfdual, iff $A_\mu(x)$ is selfdual
- charge N

- advantages:

charge 1 : dual gauge field is $U(1)$, a linear problem

byproduct: no charge 1 instantons on T^4, since no $U(1)$ instantons on $\tilde{T}^4$

unless twisted boundary conditions

related manifolds: $T^{4-n} \times \mathbb{R}^n$

$\updownarrow$

$\tilde{T}^{4-n}$: selfduality equ.s with less derivatives

still $SU(Q)$

top. charge $N \longrightarrow$ N singularities

- the ADHM formalism : all $SU(N)$ instantons on $\mathbb{R}^4$

$\sim$ inverse Nahmtransform for $n=4$ point

purely algebraic, but non-linear for higher charge

ADHM "dual" data : $\Delta_x = \begin{pmatrix} \lambda \\ B - x \end{pmatrix}$ $\leftarrow$ singularities (vector) $\leftarrow$ dual gauge "field" (matrix)

$\Delta_x^\dagger \Delta_x \overset{!}{=}$ real and invertible $\leftarrow$ "selfduality"

zero modes: $\Delta_x^\dagger v_x = 0$

original gauge field : $A_\mu(x) = v_x^\dagger \partial_\mu v_x$

e.g. CFTW class : λ contains $\rho^{(p)}$, B contains $y^{(p)}$

- other manifolds:

particular results for $T^3 \times \mathbb{R}$: van Baal

$T^2 \times \mathbb{R}^2$: Jardim; Ford, Pawlowski

Calorons

$\equiv$ instantons at finite temperature, i.e. over $S^1 \times \mathbb{R}^3$

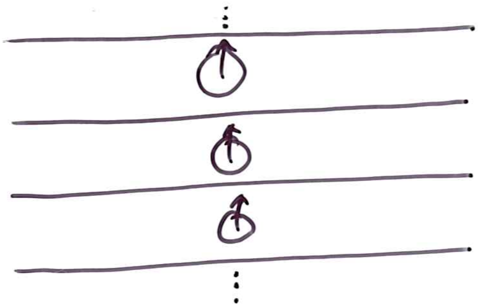
circumference $\beta = 1/k_B T$

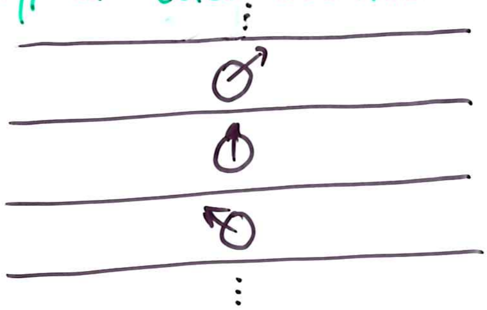
- approach from $\mathbb{R}^4$:

infinitely many copies along x_0 = charge ∞ instantons

same color orientations | different color orientations

$\beta \updownarrow$





CFTW ansatz

$\Downarrow$

Harrington–Shepard sol.n ['78]

full ADHM formalism

$\Downarrow$

calorons of nontrivial holonomy [Kraan, van Baal; Lee, Lu '98]

"dimensional reduction": β vs. ρ

$\beta \to \infty$: instantons

$\rho \to \infty$: BPS monopole [Rossi]

$$\left.\begin{array}{l} \partial_0 = 0 \\ \phi \leftrightarrow A_0 \end{array}\right\} \vec{D}\phi \leftrightarrow \vec{E}$$

$V(\phi) = 0$

strong overlap!

gauge field periodic up to gauge tr.

$\downarrow$

make periodic by x_0-dep. gauge tr.

$\downarrow$

nonvanishing $A_0(|\vec{x}| \to \infty)$

holonomy: $\mathcal{P}_\infty \equiv \lim_{|\vec{x}| \to \infty} \mathcal{P}(\vec{x}) \neq \pm \mathbb{1}_2$ nontrivial

Polyakov loop: $\mathcal{P}(\vec{x}) \equiv \mathcal{P}\exp\left(i \int_0^\beta dx_0\, A_0(\vec{x})\right)$

holonomy = background / environment for instantons
= Higgs field in the group

- **Nahm picture:**

dual gauge field $\hat{A}_\mu(z)$, $\mu = 0..3$, $z \in \tilde{S}^1_{1/\beta}$, $|Q|\times|Q|$ matrices (charge Q), N sing.s ($SU(N)$)

s.d.: $$\hat{E}_i(z) - \hat{B}_i(z) = \partial_z \hat{A}_i - [\hat{A}_0, \hat{A}_i] - i\epsilon_{ijk}[\hat{A}_j, \hat{A}_k] = \sum_{a=1}^{N} (\ldots)_a\, \delta(z - \mu_a)$$

