Fermi and Bose Liquids

Fermi Liquids

EFT for pointlike, non-relativistic neutrons

$$\mathcal{L}_{\rm eff} = \psi^\dagger \left(i\partial_0 + \frac{\nabla^2}{2M} \right) \psi - \frac{C_0}{2} (\psi^\dagger \psi)^2 + \ldots$$

Coupling constant determined by scattering length

$$C_0 = \frac{4\pi a}{M}$$

$U(1)$ symmetry $\psi \to e^{i\alpha}\psi$: Conserved charge

$$N = \int d^3x\, \psi^\dagger \psi$$

Partition function

$$Z(\mu, \beta) = \mathrm{Tr}\left[e^{-\beta(H-\mu N)} \right]$$

Path Integral representation: $\mathcal{L} \to \mathcal{L} - \mu\psi^\dagger\psi$

$$Z = \int D\psi D\psi^\dagger \exp\left(-\int_0^\beta d\tau \int d^3x\, \mathcal{L}\right)$$

Feynman rules: Propagator (Minkowski space)

$$G_0(k)_{\alpha\beta} = \delta_{\alpha\beta}\Big(\frac{\theta(k-k_F)}{k_0 - k^2/2M + i\epsilon} + \frac{\theta(k_F-k)}{k_0 - k^2/2M - i\epsilon}\Big) \qquad \frac{k_F^2}{2M} = \mu$$

particles holes

Four-Fermion Vertex

$$\Gamma_{\alpha\beta,\gamma\delta}(k_1,k_2;k_3,k_4) = i\left(\delta_{\alpha\gamma}\delta_{\beta\delta} - \delta_{\alpha\delta}\delta_{\beta\gamma}\right)\delta\left(\sum k_i\right)$$

Fermion loops (-1), Tadpoles require $e^{ik_0\eta}|_{\eta\to 0^+}$

Perturbative Results

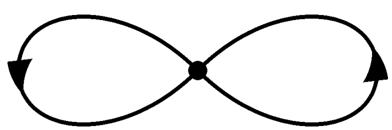
Neutron density

$$\rho = \int \frac{d^4k}{(2\pi)^4} S^0_{\alpha\alpha}(k)\, e^{ik_0\eta}\Big|_{\eta\to 0^+} = 2\int \frac{d^3k}{(2\pi)^3}\Theta(k_F - k) = \frac{k_F^3}{3\pi^2}$$

Energy density

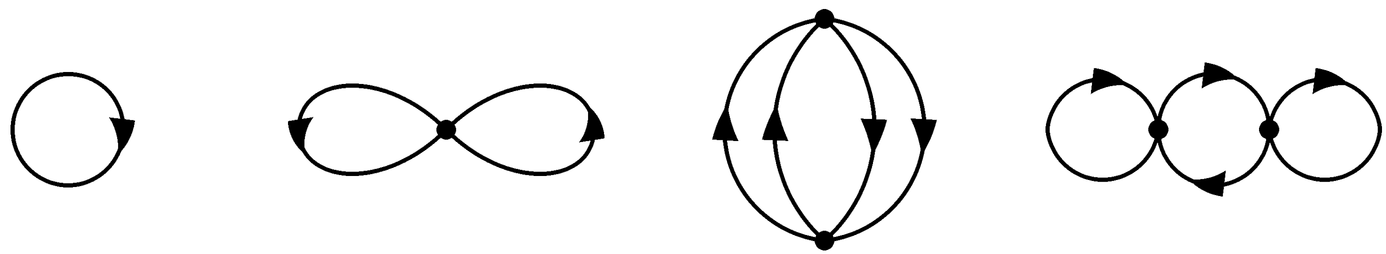
$$\mathcal{E} = 2\int \frac{d^3k}{(2\pi)^3} E_k \Theta(k_F - k) = \frac{3}{5}\rho\frac{k_F^2}{2m}$$

First order perturbative correction



$$\mathcal{E}_1 = C_0 \left(\frac{k_F^3}{6\pi^2}\right)^2$$

Higher orders: $(k_F a)$ expansion

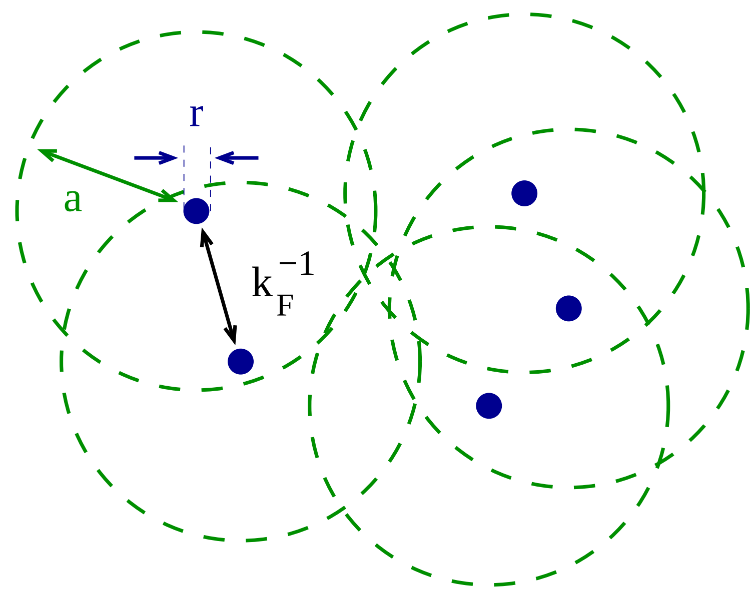


$\epsilon_F \rho$ $\quad$ $\epsilon_F \rho \, (k_F a)$ $\quad$ $\epsilon_F \rho \, (k_F a)^2$

