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Some Solutions of the Classical Isotopic Gauge Field Equations

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I. Introduction

It was pointed out[1] a number of years ago that an *isotopic gauge* can be defined in analogy with the usual electromagnetic gauge, and that the concept of local isotopic gauge invariance leads to a gauge field $\mathbf{b}_\mu$. The equations describing $\mathbf{b}_\mu$ in interaction with any source of isotopic spin are essentially uniquely determined, much like the equations describing the electromagnetic field A_μ in interaction with the electric charge. In the absence of any external sources of isotopic spin, the $\mathbf{b}_\mu$ field interacts with itself, since the $\mathbf{b}_\mu$ field possesses an isotopic spin and hence is self-generating. In this latter characteristic, the $\mathbf{b}_\mu$ field is different from the electromagnetic field, which is described by linear equations in the absence of other fields. (The nonlinear equations describing the self-generating $\mathbf{b}_\mu$ field is in some respects[2] similar to the equations of general relativity.)

We seek in this paper to find a solution of the (unquantized) $\mathbf{b}_\mu$ field in the absence of other interacting fields. Our aim is then similar to that of Born and Infeld,[3] except that they started with equations which were written down on a more or less *ad hoc* basis.

II. A Special Type of Solution

The equations for the $\mathbf{b}_\mu$ field are[1]

$$\mathbf{f}_{\mu\nu} = \mathbf{b}_{\mu,\nu} - \mathbf{b}_{\nu,\mu} - \mathbf{b}_\mu \times \mathbf{b}_\nu \tag{1}$$

$$\mathbf{f}_{\mu\nu,\nu} + \mathbf{b}_\nu \times \mathbf{f}_{\mu\nu} = 0 \tag{2}$$

$$\mathbf{b}_{\mu,\mu} = 0 \tag{3}$$

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We have chosen units for $\mathbf{b}_\mu$ so that the coupling constant ε is equal to $\frac{1}{2}$. We further adopt the convention that $x_4 = ict$, and that $_{,\mu}$ means differentiation with respect to x_μ. Also subscripts μ or ν run 1 to 4, while others run 1 to 3.

Solutions in which the $\mathbf{b}_\mu$ field lies in one isotopic direction are easily found, since for them the nonlinear terms vanish, and the $\mathbf{b}_\mu$ field equations reduce to that for the free electromagnetic field. But such solutions are of no interest to us here.

To find some special solutions of Eqs. (1)–(3) we look for a static case, so that

$$\mathbf{b}_4 = 0, \qquad \mathbf{b}_{i,4} = 0 \tag{4}$$

We write the components of $\mathbf{b}_\mu$ as $b_{\mu\alpha}$, where $\alpha = 1, 2, 3$ designates the isotopic spin index.

We shall seek for a solution of the following form

$$b_{11} = b_{22} = b_{33} = 0, \qquad b_{12} = -b_{21} = x_3 f(r)/r, \text{ etc.}$$

i.e.,

$$b_{i\alpha} = \varepsilon_{i\alpha\gamma} x_\gamma f(r)/r \tag{5}$$

where r is the length of (x_1, x_2, x_3). Equation (3) is then automatically satisfied and (1) and (2) reduce to

$$f'' + \frac{2}{r} f' - (1 + rf)\left(\frac{2f}{r^2} + \frac{f^2}{r}\right) = 0 \tag{6}$$

where the prime means d/dr.

Writing

$$\Phi(r) = 1 + rf(r) \tag{7}$$

one has

$$r^2\Phi'' - \Phi(\Phi^2 - 1) = 0 \tag{8}$$

To study this equation we observe that putting

$$r = e^\xi \tag{9}$$

one has

$$\frac{d^2\Phi}{d\xi^2} - \frac{d\Phi}{d\xi} = \Phi(\Phi^2 - 1) \tag{10}$$

or

$$\frac{d\Phi}{d\xi} = \psi \tag{11a}$$

$$\frac{d\psi}{d\xi} = \psi + \Phi(\Phi^2 - 1) \tag{11b}$$

In the Φ–ψ plane (phase plane), $(d\Phi/d\xi,\ d\psi/d\xi)$ defines a vector field given by the right-hand side of (11a, b). The curves that are tangent to this vector field are the solutions we desire. The vector vanishes at exactly three points: $(\Phi, \psi) = (0, 0), (1, 0)$, and $(-1, 0)$.