ordinary differential equation

$e^{2\pi i \mu_a} \equiv$ ev.s of $\mathcal{P}_\infty$

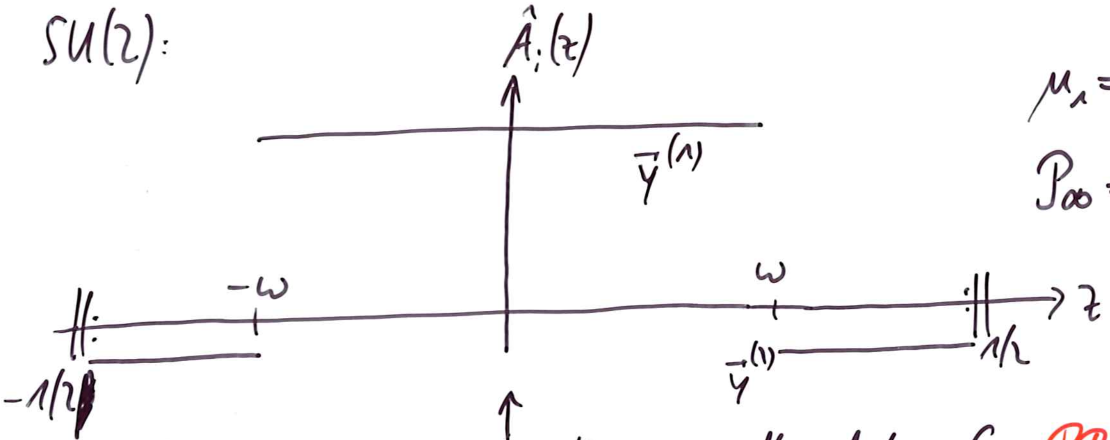
divide $\tilde{S}^1$ into N intervals

[or less if some μ_a equal]

charge 1: no commutator $\Longrightarrow$ $\hat{A}_i(z)$ piecewise constant $\Longrightarrow$

$\hat{\Psi}_x(z)$ piecewise exponential $\Longrightarrow$ $A_\mu(x)$ in closed form

SU(2):



$\mu_1 = -\mu_2 = -\omega$

$\mathcal{P}_\infty = e^{2\pi i \omega \tau_3}$

$(\beta = 1)$

Nahm's original transform: these are the data of a BPS monopole

- **"dissociation" / substructure:**

the dual data indicate that

$\Longrightarrow$ a charge 1 SU(2) caloron has 2 constituent monopoles

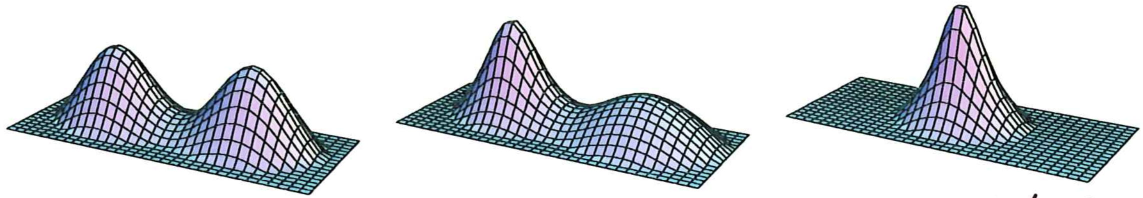
with locations $\vec{y}^{(1),(2)}$ $\leftarrow$ dual gauge field values

and masses 2ω and $1-2\omega$ $\leftarrow$ interval lengths

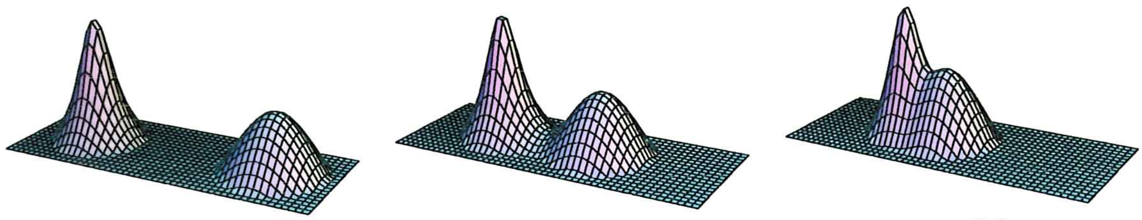
unless the holonomy is trivial

fractional charge

charge 1 SU(N): N monopoles $\sim$ "instanton quarks" calorons of charge 1 [Kraan,van Baal]

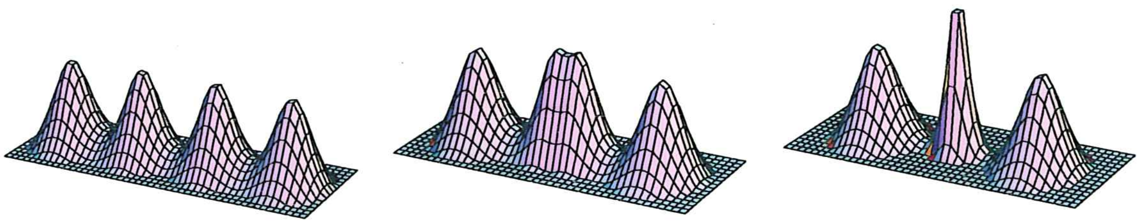


$\omega = 1/4$ $\omega = 1/8$ $\omega = 0$ (HS)