$$\frac{E}{A} = \frac{k_F^2}{2M}\left[\frac{3}{5} + \left(\frac{2}{3\pi}(k_F a) + \frac{4}{35\pi^2}(11 - 2\log(2))(k_F a)^2\right) + \ldots\right]$$

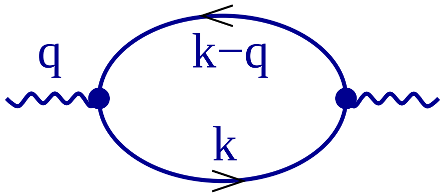
Problem: $a_{nn} \simeq -20$ fm

$\Rightarrow (k_F a) \gg 1$



Charged Fermions: Screening

Photon polarization function



$$\Pi_{00}(q) = e^2 \int \frac{d^4p}{(2\pi)^4} \frac{1}{(q_0 + p_0 - \epsilon_{p+q})(p_0 - \epsilon_p)}$$

Perform p_0 integral: particle-hole contribution

$$\Pi_{00}(q) = e^2 \int \frac{d^3p}{(2\pi)^3} \frac{n_{p+q} - n_p}{E_{p+q} - E_p - q_0}$$

Static polarization function, long distance

$$\Pi_{00}(q_0 = 0, \vec{q} \to 0) = e^2 \int \frac{d^3p}{(2\pi)^3} \frac{\partial n_p}{\partial E_p} = e^2 \frac{p_F m}{2\pi^2}$$

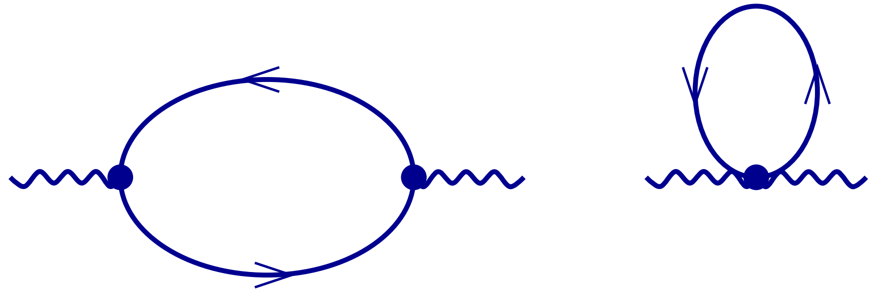
Screened potential

$$V(r) = -\frac{e}{r} \exp(-m_D r) \qquad\qquad m_D^2 = e^2 \frac{p_F m}{2\pi^2}$$

Charged Fermions: Landau Damping

transverse polarization function

$$\mathcal{L} = \vec{v} \cdot \vec{A} + \frac{1}{2m} \vec{A}^2$$



$$\Pi_{ij}(q) = e^2 m_D^2 \int \frac{d\Omega}{4\pi} \left\{ v_i v_j \frac{vk(\hat{q} \cdot \hat{p})}{q_0 - vk(\hat{q} \cdot \hat{p})} - \frac{1}{3} v^2 \delta_{ij} \right\},$$

No screening for $q_0 \to 0$

$$\Pi_{ii}(q) = m_D^2 \frac{vq_0}{2q} \log\left(\frac{q_0 - vq}{q_0 + vq} \right)$$

Imaginary part: Landau damping

$$\text{Im}\Pi_{ii}(q) = \pi m_D^2 \frac{vq_0}{q} \Theta(vq - q_0)$$

So far: Free space EFT $\Rightarrow$ System at non-zero density

Non-perturbative if $(k_F a), (k_F r), \ldots > 1$

Construct EFT for low energy excitations in dense matter

Fermions: Landau Fermi-Liquid Theory

Bosons: Broken Symmetry, Goldstone bosons

Fermi Liquid Theory

Free non-relativistic quasi-particles near Fermi surface

$$S = \int dt \int \frac{d^3p}{(2\pi)^3} \psi(p)^\dagger \left(i\partial_t - (\epsilon(p) - \epsilon_F)\right) \psi(p)$$

