SU_3 AND SU_6 SYMMETRY OF STRONG INTERACTIONS

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The aim of these lectures is to present the SU_3 and SU_6 classification schemes for the strongly interacting particles, the so-called hadrons. These symmetry schemes, based on group theory, are a generalization of the very successful isospin formalism introduced by Heisenberg (1932), and Cassen and Condon (1936) in order to account for the similarity of proton and neutron. SU_3 was proposed by Gell-Mann and Ne'eman in 1962 and combines isospin and hypercharge, while SU_6 combines SU_3 and ordinary spin. SU_6 originates from Sakita, Radicati and Gürsey.

The lectures present the methods used in formulating and calculating the symmetry properties, and illustrate them on the most significant predictions, mass formulae, magnetic moments of baryons and meson-baryon interaction. Our treatment of SU_6 will be limited to the non-relativistic case. The attempts at relativistic extensions, which are very far from being satisfactory, will not be reviewed. Our treatment is elementary and examples of simple calculations are given. A selected list of references is given at the end of the lecture notes.

I. SU_3 SYMMETRY

1. Three-dimensional spinor calculus, the group SU_3 and its representations

The formalism is a straightforward generalization of the two-dimensional isospinor calculus. A third component is added to the isospinors, so the basic element in SU_3 is a set of three complex numbers ξ_α ($\alpha = 1,2,3$) called covariant spinor:

$$\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} . \tag{1}$$

A linear transformation is given by the 3×3 matrix u acting on the covariant spinors:

$$\xi \to \xi' = u\xi \tag{2}$$

which means

$$\begin{pmatrix} \xi'_1 \\ \xi'_2 \\ \xi'_3 \end{pmatrix} = \begin{pmatrix} u_1^1 & u_1^2 & u_1^3 \\ u_2^1 & u_2^2 & u_2^3 \\ u_3^1 & u_3^2 & u_3^3 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} \tag{3}$$

or, written for the components of the spinor,

$$\xi'_\alpha = u_\alpha^\beta \xi_\beta \quad . \tag{4}$$

Summation over repeated upper and lower indices is understood here and throughout the lecture notes.

Besides the covariant spinors we define the contravariant spinors η^α (α = 1,2,3) with the index in the upper position and the spinor written as a line

$$\eta = (\eta^1 \ \eta^2 \ \eta^3) \quad . \tag{5}$$

They shall transform in such a way that

$$\eta\xi = \eta^\alpha \xi_\alpha \tag{6}$$

be an invariant. This fixes the transformation law of the contravariant spinor:

$$\eta'^\alpha \xi'_\alpha = \eta^\alpha \xi_\alpha = \eta'^\alpha u_\alpha^\beta \xi_\beta \quad , \tag{7}$$

hence

$$\eta^\beta = \eta'^\alpha u_\alpha^\beta \quad , \tag{8}$$

and finally

$$\eta^\alpha \rightarrow \eta'^\alpha = \eta^\beta (u^{-1})_\beta^\alpha \quad . \tag{9}$$

A general tensor is a quantity $\zeta^{\mu\nu}_{\alpha\beta\gamma}$ with arbitrary numbers of covariant (lower) and contravariant (upper) indices. ζ follows the covariant transformation law (4) for all lower indices and the contravariant transformation law (9) for all upper indices.

$$\zeta^{\mu\nu\cdot\cdot}_{\alpha\beta\gamma\cdot\cdot} \rightarrow \zeta'^{\mu\nu}_{\alpha\beta\gamma} = u_\alpha^{\alpha'} u_\beta^{\beta'} u_\gamma^{\gamma'} \ldots \zeta^{\mu'\nu'}_{\alpha'\beta'\gamma'\cdot\cdot} (u^{-1})^\mu_{\mu'} (u^{-1})^\nu_{\nu'} \ldots \tag{10}$$

which will be written in shorthand notation

$$\zeta \rightarrow \zeta' = U(u)\,\zeta \quad . \tag{11}$$

U(u) describes a linear transformation acting on ζ.

For our purpose of particle description we limit the transformations to the group SU_3 which is the group of linear, unitary, unimodular transformations in three dimensions. The matrices u have to satisfy the two conditions:

1. u is unitary: $u^+ u = 1$, or $(u^+)^\beta_\alpha u^\gamma_\beta = \delta^\gamma_\alpha$, where $(u^+)^\beta_\alpha = (u^*)^\alpha_\beta$.

   +: Hermitian conjugate

   *: complex conjugate

   unit matrix $\delta^\gamma_\alpha = \begin{cases} 1 & \alpha = \gamma \\ 0 & \alpha \neq \gamma \end{cases}$;

2. u is unimodular: det u = 1.

The tensors which are left <u>invariant</u> by the group SU_3 are

$$\delta^{\beta}_{\alpha} = \begin{cases} 1 \text{ for } \alpha = \beta \\ 0 \text{ for } \alpha \neq \beta \end{cases}$$

$$\epsilon_{\alpha\beta\gamma} = \begin{cases} 1 \text{ for } \alpha\beta\gamma = 123,\ 231,\ 312 \\ -1 \text{ for } \alpha\beta\gamma = 213,\ 132,\ 321 \\ 0 \text{ otherwise} \end{cases} \tag{12}$$

$$\epsilon^{\alpha\beta\gamma} = \epsilon_{\alpha\beta\gamma} \quad .$$

<u>Proof</u>:

a) δ^{β}_{α} is left invariant under every linear transformation

$$\delta^{\beta}_{\alpha} \to \delta'^{\beta}_{\alpha} = u^{\alpha'}_{\alpha}\,\delta^{\beta'}_{\alpha'}(u^{-1})^{\beta}_{\beta'} = u^{\beta'}_{\alpha}(u^{-1})^{\beta}_{\beta'} = \delta^{\beta}_{\alpha} \;;$$

b) $\epsilon_{\alpha\beta\gamma} \to \epsilon'_{\alpha\beta\gamma} = u^{\alpha'}_{\alpha} u^{\beta'}_{\beta} u^{\gamma'}_{\gamma}\,\epsilon_{\alpha'\beta'\gamma'}$

$$\epsilon'_{123} = u^{\alpha'}_{1} u^{\beta'}_{2} u^{\gamma'}_{3}\,\epsilon_{\alpha'\beta'\gamma'} = \det u = 1$$

$$\epsilon'_{\beta\alpha\gamma} = u^{\beta'}_{\beta} u^{\alpha'}_{\alpha} u^{\gamma'}_{\gamma}\,\epsilon_{\beta'\alpha'\gamma'} = u^{\alpha'}_{\alpha} u^{\beta'}_{\beta} u^{\gamma'}_{\gamma}(-1)\,\epsilon_{\alpha'\beta'\gamma'} = -\,\epsilon'_{\alpha\beta\gamma} \;.$$

So $\epsilon'_{123} = 1$ and $\epsilon'_{\alpha\beta\gamma}$ is fully antisymmetric, therefore $\epsilon'_{\alpha\beta\gamma} = \epsilon_{\alpha\beta\gamma}$ q.e.d.

The condition that u is unitary leads to the fact that the Hermitian conjugate of a covariant spinor ξ_α

$$\xi^{+\alpha} = (\xi_\alpha)^* \quad , \tag{13}$$

which is written with an index in upper position, transforms according to the contravariant transformation law (4).

(Remark: Note that Hermitian conjugation involves complex conjugation and interchanging lines and columns.)

<u>Proof</u>:

The relation

$$\xi'_\alpha = u^{\beta}_{\alpha}\xi_\beta$$

becomes by complex conjugation

$$\xi'^{*}_{\alpha} = u^{*\beta}_{\alpha}\xi^{*}_{\beta} \quad .$$

Hence, using (13)

$$\xi'^{+\alpha} = u^{+\alpha}_{\beta}\xi^{+\beta} = (u^{-1})^{\alpha}_{\beta}\xi^{+\beta} = \xi^{+\beta}(u^{-1})^{\alpha}_{\beta} \quad \text{q.e.d.}$$

The Hermitian conjugate of the general tensor (10) is

$$\zeta^{+\alpha\beta\gamma\ldots}_{\mu\nu\ldots} = \left(\zeta^{\mu\nu\ldots}_{\alpha\beta\gamma\ldots}\right)^{*} . \tag{14}$$

It is covariant in μ, $\nu\ldots$ and contravariant in α, β, $\gamma\ldots$. In the SU_3 classification, particles are associated with certain tensors, antiparticles with the respective Hermitian conjugate tensors.

Using the invariant tensors (12) one can, in an invariant way, from a given tensor construct other tensors with a smaller number of indices. This provides a simple method for the reduction of tensors. Tensors which cannot be reduced further can be shown to form the irreducible representations of the group SU_3. (The proof of this important fact will not be given here.) The general principle of reduction of a tensor ξ is to construct a tensor η with a smaller number of indices, its elements being linear combinations of the elements of ζ. This has to be an invariant operation in the sense that the reduction procedure applied to $\zeta' = U(u)\zeta$ [see (11)] must lead to $\eta' = U(u)\eta$.

The means of reduction are the invariant tensors (12), with which one has in general three possible ways of reducing a tensor.

1. One forms a trace between an arbitrary upper and an arbitrary lower index

$$\zeta^{\mu\nu\ldots}_{\alpha\beta\gamma\ldots} \cdot \delta^{\alpha}_{\mu} = \zeta^{\alpha\nu\ldots}_{\alpha\beta\gamma\ldots} \underset{\text{def}}{=} \eta^{\nu\ldots}_{\beta\gamma\ldots} . \tag{15}$$

2. One picks out the antisymmetric part relative to two arbitrary contravariant indices (here κ and λ) and replaces the two indices by a covariant one (here α')

$$\zeta^{\kappa\lambda\mu\ldots}_{\alpha\beta\gamma\ldots} \epsilon_{\alpha'\kappa\lambda} \underset{\text{def}}{=} \eta^{\mu\ldots}_{\alpha'\alpha\beta\gamma\ldots} . \tag{16}$$

3. The same as 2) with interchanged roles of contravariant and covariant indices

$$\zeta^{\kappa\lambda\mu\ldots}_{\alpha\beta\gamma\ldots} \epsilon^{\kappa'\alpha\beta} \underset{\text{def}}{=} \eta^{\kappa'\kappa\lambda\mu\ldots}_{\gamma\ldots} . \tag{17}$$

All these reductions are invariant. Let us show this for case 2).

Transform the reduced tensor:

$$\zeta^{\kappa\lambda\mu\ldots}_{\alpha\beta\gamma\ldots} \epsilon_{\alpha'\kappa\lambda} \underset{\text{def}}{=} \eta^{\mu\ldots}_{\alpha'\alpha\beta\gamma\ldots} \rightarrow u^{\tilde{\alpha}'}_{\alpha'} u^{\tilde{\alpha}}_{\alpha} u^{\tilde{\beta}}_{\beta} u^{\tilde{\gamma}}_{\gamma} \ldots \eta^{\tilde{\mu}\ldots}_{\tilde{\alpha}'\tilde{\alpha}\tilde{\beta}\tilde{\gamma}} (u^{-1})^{\mu}_{\tilde{\mu}} \ldots .$$

Reduce the transformed tensor by means of the "transformed ϵ tensor" which is allowed since ϵ is an invariant tensor. Use $(u^{-1})^{\kappa}_{\tilde{\kappa}} u^{\hat{\kappa}}_{\kappa} = \delta^{\hat{\kappa}}_{\tilde{\kappa}}$

$$\left[u^{\tilde{\alpha}}_{\alpha} u^{\tilde{\beta}}_{\beta} u^{\tilde{\gamma}}_{\gamma} \ldots \zeta^{\tilde{\kappa}\tilde{\lambda}\tilde{\mu}\ldots}_{\tilde{\alpha}\tilde{\beta}\tilde{\gamma}\ldots} (u^{-1})^{\kappa}_{\tilde{\kappa}} (u^{-1})^{\lambda}_{\tilde{\lambda}} (u^{-1})^{\mu}_{\tilde{\mu}} \ldots\right] \times \left[u^{\tilde{\alpha}'}_{\alpha'} u^{\hat{\kappa}}_{\kappa} u^{\hat{\lambda}}_{\lambda} \cdot \epsilon_{\tilde{\alpha}'\hat{\kappa}\hat{\lambda}}\right]$$

$$= u^{\tilde{\alpha}}_{\alpha} u^{\tilde{\beta}}_{\beta} u^{\tilde{\gamma}}_{\gamma} \ldots \left[\zeta^{\hat{\kappa}\hat{\lambda}\tilde{\mu}\ldots}_{\tilde{\alpha}\tilde{\beta}\tilde{\gamma}\ldots} \epsilon_{\tilde{\alpha}'\hat{\kappa}\hat{\lambda}}\right] \cdot u^{\tilde{\alpha}'}_{\alpha'} (u^{-1})^{\mu}_{\tilde{\mu}} \ldots = u^{\tilde{\alpha}'}_{\alpha'} u^{\tilde{\alpha}}_{\alpha} u^{\tilde{\beta}}_{\beta} u^{\tilde{\gamma}}_{\gamma} \ldots \left[\eta^{\tilde{\mu}\ldots}_{\tilde{\alpha}'\tilde{\alpha}\tilde{\beta}\tilde{\gamma}\ldots}\right] (u^{-1})^{\mu}_{\tilde{\mu}} \ldots \text{ q.e.d.}$$

A tensor is called irreducible when every reduction (15), (16), or (17) gives zero. The characteristic properties of irreducible tensors follow from the reduction recipe:

1. full symmetry in all upper indices

$$\zeta^{\mu\nu\cdot\cdot}_{\alpha\beta\gamma\cdot\cdot} = \zeta^{\nu\mu\cdot\cdot}_{\alpha\beta\gamma\cdot\cdot} \quad \text{etc.}$$

2. full symmetry in all lower indices

$$\zeta^{\mu\nu\cdot\cdot}_{\alpha\beta\gamma\cdot\cdot} = \zeta^{\mu\nu\cdot\cdot}_{\beta\alpha\gamma\cdot\cdot} \quad \text{etc.}$$

3. vanishing trace

$$\zeta^{\alpha\nu\cdot\cdot}_{\alpha\beta\gamma\cdot\cdot} = 0 \quad .$$

Because of 1) and 2), condition 3) means that every trace vanishes. The number of independent components of a tensor is called its dimension . In the following list of the simplest (lowest dimensional) irreducible tensors the dimension is found by subtracting from the number of components the number of equations which come from the condition of irreducibility. These irreducible tensors form the irreducible representations of SU_3 (the representations are denoted by their dimension, with bars to distinguish inequivalent representations of the same dimension; all tensors are subject to conditions 1), 2), and 3) above):

ξ_α	representation	3
ξ^α	"	$\bar{3}$
$\xi_{\alpha\beta}$	"	6
$\xi^{\alpha\beta}$	"	$\bar{6}$
ξ^β_α	"	8
$\xi_{\alpha\beta\gamma}$	"	10
$\xi^{\alpha\beta\gamma}$	"	$\overline{10}$
$\xi^\gamma_{\alpha\beta}$	"	15
$\xi^{\beta\gamma}_\alpha$	"	$\overline{15}$
$\xi^{\mu\nu}_{\alpha\beta}$	"	27
$\xi^{\kappa\mu\nu}_{\alpha\beta\gamma}$	"	64

Representation $\bar{3}$ is formed by the Hermitian conjugate tensor to representation 3. It is also important to note that representations 3 and $\bar{3}$ are inequivalent in the sense that there is no linear transformation of 3 onto $\bar{3}$ which is invariant for the group. The same holds for the other pairs of representations having the same dimension, like 6 and $\bar{6}$, 10 and $\overline{10}$, etc.