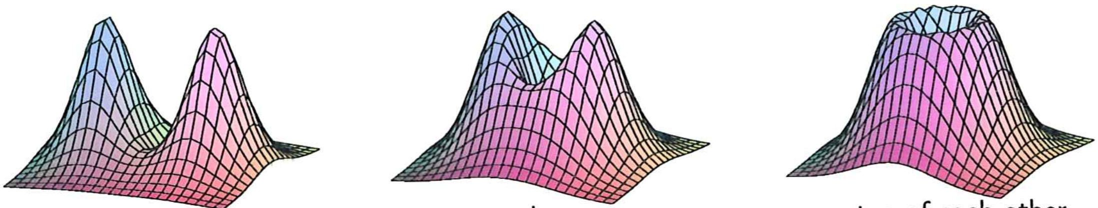


$\rho = 1.6$ $\rho = 1.2$ $\rho = 0.8$

calorons of any charge, here 2, axial [FB,van Baal]



charge 2: like charge monopoles overlap [FB,Nogradi,van Baal]



separated closer on top of each other

- $\omega \in \{0, \frac{1}{2}\}$: $\mathcal{P}_\infty = \pm 1\!\!1$, $A_0 \equiv \Phi \longrightarrow 0$, no symmetry breaking

  HS–sol.n, only one monopole (the other massless = infinitely spread)

  $\omega = \frac{1}{4}$: $\mathcal{P}_\infty = i\tau_3$ (equator of $SU(2)$), $\mathrm{tr}\,\mathcal{P}_\infty = 0 \xleftarrow{?!}$ confined phase $\langle \mathrm{tr}\,\mathcal{P} \rangle = 0$

  both monopoles of same mass

- well separated $|\vec{y}^{(1)} - \vec{y}^{(2)}| \gg \beta$: almost static, monopoles of size β

  close together $|\vec{y}^{(1)} - \vec{y}^{(2)}| \ll \beta$: strong time dependence, one lump instanton, $\rho = \sqrt{\frac{|\vec{y}^{(1)} - \vec{y}^{(2)}|\,\beta}{\pi}} \ll \beta$

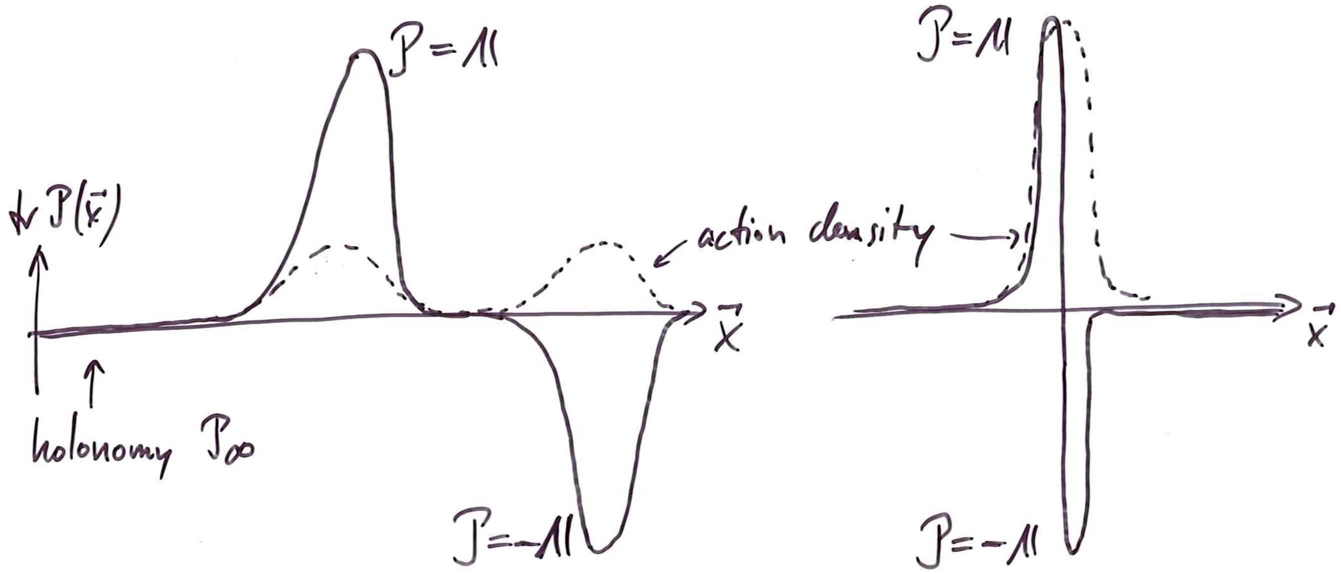
far field is abelian (along Higgs) = dipoles:

- the monopoles have opposite magnetic charges

  and opposite "electric charges" in Euclidean space

  $\Longrightarrow$ forces compensate

- the Polyakov loop $\mathcal{P}(\vec{x})$ passes through $\pm 1\!\!1$ near the monopoles:



for topological reasons: winding number of $\mathcal{P}(\vec{x})$

[Reinhardt; Jahn, Lenz; Ford et al.]