Expand momenta around Fermi momentum $\vec{p} = \vec{k} + \vec{l}$

$$\epsilon(p) - \epsilon_F = \vec{v}_F(k) \cdot \vec{l} + O(l^2)$$

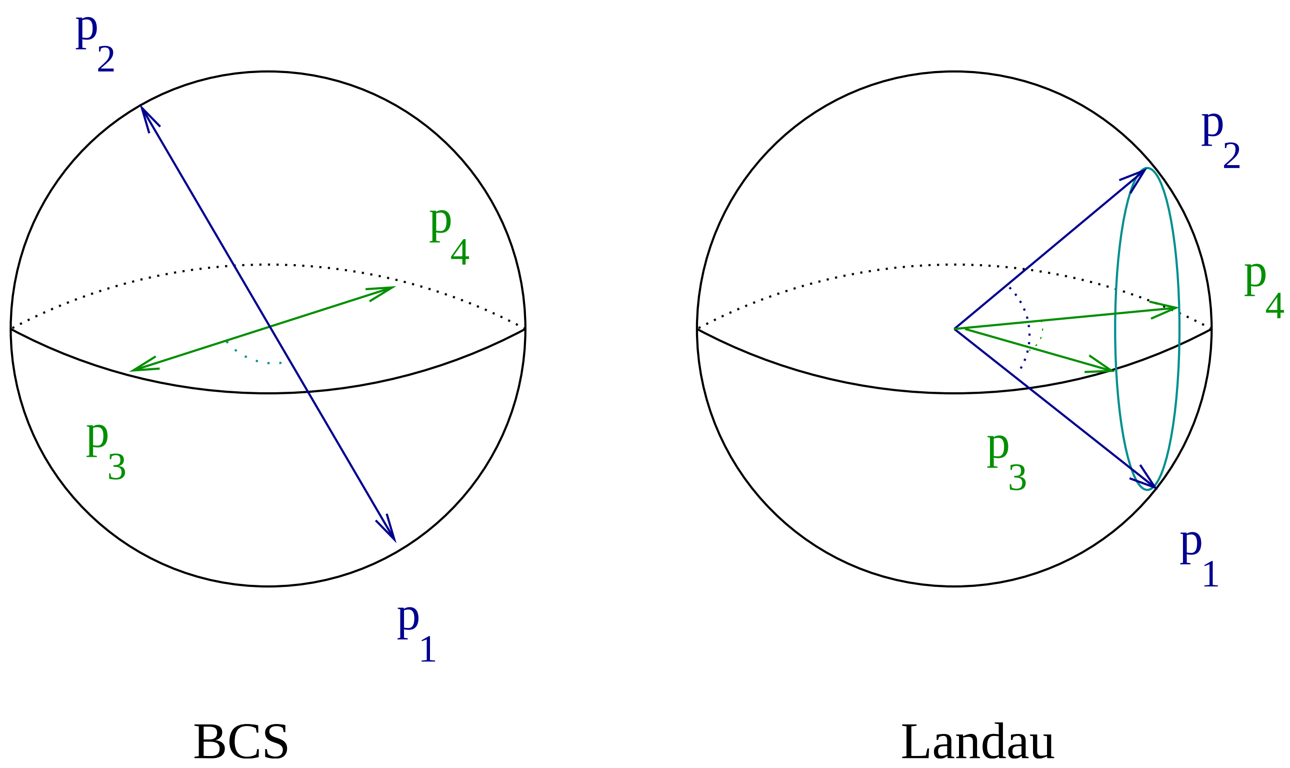
Study scaling behavior $\vec{l} \to s\vec{l}$. Scaling dimensions

$$[k] = 0, \qquad [l] = 1, \qquad [\partial_t] = 1, \qquad [d^3p] = 1, \qquad [\psi] = -\frac{1}{2}$$

Interaction

$$S_{int} = \int dt \left[\prod_{i=1}^{4} \int \frac{d^3p_i}{(2\pi)^3}\right] \psi^\dagger(p_4)\psi^\dagger(p_3)\psi(p_2)\psi(p_1)\delta^3(p_{tot})U(p_i)$$

Marginal Interactions



$$BCS: \quad U(-\hat{p}_3, \hat{p}_3, -\hat{p}_1, \hat{p}_1) = V(\hat{p}_1 \cdot \hat{p}_3) = \sum_l V_l P_l(\hat{p}_1 \cdot \hat{p}_3),$$

$$LFL: \quad U(\hat{p}_4, \hat{p}_3, \hat{p}_2, \hat{p}_1)|_{\hat{p}_1 \cdot \hat{p}_2 = \hat{p}_3 \cdot \hat{p}_4} = F(\hat{p}_1 \cdot \hat{p}_2, \phi_{12,34})$$