In general, the irreducible tensor $\zeta^{\mu\nu\cdot\cdot}_{\alpha\beta\gamma\cdot\cdot}$ with m (n) upper (lower) indices has the dimensions

$$(m+1)(n+1)\left(\frac{m+n}{2}+1\right) \quad .$$

The Hermitian conjugate of a representation is obtained by exchanging m and n. If m = n the representation is selfconjugate (e.g. $8 = \bar{8}$, $27 = \overline{27}$).

The direct product of two irreducible representations can be decomposed into a sum of irreducible representations. To do this we have to reduce the product of two irreducible tensors. For example, the product 3×3, realized by the product tensor

$$\xi_\alpha \cdot \xi'_\beta \underset{\text{def}}{=} \zeta_{\alpha\beta}$$

may be decomposed into a symmetric and an antisymmetric part:

$$\zeta_{\alpha\beta} = \zeta^{\text{sym}}_{\alpha\beta} + \zeta^{\text{anti}}_{\alpha\beta} = \frac{1}{2}\left(\zeta_{\alpha\beta}+\zeta_{\beta\alpha}\right) + \frac{1}{2}\left(\zeta_{\alpha\beta}-\zeta_{\beta\alpha}\right) \quad .$$

$\zeta^{\text{sym}}_{\alpha\beta}$ is already irreducible, it belongs to representation 6 (see list). $\zeta^{\text{anti}}_{\alpha\beta}$ is separated out by the reduction (17):

$$\zeta_{\alpha\beta}\, \epsilon^{\alpha\beta\gamma} = \zeta^{\text{anti}}_{\alpha\beta} \cdot \epsilon^{\alpha\beta\gamma} \underset{\text{def}}{=} \psi^\gamma \quad .$$

ψ^γ is irreducible and belongs to representation $\bar{3}$. One easily verifies $\zeta^{\text{anti}}_{\alpha\beta} = \frac{1}{2}\epsilon_{\alpha\beta\gamma}\psi^\gamma$, so that ψ^γ and $\zeta^{\text{anti}}_{\alpha\beta}$ are equivalent representations (i.e. related by a one-to-one, linear, invariant relation). The reduction just obtained is written

$$3\times 3 = 6+\bar{3} \quad . \tag{18}$$

Other reductions are

$$\bar{3}\times 3 = 1+8 \tag{19}$$

$$3\times 6 = 8+10 \tag{20}$$

$$8\times 8 = 1+8+8+10+\overline{10}+27 \tag{21}$$

$$10\times 10 = 1+8+27+64 \quad . \tag{22}$$

Formula (21) will be derived as an example to show in detail the reduction procedure:

a) Decomposition into symmetric and antisymmetric part

$$\begin{aligned} \xi^\alpha_{\alpha'}\eta^\beta_{\beta'} &= \frac{1}{2}\left(\xi^\alpha_{\alpha'}\eta^\beta_{\beta'} + \xi^\beta_{\beta'}\eta^\alpha_{\alpha'}\right) + \frac{1}{2}\left(\xi^\alpha_{\alpha'}\eta^\beta_{\beta'} - \xi^\beta_{\beta'}\eta^\alpha_{\alpha'}\right) \\ &\underset{\text{def}}{=} \varphi^{\alpha\beta}_{\alpha'\beta'} + \psi^{\alpha\beta}_{\alpha'\beta'} \quad . \end{aligned} \tag{23}$$

φ (ψ) is symmetric (antisymmetric) for simultaneous permutation of upper and lower indices.

b) Reduction of φ

Invariants are obtained by taking complete traces, which can be done in two ways:

$$\varphi^{\alpha'\beta'}_{\alpha'\beta'} = \xi^{\alpha'}_{\alpha'} \cdot \eta^{\beta'}_{\beta'} = 0 \quad (\xi \text{ and } \eta \text{ are traceless because they are irreducible})$$

$$\varphi^{\beta'\alpha'}_{\alpha'\beta'} = \xi^{\beta'}_{\alpha'}\eta^{\alpha'}_{\beta'} \underset{\text{def}}{=} (\xi\eta) \quad . \tag{24}$$

Decompose φ as follows

$$\varphi^{\alpha\beta}_{\alpha'\beta'} = \tilde{\varphi}^{\alpha\beta}_{\alpha'\beta'} + a\delta^{\alpha}_{\alpha'}\delta^{\beta}_{\beta'} + b\delta^{\alpha}_{\beta'}\delta^{\beta}_{\alpha'} \tag{25}$$

where a and b are so chosen that both complete traces of $\tilde{\varphi}$ are zero. This fixes the coefficients a and b, and one finds

$$\tilde{\varphi}^{\alpha\beta}_{\alpha'\beta'} = \varphi^{\alpha\beta}_{\alpha'\beta'} + \frac{(\xi\eta)}{24}\delta^{\alpha}_{\alpha'}\delta^{\beta}_{\beta'} - \frac{(\xi\eta)}{8}\delta^{\alpha}_{\beta'}\delta^{\beta}_{\alpha'} \quad . \tag{26}$$

$\tilde{\varphi}$ may have single traces which leave two indices unsaturated, thus leading to an octet representation. One finds

$$\tilde{\varphi}^{\alpha\beta}_{\alpha\beta'} = 0$$

$$\tilde{\varphi}^{\alpha\beta}_{\alpha'\alpha} = \varphi^{\alpha\beta}_{\alpha'\alpha} + \frac{(\xi\eta)}{24}\delta^{\beta}_{\alpha'} - \frac{(\xi\eta)}{8}\cdot 3\delta^{\beta}_{\alpha'} = \frac{1}{2}\left[\xi^{\alpha}_{\alpha'}\eta^{\beta}_{\alpha} + \xi^{\beta}_{\alpha}\eta^{\alpha}_{\alpha'} - {}^{2}\!/_{3}(\xi\eta)\,\delta^{\beta}_{\alpha'}\right] \underset{\text{def}}{=} \frac{1}{2}(\xi\eta)^{\beta}_{D\alpha'} \quad . \tag{27}$$

$(\xi\eta)^{\beta}_{D\alpha'}$ is called the D coupling octet formed with the two tensors ξ and η. We extract the octet part out of $\tilde{\varphi}$

$$\tilde{\varphi}^{\alpha\beta}_{\alpha'\beta'} = \tilde{\tilde{\varphi}}^{\alpha\beta}_{\alpha'\beta'} + c\tilde{\varphi}^{\nu\beta}_{\alpha'\nu}\delta^{\alpha}_{\beta'} + d\tilde{\varphi}^{\alpha\nu}_{\nu\beta'}\delta^{\beta}_{\alpha'} \quad . \tag{28}$$

The coefficients c and d are fixed by the condition that $\tilde{\tilde{\varphi}}$ has all single traces equal to zero. One gets

$$\tilde{\tilde{\varphi}}^{\alpha\beta}_{\alpha'\beta'} = \tilde{\varphi}^{\alpha\beta}_{\alpha'\beta'} - \frac{1}{6}(\xi\eta)^{\beta}_{D\alpha'}\delta^{\alpha}_{\beta'} - \frac{1}{6}(\xi\eta)^{\alpha}_{D\beta'}\delta^{\beta}_{\alpha'} \quad . \tag{29}$$

If $\tilde{\tilde{\varphi}}$ would have a part antisymmetric in α and β, this part would be antisymmetric in α' and β' too, since $\tilde{\tilde{\varphi}}$ is symmetric for the simultaneous permutation of upper <u>and</u> lower indices. So we can find this part by simultaneous antisymmetrization of upper and lower indices

$$\left(\tilde{\tilde{\varphi}}^{\text{anti}}\right)^{\gamma'}_{\gamma} = \epsilon_{\alpha\beta\gamma}\tilde{\tilde{\varphi}}^{\alpha\beta}_{\alpha'\beta'}\epsilon^{\alpha'\beta'\gamma'} \quad . \tag{30}$$

We now make use of the important identity

$$\epsilon^{\alpha\beta\gamma}\epsilon_{\alpha'\beta'\gamma'} = \delta^{\alpha}_{\alpha'}\delta^{\beta}_{\beta'}\delta^{\gamma}_{\gamma'} + \delta^{\beta}_{\alpha'}\delta^{\gamma}_{\beta'}\delta^{\alpha}_{\gamma'} + \delta^{\gamma}_{\gamma'}\delta^{\alpha}_{\beta'}\delta^{\beta}_{\gamma'} - \delta^{\alpha}_{\beta'}\delta^{\beta}_{\alpha'}\delta^{\gamma}_{\gamma'} - \delta^{\beta}_{\beta'}\delta^{\gamma}_{\alpha'}\delta^{\alpha}_{\gamma'} - \delta^{\gamma}_{\beta'}\delta^{\alpha}_{\alpha'}\delta^{\beta}_{\gamma'} \quad . \tag{31}$$

Inserting it into (30) we get zero for every summand because of the deltas and the fact that $\tilde{\tilde{\varphi}}$ has no traces unequal to zero. So $\tilde{\tilde{\varphi}}$ is symmetric in upper and lower indices, that means it is in representation 27.

c) We now have to reduce the antisymmetric part ψ from (23). The complete traces are zero

$$\psi^{\alpha\beta}_{\alpha\beta} = 0$$

$$\psi^{\alpha\beta}_{\beta\alpha} = -\psi^{\beta\alpha}_{\alpha\beta} = 0 \quad .$$

The single traces lead to one octet:

$$\psi^{\alpha\beta}_{\alpha\beta'} = 0$$

$$\psi^{\alpha\beta}_{\beta'\alpha} = \frac{1}{2}\left(\xi^{\alpha}_{\beta'}\eta^{\beta}_{\alpha} - \xi^{\beta}_{\alpha}\eta^{\alpha}_{\beta'}\right) \underset{\text{def}}{=} -\frac{1}{2}\,(\xi\eta)^{\beta}_{F\beta'} \quad . \tag{32}$$

$(\xi\eta)^{\beta}_{F\beta'}$ is called the F coupling octet formed with the tensors ξ and η. We again separate

$$\psi^{\alpha\beta}_{\alpha'\beta'} = \tilde{\psi}^{\alpha\beta}_{\alpha'\beta'} + m\psi^{\nu\beta}_{\alpha'\nu}\delta^{\alpha}_{\beta'} + n\psi^{\alpha\nu}_{\nu\beta'}\delta^{\beta}_{\alpha'} \tag{33}$$

where the coefficients m and n are fixed by the condition that $\tilde{\psi}$ must have all single traces vanishing. One finds

$$\tilde{\psi}^{\alpha\beta}_{\alpha'\beta'} = \psi^{\alpha\beta}_{\alpha'\beta'} + \frac{1}{6}\,\delta^{\alpha}_{\beta'}(\xi\eta)^{\beta}_{F\alpha'} - \frac{1}{6}\,\delta^{\beta}_{\alpha'}(\xi\eta)^{\alpha}_{F\beta'} \quad . \tag{34}$$

$\tilde{\psi}$ is antisymmetric for simultaneous permutation of upper and lower indices. If we make the decomposition

$$\tilde{\psi}^{\alpha\beta}_{\alpha'\beta'} = \frac{1}{2}\left(\tilde{\psi}^{\alpha\beta}_{\alpha'\beta'} + \tilde{\psi}^{\beta\alpha}_{\alpha'\beta'}\right) + \frac{1}{2}\left(\tilde{\psi}^{\alpha\beta}_{\alpha'\beta'} - \tilde{\psi}^{\beta\alpha}_{\alpha'\beta'}\right) \tag{35}$$

the first term is symmetric in the upper indices and therefore antisymmetric in the lower indices. The second term is antisymmetric in the upper indices and therefore symmetric in the lower ones. Without loosing information we may therefore contract the first (second) term with $\epsilon^{\alpha'\beta'\gamma'}$ ($\epsilon_{\alpha\beta\gamma}$), which gives parts belonging to representation $\overline{10}$ (10), because the resulting tensor is fully symmetric.

<u>Proof</u>:

Try to find an antisymmetric part and use (31), all traces of $\tilde{\psi}$ vanishing. One finds zero. For example

$$\frac{1}{2}\left(\tilde{\psi}^{\alpha\beta}_{\alpha'\beta'} + \tilde{\psi}^{\beta\alpha}_{\alpha'\beta'}\right)\epsilon^{\alpha'\beta'\gamma'} \cdot \epsilon_{\beta\gamma'\nu} = 0 \, .$$

We end up with the result (21)

$$8\times 8 = 1+8+8+10+\overline{10}+27 \quad .$$

2. The main multiplets of hadrons

The connection of SU_3 with physics is established through the fact that elementary particles are usefully represented by irreducible SU_3 tensors. The particles classified up till now are represented by tensors belonging to the irreducible representations 8, 10, $\overline{10}$, and the trivial representation 1 (the scalar). One should remember, however, that not all of the known resonances could yet be assigned to multiplets, and that not all quantum numbers of the already classified resonances have been established with certainty. [Note added in proof: we do not mention hereunder the X_0 meson of mass 959 MeV. It is believed to be a 0^- unitary singlet and has only small mixing with the η meson.]