- calorons of higher charge Q

charge $Q\ (>0) \Longrightarrow \hat{A}_\mu(x)$ is $Q \times Q \Longrightarrow [\hat{A}_j, \hat{A}_k] \Longrightarrow$ not piecewise const.

in each interval: Q vectors $\vec{y}$ ($\sim$ evs. of $\hat{\vec{A}}$ matrices)

$\Longrightarrow$ Q monopoles of each charge

some special solutions for arbitrary charge [FB, van Baal]

and charge 2 [FB, Nogradi, van Baal]

- counting of moduli:

instantons:		monopoles and antimonopoles:	
4D locations:	$4Q$	3D locations:	$3Q+3Q$
sizes:	Q	time locations/phases:	$Q+Q$
color orientations:	$3Q$		
	$8Q$		$8Q$

(minus global gauge rotations)

holonomy parameters (ω) do not count as moduli:

$S = \frac{8\pi^2}{g^2}|Q|$ independently of ω, hence

$\frac{\partial A_\mu}{\partial \omega}$ is a zero mode of the fluctuation operator,

but not normalisable [A_0 changes asymptotically]

- **fermionic zero modes in the caloron background**

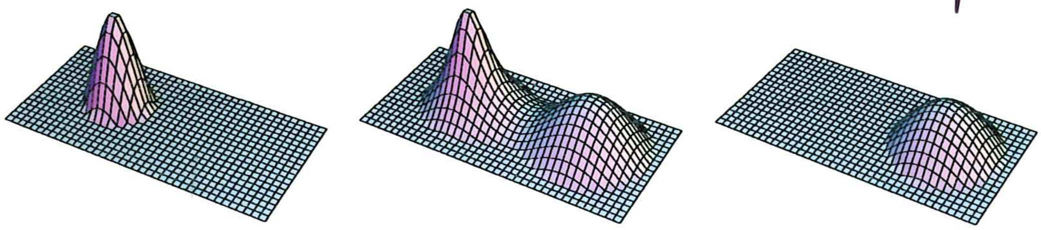
dilemma: only 1 zero mode (index = charge = 1)

but 2 monopoles to localise to (SU(2), N for SU(N))

the zero mode hops with boundary conditions in S^1 [Garcia-Perez]

let: $\Psi_z(x+\beta e_0) = e^{2\pi i z}\Psi_z(x)$, $|\Psi(z)|^2$ periodic

$\Rightarrow$ for $z \in \begin{cases} (-\omega,\omega) & \text{ind. periodic} \\ (\omega, 1-\omega) & \text{ind. antiperiodic} \end{cases}$ $\Psi_z(x)$ is exponentially localised to the corresponding monopole



periodic zm. action density antiperiodic zm.

at $z = \pm\omega$: sees both monopoles, only algebraic decay

explanation: $\Psi_z(x) = e^{-2\pi i z x_0/\beta}\Psi_z(x)$ is periodic, but z now

enters $\sigma_\mu(D_\mu - 2\pi i z\,\delta_{\mu 0})\Psi_z(x) = 0$ as a mass (4D: imag. μ)

each monopole supports a zero mode, when mass in "its Higgs range"

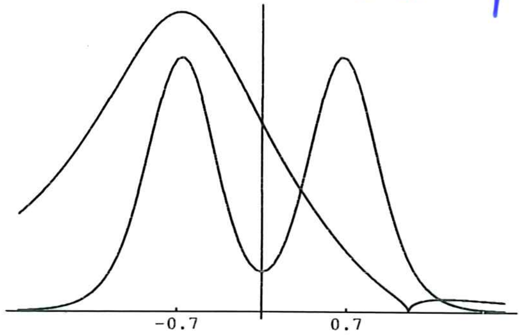
[Callias]

$\Psi_z(x)$ is used in the Nahm transform:

$$\vec{\hat{A}}(z) = \int d^4x\, \Psi_z^\dagger(x)\, \vec{x}\, \Psi_z(x) = \langle \vec{x}\rangle_{\Psi_z} \simeq \vec{y}^{(1),(2)}$$

(was $i\partial_{\vec z}$, but non-compact direction $e^{-2\pi i \vec{x}\vec{z}}$)

- **zero in the zero mode profile**



does the zero mode notice the "other" monopole at all?

YES: a zero in its profile [FB]

near the other core

top. origin...

(Some) Models for QCD/YM

explain IR phenomena:

– quarks and gluons are not observed freely (only colorless bound states)

= confinement

in pure YM: interquark potential is linear $V_{q\bar{q}}(R) \longrightarrow \sigma R$

$\sigma \simeq 1\,\text{GeV}/\text{fm}$ ... string tension

challenge: area law for large Wilson loops $\langle W(R\times T)\rangle = e^{-\sigma\cdot RT}$

below the critical temperature ...

model: dual superconductor based on magn. monopoles

same T_{crit}

– hadrons are massive: mass gap, hadron spectroscopy

– hadrons do not have parity doublers, although in $\mathcal{L}_{QCD}$ at $m=0$

L and R quarks decouple = chiral symmetry breaking

chiral condensate: $\langle \bar{\psi}\psi\rangle \simeq -(240\,\text{MeV})^3$

$$\langle \bar{\psi}\psi\rangle = -\frac{\pi}{V}\,\rho(\lambda=0) \qquad \text{[Banks, Casher]}$$

challenge: density of eigenvalues at 0 of the Dirac operator

model: instanton liquid

note: (massless) QCD is dimensionless $\Longrightarrow$

all dimensionful observables by quantum effects = 'dimensional transmutation'

phenomena are expected to be due to gauge field dynamics

which nonperturbative degrees of freedom? effective action?