Example: A Tale of Two Sounds

Zero Sound: Collective Oscillation of Fermi surface

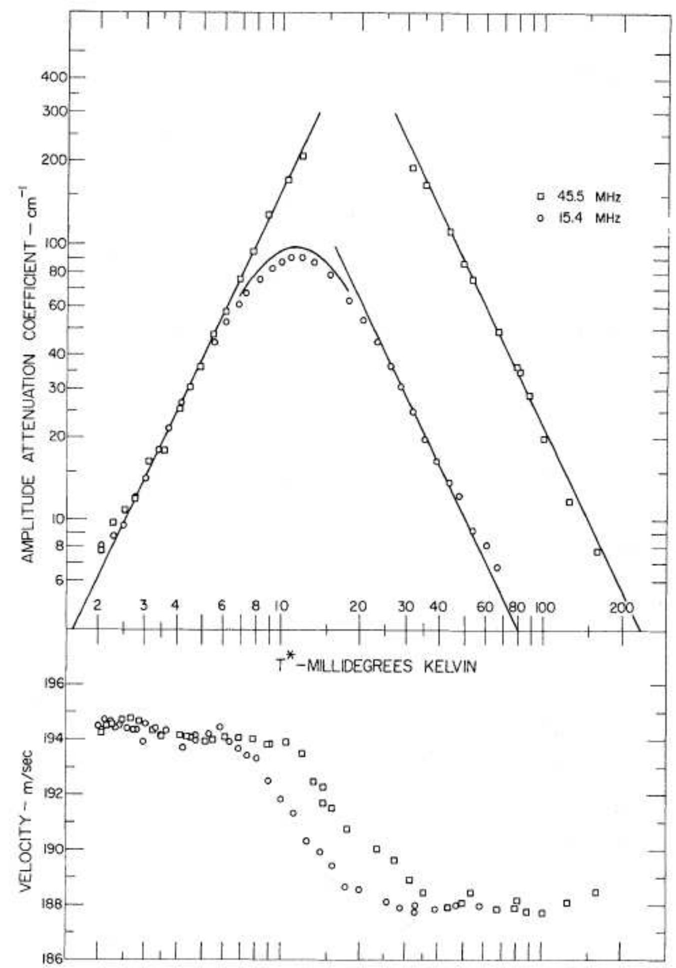
$$v_0 = s_0 v_F$$

$$\frac{s_0}{2}\log\left(\frac{s_0+1}{s_0-1}\right) = \frac{F_0+1}{F_0}$$

First Sound: Collisional Mode

$$v_1 = s_1 v_F$$

$$s_1^2 = \frac{1}{3}\frac{1+F_0}{1+F_1/3}$$



Data for ^3He, Abel et al.

Bose Condensation

Charged relativistic bosons (need $\lambda > 0$, repulsive)

$$\mathcal{L} = (\partial^\mu \phi^*)(\partial_\mu \phi) - m^2 \phi^* \phi - \lambda (\phi^* \phi)^2$$

$U(1)$ symmetry $\phi \to e^{-i\varphi}\phi$. Conserved charge

$$Q = \int d^3x\, i\, (\phi^* \partial_0 \phi - \phi \partial_0 \phi^*)$$

Path integral representation

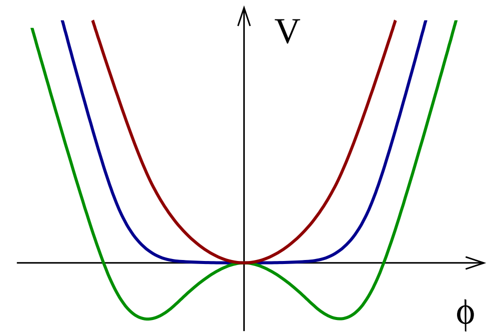
$$Z = \int D\phi D\phi^* \exp\left(i \int d^4x \mathcal{L} \right)$$

$$\mathcal{L} = (\partial_0 + i\mu)\phi^* (\partial_0 - i\mu)\phi - |\vec{\nabla}\phi|^2 - m^2|\phi|^2 - \lambda|\phi|^4$$

Effective potential

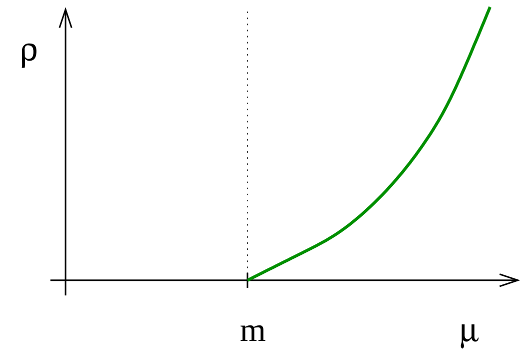
$$V(\phi) = (m^2 - \mu^2)(\phi^*\phi) + \lambda(\phi^*\phi)^2.$$

$\mu > m$: origin is unstable and



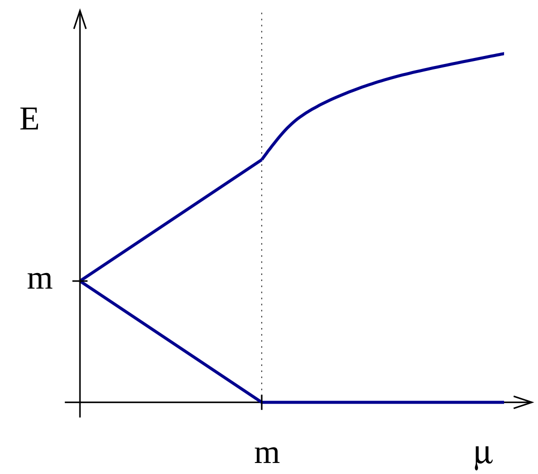
$$\langle \phi \rangle^2 = \frac{\mu^2 - m^2}{2\lambda}.$$

