II. Vortices and quark confinement in non-Abelian gauge theories

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In this talk I wish to discuss non-Abelian vortices of the type proposed by Nielsen and Olesen [1]. I shall show that the vortices must contain a *single* unit of quantized flux absorbed by a Dirac monopole at each end. The monopoles satisfy a confinement condition; if we assign quark numbers to the monopoles, we find that the model contains a natural explanation of quark confinement. The *I*-spin variables associated with the non-Abelian gauge field correspond to the colour degree of freedom.

I shall also suggest an alternative model in which (colour) charges and monopoles are interchanged. The Higgs field which breaks the degeneracy of the vacuum is replaced by an operator which creates monopoles of the type suggested by 't Hooft [2]. In such a model colour might be confined. The investigations are at a very preliminary stage, but the model appears to offer a natural explanation of confinement without the explicit introduction of monopole fields.

Nambu [3] pointed out that vortices of finite length require monopoles at their ends. In the Abelian model which he examined such monopoles would be permanently bound in pairs. Meanwhile, 't Hooft [2] showed that monopoles of strength $4\pi/g$, i.e., two Dirac units, could occur as solutions of the field equations in an SU(2) model. He therefore suggested that a vortex of finite length with two units of quantized flux could exist, even in a model where monopoles were not explicitly introduced.

However, by extending the reasoning of ref. [2] we can show that flux in an SU(2) model (or, more strictly, an O(3) model) must be quantized in single units. In a vortex solution of such a model with n units of flux, the Higgs field at larger r is rotated by $2n\pi$ as the azimuthal angle ϕ goes from 0 to 2π. If n is even, we can continuously deform the Higgs field until its direction is independent of ϕ. In other words, the boundary conditions have solutions which go continuously from two to zero units of flux as a parameter β is varied from 0 to π. An example is the following:

$$\Phi, \Psi = a\{-\cos\alpha + (1+\cos\beta)\cos\phi\cos(\phi-\alpha),\ \sin\alpha - (1+\cos\beta)\sin\phi\cos(\phi-\alpha),\ \sin\beta\cos(\phi-\alpha)\}\ , \tag{1a}$$

$$A_\phi = r^{-1}\{-\sin\beta\cos\phi,\ \sin\beta\sin\phi,\ 1+\cos\beta\}\ . \tag{1b}$$

Φ and Ψ are the two Higgs fields of ref. [1], and a and α are constants, different for Φ and Ψ. The entries in braces are *I*-spin components. It is easily checked that the scalar products Φ^2, Ψ^2, $\Phi\cdot\Psi$ are independent of the polar angle ϕ, and that the covariant derivatives of Φ and Ψ are zero, so that the zero-current boundary condition is satisfied. Further, we can show without much difficulty that the energy of the vortex decreases monotonically as β is increased from 0 to π.

We therefore conclude that we cannot have classically stable quantized vortices with two units of flux. The above reasoning does not apply to a vortex with a single unit of flux, since the Higgs fields are then rotated by 2π instead of 4π when ϕ goes from 0 to 2π, and such vortices cannot be continuously transformed to constant fields. In general, the number of units of quantized flux is only defined modulo 2.

We propose a model with finite-length vortices containing a single unit of flux, so that Dirac monopoles must be explicitly introduced. While we shall carry out the discussion in terms of an SU(2) model, we can readily extend it to the more interesting SU(3) model, where the number of units of flux is defined modulo 3.

The monopoles will be treated by the method introduced by Dirac [4]. A "string" of arbitrary shape ends at each monopole. In the Abelian case, an operator Φ for a changed field undergoes a phase change of 2π when the field point is taken around a small curve surrounding the string. The potential A has a singularity of the form

$$A_\phi = 2\pi e^{-1} r^{-1} , \tag{2}$$

where ϕ is the azimuthal angle in a local polar co-ordinate system with the direction of the string taken as the z-axis. The covariant derivatives of Φ remain finite in the neighbourhood of the string, and the fields $F_{\mu\nu}$ have singularities at the end of the string only.

The singularities at strings associated with non-Abelian monopoles will be similar. All fields undergo an SU(2) rotation through 2π when the field point is taken around a small curve surrounding the string. At the same time, the potentials must have a singularity analogous to (2), so that the co-variant derivatives of the fields $F^\alpha_{\mu\nu}$ remain finite. As an example, we may have

$$A^3_\phi = 2\pi g^{-1} r^{-1}, \qquad A^1 = D\cos\phi, \qquad A^2 = D\sin\phi . \tag{3}$$

From (3), we notice that the ϕ components of the potential in a particular I-spin direction have the Dirac-string singularity. The I-spin direction of the singular component will depend on the gauge chosen. In any particular gauge, the variation of the I-spin direction along the string can be calculated from the requirement that $F^\alpha_{\phi z}$ remain finite. In other words

$$\frac{\partial A^\alpha_\phi}{\partial z} + \epsilon^{\alpha\beta\gamma} A^\beta_\phi A^\gamma_z = \text{finite} . \tag{4}$$

(Again we are taking local co-ordinates with the z-direction along the string.) We conclude that we need specify the singular I-spin direction at only one point ξ_0 on the string. We shall take this point to be the point at infinity.