We shall write particles belonging to one multiplet in the same order as one writes the corresponding tensor elements. This is analogous to the condensed form of writing proton and neutron as an isodoublet:

$$p = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad n = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Bigg\} \rightarrow \begin{pmatrix} p \\ n \end{pmatrix} .$$

In order to obtain the tensor belonging to a given particle, one has to replace the symbol of that particle by one and the symbols of all other particles by zero. The normalization will be throughout such that for each particle the corresponding tensor verifies

$$\zeta^{+\alpha\beta..}_{\gamma..} \zeta^{\gamma}_{\alpha\beta..} = \sum_{\alpha\beta\gamma} | \zeta^{\gamma..}_{\alpha\beta..} |^2 = 1 . \tag{36}$$

<u>$\frac{1}{2}^+$ baryon octet ζ^{β}_{α} (α labels the lines, β the columns)</u>

$$\zeta^{\beta}_{\alpha} = \begin{pmatrix} \frac{\Lambda^0}{\sqrt{6}} + \frac{\Sigma^0}{\sqrt{2}} & \Sigma^+ & p \\ \Sigma^- & \frac{\Lambda^0}{\sqrt{6}} - \frac{\Sigma^0}{\sqrt{2}} & n \\ \Xi^- & \Xi^0 & \frac{-2\Lambda^0}{\sqrt{6}} \end{pmatrix} \tag{37}$$

<u>$\frac{3}{2}^+$ baryon decuplet $\zeta_{\alpha\beta\gamma}$</u>

	<u>Mass</u>
$\zeta_{111} = N^{*++}$; $\zeta_{112} = \frac{N^{*+}}{\sqrt{3}}$; $\zeta_{122} = \frac{N^{*o}}{\sqrt{3}}$; $\zeta_{222} = N^{*-}$	1237 MeV
$\zeta_{113} = \frac{Y_1^{*+}}{\sqrt{3}}$; $\zeta_{123} = \frac{Y_1^{*o}}{\sqrt{6}}$; $\zeta_{223} = \frac{Y^{*-}}{\sqrt{3}}$	1385 MeV
$\zeta_{133} = \frac{\Xi^{*o}}{\sqrt{3}}$; $\zeta_{233} = \frac{\Xi^{*-}}{\sqrt{3}}$	1533 MeV
$\zeta_{333} = \Omega^-$	1680 MeV

(38)

<u>0^- meson octet ζ_α^β</u>

$$\zeta_\alpha^\beta = \begin{pmatrix} \frac{\eta^0}{\sqrt{6}} + \frac{\pi^0}{\sqrt{2}} & \pi^+ & K^+ \\ \pi^- & \frac{\eta^0}{\sqrt{6}} - \frac{\pi^0}{\sqrt{2}} & K^0 \\ K^- & \overline{K}^0 & \frac{-2\eta^0}{\sqrt{6}} \end{pmatrix} \tag{39}$$

<u>1^- meson singlet $\overset{\circ}{\varphi}_0$ and 1^- meson octet ζ_α^β</u>

$$\zeta_\alpha^\beta = \begin{pmatrix} \frac{\overset{\circ}{\omega}_0}{\sqrt{6}} + \frac{\rho^0}{\sqrt{2}} & \rho^+ & K^{*+} \\ \rho^- & \frac{\overset{\circ}{\omega}_0}{\sqrt{6}} - \frac{\rho^0}{\sqrt{2}} & K^{*0} \\ K^{*-} & \overline{K}^{*0} & \frac{-2\overset{\circ}{\omega}_0}{\sqrt{6}} \end{pmatrix} \tag{40}$$

The meaning of $\overset{\circ}{\omega}_0$ is as follows: the observed vector mesons ω^0 and φ^0 both have the quantum numbers I = Y = 0. It is possible therefore to form coherent superpositions $\overset{\circ}{\omega}_0$ and $\overset{\circ}{\varphi}_0$:

$$\left.\begin{aligned} |\overset{\circ}{\omega}_0 \rangle &= \sin\vartheta \, |\omega^0\rangle - \cos\vartheta \, |\varphi^0\rangle \\ |\overset{\circ}{\varphi}_0 \rangle &= \cos\vartheta \, |\omega^0\rangle + \sin\vartheta \, |\varphi^0\rangle \end{aligned}\right\} \tag{41}$$

$$\left.\begin{aligned} |\omega^0 \rangle &= \sin\vartheta \, |\overset{\circ}{\omega}_0\rangle + \cos\vartheta \, |\overset{\circ}{\varphi}_0\rangle \\ |\varphi^0 \rangle &= -\cos\vartheta \, |\overset{\circ}{\omega}_0\rangle + \sin\vartheta \, |\overset{\circ}{\varphi}_0\rangle \end{aligned}\right\} \tag{42}$$

It appears that if one chooses an appropriate mixing angle ϑ the superposition $\overset{\circ}{\omega}_0$ belongs to the octet and $\overset{\circ}{\varphi}_0$ forms an SU_3 singlet. From experiment the angle ϑ comes out to be approximately (see Section 5):

$$\vartheta \simeq 35°; \quad \sin\vartheta \simeq \frac{1}{\sqrt{3}}; \quad \cos\vartheta \simeq \sqrt{\frac{2}{3}} \ . \tag{43}$$

Using these values of $\sin\vartheta$ and $\cos\vartheta$ it is possible to place all nine vector mesons into one tensor. Add the octet and the singlet

$$\frac{\overset{\circ}{\varphi}_0}{\sqrt{3}} \cdot \delta_\alpha^\beta + \zeta_\alpha^\beta$$

(where the first term is the diagonal tensor representing the singlet) and convert $\overset{\circ}{\varphi}_0$ and $\overset{\circ}{\omega}_0$ into φ^0 and ω^0 by using (41) and (43). One finds the very simple form

$$\begin{pmatrix} \frac{\omega^0 + \rho^0}{\sqrt{2}} & \rho^+ & K^{*+} \\ \rho^- & \frac{\omega^0 - \rho^0}{\sqrt{2}} & K^{*0} \\ K^{*-} & \overline{K}^{*0} & \varphi^0 \end{pmatrix} \tag{44}$$

Note that this tensor has a non-vanishing trace. Remember that Φ^0 and ω^0 are the physical particles, so nature has chosen not to form particles belonging to a singlet and an octet, respectively, but a special mixture of them. SU_6 symmetry will provide an explanation for this fact.

The antiparticles of the particles in a multiplet are represented by the Hermitian conjugate tensor elements. Thus, antibaryons are in $\bar{8} = 8$ for spin parity $\frac{1}{2}^+$, and in $\overline{10}$ for $\frac{3}{2}^+$. For the 0^- and 1^- mesons, particles and antiparticles are in the same multiplets, which consequently are self-conjugate ($1 = \bar{1}$, $8 = \bar{8}$).

3. Infinitesimal transformations - Operators for isospin and hypercharge

From general principles of quantum mechanics it is known that any observable is represented by a Hermitian operator. We can introduce such operators acting on the tensor indices. Some of them will represent isospin and hypercharge. Let $\zeta^{\gamma\delta\cdot\cdot}_{\alpha\beta\cdot\cdot}$ be an irreducible tensor and h^β_α an arbitrary 3×3 matrix. We define the linear transformation

$$\zeta \to \zeta' = \Lambda(h)\,\zeta \tag{45}$$

by the formula

$$\zeta'^{\gamma\delta\cdot\cdot}_{\alpha\beta\cdot\cdot} = h^{\alpha'}_{\alpha}\zeta^{\gamma\delta\cdot\cdot}_{\alpha'\beta\cdot\cdot} + h^{\beta'}_{\beta}\zeta^{\gamma\delta\cdot\cdot}_{\alpha\beta'\cdot\cdot} + \cdot\cdot - \zeta^{\gamma'\delta\cdot\cdot}_{\alpha\beta\cdot\cdot}h^{\gamma}_{\gamma'} - \zeta^{\gamma\delta'\cdot\cdot}_{\alpha\beta\cdot\cdot}h^{\delta}_{\delta'}\cdots \tag{46}$$

If the matrix h is Hermitian, $\Lambda(h)$ is Hermitian too, as is easily verified if one remembers that the scalar product of two tensors of the same representation is defined by

$$< \zeta \mid \eta > = \zeta^{+\alpha\beta\cdot\cdot}_{\gamma\cdot\cdot}\,\eta^{\gamma\cdot\cdot}_{\alpha\beta\cdot\cdot} \quad . \tag{47}$$

We give the proof for two indices:

$$< \Lambda\zeta \mid \eta > = (\Lambda\zeta)^{+\alpha}_{\beta}\,\eta^{\beta}_{\alpha} = \left[(\Lambda\zeta)^{\beta}_{\alpha}\right]^*\eta^{\beta}_{\alpha} = \left(h^{\alpha'}_{\alpha}\zeta^{\beta}_{\alpha'} - \zeta^{\beta'}_{\alpha}h^{\beta}_{\beta'}\right)^*\eta^{\beta}_{\alpha} = h^{\alpha}_{\alpha'}\left(\zeta^{\beta}_{\alpha'}\right)^*\eta^{\beta}_{\alpha} - \left(\zeta^{\beta'}_{\alpha}\right)^* h^{\beta'}_{\beta}\eta^{\beta}_{\alpha}$$

$$= < \zeta \mid \Lambda\eta > \quad .$$

An arbitrary 3×3 matrix h^β_α can be written as a linear combination of nine matrices ${}^{(\nu\mu)}h^\beta_\alpha$, where ${}^{(\nu\mu)}h^\beta_\alpha$ is defined as having the element 1 in the ν^{th} column and μ^{th} line, all other elements being zero

$${}^{(\nu\mu)}h^{\beta}_{\alpha} = \delta^{\nu}_{\alpha}\delta^{\beta}_{\mu} \quad . \tag{48}$$

For these nine matrices one defines nine transformations, $\zeta \to \zeta' = \Lambda^\nu_\mu\zeta$, simply by inserting the special matrix (48) into the equation (46). Thus,

$$\Lambda^{\nu}_{\mu} = \Lambda\left({}^{(\nu\mu)}h\right) = \Lambda\left(\delta^{\nu}_{\alpha}\delta^{\beta}_{\mu}\right) \quad . \tag{48a}$$

In terms of the nine transformations Λ_{μ}^{ν} every transformation $\Lambda(h)$, by using the identity

$$h_{\alpha}^{\beta} = h_{\nu}^{\mu}\delta_{\alpha}^{\nu}\delta_{\mu}^{\beta} \quad , \tag{49}$$

may be written in the form

$$\Lambda\left(h_{\alpha}^{\beta}\right) = \Lambda\left(h_{\nu}^{\mu}\delta_{\alpha}^{\nu}\delta_{\mu}^{\beta}\right) = h_{\nu}^{\mu}\Lambda_{\mu}^{\nu} \quad . \tag{50}$$

The action of the transformation Λ_{μ}^{ν} is easily understood: as far as a lower index is concerned Λ_{μ}^{ν} puts the μ^{th} line in the position of the ν^{th} line, all other lines are put to zero. As far as an upper index is concerned, Λ_{μ}^{ν} puts the ν^{th} column in the position of the μ^{th} column, all other columns are set zero.

If h_{α}^{β} is a Hermitian matrix with vanishing trace, then $[1 + i\epsilon\Lambda(h)]$ is an infinitesimal transformation of the group SU_3, and conversely. Indeed, let $u \epsilon SU_3$ be an infinitesimal transformation:

$$u_{\alpha}^{\alpha'} = \delta_{\alpha}^{\alpha'} + i\epsilon h_{\alpha}^{\alpha'} \quad . \tag{51}$$

The necessary and sufficient conditions that det u = 1 and u = unitary are respectively Sp h = 0 and h Hermitian.

<u>Proof</u>:

Note:- $(u^{-1})_{\alpha}^{\alpha'} = \delta_{\alpha}^{\alpha'} - i\epsilon h_{\alpha}^{\alpha'}$.

$$u^{+} = u^{-1}$$

$$\left.\begin{aligned} (u^{+})_{\alpha}^{\alpha'} &= u_{\alpha'}^{*\alpha} = \delta_{\alpha'}^{\alpha} - i\epsilon h_{\alpha'}^{*\alpha} = \delta_{\alpha}^{\alpha'} - i\epsilon h_{\alpha}^{+\alpha'} \\ &= (u^{-1})_{\alpha}^{\alpha'} = \delta_{\alpha}^{\alpha'} - i\epsilon h_{\alpha}^{\alpha'} \end{aligned}\right\} \curvearrowright$$

$$\curvearrowright h^{+} = h$$

$$\begin{aligned} \det u &= u_1^{\alpha}u_2^{\beta}u_3^{\gamma}\,\epsilon_{\alpha\beta\gamma} \\ &= \left(\delta_1^{\alpha} + i\epsilon h_1^{\alpha}\right)\left(\delta_2^{\beta} + i\epsilon h_2^{\beta}\right)\left(\delta_3^{\gamma} + i\epsilon h_3^{\gamma}\right)\epsilon_{\alpha\beta\gamma} \\ &= \left\{\delta_1^{\alpha}\delta_2^{\beta}\delta_3^{\gamma} + i\delta_1^{\alpha}\delta_2^{\beta}\epsilon h_3^{\gamma} + i\delta_2^{\beta}\delta_3^{\gamma}\epsilon h_1^{\alpha} + i\delta_3^{\gamma}\delta_1^{\alpha}\epsilon h_2^{\beta}\right\} \cdot \epsilon_{\alpha\beta\gamma} \\ &= 1 + i\epsilon\,(h_3^3 + h_1^1 + h_2^2) = 1 \end{aligned}$$

$$\curvearrowright Sp\,h = 0 \quad .$$

All this is calculated to first order in ϵ. By using (10) and (11), and (45) and (46) it is immediately seen that for infinitesimal u as in (51) one has

$$U(u) = 1 + i\epsilon\Lambda(h) \quad , \qquad (51a)$$

1 being the unit transformation.

Corresponding to the eight linearly independent matrices h with vanishing trace there are eight linearly independent transformations (51a). As will be seen later [Eq. (62)] they form representation 8 of SU_3 (this representation, formed by the infinitesimal transformations of the group, is usually called the adjoint or the regular representation). One usually chooses, following Gell-Mann, the following basic h matrices (dots stand for 0):

$$\lambda_1 = \begin{pmatrix} \cdot & 1 & \cdot \\ 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}; \quad \lambda_2 = \begin{pmatrix} \cdot & -i & \cdot \\ i & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}; \quad \lambda_3 = \begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$$

$$\lambda_4 = \begin{pmatrix} \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot \end{pmatrix}; \quad \lambda_5 = \begin{pmatrix} \cdot & \cdot & -i \\ \cdot & \cdot & \cdot \\ i & \cdot & \cdot \end{pmatrix}; \quad \lambda_6 = \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 \\ \cdot & 1 & \cdot \end{pmatrix}$$

$$\lambda_7 = \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & -i \\ \cdot & i & \cdot \end{pmatrix}; \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & -2 \end{pmatrix} .$$

The operators for isospin and hypercharge are the operators $\Lambda(h)$ for the following special choices of h:

$$\text{Isospin} \quad : \quad I_i = \Lambda(h_{I_i}); \quad h_{I_i} = \frac{1}{2}\lambda_i; \quad i = 1,2,3 \qquad (52)$$

$$\text{Hypercharge} \quad : \quad Y = \Lambda(h_y); \quad h_y = \frac{1}{\sqrt{3}}\lambda_8 \quad . \qquad (53)$$

Here the composite structure of SU_3 is seen. In the first and second line and column of λ_i one has the Pauli matrices representing isospin. The third line and column refer to hypercharge. Thus, isospin and hypercharge commute because λ_8 commutes with $\lambda_{1,2,3}$.

The operator for the electric charge is

$$Q = \Lambda(h_Q) \qquad (54)$$

where

$$h_Q = h_{I_3} + \frac{1}{2} h_y = \frac{1}{2}\lambda_3 + \frac{\frac{1}{\sqrt{3}}\lambda_8}{2} = \frac{1}{3}\begin{pmatrix} 2 & \cdot & \cdot \\ \cdot & -1 & \cdot \\ \cdot & \cdot & -1 \end{pmatrix} \qquad (55)$$

in accordance with the relation of Gell-Mann and Nishijima

$$Q = I_3 + \frac{Y}{2} \quad . \qquad (56)$$

With these operators the assignments of isospin and hypercharge for the particles in the multiplets (37 - 40) may be checked. For example, the hypercharge of the proton is verified to be one:

$$Y \cdot p = 1 \cdot p$$

$$\frac{1}{3}\begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & -2 \end{pmatrix} \cdot \begin{pmatrix} \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} = \begin{pmatrix} \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} .$$

Another example is the calculation of the fractional electric charge $^2/_3$, $-^1/_3$, $-^1/_3$ of hypothetical particles which would be associated with the representation 3, the so-called quarks.