Bose condensate: charge density



$$\rho = \frac{\mu}{\lambda}(\mu^2 - m^2).$$

Spectrum: write $\phi = \langle \phi \rangle + \chi_1 + i\chi_2$

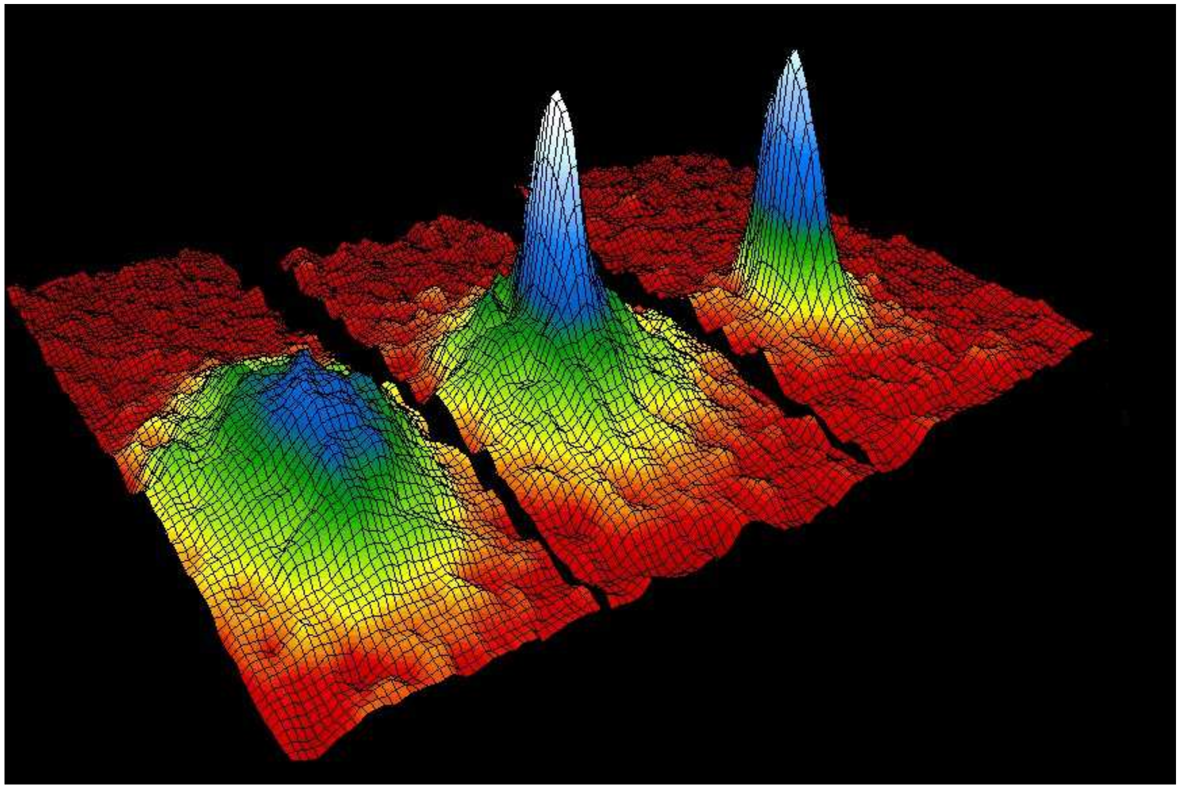


$$\mu < m : \qquad E_{1,2}(\vec{p}{=}0) = m \pm \mu$$

$$\mu > m : \qquad E_{1,2}(\vec{p}{=}0) = \begin{cases} 0 \\ \sqrt{6\mu^2 - 2m^2} \end{cases}$$

with $v_{GB}^2 = (\mu^2 - m^2)/(3\mu^2 - m^2)$

BEC in Dilute Atomic Gas



Velocity distribution in trapped Ru atoms, Cornell & Wiemann (1995)

Effective Action for Superfluid

Goldstone boson field

$$\Phi = |\Phi| e^{i\varphi}$$

Promote $U(1)$ to a gauge symmetry. Effective action

$$\Gamma[A_\mu, \varphi] = \Gamma[A_\mu + \partial_\mu \alpha, \varphi + \alpha]$$

Gauge invariance implies

$$\Gamma[A_\mu, \phi] = \int d^4x \, \mathcal{L}_{eff}(D_\mu \varphi) \qquad D_\mu \varphi = \partial_\mu \phi - A_\mu$$

Chemical potential enters as a constant gauge field $A_\mu = (\mu, 0)$

$$\mathcal{L}_{eff}(-A_\mu) = P\left((A_\mu A^\mu)^{1/2} \right)$$

$$\mathcal{L}_{eff}(A_\mu, \varphi) = P\left((D_\mu \varphi D^\mu \varphi)^{1/2} \right)$$

Here: $P(\mu) = (m^2 - \mu^2)^2/(2\lambda) \sim \mu^4/(2\lambda)$ ($\mu \gg m$)

$$\mathcal{L}_{eff} = \frac{1}{2\lambda}\Big\{\mu^4 - 4\mu^3\partial_0\varphi + 6\mu^2(\partial_0\varphi)^2 - 2\mu^2(\partial_i\varphi)^2 - 4\mu\partial_0\varphi(\partial_\mu\varphi)^2 + (\partial_\mu\varphi)^4\Big\}$$