- semiclassics in QCD: the instanton liquid

expansion $A_\mu = A_\mu^{cl} + a_\mu$ plus Gaussian integration like for the kink

several subtleties:

(i) $\{A_\mu(x)\}$ too big: contains equivalent config.s ~ the local gauge group

$\Longrightarrow$ fix the gauge $\leftarrow$ here $D_\mu(A^{cl})\,a_\mu = 0$

include Faddeev–Popov determinant = 'ghosts' $\leftarrow$ here $-D_\mu^2(A^{cl})$

(ii) take A_μ^{cl} as superposed instantons/antiinstantons = approx. solutions

top. charges cancel to a few units $Q = 0, \pm 1, \ldots$

not a minimum, no strict separation from perturbative fluctuations

(iii) moduli: locations y_μ, sizes ρ, color orientations Ω: explicit integration

(iv) diluteness?

kink: $\phi(x) \sim e^{-m|x-y|}$ exponentially

inst.: $A_\mu(x) \sim \dfrac{\rho^2}{(x-y)^3}$ only algebraically, a priori all values of ρ

$\rightsquigarrow$ interactions more important

one instanton weight:

$$\int d^4y\, d\rho\, d^3\Omega\ \text{Jacobian} \cdot e^{-\frac{8\pi^2}{g^2}} \cdot \frac{\det(-D_\mu^2)}{\sqrt{\det{}'(-D_\alpha^2\delta_{\mu\nu} + 2i[F_{\mu\nu},\ldots])}}$$

in the instanton background, regularisation ['t Hooft '76]

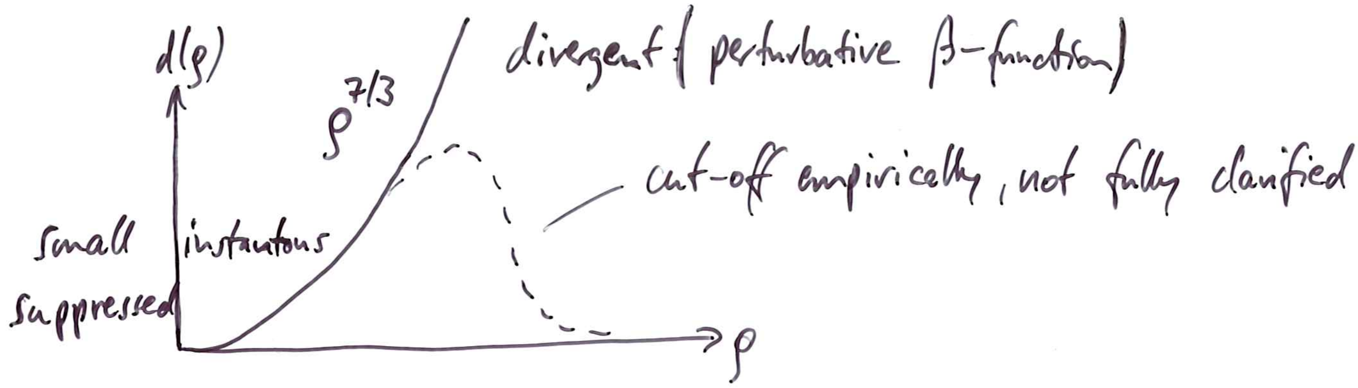
$$\sim \int d^4y\, d\rho \cdot d(\rho)$$

instanton size distribution: $d(\rho) = \dfrac{1}{\rho^5} \cdot \exp\left(-\dfrac{8\pi^2}{g(\rho)}\right)$

one loop β-function

$$\underline{\underline{d(\rho) \sim \rho^{b-5}}} \qquad b_{\text{pure } SU(2)} = \frac{22}{3}$$

scale invariance broken by quantum effects



interactions: $S[n \text{ instantons and antiinstantons}] \equiv n \frac{8\pi^2}{g^2} + S_{int}$

hard core [Ilgenfritz, Müller-Preussker; Münster]

variational principle [Diakonov, Petrov]

interaction depends on rel. color orientation, on average repulsive

$\Longrightarrow d(\rho) \sim \rho^{7/3} \exp\left(-\# \sqrt{\frac{n}{V}}\, \rho^2\right)$ peaked around $\rho = \bar{\rho}$

phenomenological values in the instanton liquid model [Shuryak '82]

$\bar{\rho} \simeq \frac{1}{3}\,\mathrm{fm}$ average instanton size

$\bar{R} = \left(\frac{n}{V}\right)^{-1/4} \simeq 1\,\mathrm{fm}$ average separation

packing fraction: $\pi^2 \bar{\rho}^4 / \bar{R}^4 \sim \frac{1}{8}$ fairly dilute

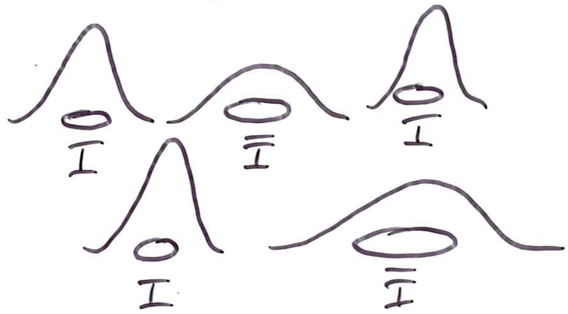
- chiral condensate: ✓ (see below)
- hadronic properties
- confinement: ?? only with special orientations or large instantons
  but: finite temperature $\longrightarrow$ calorons
- top. susceptibility: $\chi_{top} = \frac{\langle Q^2 \rangle}{V}$