4. Transformation of operators under SU_3

Let ζ_1 and ζ_2 be two irreducible tensors, and let u be a transformation of SU_3. It transforms ζ_1 and ζ_2 into

$$\zeta_1' = U(u)\,\zeta_1 ; \quad \zeta_2' = U(u)\,\zeta_2 \quad . \tag{57}$$

We define the effect of u on an operator A to be $A \to A'$, where A' verifies

$$< \zeta_2' \mid A' \mid \zeta_1' > = < \zeta_2 \mid A \mid \zeta_1 > \tag{58}$$

for arbitrary ζ_1 and ζ_2. Using unitarity of U(u) one finds

$$< \zeta_2' \mid A' \mid \zeta_1' > = < U(u)\,\zeta_2 \mid A' \mid U(u)\,\zeta_1 >$$

$$= < \zeta_2 \mid U^{-1}(u)\; A' \; U(u) \mid \zeta_1 > = < \zeta_2 \mid A \mid \zeta_1 >$$

$$\curvearrowright A' = U(u)\; A\; U^{-1}(u) \tag{59}$$

where

$$U^{-1}(u) = U(u^{-1}) \quad . \tag{60}$$

The application of the transformation law (59) to the operator $\Lambda(h)$ defined in (45) gives

$$\Lambda(h) \to U(u)\; \Lambda(h)\; U^{-1}(u) \tag{61}$$

which we state to be equal to

$$U(u)\; \Lambda(h)\; U^{-1}(u) = \Lambda(uhu^{-1}) \; , \tag{62}$$

where

$$(uhu^{-1})_\alpha^\beta = u_\alpha^\kappa h_\kappa^\lambda (u^{-1})_\lambda^\beta \; . \tag{63}$$

To prove (62) one applies the operators proposed to be equal [$U\Lambda U$ and $\Lambda(uhu^{-1})$] to a tensor and carries out the transformations U(u) and Λ, (11) and (45).

The application of the transformation law (59) to the nine operators Λ^{ν}_{μ} [see (50)] gives

$$U(u)\ \Lambda^{\nu}_{\mu}\ U^{-1}(u) = (u^{-1})^{\kappa}_{\mu}\ \Lambda^{\lambda}_{\kappa} u^{\nu}_{\lambda} \quad . \tag{64}$$

Proof:

Use the definition of Λ^{ν}_{μ}, see (48) and thereafter

$$U\Lambda^{\nu}_{\mu}U^{-1} = U\Lambda\left(\delta^{\nu}_{\alpha}\delta^{\beta}_{\mu}\right)U^{-1} = \Lambda\left(u^{\alpha'}_{\alpha}\delta^{\nu}_{\alpha'}\delta^{\beta'}_{\mu}(u^{-1})^{\beta}_{\beta'}\right) = \Lambda\left(u^{\nu}_{\alpha}(u^{-1})^{\beta}_{\mu}\right) \quad .$$

With help of (49) the right-hand side of (64) can be written

$$(u^{-1})^{\kappa}_{\mu}\ \Lambda^{\lambda}_{\kappa}u^{\nu}_{\lambda} = u^{\nu}_{\lambda}(u^{-1})^{\kappa}_{\mu}\ \Lambda^{\lambda}_{\kappa} = \Lambda\left(u^{\nu}_{\alpha}(u^{-1})^{\beta}_{\mu}\right) \text{ q.e.d.}$$

For subsequent use we want to define a further operator $\Lambda'(h)$ which is linear in h but quadratic in Λ^{β}_{α}

$$\Lambda'(h) = h^{\beta}_{\alpha}\Lambda^{\gamma}_{\beta}\Lambda^{\alpha}_{\gamma} \quad . \tag{65}$$

For this operator again we find the transformation law

$$U(u)\ \Lambda'(h)\ U^{-1}(u) = \Lambda'(uhu^{-1}) \quad . \tag{66}$$

Proof:

$$U\Lambda'U^{-1} = U(u)\cdot h^{\beta}_{\alpha}\Lambda^{\gamma}_{\beta}\left(U(u^{-1})\cdot U(u)\right)\Lambda^{\alpha}_{\gamma}\ U(u^{-1}) = h^{\beta}_{\alpha}\left[U(u)\ \Lambda^{\gamma}_{\beta}\ U(u^{-1})\right]\left[U(u)\ \Lambda^{\alpha}_{\gamma}\ U(u^{-1})\right]$$

$$= h^{\beta}_{\alpha}(u^{-1})^{\beta'}_{\beta}\ \Lambda^{\gamma'}_{\beta'}\left(u^{\gamma}_{\gamma'}(u^{-1})^{\gamma''}_{\gamma}\right)\Lambda^{\alpha''}_{\gamma''}u^{\alpha}_{\alpha''} = \left[u^{\alpha}_{\alpha''}h^{\beta}_{\alpha}(u^{-1})^{\beta'}_{\beta}\right]\Lambda^{\gamma''}_{\beta'}\Lambda^{\alpha''}_{\gamma''} = \Lambda'(uhu^{-1}) \quad .$$

5. Mass splitting operator. Mass formula of Gell-Mann and Okubo

Particles in the same isospin multiplet show almost the same mass. The small mass splittings can probably be accounted for entirely in terms of electromagnetic interaction.

In contrast thereto, the mass splittings inside an SU_3 multiplet are very large. The mass values nevertheless give great support to the SU_3 symmetry scheme. This is due to the fact that they appear to have simple group theoretical properties.

The mass splitting is attributed to a so-called semi-strong interaction, which, in contrast to the SU_3 invariant very strong interaction, is not SU_3 invariant. How it transforms under SU_3 is guessed on the basis of the experimental fact that the masses depend only on the hypercharge. (We neglect electromagnetic mass differences.) One therefore tries the simple assumption that the mass splitting operator ΔM, which is responsible for the mass splittings, transforms like the hypercharge operator. This assumption turns out to be remarkably successful. We proceed with the precise formulation of the properties of ΔM.

We demand the existence of a class of operators $\Delta M(h)$ which is defined over all traceless 3×3 matrices h^{β}_{α}, is linear in h, and transforms according to

$$U(u)\ \Delta M(h)\ U^{-1}(u) = \Delta M(uhu^{-1}) \quad . \tag{67}$$

That is, $\Delta M(h)$ obeys the same transformation law as do the operators Λ and Λ' defined above. All these operators transform under SU_3 like the traceless matrix h^{β}_{α}, i.e. like the octet representation of SU_3. It is assumed that the mass splitting operator ΔM is the member $\Delta M(\lambda_8)$ of the class. From (53) we know indeed that λ_8 gives the hypercharge (the factor $\sqrt{3}$ is immaterial).

If the symmetry scheme SU_3 is to make sense, the symmetry violating semi-strong interaction has to be small compared with the very strong symmetry conserving interaction. Therefore, we expect to get the actual mass splittings from a first order perturbation calculation. This will be found to work. In first order perturbation we have to find the eigenvalues of the operator ΔM

$$< \zeta_2 \mid \Delta M \mid \zeta_1 > = < \zeta_2 \mid \Delta M(\lambda_8) \mid \zeta_1 > \quad , \tag{68}$$

i.e. to diagonalize the matrix (68) taken between irreducible tensors. Before deriving in this way the famous Gell-Mann-Okubo mass formula, we make a few general comments.

In the case of the pseudoscalar meson octet ΔM should be regarded as an operator which gives the splitting among the squared masses. There is no really convincing theoretical explanation of this fact, it simply turns out empirically that the masses squared rather than the masses themselves fit the mass formula*). For the other multiplets the formula applies equally well to masses or squared masses.

For the baryon octet and decuplet the mass formula derived in a moment holds very well. The precision is of the order of a few percent, the mass splittings themselves being of the order of 10 - 30%. For the pseudoscalar mesons the predictions are less well satisfied, but the relative splittings are also much larger*). In all cases the discrepancy between predictions and experimental masses gives an estimate of the importance which second order perturbation effects of ΔM may have. One concludes that these effects must be, very roughly, a factor 10 or 20 smaller than first order splittings. We now proceed with the mathematical derivations.

Consider the matrix element

$$< \zeta_2 \mid \Delta M(h) \mid \zeta_1 > \tag{68a}$$

of which (68) is a special case. It is left invariant if ζ_1, ζ_2 and h are all transformed by a transformation U(u) from SU_3. Indeed, because of (67), the operator $\Delta M(h)$ undergoes a transformation which leaves the matrix element invariant, namely (59). Since the matrix element (68a), as far as indices are concerned, is of the structure

$$(\zeta_2^{+})^{\alpha\beta\cdot\cdot}_{\gamma\delta\cdot\cdot}\ h^{\beta''}_{\alpha''}(\zeta_1)^{\gamma'\delta'\cdot\cdot}_{\alpha'\beta'\cdot\cdot} \tag{69}$$

we have to construct invariants out of expressions like (69). An invariant is obtained by contracting all indices with the help of the invariant tensors δ and ϵ, (12). We deal first

*) Note added in proof: this difficulty can be resolved by mixing η and X_0 (see note p. 8).

with the case that ζ_1 and ζ_2 belong to the same multiplet, i.e. have the same numbers of upper and lower indices. The number of upper indices in (69) is then the same as the number of lower ones. Because of identiy (31) all contractions can be done with δ. Remembering that ζ_1 and ζ_2 are traceless, one is left with three posibilities:

$$1: \quad (\zeta_1^+)^{\alpha\beta..}_{\gamma\delta..}\, h^{\alpha''}_{\alpha''}(\zeta_2)^{\gamma\delta..}_{\alpha\beta..} = h^{\alpha''}_{\alpha''} < \zeta_1 \mid \zeta_2 > \tag{70}$$

(This invariant gives zero when the trace of h vanishes.)

$$2: \quad (\zeta_2^+)^{\alpha\beta..}_{\gamma\delta..}\, h^{\alpha'}_{\alpha}(\zeta_1)^{\gamma\delta..}_{\alpha'\beta..} \tag{71}$$

$$3: \quad (\zeta_2^+)^{\alpha\beta..}_{\gamma\delta..}\, h^{\gamma}_{\gamma'}(\zeta_1)^{\gamma'\delta..}_{\alpha\beta..} \tag{72}$$

If ζ_1 and ζ_2 possess only lower or only upper indices (case A), it is possible only to construct one of the two invariants [(71) for only lower - (72) for only upper indices] .

If ζ_1 and ζ_2 possess both lower and upper indices, both the invariants (71) and (72) can be constructed (case B).

So the invariant matrix element (68) has to be a linear combination of the invariant (70), and

in case A) - one other invariant (71) or (72)
in case B) - the two other invariants (71) and (72).

The same argument now holds for two matrix elements:

$$< \zeta_2 \mid \Lambda(h) \mid \zeta_1 > \text{ and } < \zeta_2 \mid \Lambda'(h) \mid \zeta_1 > \tag{73}$$

because $\Lambda(h)$ and $\Lambda'(h)$ transform as $\Delta M(h)$. In case A, (70) and any one of the invariants (73) are linearly independent; in case B, (70) and both invariants (73) are linearly invariant; this is easily checked because Λ and Λ' are known explicitly. Consequently:

<u>Case A</u> - The one existing invariant (71) or (72) may be expressed by the invariants

$$h^{\alpha}_{\alpha} < \zeta_2 \mid \zeta_1 > \text{ and } < \zeta_2 \mid \Lambda \mid \zeta_1 > \tag{74}$$

or as well by the invariants

$$h^{\alpha}_{\alpha} < \zeta_2 \mid \zeta_1 > \text{ and } < \zeta_2 \mid \Lambda' \mid \zeta_1 > \quad , \tag{75}$$

thus leading to

$$< \zeta_2 \mid \Delta M(h) \mid \zeta_1 > = c_0 h^{\alpha}_{\alpha} < \zeta_2 \mid \zeta_1 > + c_1 < \zeta_2 \mid \Lambda(h) \mid \zeta_1 > \tag{76}$$

$$= c_0' h^{\alpha}_{\alpha} < \zeta_2 \mid \zeta_1 > + c_1' < \zeta_2 \mid \Lambda'(h) \mid \zeta_1 > \tag{77}$$

where c_0, c_1, c_0' and c_1' are constants independent of h and of the members ζ_1 and ζ_2 of the multiplet considered (their values can change from multiplet to multiplet).

<u>Case B</u> - Both invariants (73) are needed to eliminate the two invariants (71) and (72), thus leading to

$$< \zeta_2 \mid \Delta M(h) \mid \zeta_1 > = d_0 h^\alpha_\alpha < \zeta_2 \mid \zeta_1 > + d_1 < \zeta_2 \mid \Lambda(h) \mid \zeta_1 > + d_2 < \zeta_2 \mid \Lambda'(h) \mid \zeta_1 > , \tag{78}$$

where d_0, d_1 and d_2 are again constant within each multiplet.

We now turn to the calculation of the mass splittings, i.e. we put $h = \lambda_8$ and remember that $\Delta M = \Delta M(\lambda_8)$ is the mass splitting operator. Formula (76) gives with $h = \lambda_8$

$$< \zeta_2 \mid \Delta M \mid \zeta_1 > = c_1 < \zeta_2 \mid \Lambda(\lambda_8) \mid \zeta_1 > = c_1 \sqrt{3} < \zeta_2 \mid Y \mid \zeta_1 > = c_1 \sqrt{3}\, \delta_{\zeta_2\zeta_1} Y_{\zeta_1} , \tag{79}$$

[for Y see (53)]. Because of (74) this is sufficient to deal with case A. The relevant example is the baryon decuplet (38). All mass splittings inside the decuplet are expressed in terms of one constant; the mass differences are equal from line to line in (38). This is in excellent agreement with the observed masses. Using this formula it was possible to predict the mass of the then unobserved Ω^-, and to predict that it decays only through weak interactions. These predictions are now confirmed.

To treat case B, we first want to express $\Lambda'(\lambda_8)$ in terms of the operators $I_1^2 + I_2^2 + I_2^2$ and Y

$$h_y = \frac{1}{\sqrt{3}} \lambda_8 \tag{53}$$

$$\Lambda'(h_y) = (h_y)^\beta_\alpha \Lambda^\gamma_\beta \Lambda^\alpha_\gamma = \frac{1}{3} (\Lambda^\gamma_1 \Lambda^1_\gamma + \Lambda^\gamma_2 \Lambda^2_\gamma - 2\Lambda^\gamma_3 \Lambda^3_\gamma) = \frac{1}{3} \Lambda^\gamma_\alpha \Lambda^\alpha_\gamma - \Lambda^\gamma_3 \Lambda^3_\gamma \tag{80}$$

$$\Lambda(h_y) = Y = (h_y)^\beta_\alpha \Lambda^\alpha_\beta = \frac{1}{3} (\Lambda^1_1 + \Lambda^2_2 - 2\Lambda^3_3) = \frac{1}{3} \Lambda^\gamma_\gamma - \Lambda^3_3 \tag{81}$$

$$\curvearrowright \Lambda^3_3 = \frac{1}{3} \Lambda^\gamma_\gamma - Y \tag{82}$$

$$I_1 = \Lambda(h_{I_1}) = (h_{I_1})^\beta_\alpha \Lambda^\alpha_\beta = \frac{1}{2} (\Lambda^1_2 + \Lambda^2_1) \tag{83}$$

$$I_2 = \cdots\cdots\cdots = i/2(\Lambda^2_1 - \Lambda^1_2) \tag{84}$$

$$I_3 = \cdots\cdots\cdots = \frac{1}{2} (\Lambda^1_1 - \Lambda^2_2) \tag{85}$$

$$|I|^2 = I_1^2 + I_2^2 + I_3^2 = \frac{1}{2} \Lambda^j_i \Lambda^i_j - \frac{1}{4} (\Lambda^1_1 + \Lambda^2_2)^2 \quad . \tag{86}$$

(Our index convention is $i,j = 1,2$ and $\alpha,\beta,\gamma = 1,2,3$.)