Quadratic terms

$$\mathcal{L}_{eff} = \frac{3\mu^2}{\lambda}\Big\{(\partial_0\varphi)^2 - \frac{1}{3}(\partial_i\varphi)^2 + \ldots\Big\}$$

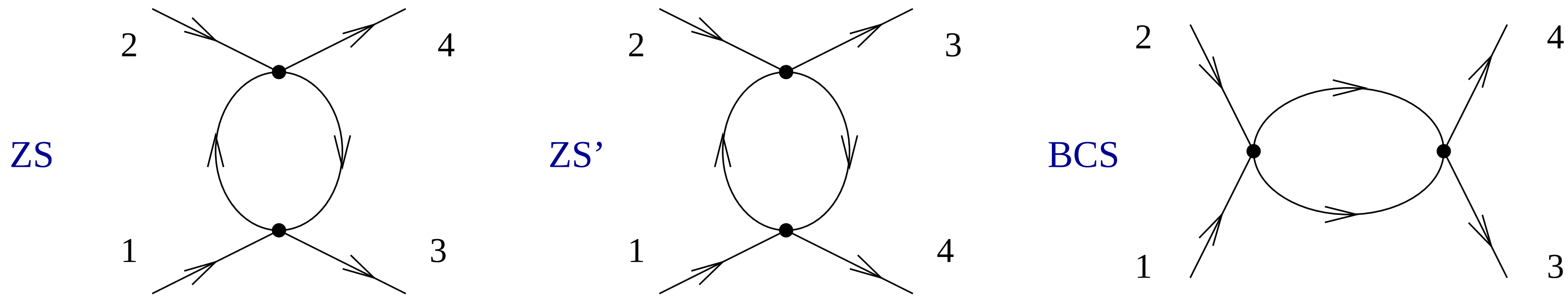
Goldstone boson velocity $v_{GB}^2 = 1/3$. In general

$$v_{GB}^2 = \frac{\partial P}{\partial \epsilon}$$

Also get: non-linear & topological terms

BCS Instability

Loop corrections to scattering near Fermi surface



BCS graph is special: Consider $\vec{p}_{1,2} = \pm\vec{p}$

$$\Gamma = C_0^2 \int \frac{d^4q}{(2\pi)^4} \frac{1}{(E+q_0-\epsilon_q)(E-q_0-\epsilon_q)} = -C_0^2 \left(\frac{p_F m}{2\pi^2}\right) \log\left(\frac{\Lambda}{E}\right)$$

Effective, energy dependent coupling

$$E\frac{dC_0}{dE} = C_0^2 \left(\frac{p_F m}{2\pi^2}\right)$$

Evolution of effective coupling ($N = (p_F m)/(2\pi^2)$)

$$C_0(E) = \frac{C_0(\Lambda)}{1 + NC_0(\Lambda)\log(E_0/E)}$$

Effective coupling

$C_0(\Lambda) > 0 \qquad C_0(E \to 0) \to 0$

$C_0(\Lambda) < 0 \qquad C_0(E \to E_{crit}) \to \infty \qquad E_{crit} \sim \Lambda e^{-1/(N|C_0(\Lambda)|)}$

What happens when C_0 reaches Landau pole?

Pair Condensate $\langle \psi(-\vec{p})\psi(\vec{p}) \rangle$

BCS Calculation of Pair Condensate

Step 1: Fierz rearrange

$$\frac{C_0}{2}(\psi^\dagger\psi)^2 = \frac{C_0}{4}(\psi^\dagger\sigma_2\psi^\dagger)(\psi\sigma_2\psi)$$

Step 2: Hubbard-Stratonovich trick

$$1 = Z^{-1}\int D\Delta \exp((\Delta^*\Delta)/C_0))$$

Step 3: Shift $\Delta \to \Delta - C_0(\psi\sigma_2\psi)$

$$\mathcal{L}_I = \psi\sigma_2\Delta\psi + h.c. + (\Delta^*\Delta)/C_0$$

Step 4: Nambu-Gorkov field $\Psi = (\psi, \psi^\dagger\sigma_2)$

$$\mathcal{S} = \frac{1}{2}\int \frac{d^4p}{(2\pi)^4}\Psi^\dagger \begin{pmatrix} p_0 - \epsilon_p & \Delta \\ \Delta^* & p_0 + \epsilon_p \end{pmatrix} \Psi.$$

Step 5: Integrate out Ψ

$$\mathcal{L} = \frac{1}{2}\mathrm{Tr}\Big[\log\Big(G_0^{-1}G\Big)\Big] + \frac{1}{C_0}|\Delta|^2$$

$$G(p) = \frac{1}{p_0^2 - \epsilon_p^2 - |\Delta|^2}\begin{pmatrix} p_0 + \epsilon_p & \Delta^* \\ \Delta & p_0 - \epsilon_p \end{pmatrix}$$