The extra term in the Hamiltonian associated with the Dirac string will be

$$\frac{2\pi}{g} \int \epsilon^{ijk}\, \mathrm{d}\xi^i\, F^{\alpha_s(\xi)}_{0j}(\xi)\, J^{\alpha_s(\xi_0)}_k(x) \tag{5}$$

where the component $\alpha_s(\xi)$ refers to the I-spin direction of the singular component of A at the point ξ. The integral is to be taken over the Dirac string leading to the point x.

In my previous paper, I stated that monopoles were isotopic singlets, since the topological character of the Dirac-string singularity was described by a single number. From the analysis just given I no longer believe that conclusion to be correct. Under a globel SU(2) transformation the monopole field will transform like an SU(2) spinor. However, one cannot really apply local gauge invariance to the monopole fields. In eq. (5), the two super-scripts $\alpha_s(\xi)$ and $\alpha_s(\xi_0)$ are not the same, and the relation between them depends on the gauge.

The classical theory of non-Abelian monopoles is relativistically covariant, since it depends only on the field equations and the conditions at the end of the Dirac string. It is reasonable to suppose that the quantum theory is also covariant, but we have not verified this fact.

Now let us discuss the question of confinement. The mechanism is the same as in the Abelian theory. The Hamiltonian contains a term

$$|\{(\partial/\partial x^{\mu})\delta^{\alpha\gamma} + g\epsilon^{\alpha\beta\gamma}A^{\beta}_{\mu}\}\Phi^{\gamma}|^2 \; . \tag{6}$$

If, asymptotically

$$A \to r^{-1}, \qquad \Phi \to \text{const.} \, ,$$

the energy will diverge, unless the direction of Φ can adjust itself so that the two terms in the covariant derivative cancel. We thus require that the expression

$$g\epsilon^{\alpha\beta\gamma}A^{\beta}_{\mu}\Phi^{\gamma}\,\mathrm{d}x^{\mu} \tag{7}$$

be a perfect differential. It follows that the transformation

$$T = \Pi\{1 + \epsilon^{\alpha\beta\gamma}A^{\beta}_{\mu}\,\mathrm{d}x^{\mu}\} \tag{8}$$

must leave Φ unaffected when x goes round a closed curve. Since Nielsen and Olesen found it necessary to break the SU(2) symmetry completely by introducing two Higgs fields which select out different directions in *I*-space, we conclude that the transformation (8) must be a rotation through $2n\pi$.

In the presence of a monopole we shall take the Dirac string in the $+z$ direction and shall examine curves with constant polar angle θ. For θ near zero the string condition implies T is a rotation through 2π while, for θ near π, $T = 1$. As θ varies from 0 to 2π, T cannot always be a rotation through $2n\pi$ without violating continuity, and the condition for finite energy is not maintained. This is simply a manifestation of the Meissner effect; the flux from the monopole is forced into the superconducting vacuum.

The reasoning does not apply to monopoles of strength $4\pi/g$, since the transformations along the circles at $\theta = \epsilon$, $\theta = \pi - \epsilon$ may now be continuously transformed into one another. In fact, the topological properties of the group O(3) imply that a Dirac string of double strength can be gauged away. The confinement condition is thus that the net number of monopoles must be $2n$ ($3n$ in SU(3)). Since the SU(2) symmetry is broken by the Higgs field, the question of colour confinement does not arise.

In spite of the fact that an obvious extension of the Nielsen–Olesen ideas leads directly to quark confinement, it appears to possess a difficulty associated with renormalization. The Dirac quantization condition arises from the field equations, and it therefore involves *unrenormalized* coupling constants. Thus

$$g_{0,\mathrm{C}}\,g_{0,\mathrm{M}} = 2\pi \, . \tag{9}$$

Asymptotic freedom implies that $g_{0,\mathrm{C}}$ is zero. It could happen that the product of the renormalization constants associated with $g_{0\mathrm{C}}$ and $g_{0\mathrm{M}}$ was finite, but we see no reason to believe that such is the case. Even if it were, we should have to give up asymptotic freedom for magnetic monopoles.

We therefore propose another model in which the roles of charges and monopoles are interchanged. We do not introduce monopole fields explicitly, but we replace the Higgs field by an operator which creates 't Hooft-type monopoles. With a suitable choice of gauge, the potentials in such monopoles, at large r, are identical to those of an Abelian monopole; they are all in the same SU(2) direction and they have a Dirac-string singularity. 't Hooft introduced a Higgs field and showed that one could obtain a static solution of the classical equations. For our purposes the Higgs field is unnecessary. We do not require a classical solution of the equation, but simply a non-singular set of potentials which gives rise to a monopole field at infinity.