It is not difficult to study the vector field. One finds that there are only five solutions of Eqs. (11a, b) which are finite for all $0 < r < \infty$:

(*a*) $\Phi = 0$ (12a)

(*b*) $\Phi = +1$ (12b)

(*c*) $\Phi = -1$ (12c)

(*d*) $\Phi = 1 - \dfrac{c}{r} + O\left(\dfrac{1}{r^2}\right)$ as $r \to \infty$, $(c > 0)$; $\Phi \to 0$ as $r \to 0$; (12d)

(*e*) The same as (*d*) except Φ changes sign. (12e)

To discuss the meaning of these solutions, we split the 6-vector $\mathbf{f}_{\mu\nu}$ into "electric" and "magnetic" components:

$$\mathbf{E}_j = i\mathbf{f}_{j4}, \qquad -\varepsilon_{ijk}\mathbf{H}_k = \mathbf{f}_{ij}$$

For the special type of solution satisfying (5),

$$\begin{aligned} \mathbf{E}_j &= 0, \\ -H_{j\alpha} &= -\delta_{j\alpha}\Phi'/r + x_j x_\alpha r^{-3}[\Phi' - r^{-1}\Phi^2 + r^{-1}] \\ &= -\delta_{j\alpha}\psi/r^2 + x_j x_\alpha r^{-4}[\psi - \Phi^2 + 1] \end{aligned} \tag{13}$$

It is clear that for (12b) and (12c), $\mathbf{H}_j = 0$ and the solutions are merely complicated ways of describing the vacuum $\mathbf{f}_{\mu\nu} = 0$. The nontrivial solutions are thus the three tabulated below.

1. Solution (12a)

$$\begin{aligned} \Phi &= 0, & f &= -r^{-1} \\ E_{j\alpha} &= 0, & H_{j\alpha} &= -x_j x_\alpha r^{-4} \end{aligned} \tag{14}$$

2. Solution (12d)

This solution has the following asymptotic behavior:

$r \to \infty$,

$$\begin{aligned} \Phi &= 1 - \frac{c}{r} + O\left(\frac{1}{r^2}\right), & f &= -\frac{c}{r^2} + O\left(\frac{1}{r^3}\right) \\ E_{j\alpha} &= 0, & H_{j\alpha} &= \frac{-c}{r^3}\left[-\delta_{j\alpha} + \frac{3x_j x_\alpha}{r^2}\right] + O\left(\frac{1}{r^4}\right) \end{aligned} \tag{15}$$

$r \to 0$,

$\Phi \to 0$, $\psi \to 0$, as oscillatory functions of r with minima and maxima $= O(r^{1/2})$. (16)

$$E_{j\alpha} = 0, \qquad H_{j\alpha} = -x_j x_\alpha r^{-4} + O(r^{-3/2})$$

This solution is actually a one-parameter family of solutions with the parameter c of the dimension of a length. Numerical results are given in Table I.

TABLE I. Φ as a Function of r for Solution (12d),[a] with $c = 1$

r	Φ
∞	1
9.880×10	9.898×10^{-1}
1.095×10	9.136×10^{-1}
3.297	7.510×10^{-1}
1.098	4.584×10^{-1}
6.141×10^{-2}	-9.229×10^{-2} = min.
1.617×10^{-3}	1.498×10^{-2} = max.
4.296×10^{-5}	-2.441×10^{-3} = min.
1.142×10^{-6}	3.980×10^{-4} = max.
3.036×10^{-8}	-6.489×10^{-5} = min.
8.070×10^{-10}	1.058×10^{-5} = max.
2.15×10^{-11}	-1.725×10^{-6} = min.

[a] This table is obtained by numerical integration. As $r \to 0$, Φ oscillates with damped amplitude. The first four minima and first three maxima are tabulated.

3. *Solution (12e)*

This solution has the following asymptotic behavior:

$r \to \infty$,

$$\Phi = -1 + \frac{c}{r} + O\left(\frac{1}{r^2}\right), \qquad f = \frac{-2}{r} + \frac{c}{r^2} + O\left(\frac{1}{r^3}\right)$$

$$E_{j\alpha} = 0, \qquad H_{j\alpha} = \frac{-c}{r^3}\left[\delta_{j\alpha} + \frac{x_j x_\alpha}{r^2}\right] + O\left(\frac{1}{r^4}\right) \tag{17}$$

$r \to 0$,

Φ, $\frac{f}{r} \to 0$ as oscillatory functions of r with maxima and minima $= O(r^{1/2})$.

$$E_{j\alpha} = 0, \qquad H_{j\alpha} = -x_j x_\alpha r^{-4} + O(r^{-3/2}) \tag{18}$$

This solution is again a one-parameter family of solutions with the parameter c of the dimension of a length.

Notice that the three types of solutions (12a), (12d), and (12e) share the same dominant asymptotic form as $r \to 0$.

III. Energy

The Hamiltonian of the **b** field was given in reference 1. For the present cases,

$$H = \frac{1}{4}\int H_{j\alpha} H_{j\alpha}\, d^3x = \pi \int_0^\infty [2\Phi'^2 + (\Phi^2 - 1)^2 r^{-2}]\, dr \tag{19}$$

This integral is divergent at $r \simeq 0$.