Step 5: Mean Field (Classical) Approximation $(\delta S)/(\delta\Delta) = 0$

$$1 = \frac{|C_0|}{2}\int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{\epsilon_p^2 + \Delta^2}}$$

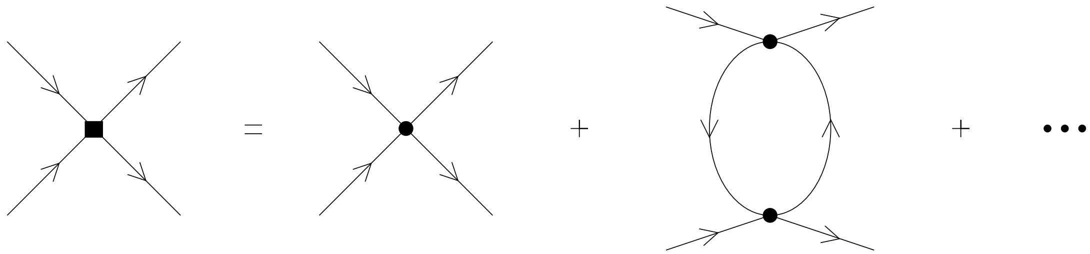
Step 6: Regularized gap equation

$$-\frac{m}{4\pi a} = \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\left\{\frac{1}{\sqrt{\epsilon_p^2 + \Delta^2}} - \frac{1}{E_p}\right\}.$$

Step 7: Solve gap equation

$$\Delta = \frac{8E_F}{e^2}\exp\left(-\frac{\pi}{2p_F|a|}\right) \qquad F = \left(\frac{mp_F}{2\pi^2}\right)\Delta^2$$

Step 8: Higher order corrections



$$C_{eff} = C_0 + C_0^2\left\langle \Pi_{ph}(0, \sqrt{2}p_F(1-\cos(\theta))\right\rangle$$

$$\Delta = \frac{1}{(4e)^{1/3}}\frac{8E_F}{e^2}\exp\left(-\frac{\pi}{2p_F|a|}\right)$$

Pairing Gap: Numerical Estimate

Nuclear matter saturation density $\rho_0 \simeq 0.15\ \mathrm{fm}^{-3}$

$$p_F \simeq 250\ \mathrm{MeV} \qquad\qquad E_F \simeq 35\ \mathrm{MeV}$$

This suggest that $\Delta_{nn} \sim 30$ MeV!

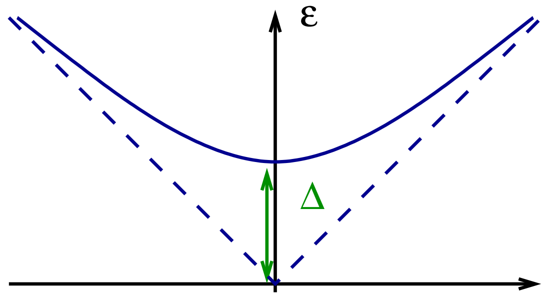
Higher order effects cut this down by $\sim 1/2$.

More importantly, have to go beyond the scattering length

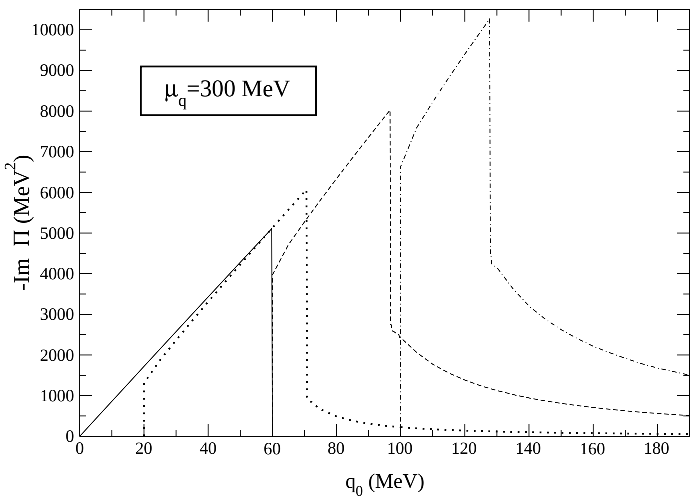
$$\Delta = \frac{8E_F}{e^2}\exp(-\frac{\pi}{2}\cot\delta(p_F))$$

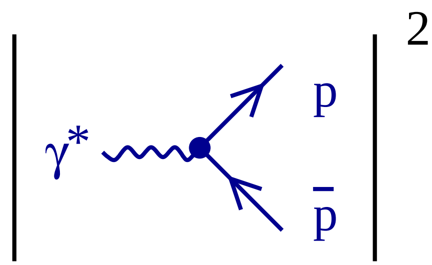
$$\Delta_{nn} \simeq (1-2)\ \mathrm{MeV}$$