Add and subtract special terms:

$$|I|^2 = \frac{1}{2}\Lambda^\gamma_\alpha\Lambda^\alpha_\gamma - \frac{1}{2}\Lambda^\gamma_3\Lambda^3_\gamma - \frac{1}{2}\Lambda^3_\gamma\Lambda^\gamma_3 + \frac{1}{2}\Lambda^3_3\Lambda^3_3 - \frac{1}{4}(\Lambda^\gamma_\gamma - \Lambda^3_3)^2 \quad . \tag{87}$$

Use the commutator

$$\left[\Lambda^{\alpha'}_{\beta'}, \Lambda^\alpha_\beta\right] = \delta^\alpha_{\beta'}\Lambda^{\alpha'}_\beta - \delta^{\alpha'}_\beta\Lambda^\alpha_{\beta'} \quad . \tag{88}$$

Equation (88) can be proved directly by using the definition of Λ^α_β [see (59)].

$$\Lambda^3_\gamma\Lambda^\gamma_3 = \Lambda^\gamma_3\Lambda^3_\gamma + \left[\Lambda^3_\gamma, \Lambda^\gamma_3\right] = \Lambda^\gamma_3\Lambda^3_\gamma + 3\Lambda^3_3 - \Lambda^\gamma_\gamma \quad . \tag{89}$$

Put (89) and (82) into (87) and solve for

$$\Lambda^\gamma_3\Lambda^3_\gamma \quad ,$$

insert this into (80). It follows that

$$\Lambda'(h_y) = a \cdot \Lambda^\gamma_\alpha\Lambda^\alpha_\gamma + b\left(\Lambda^\gamma_\gamma\right)^2 + cY + \left(I^2 - \frac{Y^2}{4}\right) \quad ,$$

where a, b, and c are constants. We put this into the mass equation (78) with the result

$$< \zeta_2 \mid \Delta M \mid \zeta_1 > = a' < \zeta_2 \left| \left(I_1^2 + I_2^2 + I_3^2 - \frac{Y^2}{4}\right) \right| \zeta_1 > + b' < \zeta_2 \mid Y \mid \zeta_1 > + c' < \zeta_2 \mid \zeta_1 > \quad , \tag{90}$$

a′, b′, c′ are constants independent of the members ζ_1 and ζ_2 of the multiplet. The operator representing the total mass (or the total mass squared) has the form

$$M = M_0 + \Delta M \tag{91}$$

where M_0 is the SU_3 invariant part with a common value for all members of a multiplet, and ΔM is the mass splitting operator as before. From (90), replacing the operators on the right-hand side by their eigenvalues, we obtain the Gell-Mann-Okubo mass formula relating the mass of a multiplet member to its hypercharge and isospin

$$m = a_0\left(I(I+1) - \frac{Y^2}{4}\right) + b_0 Y + c_0 \quad , \tag{92}$$

a_0, b_0, c_0 are constants within the multiplet.

Let us apply (92) to the baryon octet. Using the four masses of the four isospin multiplets involved (N, Λ, Σ, Ξ) one can eliminate the three constants in (92) and one relation between the masses is obtained

$$\frac{1}{2}\left(m_N + m_\Xi\right) = \frac{1}{4}\left(m_\Sigma + 3m_\Lambda\right) \quad . \tag{93}$$

The analogous relation for the octet of pseudoscalar mesons [use masses squared, cf. paragraph after (68)] reads (remembering $m_K = m_{\bar{K}}$)

$$m_K^2 = \frac{1}{4}\left(m_\pi^2 + 3m_\eta^2\right) \quad . \tag{94}$$

The analogous relation for the octet of vector mesons gives

$$m_{K^*} = \frac{1}{4}\left(m_\rho + 3 < \omega_0^0 \,|M|\, \omega_0^0 >\right) \quad . \tag{95}$$

We used again the masses themselves. The last term in (95) is the mass expectation value of the I = Y = 0 octet member ω_0^0. With vector mesons we have an example of "mixing", in the sense that the operator ΔM has also a non-vanishing matrix element between different multiplets, namely the element $< \omega_0^0 \,|\Delta M|\, \varphi_0^0 >$ between ω_0^0 and the singlet φ_0^0. Of course, $< \omega_0^0 \,|M_0|\, \varphi_0^0 >$ vanishes. As we show presently, consideration of this matrix element and of (95) allows us to determine from the experimental masses the mixing angle ϑ appearing in (41) and (42). By the same token, SU_3 predicts no mass relation between the nine vector mesons. Such a relation, i.e. the value of ϑ, will be predicted by SU_6.

The operator M in the Y = I = 0 subspace spanned by $|\,\omega_0^0>$, $|\,\varphi_0^0>$ is given by the matrix

$$\begin{pmatrix} < \omega_0^0 \,|M|\, \omega_0^0 > & < \omega_0^0 \,|M|\, \varphi_0^0 > \\ < \varphi_0^0 \,|M|\, \omega_0^0 > & < \varphi_0^0 \,|M|\, \varphi_0^0 > \end{pmatrix} \tag{96}$$

$$= \begin{pmatrix} m_0 + \delta & \mu \\ \mu & m_0 - \delta \end{pmatrix} \tag{97}$$

The relative phase between $|\,\omega_0^0>$ and $|\,\varphi_0^0>$ may be chosen in such a way that

$$< \omega_0^0 \,|M|\, \varphi_0^0 > \; = \; < \varphi_0^0 \,|M|\, \omega_0^0 > \; = \mu = \text{real}, \tag{98}$$

a relation already used in (97). Recall that M is hermitian. The eigenvalues of the matrix (97) are

$$m_{1,2} = m_0 \pm \sqrt{\delta^2 + \mu^2} \quad . \tag{99}$$

They are the known masses of the physical particles

$$m_1 = m_{\varphi^0} \; ; \quad m_2 = m_{\omega^0} \quad . \tag{100}$$

The parameter $m_0 + \delta = < \omega_0^0 \,|M|\, \omega_0^0 >$ is known from the mass formula (95), m_0 and $\delta^2 + \mu^2$ follow from (99). Since one knows all matrix elements of (97) it is then possible to find the eigenstates (for example, the ω^0)

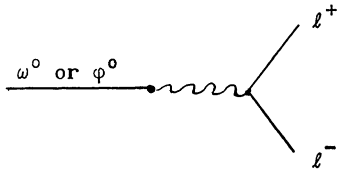
$$\begin{pmatrix} m_0 + \delta & \mu \\ \mu & m_0 - \delta \end{pmatrix} \begin{pmatrix} \sin\vartheta \\ \cos\vartheta \end{pmatrix} = \left[m_0 - \sqrt{\delta^2 + \mu^2} \right] \begin{pmatrix} \sin\vartheta \\ \cos\vartheta \end{pmatrix} , \tag{101}$$

(the − sign has to be taken in the eigenvalue because $m_{\omega^0} < m_{\varphi^0}$). This simple calculation leads to the angle $\vartheta \simeq 35°$ mentioned in (43).

An experimental check of this result is possible by an independent determination of the octet and singlet parts in the physical particles ω^0 and φ^0. Consider the transitions

$$\omega^0 \to \text{virtual } \gamma; \quad \varphi^0 \to \text{virtual } \gamma \tag{102}$$

which control, for instance, the rare decay processes



ℓ denotes a charged lepton (electron or muon). The matrix element for this process would be

$$< \gamma \mid A_\mu \mid 0 > < 0 \mid j_\mu \mid {}^{\omega^0}_{\text{or } \varphi^0} > \quad . \tag{103}$$

The electric current operator j_μ transforms under SU_3 like the operator of the electric charge, see (54), i.e. like an octet member (for more details see next paragraph). Since the matrix element

$$< 0 \mid j_\mu \mid {}^{\omega^0}_{\text{or } \varphi^0} > \tag{104}$$

is SU_3 invariant, we have to construct an invariant out of

an octet (the electric current j_μ),
a singlet (the vacuum 0), and
a mixture (42) of octet and singlet,
namely the physical particles ω^0 or φ^0.

An invariant is obtained only when the matrix element (104) picks out the octet part from the physical particles ω^0, φ^0. It follows

$$< 0 \mid j_\mu \mid \omega^0 > = \sin\vartheta < 0 \mid j_\mu \mid \omega_0^0 >$$

$$< 0 \mid j_\mu \mid \varphi^0 > = -\cos\vartheta < 0 \mid j_\mu \mid \omega_0^0 > \quad .$$

Therefore, the ratio of the two decay rates (102) will be

$$\frac{R(\omega^0 \to \ell^+\ell^-)}{R(\varphi^0 \to \ell^+\ell^-)} = \frac{\sin^2\vartheta}{\cos^2\vartheta} \simeq \frac{1/3}{2/3} = \frac{1}{2} \quad .$$

This ratio has not yet been measured.

6. Electromagnetic properties. U spin

The basic consideration leading to predictions of the electromagnetic properties of the particles grouped in irreducible tensors is the same as in the case of mass splittings. One knows the transformation law for the electric charge and electric current, since they are operators in the octet representation. This is shown by (54), (61) and (62) for the charge operator Q. One assumes that similarly the electromagnetic current operator j_μ is a special case $j_\mu = j_\mu(h_Q)$ of an operator depending linearly on a traceless matrix h_α^β, with the usual transformation law

$$U(u)\, j_\mu(h)\, U(u^{-1}) = j_\mu(uhu^{-1}) \ .$$

Therefore, the matrix element

$$< \zeta_2 \mid j_\mu(h) \mid \zeta_1 > \tag{105}$$

is an invariant, under SU_3, and one constructs invariants out of the parts forming the matrix element (105). As in the case of the mass splitting, the number of linearly independent invariants is zero, one, or two. An example of application of the method was given at the end of the preceding section.

A very convenient tool to discuss electromagnetic properties is the reclassification of particles by means of a quantum number called U spin, introduced by Lipkin. The idea behind it can be described as follows.

The isospin $\vec{I}$ commutes with Y, so all particles in the same isospin multiplet have the same properties as far as Y is concerned. The U spin is now constructed to play with respect to Q the same role as I plays for Y. Thus, the operators $\vec{U}$ will commute with Q, the electric charge. Therefore, all particles in the same U-spin multiplet will have the same electromagnetic properties.

Consider the special SU_3 transformation

$$u_0 = \begin{pmatrix} \cdot & \cdot & 1 \\ 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \end{pmatrix} \ . \tag{106}$$

For a covariant spinor it gives a circular permutation of the components

$$\zeta \to \zeta' = u_0 \zeta \ ; \quad \begin{pmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{pmatrix} \to \begin{pmatrix} \zeta_3 \\ \zeta_1 \\ \zeta_2 \end{pmatrix} \tag{107}$$

and, as is easily verified, it satisfies

$$u_0 h_y u_0^{-1} = u_0 \frac{\lambda_8}{\sqrt{3}} u_0^{-1} = - \lambda_Q \ . \tag{108}$$

Hence, the transformation $U(u_0) = U_0$ corresponding to u_0 transforms the hypercharge in the opposite of the charge

$$U_0 Y U_0^{-1} = - Q \ . \tag{109}$$

The transformed of the isospin $I_{1,2,3}$,

$$U_{1,2,3} = U_0 \, I_{1,2,3} \, U_0^{-1} \tag{110}$$

is by definition the U spin. It has the same commutation rules as the isospin

$$\left[U_i, U_j \right] = i U_k \quad (i,j,k = \text{circ.perm. of } 1,2,3)$$

and commutes with the electric charge.

Instead of using Y and I_3, one can now classify particles by means of the commuting additive quantum numbers

$$\left. \begin{aligned} Q &= I_3 + \frac{1}{2} Y \\ U_3 &= \frac{1}{2} I_3 + \frac{3}{4} Y \end{aligned} \right\} . \qquad (111)$$

The two classifications for the baryon octet and decuplet are given in the diagrams on p. 24. The isospin multiplets are found on the lines Y = const, the U-spin multiplets on the lines Q = const. The only complication arising is the $Y = I_3 = Q = U_3 = 0$ pair of particles occuring in the octet. The physical particles are Λ^o and Σ^o with isospin 0 and 1, respectively. The eigenstates of total U spin are, for U spin 0 and 1, respectively, given by

$$\left. \begin{aligned} \Lambda^o_u &= U_0 \Lambda^o = -\frac{1}{2} \Lambda^o - \frac{\sqrt{3}}{2} \Sigma^o \\ \Sigma^o_u &= U_0 \Sigma^o = \frac{\sqrt{3}}{2} \Lambda^o - \frac{1}{2} \Sigma^o \end{aligned} \right\} \qquad (112)$$

As an example of SU_3 predictions for electromagnetic properties we discuss the magnetic moments of the baryon octet. In addition to the eight magnetic moments μ_p, μ_n, μ_Λ, μ_{Σ^+}, μ_{Σ^o}, μ_{Σ^-}, μ_{Ξ^o}, μ_{Ξ^-}, one must consider the transition moment $\mu_{\Lambda\Sigma}$ between Λ^o and Σ^o, which determines the Σ^o lifetime for its main decay $\Sigma^o \to \Lambda^o + \gamma$. Because of time-reversal invariance, all nine quantities are real, and $\mu_{\Lambda\Sigma}$ describes both transitions $\Lambda^o \longleftrightarrow \Sigma^o$. The predictions can be grouped as follows:

a) Using (56), isospin invariance, and the fact that Y is an isoscalar, one obtains

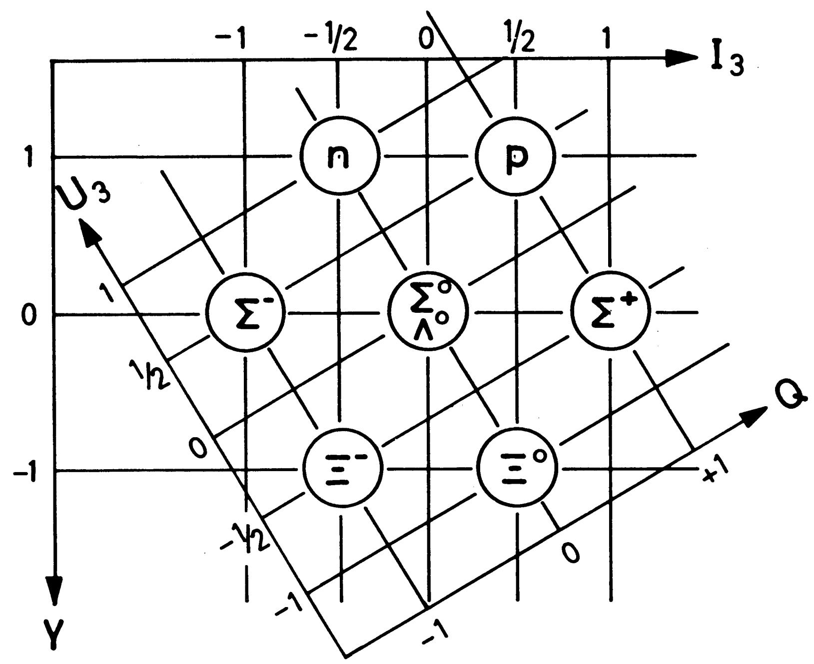
$$\mu_{\Sigma^+} + \mu_{\Sigma^-} = 2\mu_{\Sigma^o} . \qquad (113)$$

b) From U-spin invariance of strong and electromagnetic interactions one has

$$\left. \begin{aligned} \mu_{\Sigma^+} &= \mu_p \\ \mu_{\Xi^-} &= \mu_{\Sigma^-} \\ \mu_{\Xi^o} &= \mu_{\Sigma^o_u} = \mu_n \\ \mu_{\Lambda_u \Sigma_u} &= 0 \end{aligned} \right\} . \qquad (114)$$

c) From SU_3 invariance, especially (109) and (112)