If one replaces $\Phi \to \Phi + \delta\Phi$ in (19) for variations $\delta\Phi$ which are zero in an interval $r = 0 \to r_0$, (19) gives a stationary value at the solutions Φ discussed in Sections II-1, II-2, and II-3, above [cf. Eq. (8)]. The second variation gives

$$\delta^2 H = \pi \int_{r_0}^\infty [2(\delta\Phi')^2 + (\delta\Phi)^2(6\Phi^2 - 2)r^{-2}]\, dr \tag{20}$$

For sufficiently small r_0, $6\Phi^2 - 2 \sim -2$ near $r = r_0$ and (20) is not positive definite.

IV. Source

Are the solutions above really sourceless? The answer is clearly yes except at $r = 0$. For $r = 0$, however, the solutions are singular and this question will have to be examined in greater detail.

Another way of asking the same question is whether the solutions exhibited above do satisfy the field equation (2). To discuss this we define

$$-\mathbf{J}_\mu = \mathbf{f}_{\mu\nu,\nu} + \mathbf{b}_\nu \times \mathbf{f}_{\mu\nu} \tag{21}$$

Equation (5) leads to

$$J_{4\alpha} = 0, \qquad J_{i\alpha} = \varepsilon_{i\alpha j} x_j r^{-1} J \tag{22}$$

where

$$J = -f'' - \frac{2}{r}f' + (1 + rf)\left(\frac{2f}{r^2} + \frac{f^2}{r}\right) \tag{23}$$

At $r \neq 0$, this is clearly zero, by Eq. (6). At $r \simeq 0$, the dominant term in f is $-r^{-1}$ for all three types of solutions (12a), (12d), and (12e). Thus

$$J = -4\pi\delta^3(x) \tag{24}$$

i.e.,

$$J_{4\alpha} = 0, \qquad J_{i\alpha} = -4\pi\varepsilon_{i\alpha j} x_j r^{-1}\delta^3(x) \tag{25}$$

The source function (25) is, in the sense of Dirac's definition of δ functions or in the sense of the theory of distributions, equal to zero. We thus conclude that the solutions indeed represent classical sourceless gauge fields.

V. Total Isotopic Spin

The total isotopic spin was given in reference 1:

$$\mathbf{T} = \int \mathbf{b}_\nu \times \mathbf{f}_{4\nu}\, d^3x$$

which is equal to zero for the present solutions.

Generalizations of the above solutions are in progress.

Acknowledgment

One of the authors (T. T. Wu) would like to take this opportunity to thank Professor A. Pais and the Rockefeller University for the hospitality extended him during his visit when this work was carried out.

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Eigenvalues of Casimir Operators*

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The original Casimir operator is the invariant bilinear form composed of generators of a semi-simple Lie group.[1] The idea has been generalized to include all invariant operators of homogeneous degree that commute with the generators of the group.[2] In the following we shall be concerned primarily with operators of degree 4 and lower for the general linear group. The results apply, of course, to the unitary groups, $U(N)$, since their generators are the same as for $GL(N)$.

The Casimir operator encountered most frequently in mathematical physics is the quadratic form of generators of $O(3)$. The latter are essentially components of orbital angular momentum of a single particle. Let

$$\mathbf{l} = \mathbf{r} \times \nabla \tag{1}$$

The invariant operator is

$$\mathbf{l}^2 = r^2\nabla^2 - (r, \nabla)^2 - (r, \nabla) \tag{2}$$

Operating upon a harmonic polynomial of degree n, $H_i(n)$, $\mathbf{l}^2$ has the well-known eigenvalues

$$\mathbf{l}^2 H_i(n) = -n(n+1)H_i(n) \tag{3}$$

when $\nabla^2 H_i(n) = 0$. The $2n+1$ independent polynomials $H_i(n)$ constitute a basis for an irreducible representation of the orthogonal group in three dimensions.

Irreducible bases of O_3 may also be constructed from a given number of particles all of which are in p-waves. The generators are then most conveniently expressed in the form given by Racah.[3] Let $x^\varkappa, y^\varkappa, z^\varkappa$ be the coordinates of the $\varkappa$th particle and define

$$x_1^\varkappa = (1/\sqrt{2})(x^\varkappa + iy^\varkappa); \qquad x_0^\varkappa = z^\varkappa; \qquad x_{-1}^\varkappa = (1/\sqrt{2})(x^\varkappa - iy^\varkappa) \tag{4}$$

Then the generators have the form

$$X_k^i = -X_i^k = \sum \left(x_i^\varkappa \frac{\partial}{\partial x_{-k}^\varkappa} - x_k^\varkappa \frac{\partial}{\partial x_{-i}^\varkappa} \right) \qquad i, k = 0, \pm 1 \tag{5}$$

... Energy Commission.