Pairing: Gap in Excitation Spectrum

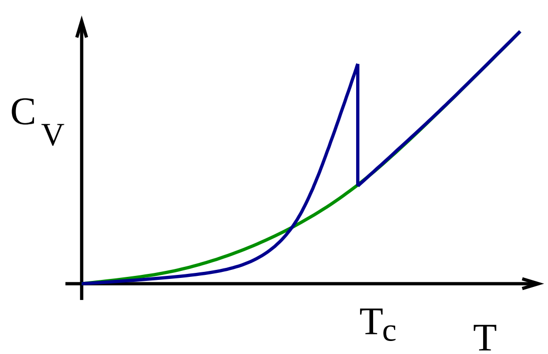


$$\epsilon_p = \sqrt{(p-p_F)^2 + \Delta^2}$$





$$\sim \mathrm{Im}\ \Pi(q_0, \vec{q})$$



specific heat

$$C_V \sim \exp(-\Delta/T)$$

Charged Fermions: Superconductivity

Order parameter $\Phi = \langle \epsilon^{\alpha\beta}\psi_\alpha\psi_\beta\rangle$ transforms as

$$\Phi \to \exp(2ie\alpha)\Phi \qquad\qquad A_\mu \to A_\mu + \partial_\mu\alpha$$

Define Goldstone boson field $\phi(x)$

$$\Phi(x) = \exp(2ie\phi(x))\tilde{\Phi}(x) \qquad\qquad \phi(x) \to \phi(x) + \alpha(x)$$

Gauge invariance determines structure of the effective lagrangian

$$L = -\frac{1}{4}\int d^3x\, F_{\mu\nu}F_{\mu\nu} + L_s(A_\mu - \partial_\mu\phi)$$

Stability requires L_s to have a minimum at the origin

Action is minimized by $A_\mu = \partial_\mu \phi$

This explains the two main properties of superconductors!

Meissner effect: Magnetic field

$$\vec{B} = \vec{\nabla} \times \vec{A} = 0$$

Perfect Conductor: Potential

$$V(x) = \dot{\phi}(x)$$

stationary current $\vec{j} \sim \vec{\nabla}\phi$ requires $V = const$

Landau-Ginzburg Theory

Consider time-independent, small, slowly varying $\Phi(x)$

$$L_s = \int d^3x \left\{ -\frac{1}{2} \left| \left(\nabla - 2ie\vec{A} \right) \Phi \right|^2 + \frac{1}{2} m_H^2 \left(\Phi^* \Phi \right)^2 - \frac{1}{4} g \left(\Phi^* \Phi \right)^4 + \ldots \right\}$$

Decompose $\Phi = \rho \exp(2ie\phi)$. Effective potential for ρ

$$V(\rho) = -\frac{1}{2} m_H^2 \rho^2 + \frac{1}{4} g \rho^4$$

Parameters $m_H, g \leftrightarrow \langle \Phi \rangle, E_0$

Equations of motion

$$\vec{\nabla} \times \vec{B} = 4e^2 \rho^2 \left(\nabla \phi - \vec{A} \right)$$

$$\nabla^2 \rho = -m_H^2 \rho^2 + g\rho^3 + 4e^2 \rho \left(\vec{\nabla} \phi - \vec{A} \right)$$

This implies

$$\nabla^2 \vec{B} = -4e^2\rho^2\vec{B} \qquad B(z) = B_0 e^{-z/\lambda}$$

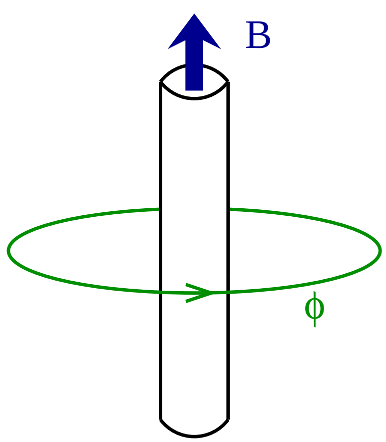
penetration depth λ

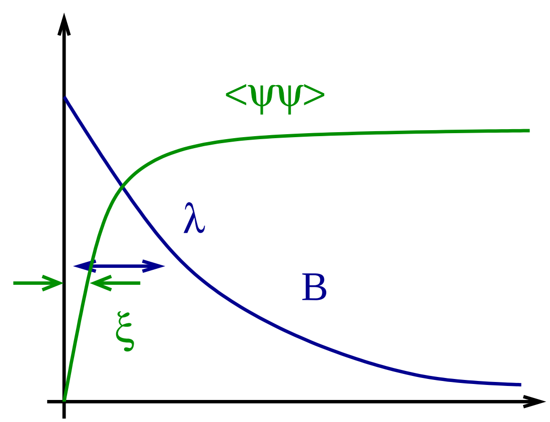
$$\nabla^2 \rho = -m_H^2 \rho + \ldots \qquad \rho(z) = \rho_0 e^{-z/\xi}$$

coherence length ξ

Type II materials: $\xi < \lambda$. Magnetic flux goes through vortices

$$\int_A \vec{B}\cdot d\vec{S} = \oint_{\partial A} \vec{A}\cdot d\vec{l} = \oint_{\partial A} \vec{\nabla}\phi \cdot d\vec{l} = \frac{n\pi\hbar}{e}$$





Vortices in Dilute Fermi Liquid