$$\mu_{\Lambda_u} = - \mu_\Lambda^{(Y)} \qquad (115)$$


I3
Y
U3
Q
-1
-1/2
0
1/2
1
1
0
-1
1
1/2
0
-1/2
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+1
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Σ-
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Λ°
Σ+
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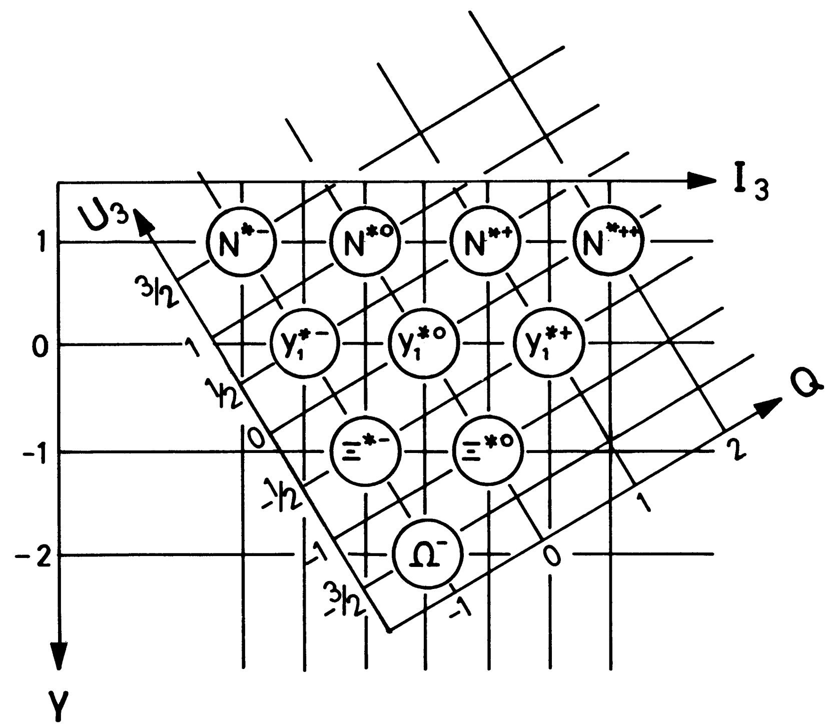
I3
Y
U3
Q
1
0
-1
-2
3/2
1
1/2
0
-1/2
-1
-3/2
2
1
0
-1
N*-
N*o
N*+
N*++
Y1*-
Y1*o
Y1*+
Ξ*-
Ξ*o
Ω-

where $\mu_{\Lambda}^{(Y)}$ is the contribution of the hypercharge to the Λ magnetic moment. From (56)

$$\mu_{\Lambda} = \mu_{\Lambda}^{(I_3)} + \frac{1}{2}\mu_{\Lambda}^{(Y)} \quad , \qquad (116)$$

where $\mu_{\Lambda}^{(I_3)}$ is the isospin contribution to μ_{Λ}. Since Λ is an isoscalar and I_3 an isovector, this latter contribution vanishes, and (115) becomes

$$\mu_{\Lambda_u} = - 2\mu_{\Lambda} \quad . \qquad (117)$$

Equation (112) gives

$$\left.\begin{aligned} \mu_{\Sigma^o_u} &= \frac{3}{4}\mu_{\Lambda} + \frac{1}{4}\mu_{\Sigma^o} - \frac{\sqrt{3}}{2}\mu_{\Lambda\Sigma} \\ \mu_{\Lambda_u} &= \frac{1}{4}\mu_{\Lambda} + \frac{3}{4}\mu_{\Sigma^o} + \frac{\sqrt{3}}{2}\mu_{\Lambda\Sigma} \\ \mu_{\Lambda_u\Sigma_u} &= -\frac{\sqrt{3}}{4}\mu_{\Lambda} + \frac{\sqrt{3}}{4}\mu_{\Sigma^o} - \frac{1}{2}\mu_{\Lambda\Sigma} \end{aligned}\right\} \quad . \qquad (118)$$

We have enough equations to express all μ's in terms of two of them. This was predictable since the representation 8 to which the electric current belongs occurs twice in the reduction of the antibaryon-baryon combination 8×8. Expressing all μ's in terms of the proton and neutron moments one finds

$$\left.\begin{aligned} \mu_{\Sigma^+} &= \mu_p \\ \mu_{\Xi^o} &= 2\mu_{\Lambda} = -2\mu_{\Sigma^o} = -\frac{2}{\sqrt{3}}\mu_{\Lambda\Sigma} = \mu_n \\ \mu_{\Sigma^-} &= \mu_{\Xi^-} = -\mu_n - \mu_p \end{aligned}\right\} \quad . \qquad (119)$$

The experimental value of μ_{Λ} agrees with this prediction within the large experimental error ($\gtrsim 30\%$) (see bibliography). By similar methods one can derive SU_3 predictions for electromagnetic mass differences, which are of second order in the electromagnetic interaction. The most interesting one is the formula given by Glashow and Coleman

$$m_n - m_p + m_{\Sigma^+} - m_{\Sigma^-} + m_{\Xi^-} - m_{\Xi^o} = 0 \qquad (120)$$

which is well verified by experiment. One can also combine mass splitting effects due to electromagnetic and semi-strong interactions.

The U-spin formalism can be profitably used for many other cases. An example is the photo-excitation of octet baryons leading to the baryon resonances in the decuplet. The relevant matrix element is

$$< b_{\alpha\beta\gamma} \mid j_{\mu}{}^{\kappa}_{\lambda} \mid b^{\nu}_{\mu} > \quad ,$$

where j_{μ} is the electromagnetic current. j_{μ} is an invariant for U spin, hence the transition $b^{\nu}_{\mu} \rightarrow b_{\alpha\beta\gamma}$ must conserve U spin. U spin is also very useful in the discussion of strong interaction processes.

7. Coupling between baryons and pseudoscalar mesons

Let p^{β}_{α} and b^{β}_{α} be the traceless tensors representing the 0^- meson and baryon octets, respectively. We want to write down the most general $\bar{b}\, b'\, p$ interaction which is SU_3 invariant. $\bar{b}$ transforms under SU_3 as the Hermitian conjugate tensor b^+. With b^+ and b' the following octets can be constructed

$$(b^+b')^{\beta}_{F\alpha} = b^{+\beta}_{\gamma} b'^{\gamma}_{\alpha} - b^{+\gamma}_{\alpha} b'^{\beta}_{\gamma} \tag{121}$$

$$(b^+b')^{\beta}_{D\alpha} = b^{+\beta}_{\gamma} b'^{\gamma}_{\alpha} + b^{+\gamma}_{\alpha} b'^{\beta}_{\gamma} - \frac{2}{3}\,\delta^{\beta}_{\alpha} b^{+\gamma}_{\delta} b'^{\delta}_{\gamma} \tag{122}$$

[see (32) and (27)]. The most general invariant $\bar{b}\, b\, p$ interaction has the form

$$\left[f(b^+b')^{\beta}_{F\alpha} + d(b^+b')^{\beta}_{D\alpha} \right] p^{\alpha}_{\beta} \quad . \tag{123}$$

The two terms are called F-type and D-type couplings, respectively. Experiment gives rough indications on the value of the f/d ratio. It appears to be of the order

$$\frac{1}{3} \lesssim \frac{f}{d} < 1 \quad . \tag{124}$$

Similarly, if $\Delta_{\alpha\beta\gamma}$ is the fully symmetric tensor representing the baryon decuplet, the most general SU_3 invariant $\bar{b}\,\Delta\, p$ interaction is given by

$$g\, b^{+\beta}_{\alpha}\, \Delta_{\beta\gamma\delta}\, p^{\gamma}_{\varphi}\, \epsilon^{\alpha\delta\varphi} \quad . \tag{125}$$

Hence, there is only one coupling constant governing all decays $\Delta \to b + p$. Experiment does not agree too well with this prediction.

II. SU_6 SYMMETRY

The SU_6 symmetry scheme was developed by Sakita, Radicati and Gürsey. The idea is to combine ordinary spin and SU_3 symmetry. This is a concept going back to the Wigner theory of supermultiplets in nuclear physics. [Wigner (1937) combined spin and isospin of the nucleon, introducing the group SU_4.] Only in the non-relativistic limit is it possible to separate the angular momentum of a particle into a spin and an orbital momentum part in an invariant way, independent of the co-ordinate system. It is for this reason that SU_6 has to be regarded as a non-relativistic theory. Despite many attempts, very little success has been met in the direction of relativistic extensions of SU_6.

1. Non-relativistic spin and the group SU_2

Following Pauli, non-relativistic spin is described by tensors in two-dimensional unitary space, the corresponding group being SU_2. As is well known, this group is locally isomorphic to the three-dimensional rotation group SU_3; it is its covering group.

The covariant spinors are

$$\chi = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} \ , \tag{126}$$

and the SU_2 transformations are

$$\chi \to \chi' = u\chi \quad \text{or} \quad \chi'_i = u^j_i \chi_j \qquad (i,j = 1,2)$$
$$u^+ u = 1, \quad \det u = 1 \ . \tag{127}$$

The contravariant spinors

$$\eta = (\eta^1 , \eta^2) \tag{128}$$

transform as

$$\eta \to \eta' = \eta u^{-1}; \quad \eta'^i = \eta^j (u^{-1})^i_j \ . \tag{129}$$

The Hermitian conjugate χ^+ of χ with components

$$(\chi^+)^i = (\chi_i)^* \tag{130}$$

transforms contravariantly.

The general tensor $\chi^{k\ell ..}_{ij..}$ follows the transformation law (127) for each covariant index, and (129) for each contravariant index, in the same manner as in SU_3 [see (10)].

The invariant tensors are

$$\delta^j_i = \begin{cases} 1 \text{ for } i=j \\ 0 \text{ for } i\neq j \end{cases}$$
$$\epsilon_{ij} = \epsilon^{ij} = \begin{cases} 0 \text{ for } i=j \\ 1 \text{ for } i=1,\ j=2 \\ -1 \text{ for } i=2,\ j=1 \end{cases} \tag{131}$$

The reduction of tensors is carried out, as was shown for SU_3, with the invariant tensors (131). There are three ways of reducing a tensor, i.e. of reducing its number of indices:

a) use the δ^j_i (i.e. form a trace);

b) use the ϵ^{ij} (i.e. take the antisymmetric part with respect to two lower indices);

c) use the ϵ_{ij} (i.e. take the antisymmetric part with respect to two upper indices).

The irreducible tensors are the ones which give zero for every reduction. They have the properties specified already in Chapter I-1 : All their traces vanish, and they are fully symmetric in upper and lower indices.

Because of the dimension two a further simplification arises: one can raise lower indices, or lower upper indices, with the help of the tensor ϵ^{ij} or ϵ_{ij}, respectively:

$$\chi_{ij..}^{k\ell..} \to \chi_{ij..}^{k\ell..} \cdot \epsilon^{im} = \tilde{\chi}_{j..}^{mk\ell..} \qquad (132)$$

$$\chi_{ij..}^{k\ell..} \to \chi_{ij..}^{k\ell..} \cdot \epsilon_{km} = \tilde{\chi}_{mij..}^{\ell..} \qquad (133)$$

Because of this fact all irreducible tensors with the same total number of indices (upper and lower) are equivalent to each other, i.e. can be transformed linearly into each other in SU_2 invariant way. Thus, a complete set of inequivalent irreducible tensors is given by

$$\chi_{i_1..i_n}; \text{ fully symmetric in } (i_1...i_n) \ . \qquad (134)$$

The dimension of (134), i.e. its number of independent components, is $n+1$. Expression (134) represents spin $s = n/2$, with its $2s+1 = n+1$ possible orientations.

For spin 1, (134) specializes to the symmetric tensor χ_{ij}. One can use equivalently the traceless mixed tensor

$$\eta_j^i = \chi_{kj} \cdot \epsilon^{ik}; \quad \eta_i^i = 0 \ . \qquad (135)$$

Verify that the traceless condition [2nd equation (135)] implies and is implied by the symmetry of χ_{ij}. This holds for arbitrary number of indices.

2. The group SU_6

We now introduce indices which combine spin and SU_3

$$A = (i,\alpha) \left\{ \begin{array}{l} i = 1,2 \text{ is the spin index} \\ \alpha = 1,2,3 \text{ is the } SU_3 \text{ index} \\ A = 1,..6 \end{array} \right\} . \qquad (136)$$

The basic elements of the SU_6 scheme are six-dimensional covariant spinors

$$t = \begin{pmatrix} t_1 \\ \vdots \\ t_6 \end{pmatrix} . \qquad (137)$$

The contravariant spinors are

$$s = (s^1..s^6) \ . \qquad (138)$$

The transformations are given by 6×6 matrices, which act on the spinors (137) and (138) in the same manner as in SU_2 and SU_3

$$t_A \to t'_A = u_A^B t_B \qquad (139)$$

$$s^A \to s'^A = s^B (u^{-1})_B^A \ . \qquad (140)$$

The general tensor $t_{ABC..}^{DE..}$ is transformed covariantly, see (139), with respect to the lower indices, and contravariantly, see (140), with respect to the upper indices [see (10)]. Its Hermitian conjugate is defined as in (14)

$$t_{DE..}^{+ABC..} = \left(t_{ABC..}^{DE..} \right)^{*} .$$

The invariant tensors are

$$\delta_A^B = \begin{cases} 1 \text{ for } A=B \\ 0 \text{ for } A\neq B \end{cases} \tag{141}$$

$$\epsilon_{A_1..A_6} = \epsilon^{A_1..A_6} \quad \begin{Bmatrix} \epsilon \text{ fully antisymmetric} \\ \epsilon_{1..6} = 1 \end{Bmatrix} . \tag{142}$$

Again one carries out the reduction of tensors by means of the invariant tensors (141) and (142). Due to the higher dimension now there are four reduction procedures:

a) use the δ_A^B (i.e. form a trace);

b) use the $\epsilon^{...}$ (i.e. take the antisymmetric part with respect to four or more lower indices);

c) use the $\epsilon_{...}$ (i.e. take the antisymmetric part with respect to four or more upper indices);

d) decompose into summands with definite symmetry character in upper and/or lower indices.

In the cases of SU_2 and SU_3, point (d) was done automatically by means of contractions with ϵ. Now it has to be done separately. Only (a), (b) and (c) reduce the number of indices: (d) does not, but separates a tensor in an invariant way in tensors with a smaller number of independent components. For example, it is seen that t_{AB} cannot be treated according to procedures (a), (b), or (c). Its reduction according to (d) is

$$t_{AB} = t_{AB}^{(+)} + t_{AB}^{(-)}, \quad t_{AB}^{(\pm)} = \frac{1}{2}\left(t_{AB} \pm t_{BA}\right) .$$

There are two cases which are at present important for physics, the tensors

$$t_A^B \text{ and } t_{ABC} .$$

We give their reduction:

1. Only reduction (a) applies to t_A^B. It separates the traceless part

$$t_A^B = \frac{1}{6}\delta_A^B t_C^C + \left(t_A^B - \frac{1}{6}\delta_A^B t_C^C\right) \tag{143}$$

written as

$$6 \times 6 = 1 + 35 . \tag{144}$$

The first term represents the invariant tensor δ. The second one, 35, is the traceless tensor which has 35 independent components.

2. t_{ABC} has 216 elements. Only the procedure (d) can be applied. The symmetry classes existing for three indices are:

i) t^s_{ABC} - fully symmetric in A, B and C, has 56 independent components; it therefore belongs to representation 56;

ii) t^a_{ABC} - fully antisymmetric in A, B and C, has 20 independent components; belongs to representation 20;

iii) t^m_{ABC} - of mixed symmetry, with the following characteristic properties: it is antisymmetric in A and B

$$t^m_{ABC} = - t^m_{BAC} \quad ,$$

and the sum over the circular permutations gives zero

$$t^m_{ABC} + t^m_{BCA} + t^m_{CAB} = 0 \quad .$$

t^m has 70 independent components and belongs to representation 70.