$\langle Q \rangle = \langle n_I - n_{\bar{I}} \rangle = 0$ CP invariance

$\chi_{top} = \frac{\langle (n_I - n_{\bar{I}})^2 \rangle}{V} \simeq \frac{\langle n \rangle}{V} = \bar{\rho}$ number variation in a grandcanon. ensemble

• instantons and $\rho(\lambda=0)$

idea:



3 inst.s, 2 antiinst.s, dilute: each has its own zero mode

$Q=1 \Rightarrow 1$ exact chiral zero mode

in add.: 4 near zero modes $\lambda \simeq 0$

analogy: atoms that have a bound state for electrons (localised)

finite density $\longrightarrow$ bands in the spectrum (delocalised, conductivity)

eigenvalue splitting for an $I\bar{I}$-pair from pert. theory:

$$\lambda = 0 + \mathrm{ev}\left(\langle \text{unperturbed}^{(i)} | \text{perturbation} | \text{unperturbed}^{(j)} \rangle\right)$$

$$\lambda = \pm T_{I\bar{I}} \qquad T_{I\bar{I}} = \int d^4x \; \psi_I^\dagger(x-y_I)\, \gamma^\mu \partial_\mu\, \psi_{\bar{I}}(x-y_{\bar{I}})$$

overlap integral

spread of the band:

$$\langle |T_{I\bar{I}}|^2 \rangle_{\substack{\text{locations}\\ \text{orientations}}} = \# \cdot \frac{n}{V} \cdot \rho^2$$

shape of the band (random matrix theory)

$$\frac{1}{V}\rho(\lambda=0) = \# \sqrt{\frac{n}{V}} \cdot \frac{1}{\rho} \simeq \# \cdot \frac{1}{\bar{R}^2} \cdot \frac{1}{\bar{\rho}}$$

$$\Rightarrow \underline{\underline{\langle \bar{\psi}\psi \rangle \simeq -(250\,\mathrm{MeV})^3}} \qquad \text{quite good!}$$

remark: each object with a zero mode in a gas of finite density has the potential to generate $\rho(\lambda=0)$

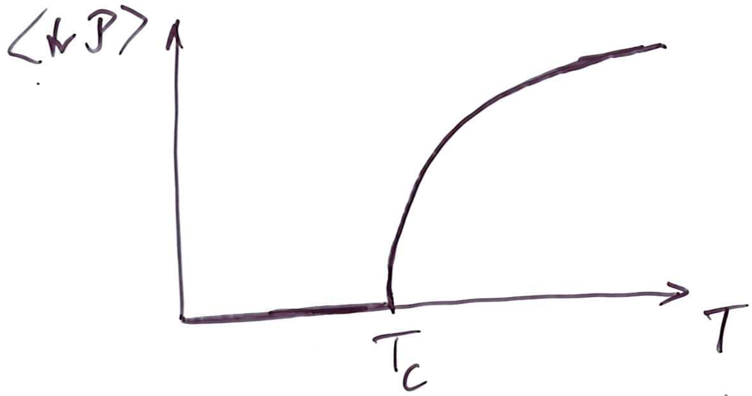
monopoles, vortices ...

• new aspects by calorons:

modifications of instantons at finite temperature:

large calorons are monopole pairs $\longleftrightarrow$ diff. suppression mechanism?

depend on asymptotic Polyakov loop $\longleftrightarrow$ sensitive to order parameter



equal mass monopoles   one monopole becomes light

[• gluino condensate]

• quantum weight of the caloron [Diakonov et al. '04]

non-perturbative contribution of a caloron ensemble to $V_{\mathrm{eff}}^{1\text{-loop}}(\mathrm{tr}\,P)$

$P = \pm 1\!\!1$, which is favoured perturbatively at high T,

becomes unstable: onset of confinement

• numerical simulation of a caloron ensemble [Gerhold et al. '06]

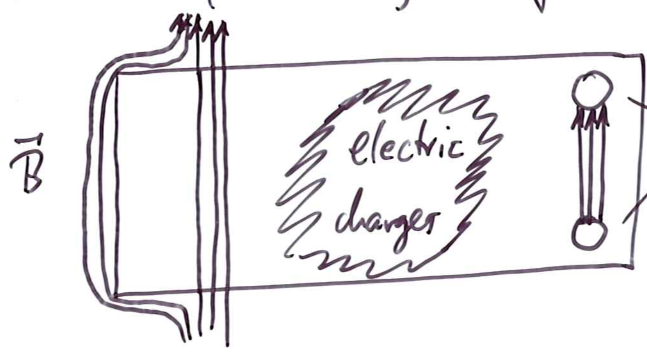
fixed holonomy — trivial $\Rightarrow$ deconfinement; nontrivial $\Rightarrow$ confinement

contact to Dual superconductor

• the Dual Superconductor picture

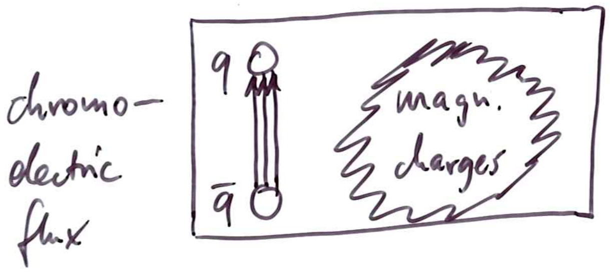
ordinary superconductor: Cooper pairs (2 electrons) condense

Meissner effect: magnetic flux tubes



[Nambu '74, Parisi, Mandelstam, 't Hooft]

QCD vacuum $\stackrel{!}{=}$ Dual Superconductor



Force$_{\bar{q}q}$ = const.