We assume that there exists a quantum state $|M\rangle$ (not necessarily a stationary state) with the same properties as 't Hooft's monopole. We then introduce an operator $\Psi(\boldsymbol{r}, \hat{\boldsymbol{n}})$ which creates such a state from the vacuum. Thus

$$|M\rangle = \Psi^\dagger(\boldsymbol{r}, \hat{\boldsymbol{n}})|0\rangle . \tag{10}$$

The co-ordinate $\boldsymbol{r}$ represents the center of the monopole, and $\hat{\boldsymbol{n}}$ is the SU(2) direction of the fields at infinity. In order to define fully the operator Ψ, we must also define the product

$$\{\Psi^\dagger(\boldsymbol{r}_1, \hat{\boldsymbol{n}}_1)\Psi^\dagger(\boldsymbol{r}_2, \hat{\boldsymbol{n}}_2)\cdots\}|0\rangle . \tag{11}$$

We shall not specify such a state completely; it must possess Dirac strings leading from all points $\hat{\boldsymbol{n}}_1, \hat{\boldsymbol{n}}_2 \dots$ and, if the co-ordinates are well separated, it must be the state with N monopoles at the points $\boldsymbol{r}_1, \boldsymbol{r}_2, \dots$.

We now propose the following state as the magnetic analogue of the Higgs vacuum:

$$|0\} = \exp\{\mathrm{i}\lambda \int \mathrm{d}^3\boldsymbol{r}_0 [\Psi(3, \boldsymbol{r}_0) + \Psi^\dagger(3, \boldsymbol{r}_0)]\}|0\rangle , \tag{12}$$

where the symbol $\Psi(3, \boldsymbol{r}_0)$ indicates that $\hat{\boldsymbol{n}}$ is in the 3-direction. The SU(2) symmetry is thus reduced to the U(1) symmetry of rotations about the 3-axis. Of course, we do not mean to imply that the actual true vacuum has the simple form given in (12), but merely that it has similar properties. For the moment we defer the question whether the energy may be minimized at a non-zero value of λ.

Suppose now that we introduce a classical electric field $F^3_{0\mu}$ at a large distance from the point $\boldsymbol{r}$. If the Dirac strings do not pass through the field, the integrand in (12) will not be modified near the point $\boldsymbol{r}$. Next let us make a gauge transformation so as to deform the Dirac strings into the electric field. We then find that (12) must be modified as follows:

$$|0\} = \exp\{\mathrm{i}\lambda \int \mathrm{d}^3\boldsymbol{r}_0 [\mathrm{e}^{\mathrm{i}\chi}\Psi(3, \boldsymbol{r}_0) + \mathrm{e}^{-\mathrm{i}\chi}\Psi^\dagger(3, \boldsymbol{r}_0)]\} , \tag{13a}$$

$$\chi(\boldsymbol{r}_2) - \chi(\boldsymbol{r}_1) = -\frac{1}{g}\int_{\boldsymbol{r}_1}^{\boldsymbol{r}_2} \mathrm{d}\xi_i B_i(\xi) , \tag{13b}$$

$$B_i(x) = -\epsilon^{ijk}\int^x \mathrm{d}\xi_j F^3_{0k}(\xi) . \tag{13c}$$

In other words, the phase of Ψ is related to the field $\widetilde{F}^3_{ij}$ in exactly the same way as the phase of an Abelian changed field was related to F_{ij}. The reasoning of ref. [1] and of the first part of this talk may be repeated, with all quantities replaced by their duals. We reach the conclusion that such a vacuum can confine quantized vortices of electric flux associated with the field F_{0i}, and that the 3rd component of I-spin is confined to the value zero. In particular, the total number of quarks must be even (or $3n$ in SU(3)). Since the SU(2) symmetry has been broken we do not have complete color confinement.

Instead of (12), we might try the following vacuum, which does not break the SU(2) symmetry:

$$|0\} = \exp\{i\lambda \int d^3\boldsymbol{r}_0 \, d^2\hat{\boldsymbol{n}} \, [\Psi(3, \boldsymbol{r}_0) + \Psi^\dagger(3, \boldsymbol{r}_0)]\} |0\rangle \,. \tag{14}$$

In such a vacuum we could not have a classical electric field, in any direction in I-space, which behaved like $\boldsymbol{r}/r^3$ at infinity, without an infinite increase of energy. It is therefore plausible that we would have complete color confinement.

Let us finally discuss the question whether a state such as (12) or (14) could be the state of minimum energy. This may be the case due to the very fact that such a vacuum can support vortices, since intermediate states of two vortices will decrease the energy.

While the above speculations are, at the moment, very vague, they do suggest that it may be possible for a magnetic analogue of the Higgs vacuum to confine color.

References

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