The most general tensor t_{ABC} can be decomposed in a sum of one fully symmetric tensor t^s, one fully antisymmetric t^a, and two distinct tensors of mixed symmetry t^m and t'^m. So one writes

$$6 \times 6 \times 6 = 20 + 56 + 70 + 70 \quad . \qquad (145)$$

Verify that the total number of dimensions is correct.

3. SU_6 multiplets of hadrons

We consider a covariant SU_6 spinor, its index written in the form (136)

$$t_{i,\alpha} \qquad (146)$$

i is the spin index, α the SU_3 index. Now we apply two transformations v^j_i and u^β_α, where v belongs to SU_2 and acts only on the spin indices, while u, belonging to SU_3, acts only on the SU_3 indices:

$$t_{i\alpha} \rightarrow t'_{i\alpha} = v^j_i u^\beta_\alpha t_{j\beta} \quad , \qquad (147)$$

where i,j = 1,2 and α,β = 1,2,3. This is a special type of SU_6 transformation w

$$t_A \rightarrow t'_A = w^B_A t_B \qquad (148)$$

with

$$w^B_A = v^j_i u^\beta_\alpha \quad , \qquad (149)$$

where A = (i,α) and B = (j,β).

It is clear that such direct products of SU_2 transformations with SU_3 transformations form a subgroup G of SU_6

$$SU_2 \otimes SU_3 \underset{\text{déf}}{=} G \subset SU_6 \quad . \qquad (150)$$

The dimension of G is 3×8, smaller than the dimension 35 of SU_6. It is clear that assuming SU_6 invariance of the very strong interactions will lead to predictions which go beyond the SU_2 and the SU_3 invariance predictions.

The SU_6 spinor or "sextet" t_A ("sextet" because it belongs to representation 6) is seen to transform under the subgroup G as a doublet for spin (one index) and as a triplet for SU_3 (one index). This $SU_2 \otimes SU_3$ structure of the sextet is usually written in the form

$$6 = (2,3) \quad . \tag{151}$$

We proceed to the tensor m_A^B, which, if traceless, forms the irreducible representation 35 of SU_6. What is its physical content in terms of SU_2 and SU_3? To find out we have to reduce this tensor with respect to the subgroup G. The tensor is

$$m_A^B = m_{i\alpha}^{j\beta} \text{ with } m_A^A = m_{i\alpha}^{i\alpha} = 0; \quad \left.\begin{array}{l} i,j = 1,2 \\ \alpha,\beta = 1,2,3 \end{array}\right\} \quad . \tag{152}$$

Reduction with respect to SU_2: one extracts a part with vanishing trace, which gives a singlet corresponding to spin 0, and leaves a traceless part, i.e. a triplet corresponding to spin 1. We write this out

$$m_{i\alpha}^{j\beta} = \left(m_{i\alpha}^{j\beta} - \frac{1}{2}\,\delta_i^j m_{k\alpha}^{k\beta} \right) + \frac{1}{2}\,\delta_i^j m_{k\alpha}^{k\beta} \quad . \tag{153}$$

spin 1 spin 0

Reduction of these two parts with respect to SU_3: the spin 0 part has automatically a vanishing trace for SU_3

$$\left(\frac{1}{2}\,\delta_i^j m_{k\alpha}^{k\beta} \right) \cdot \delta_\beta^\alpha = \frac{1}{2}\,\delta_i^j m_{k\alpha}^{k\alpha} = 0 \quad , \tag{154}$$

so it is an SU_3 octet. The spin 1 part still has a non-vanishing SU_3 trace. By separating it we obtain an SU_3 singlet and an SU_3 octet.

$$\left(m_{i\alpha}^{j\beta} - \frac{1}{2}\,\delta_i^j m_{k\alpha}^{k\beta} \right) = \underbrace{\left[\left(m_{i\alpha}^{j\beta} - \frac{1}{2}\,\delta_i^j m_{k\alpha}^{k\beta} \right) - \frac{1}{3}\,\delta_\alpha^\beta m_{i\gamma}^{j\gamma} \right]}_{SU_3\text{ octet}} + \underbrace{\frac{1}{3}\,\delta_\alpha^\beta m_{i\gamma}^{j\gamma}}_{SU_3\text{ singlet}} \quad . \tag{155}$$

In analogy to (151) we write the result (153) and (155) in the form

$$35 = (3,8) + (3,1) + (1,8) \quad . \tag{156}$$

The first number in each bracket refers to the spin, the second to the SU_3 representation. We find that representation 35 has exactly the $SU_2 \otimes SU_3$ content needed to accommodate the 8 pseudoscalar mesons and the 9 vector mesons classified in SU_3 [see (39) and (40)]. This is the first appealing property of SU_6. Note that in (156) the dimensions come out correctly:

$$35 = 3\times 8 + 3\times 1 + 1\times 8 \; .$$

Now we examine the irreducible tensor b_{ABC} of representation 56

$$b_{ABC} = b_{i\alpha,j\beta,k\gamma} \text{ fully symmetric in A,B,C} \tag{157}$$

and, most remarkably, we will find that it accommodates exactly the baryon octet and decuplet. The reduction of (157) with respect to G is

$$b_{ABC} = \chi^{s}_{ABC} + \chi^{m}_{ABC} \ , \tag{158}$$

where χ^{s}_{ABC} is the part fully symmetric in spin and SU_3 indices separately, and χ^{m}_{ABC} is the rest. Consider the fully symmetric part of (157) with respect to the spin indices: it is the expression

$$b^{s}_{i\alpha,j\beta,k\gamma} = \frac{1}{6}\left[b_{i\alpha,j\beta,k\gamma} + b_{j\alpha,k\beta,i\gamma} + b_{k\alpha,i\beta,j\gamma} + b_{i\alpha,k\beta,j\gamma} + b_{k\alpha,j\beta,i\gamma} + b_{j\alpha,i\beta,k\gamma} \right] . \tag{159}$$

Since b_{ABC} is fully symmetric for exchange of pairs $(i\alpha)$, $(j\beta)$, $(k\gamma)$, b^{s} is symmetric not only in the Latin spin indices but in the Greek SU_3 indices as well. So,

$$b^{s}_{i\alpha,j\beta,k\gamma} = \chi^{s}_{ABC} \ . \tag{160}$$

With respect to SU_2, χ^{s}_{ABC} belongs to representation 4 (spin 3/2). With respect to SU_3, χ^{s}_{ABC} with three lower indices belongs to representation 10 (see the list of representations in Chapter I-1). Hence, the $SU_2 \otimes SU_3$ structure of χ^{s} is (4,10) and it can accommodate the spin 3/2 baryon decuplet.

The rest $\chi^{m}_{i\alpha,j\beta,k\gamma}$ is of mixed symmetry. We first consider the SU_2 behaviour. For reasons of simplicity we now omit in most equations the SU_3 indices. The fully symmetric part in i,j,k has already been extracted, so symmetrization has to give zero

$$\sum_{\text{perm }(i,j,k)} \chi^{m}_{ijk} = 0 \ . \tag{161}$$

Since i, j, and k can take only the values 1 and 2, the antisymmetrization has to give zero too, because always two of the three indices i,j,k are equal.

$$\sum_{\text{perm }(i,j,k)} (-1)^{p} \chi^{m}_{ijk} = 0 \ , \tag{162}$$

(where p = parity of permutation). From (161) and (162) it follows that the sum over the even permutations as well as the sum over the odd permutations are zero

$$\sum_{\text{even perm }(i,j,k)} \chi^{m}_{ijk} = \sum_{\text{odd perm }(i,j,k)} \chi^{m}_{ijk} = 0 \ . \tag{163}$$

χ^m can have mixed symmetry only in i,j,k. Instead of χ^m_{ijk} we consider three spin ½ parts:

$$\left.\begin{array}{l} \epsilon^{ij}\chi^m_{ijk} \overset{=}{\text{def}} \zeta_k \\ \epsilon^{ij}\chi^m_{jki} \overset{=}{\text{def}} \eta_k \\ \epsilon^{ij}\chi^m_{kij} \overset{=}{\text{def}} \xi_k \end{array}\right\} . \qquad (164)$$

Instead of χ^m_{ijk} we may also write the equations (164) with the original tensor b_{ABC}. Since the ϵ^{ij} only affects the antisymmetric part, the addition of the fully symmetric χ^s_{ABC} (158) does not change the values of ζ,η,ξ.

ζ_k, η_k and ξ_k are linearly dependent

$$\zeta_\kappa + \eta_\kappa + \xi_\kappa = \epsilon^{ij}\left(\sum_{\text{even perm}} \chi^m_{ijk}\right) = 0 \qquad (165)$$

that is, there are only two linearly independent spin ½ parts in b_{ABC}. This was predictable since combination of 3 spins ½ gives spin ½ only twice. We may reverse the equations (164) by multiplication with ϵ_{ij} and renaming of indices, which gives

$$\epsilon_{ij}\zeta_k + \epsilon_{ki}\eta_j + \epsilon_{jk}\xi_i = 3\chi^m_{ijk} - \sum_{\text{odd perm}} \chi^m_{ijk} \qquad (166)$$

the last term being zero because of equation (163).

We replace χ^m by b in (164) and insert (164) into (166):

$$\epsilon_{ij}\epsilon^{mn}b_{m\alpha,n\beta,k\gamma} + \epsilon_{ki}\epsilon^{mn}b_{n\alpha,j\beta,m\gamma} + \epsilon_{jk}\epsilon^{mn}b_{i\alpha,m\beta,n\gamma} = 3\chi^m_{i\alpha,j\beta,k\gamma} . \qquad (167)$$

The three terms on the left-hand side are essentially the same because of full symmetry of b for exchange of pairs of indices

$$\left.\begin{array}{l} \epsilon^{mn}b_{n\alpha,j\beta,m\gamma} = \epsilon^{mn}b_{m\gamma,n\alpha,j\beta} \\ \epsilon^{mn}b_{i\alpha,m\beta,n\gamma} = \epsilon^{mn}b_{m\beta,n\gamma,i\alpha} \end{array}\right\} . \qquad (168)$$

In other words they are all expressible in the single tensor

$$\epsilon^{mn}b_{m\alpha,n\beta,k\gamma} . \qquad (169)$$

What is the SU_3 character of this tensor? **Formula (169) contains only the antisymmetric** part in m and n. Hence, it has to be antisymmetric in α and β. Therefore no information is lost if we contract these two indices with an SU_3 $\epsilon^{\alpha\beta\gamma'}$, giving

$$\epsilon^{mn}b_{m\alpha,n\beta,k\gamma} \cdot \epsilon^{\alpha\beta\gamma'} \overset{=}{\text{def}} \sqrt{2}\,\psi_{k;\gamma}^{\gamma'} . \qquad (170)$$

The $\sqrt{2}$ is introduced for later convenience, see (173). We now prove that the SU_3 trace of $\psi_{k;\gamma}^{\ \ \gamma'}$ is zero, so that (170) is an SU_3 octet. We have

$$\psi_{k;\gamma}^{\ \ \gamma} = \epsilon^{mn} b_{m\alpha,n\beta,k\gamma} \epsilon^{\alpha\beta\gamma} \quad . \tag{171}$$

$\epsilon^{\alpha\beta\gamma}$ picks out the antisymmetric part in α, β, γ. Due to the symmetry of b in A, B, C, this part is antisymmetric in m, n, k, too. This however is zero, as we saw in (162), because of the dimension 2 of SU_2. Hence, (171) vanishes. Equation (170) can be inverted by means of (31). The result is

$$\epsilon^{mn} b_{m\alpha,n\beta,k\gamma} = \frac{1}{\sqrt{2}} \epsilon_{\alpha\beta\gamma'} \psi_{k;\gamma}^{\ \ \gamma'} \quad .$$

Putting this into (167) gives

$$\chi^m_{i\alpha,j\beta,k\gamma} = \frac{1}{\sqrt{18}} \left[\epsilon_{ij} \epsilon_{\alpha\beta\gamma'} \psi_{k;\gamma}^{\ \ \gamma'} + \epsilon_{ki} \epsilon_{\gamma\alpha\beta'} \psi_{j;\beta}^{\ \ \beta'} + \epsilon_{jk} \epsilon_{\beta\gamma\alpha'} \psi_{i;\alpha}^{\ \ \alpha'} \right] , \tag{172}$$

where the coefficient is such as to give the same normalization for χ^m and ψ

$$\sum_{A,B,C} | \chi^m_{ABC} |^2 = \sum_{k,\gamma,\delta} | \psi_{k;\gamma}^{\ \ \delta} |^2 \quad . \tag{173}$$

Since $\psi_{k;\gamma}^{\ \ \gamma'}$ is a spin $\frac{1}{2}$ octet, the same property holds for χ^m. We now have the total $SU_2 \otimes SU_3$ content of 56:

$$56 = (4,10) + (2,8) \tag{174}$$

which indeed allows **allows the accommodation in 56 of the baryon octet (37) and decuplet (38).**

SU_6 symmetry can be used to make predictions going beyond those of SU_3, with the restriction, however, that it has to be applied to non-relativistic situations. Much work has been done recently on relativistic extensions of SU_6. All schemes proposed have the property that the higher symmetry is violated by the terms containing the energy-momentum of virtual particles in intermediate states (at least when the number of virtual particles is two or more), and it is quite unclear at present what the validity of the corresponding relativistic theories may be.

Predictions on mass formulae are unambiguous in one case only: the baryon 56-plet, for which one obtains a mass formula going beyond (92):

$$m = a_0 + b_0 Y + c_0 \left(I(I+1) - \frac{1}{4} Y^2 \right) + d_0 \ J(J+1) \ , \tag{175}$$

where J is the spin, and a_0, b_0, c_0 and d_0 are constants in the 56-plet. This formula is well verified. For mesons one can, with some ad hoc assumptions, derive the relation

$$m_K^2 - m_\pi^2 = m_{K*}^2 - m_\rho^2 \tag{176}$$

which is also in good agreement with the facts. Also the mixing angle (43) between ω and φ

can be derived from the theory. The idea is to use a new SU_2 subgroup of SU_6 defined by

$$t_{i\alpha} \to t'_{i\alpha} = \begin{cases} t_{i\alpha} & \text{for } i = 1,2;\ \alpha = 1,2 \\ v_i{}^j t_{j\alpha} & \text{for } i = 1,2;\ \alpha = 3 \end{cases} \tag{177}$$

("λ quark spin" group of Lipkin). As seen from (44) the physical ω_0 is invariant for this subgroup, whereas the physical φ_0 belongs to its representation 3.

Most further predictions of SU_6 symmetry deal with the strong, electromagnetic and weak interactions of baryons. Some of them are given in the next section.