$V_{\bar{q}q} \sim R$ ← confinement!

how to obtain magnetic monopoles in QCD?

idea: gauge fix the non-abelian part ['t Hooft '81]

choice: auxiliary field $\varphi(x)$ transforming in the adjoint repr.

Abelian gauge: use $\Omega(x)$ such that $\Omega^\dagger \varphi \Omega \sim \tau_3$ ... diagonal

residual gauge freedom: local $U(1)$ around τ_3 $[SU(N): U(1)^{N-1}]$

split: $A_\mu = \{A_\mu^{1,2}, A_\mu^3\}$

↳ A_μ^3: residual "photon"

↳ $A_\mu^{1,2}$: like matter field, massive by quantum effects

Abelian projection:

neglect $A_\mu^{1,2}$

⟹ almost a local $U(1)$ theory, but defects as remnants of the non-abelian nature the gauge fixing is ambiguous at lines $\varphi^a(x)=0$

see monopole as a static YM configuration, $\varphi \equiv \phi$

"combing" $\phi^a(x)$ to $(0,0,1)$ fails at $\phi(\vec{x}=\vec{y})=0$ because hedgehog

the Abelian gauge fixing induces (closed) worldlines of magn. monopoles

picture: in the QCD vacuum monopole worldlines percolate (~ condensation)

facts:

(0) there are many Abelian gauges [MAG: 't Hooft, LAG: van der Sijs] which on the same configuration can induce different defects

(1) continuum: topology predicts existence, but not precise realisation of defects

findings: single instanton $\rightsquigarrow$ small mon. loop
instanton ensemble $\rightsquigarrow$ large mon. loops ?! [Hart, Teper]

(2) lattice: $\exists$ procedure to identify monopoles [DeGrand, Toussaint]
$\exists$ lattice variants of Abelian gauges [Kronfeld et al.]

empirical:

Abelian dominance: only $A_\mu^3 \Longrightarrow \sigma_{Abel} = 92\% \cdot \sigma_{full}$

monopole dominance: only monopole (sing.) part of $A_\mu^3 \Longrightarrow \sigma_{mon} = 95\% \, \sigma_{Abel}$
[Suzuki et al., Stack et al.]

problems:

- results depend on the choice of gauge
- not an effective theory: MC sampling with full field, reduction only in observables; what is the guiding principle / small parameter?
- repr.-dependence of σ comes out wrong [see Greensite '03]

competing model: center vortices

Topological objects in lattice gauge theory

lattice simulations: many important quantitative results,

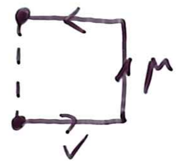
but how to understand in terms of continuum objects?

a typical lattice configuration is dominated by UV fluctuations!

- cooling

iterative procedure : $U_\mu \longrightarrow P\left(\sum_{\nu\neq\mu} U_\mu^{(\nu)}\right)$ [Berg '81; Hoek et al.]

P = projection onto $SU(2)$ $\quad \tilde{U}_\mu^\nu(x) = U_\nu(x)\, U_\mu(x+\hat\nu)\, U_\nu^\dagger(x+\hat\mu)$

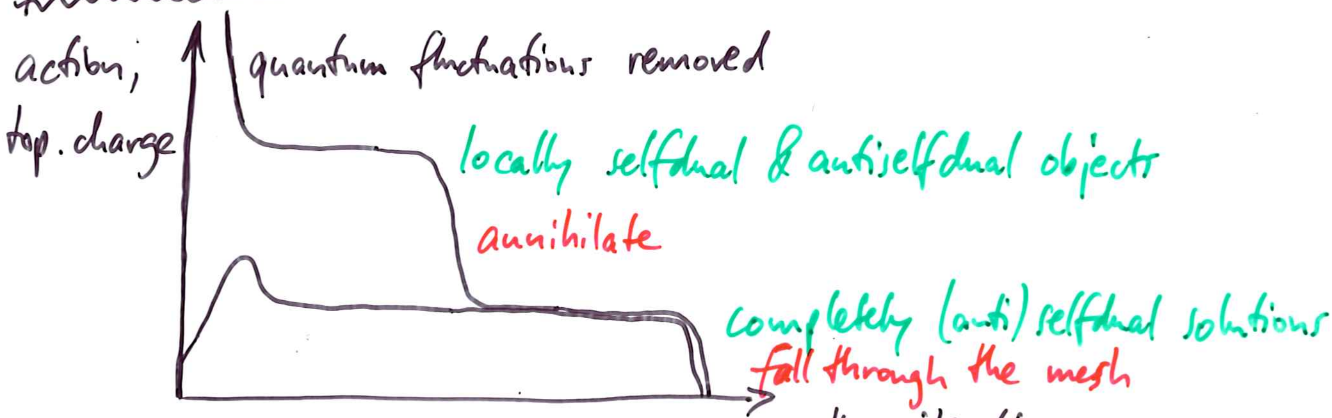
staple: 

local

reduces the action, fixed point = solution of (lattice) e.o.m.