4. SU_6 predictions for baryon interactions

The interactions of baryons with mesons, photons and leptons all involve a baryon vertex of form $\bar{b}\, b'$. The SU_6 structure of the most general $\bar{b}\, b'$ system is

$$b^{+\,A'B'C'}\, b'_{ABC} \quad . \tag{178}$$

The basic assumption made is that (with one exception referring to vector mesons, see below) all interactions mentioned above involve the part of (178) which belongs to the adjoint representation 35 of SU_6. This is supposed to hold in the non-relativistic limit. Expression (178) contains 35 only once, namely in the form

$$t_A^B = b^{+BCD} b'_{ACD} - \frac{1}{6}\, \delta_A^B \left(b^{+ECD} b'_{ECD} \right) \quad . \tag{179}$$

We shall give the predictions resulting from the above basic assumption for the case of the baryon octet. (The cases of the decuplet couplings and of the octet-decuplet transitions can be treated in the same manner.) Since the baryons are then contained in χ^m_{ABC}, see (158) and (172), we may insert the latter equation into (179). Using the special case where spin and SU_3 dependences factor out in ψ,

$$\psi_{k;\gamma}{}^{\gamma'} = \eta_k \cdot \zeta_\gamma^{\gamma'} \tag{179a}$$

and introducing the abbreviation

$$\eta^+\eta = \eta^{+i}\eta_i ; \quad \zeta^+\zeta = \zeta_\beta^{+\alpha}\zeta_\alpha^\beta$$

we get

$$\left(t_A^B \right)_{\text{octet}} = \frac{1}{18}\Big[3\delta_i^j \cdot \eta^+\eta' \cdot (\zeta^+\zeta')^\beta_{F\alpha} + 4\left(\eta^{+j}\eta'_i - \frac{1}{2}\,\delta_i^j \cdot \eta^+\eta'\right)(\zeta^+\zeta')^\beta_{F\alpha} + 6\left(\eta^{+j}\eta'_i - \frac{1}{2}\,\delta_i^j \cdot \eta^+\eta'\right)(\zeta^+\zeta')^\beta_{D\alpha} + 2\left(\eta^{+j}\eta'_i - \frac{1}{2}\,\delta_i^j \cdot \eta^+\eta'\right)\delta_\alpha^\beta \cdot \zeta^+\zeta' \Big] \quad . \tag{180}$$

For the F and D couplings see (32) and (27).

Take first the baryon meson coupling corresponding to the transitions $b \leftrightarrow b' + m$. In the non-relativistic limit, the 0^- and 1^- meson couplings

$$i\varphi\bar{b}\gamma_5 b', \quad -i\varphi_\mu \bar{b}\gamma_\mu b' \tag{181}$$

take the limiting forms

$$\frac{1}{2m_b}\vec{\nabla}\varphi \cdot (b^+\vec{\sigma}b'), \quad \varphi_0 b^+ b' , \tag{182}$$

with φ, φ_μ the 0^-, 1^- meson fields, and m_b the baryon mass supposed to have a unique value. $\vec{\sigma}$ are the Pauli spin matrices, so that the actual spin is $\frac{1}{2}\vec{\sigma}$. Units are such that $\hbar = c = 1$. (182) suggests taking the spin 0 part of (180) for the vector meson coupling, and the spin 1 part for the 0^- meson coupling. The latter is given by the second and third lines of (180) and therefore gives the value $\frac{2}{3}$ to the f/d ratio defined in Chapter I-7. The vector meson couplings are given by the first line of (180); they affect the 1^- meson octet only and are of F type (as is usually believed to be the case on theoretical grounds). If one decides somewhat arbitrarily to attribute the same norm to $(\vec{\sigma}\cdot\vec{\nabla}\varphi)/2m_b$ and φ_0, one can calculate the 1^- meson coupling strength from the 0^- meson coupling constant by using the ratio $\frac{3}{4}$ between the coefficients of the first and second lines of (180). In doing such a calculation one uses the identity

$$\eta^{+j}\eta'_i - \frac{1}{2}\delta^j_i \eta^+\eta' = (\eta^+ \vec{\sigma}\eta')\cdot(\vec{\sigma})^j_i , \tag{183}$$

which follows from the usual definition of the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{184}$$

The result of this calculation is compatible with our rough knowledge of vector meson couplings.

The coupling of the vector meson singlet φ_0^0 with the baryons falls outside representation 35, as is clear if one notes that in the non-relativistic limit this coupling is a singlet both for spin and for SU_3; this coupling involves the baryon invariant

$$b^{+ABC} b'_{ABC} \tag{185}$$

and has an independent coupling strength. One can construct additional arguments, however, to derive this coupling from (180). They, in fact, state that the $\bar{p}p\varphi_0$, $\bar{n}n\varphi_0$ couplings vanish (p = proton, n = neutron, φ_0 = physical φ meson), because $\bar{p}p$ and $\bar{n}n$ are invariant for the subgroup (177) (no SU_3 index 3 appears in the corresponding b_{ABC}) whereas φ_0 is not.

Finally, the last line of (180) is not involved in the meson-baryon coupling discussed here; it could be used to describe baryon coupling to a 0^- meson which would be an SU_3 singlet, possibly the X_0 meson of mass 960 MeV.

Although the physical applicability of the non-relativistic limit is doubtful in this case, one could repeat the above considerations for the $\bar{b}b' \leftrightarrow m$ transitions. The spin properties of the baryon vertex are now reversed in the non-relativistic limit; 0^- mesons become coupled to the spin 0 part of (180), and the 1^- mesons to the spin 1 part. It seems impossible to check the resulting predictions against experimental facts.

The coupling of baryons to the electromagnetic field is also assumed to involve exclusively the irreducible tensor (179). This is a natural assumption because the electromagnetic current, being conserved, is naturally included in the adjoint representation of the symmetry group. The electric charge in terms of SU_3 is given by the operator (54) with

$$h = \lambda_Q = \left(\lambda_Q\right)_\alpha^\beta = \begin{pmatrix} 2/3 & 0 & 0 \\ 0 & -1/3 & 0 \\ 0 & 0 & -1/3 \end{pmatrix} . \qquad (186)$$

The baryon electromagnetic coupling is given by the contraction

$$3\lambda_{Q\beta}^{\alpha} \; t_{i\alpha}^{j\beta} \; . \qquad (187)$$

The factor 3 is justified later. Expression (187) contains a spin 0 part (obtained by taking the trace over i,j) which should describe the electric charge of the baryons, and a remaining part of spin 1 describing the magnetic moment. This identification is easily verified for spin ½ baryons by taking the electromagnetic interaction

$$-ie\bar{b}\gamma_\mu b' A_\mu + \frac{1}{2i}\kappa \bar{b}\left(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu\right) b' \partial_\nu A_\mu \; , \qquad (188)$$

(e = electric charge; κ = anomalous magnetic moment) and reducing it to the non-relativistic limit

$$eb^+b'A_0 - \left(\frac{e}{2m_b} + \kappa\right)(b^+\vec{\sigma}\, b')\cdot(\vec{\nabla}\times\vec{A}),\; A_4 = iA_0 \qquad (189)$$

with the same notation as in (182).

Taking into account the factor 3 in (187) the baryon charges obtained in this way are identical to the charges given by the Gell-Mann-Nishijima formula. This is immediately verified from (187) and (179) for the baryon decuplet. The same verification is obtained from (187) and (180) for the octet; it relies in this case on the important fact that, for the charge operator $\Lambda(\lambda_Q)$ and more generally for any operator $\Lambda(h)$ as defined in (45), the matrix element $< \zeta \mid \Lambda(h) \mid \zeta' >$, when ζ and ζ' are octet members, takes the form

$$h_\beta^\alpha(\zeta^+\zeta')_{F\alpha}^\beta \qquad (190)$$

corresponding to F-type coupling; this form indeed occurs in the spin 0 part of (180).

The spin 1 part of (187), when used to describe the magnetic moments, allows us to express the magnetic moments for all baryon octet and decuplet states, as well as all transition moments, in terms of a single one of these quantities. The most obvious ratio to calculate in order to test this prediction is μ_n/μ_p, the ratio of neutron to proton magnetic moments. The result is

$$\mu_n/\mu_p = -\frac{2}{3} \qquad (191)$$

in excellent agreement with experiment ($\mu_n = -1.91$, $\mu_p = 2.79$ in proton magnetons $e/2m_p$).

An SU_6 analysis of electromagnetic mass splittings has also been carried out. It is not discussed here.

Leptonic weak interactions of baryons can be discussed along the same lines. The matrix element of the weak current between spin $\frac{1}{2}$ baryons b and b'

$$< b \mid j_\mu \mid b' > = < b \mid j_\mu^V + j_\mu^A \mid b' > \tag{192}$$

has the well-known non-relativistic limit (we neglect mass differences)

$$\text{vector current:} \qquad < b \mid j_\mu^V \mid b' > = \begin{cases} 0 & \text{for } \mu=1,2,3 \\ i f_V b^+ b' & \text{for } \mu=4 \end{cases} \tag{193}$$

$$\text{axial vector current:} \quad < b \mid j_\mu^A \mid b' > = \begin{cases} -f_A b^+ \vec{\sigma} b' & \text{for } \mu=1,2,3 \\ 0 & \text{for } \mu=4 \end{cases} \tag{194}$$

This is identical to the non-relativistic limit of $i\bar{b}(f_V\gamma_\mu + f_A\gamma_\mu\gamma_5)b$. Experimentally, for $b = p$ and $b' = n$, one has

$$f_A/f_V = 1.2 \quad . \tag{195}$$

In SU_6, one postulates that in the non-relativistic limit $i < b \mid j_\mu^V \mid b' >$ and $< b \mid j_\mu^A \mid b' >$ become respectively proportional to the spin 0 and spin 1 parts of a matrix analogous to (187):

$$\lambda_{W\beta}^{\ \alpha} \, t_{i\alpha}^{j\beta} \tag{196}$$

where $\lambda_{W\beta}^{\ \alpha}$ is an SU_3 matrix of the form proposed by Cabibbo

$$\lambda_W = \begin{pmatrix} 0 & \cos\vartheta & \sin\vartheta \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad . \tag{197}$$

The angle ϑ is known from experiment to have a value $\vartheta \simeq 0.26$ radians. Among the resulting predictions we find from (180) that the vector leptonic coupling to baryons is of F type (as is expected from the conserved vector current hypothesis), that the axial vector has the d/f ratio $\frac{3}{2}$ (which is in good agreement with the experimental value of 1.7 ± 0.35), and that the ratio f_A/f_V is $(4+6)/(3\times 2) = \frac{5}{3}$ instead of (195). The latter result is obtained by introducing in (180) the p^+n coupling matrices

$$(p^+n)_F = (p^+n)_D = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad . \tag{198}$$

One cannot claim, however, that the sign of f_A/f_V is predicted from theory; only the absolute value is. Many other predictions are possible and have been worked out; for example, the amplitudes of all neutrino reactions of type

$$\nu + b \rightarrow \ell + b^* \tag{199}$$

where b and b* are members of the baryon octet and decuplet, respectively, and ℓ is a charged lepton.

An SU_6 treatment of non-leptonic decays of hyperons has been carried out. It is not presented here.

5. Definition of quarks

In view of the great success of SU_3 and SU_6, the problem has been raised of whether particles might exist which would belong to the lowest representations of these groups. These particles would have spin $1/2$ and form an SU_3 triplet. Gell-Mann has called them quarks. From the Gell-Mann-Nishijima relation (54-56) their charges would be

$$2/3,\ -1/3,\ -1/3 \qquad \text{in representation } 3 \text{ (quarks)}$$

$$-2/3,\ 1/3,\ 1/3 \qquad \text{in representation } \bar{3} \text{ (antiquarks).}$$

Mesons could be regarded as bound states of a quark and an antiquark, and baryons as bound states of three quarks. This would imply that quarks have baryon number $1/3$. Experimentally, no quarks have been found so far.

BIBLIOGRAPHY

The following bibliography presents a selected list of papers dealing with the main items treated in the lectures. It is very far from being a complete bibliography of SU_3 and SU_6.

I. SU_3 SYMMETRY

1. General

An excellent collection of papers on SU_3 have been reprinted in M. Gell-Mann and Y. Ne'eman, The eightfold way, W.A. Benjamin, Inc. (New York, Amsterdam) 1964.

Most of the following papers are contained in this work:

M. Gell-Mann, California Institute of Technology Synchrotron Laboratory, Report CTSL-20 (1961); unpublished.
Y. Ne'eman, Nuclear Phys. 26, 222 (1961).
S. Okubo, Progr.Theoret.Phys. (Kyoto) 27, 949 (1962).
J.J. de Swart, Revs.Modern Phys. 35, 916 (1963).

2. Particle assignments

J.J. Sakurai, Phys.Rev.Letters 9, 472 (1962).
S.L. Glashow and A.H. Rosenfeld, Phys.Rev.Letters 10, 192 (1963).

3. Electromagnetic properties and U spin

S. Coleman and S.L. Glashow, Phys.Rev.Letters 6, 423 (1961).
S. Meshkov, C.A. Levinson and H.J. Lipkin, Phys.Rev.Letters 10, 361 (1963).
H.J. Lipkin, Unitary symmetry for pedestrians. A seminar lecture. Argonne National Laboratory, Physics Division, Informal report (August 1963).
G. Feldman and P.T. Matthews, U spin and the mass splitting of unitary multiplets, Ann.Phys. (to be published).

Measurement of Λ magnetic moment:

Lausanne-Bristol-CERN-Munich Collaboration, Physics Letters 15, 66 (1965), $= -0.5 \pm 0.28$ nucleon magneton.
D.A. Hill, K.K. Li, E.W. Jenkins, T.F. Kycia and H. Ruderman, Phys.Rev.Letters 15, 85 (1965), $= -0.77 \pm 0.27$.

4. Meson-baryon interaction

R.E. Cutkosky, Ann.Phys. 23, 415 (1963).
A.W. Martin and K.C. Wali, Phys.Rev. 130, 2455 (1963), and Nuovo Cimento 31, 1324 (1964).

5. Quarks

M. Gell-Mann, Physics Letters 8, 214 (1964).
G. Zweig, CERN Preprints No. 8182/TH.401 and No. 8419/TH.412 (1964), unpublished.

II. SU_6 SYMMETRY

1. General

F. Gürsey and L.A. Radicati, Phys.Rev.Letters 13, 173 (1964).
B. Sakita, Phys.Rev. 136, B.1756 (1964).
F. Gürsey, A. Pais and L.A. Radicati, Phys.Rev.Letters 13, 299 (1964).

2. Mass formulae

A. Pais, Phys.Rev.Letters 13, 175 (1964).
M.A.B. Bég and V. Singh, Phys.Rev.Letters 13, 418 (1964).

3. Interactions

F. Gürsey, A. Pais and L.A. Radicati, Phys.Rev.Letters 13, 299 (1964).
M.A.B. Bég, B.W. Lee and A. Pais, Phys.Rev.Letters 13, 514 (1964).
B. Sakita, Phys.Rev.Letters 13, 643 (1964).
H.J. Lipkin, Phys.Rev.Letters 13, 590 (1964).

Experimental result on F/D for weak interactions:

W. Willis et al., Phys.Rev.Letters 13, 291 (1964).

4. Relativistic extensions of SU_6

Simultaneously and independently of each other, many authors or groups of authors proposed relativistic extensions of SU_6. All these extensions have the defect of being intrinsically broken symmetry schemes in which the symmetry breaking may well become overwhelming in the relativistic region. The difficulties are closely connected with the requirements of unitarity and crossing symmetry. We only quote a few of the early papers dealing with the relativistic extensions.

M.A.B. Bég and A. Pais, Phys.Rev.Letters 14, 509 (1965).
R. Delbourgo, A. Salam and J. Strathdee, Proc.Roy.Soc.(London) A 284, 146 (1965).
W. Rühl, Physics Letters 14, 346 (1965); Nuovo Cimento 37, 301 and 319 (1965).
B. Sakita and K.C. Wali, Phys.Rev.Letters 14, 404 (1965).