variants: smearing (averaging with the old link), improved cooling ...

typical history:



tool to investigate classical solutions [intro]

insight into QCD vacuum?

trusted w.r.t. global observables like Q and top. suscept. $\frac{\langle Q^2\rangle}{V}$

local objects: density and sizes modified

biased to class. sol.ns; consistent with ILM, but not a proof

stop when? monitor IR features like string tension

- fermionic techniques

use Dirac operators with good chiral properties (GW, DeGrand's lecture)

as a tool to investigate lattice configurations

localisation:

low-lying eigenmodes are smooth: small 'energy' forbids large momenta

& zero modes of instantons/monopoles are localised to the cores

$\Longrightarrow$ shall find the relevant continuum objects

- zero modes at finite temperature hop as for calorons [Gattringer]
- which dimensionality? scaling of the profile (IPR) with a:

  inconclusive [MILC, Polikarpov et al.]

fermionic topological charge:

$Q_{ferm} \equiv n_L - n_R$ even locally:

$$q_{ferm}(x) \equiv \mathrm{tr}_{\substack{color\\spin}}\, \gamma_5 \left(\frac{1}{2} D_{x,x} - 1\right) = \sum_n \left(\frac{\lambda_n}{2} - 1\right) \psi_n^+(x)\, \gamma_5\, \psi_n(x)$$

[Niedermayer '99]

as a filter: truncate at small number of modes

similar: links from eigenmodes of the lattice Laplace operator [FB et al.]

- evidence for 3D structures [Horvath et al.]
- yes for low cuts in $|q|$, lower-dim. structures at high $|q|$ [Ilgenfritz et al.]

Comparison of filters [FB et al.]

smearing, fermionic and Laplacian modes see the same structures of topological charge

No conclusions.

$\longrightarrow$ Stay tuned! $\longleftarrow$

hopping of the zero mode in a thermalised configuration [Gattringer, Schaefer]

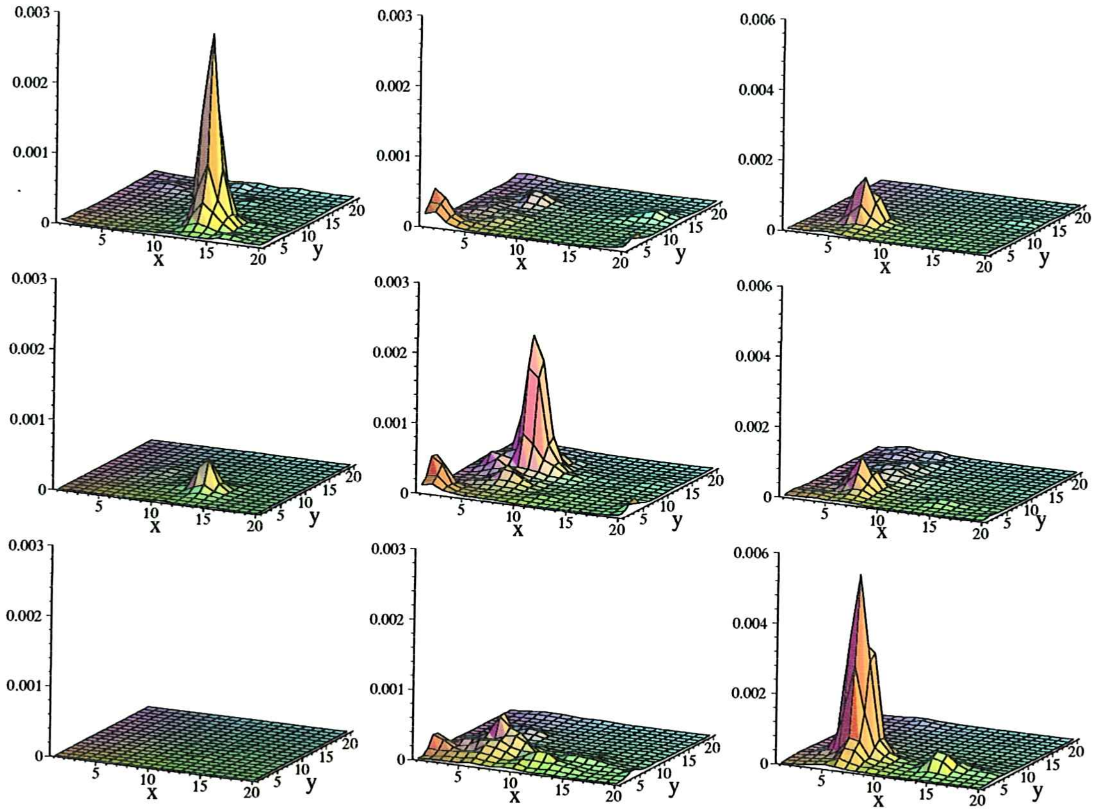


Figure 1: Slices of the scalar density for $6 \times 20^3, \beta = 8.20$, configuration 125. We show x, y-slices at $t = 5$, $z = 9$ (left column), at $t = 2$, $z = 19$ (center column) and $t = 5$, $z = 18$ (right column). The values for ζ are $\zeta = 0.05, 0.3, 0.65$ (from top to bottom).

topological charges after different filterings [FB et al.]