SPIN-ONE POLARIZATION VECTORS

Particle at rest with spin projection referred to the z-axis:

$$\varepsilon_R^\mu = (0, \underset{\sim}{\varepsilon}_R) \quad \text{where} \quad \underset{\sim}{\varepsilon}_R(\pm 1) = \frac{1}{\sqrt{2}}(\mp 1, -i, 0)$$

$$\underset{\sim}{\varepsilon}_R(0) = (0, 0, 1)$$

where $\underset{\sim}{\varepsilon}_R$ are orthonormal eigenvectors of $S_z = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

Particle at rest with spin projection referred to an arbitrary direction (θ, ϕ):

$$\underset{\sim}{\varepsilon}_R(\pm 1) = \frac{e^{\pm i\phi}}{\sqrt{2}}\left(\mp\cos\theta\cos\phi + i\sin\phi, \mp\cos\theta\sin\phi - i\cos\phi, \pm\sin\theta\right)$$ *

$$\underset{\sim}{\varepsilon}_R(0) = \left(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta\right)$$

A general polarization vector ε^μ can be obtained by boosting the polarization vector at rest ε_R^μ:

$$\varepsilon^\mu(\underset{\sim}{p}\lambda) = \left(\frac{\underset{\sim}{p}\cdot\underset{\sim}{\varepsilon}_R}{m}, \underset{\sim}{\varepsilon}_R + \left(\frac{E-m}{m}\right)\hat{\underset{\sim}{p}}\cdot\underset{\sim}{\varepsilon}_R\hat{\underset{\sim}{p}}\right)$$

where m is the mass, $E = \sqrt{\underset{\sim}{p}^2 + m^2}$ and $\underset{\sim}{p} = |p|\hat{\underset{\sim}{p}}$.

The HELICITY polarization vectors are obtained by boosting along the spin polarization axis. One obtains:

$\varepsilon_H^\mu(p\lambda) \Rightarrow$ HELICITY pol. vectors

$$\varepsilon_H^\mu(p\pm 1) = \frac{e^{\pm i\phi}}{\sqrt{2}}\left(0, \mp\cos\theta\cos\phi + i\sin\phi, \mp\cos\theta\sin\phi - i\cos\phi, \pm\sin\theta\right) = (0, \underset{\sim}{\varepsilon}_R)$$

$$\varepsilon_H^\mu(p\,0) = \left(\frac{p}{m}, \frac{E}{m}\hat{\underset{\sim}{p}}\right) = \frac{1}{m}\left(p, E\sin\theta\cos\phi, E\sin\theta\sin\phi, E\cos\theta\right)$$

APPENDIX 1: VECTOR SPHERICAL HARMONICS

The vector spherical harmonics $\{\vec{Y}_{j\ell m}(\hat{\Omega})\}$ are very useful functions for describing vector fields confined to a cavity. They are defined as the Clebsch-Gordon product of ordinary spherical harmonics $\{Y_{\ell m}\}$ and the spin one spherical unit vectors $\{\vec{e}_m\}$;

$$\vec{Y}_{j\ell m}(\hat{\Omega}) = \sum_{\mu} \langle \ell\mu, 1\, m-\mu | jm \rangle \, Y_{\ell\mu}(\hat{\Omega})\, \vec{e}_{m-\mu} \qquad \text{(A1.1)}$$

They are eigenfunctions of $\vec{J}^2$, $\vec{L}^2$, and $\vec{S}^2$ with eigenvalues $j(j+1)$, $\ell(\ell+1)$ and 2, respectively.

They are most useful in manipulations involving $\vec{\nabla}$ and in doing angular integrals, due to the properties [22]

$$\vec{\nabla}\,(f(r)Y_{jm}) = \sqrt{\frac{j}{2j+1}}\left(\frac{df}{dr}+\frac{j+1}{r}f\right)\vec{Y}_{j\,j-1\,m} - \sqrt{\frac{j+1}{2j+1}}\left(\frac{df}{dr}-\frac{j}{r}f\right)\vec{Y}_{j\,j+1\,m} \qquad \text{(A1.2)}$$

$$\vec{\nabla}\cdot(f(r)\vec{Y}_{j\,j+1\,m}) = -\sqrt{\frac{j+1}{2j+1}}\left(\frac{df}{dr}+\frac{j+2}{r}f\right)Y_{jm} \qquad \text{(A1.3)}$$

$$\vec{\nabla}\cdot(f(r)\vec{Y}_{jjm}) = 0 \qquad \text{(A1.4)}$$

$$\vec{\nabla}\cdot(f(r)\vec{Y}_{j\,j-1\,m}) = \sqrt{\frac{j}{2j+1}}\left(\frac{df}{dr}-\frac{j-1}{r}f\right)Y_{jm} \qquad \text{(A1.5)}$$

$$\vec{\nabla}\times(f(r)\vec{Y}_{j\,j+1\,m}) = i\sqrt{\frac{j}{2j+1}}\left(\frac{df}{dr}+\frac{j+2}{r}f\right)\vec{Y}_{jjm} \qquad \text{(A1.6)}$$

$$\vec{\nabla}\times(f(r)\vec{Y}_{jjm}) = \frac{i}{\sqrt{2j+1}}\left\{\sqrt{j}\left(\frac{df}{dr}-\frac{j}{r}f\right)\vec{Y}_{j\,j+1\,m} + \sqrt{j+1}\left(\frac{df}{dr}+\frac{j+1}{r}f\right)\vec{Y}_{j\,j-1\,m}\right\} \qquad \text{(A1.7)}$$

$$\vec{\nabla}\times(f(r)\vec{Y}_{j\,j-1\,m}) = i\sqrt{\frac{j+1}{2j+1}}\left(\frac{df}{dr}-\frac{j-1}{r}f\right)\vec{Y}_{jjm} \qquad \text{(A1.8)}$$

$$\vec{Y}^{\,*}_{j\ell m} = (-)^{j+\ell+m+1}\,\vec{Y}_{j\ell -m} \qquad \text{(A1.9)}$$

$$\int \vec{Y}^{\,*}_{j\ell m}\cdot\vec{Y}_{j'\ell'm'}\,d\Omega = \delta_{jj'}\delta_{\ell\ell'}\delta_{mm'} \qquad \text{(A1.10)}$$

A few specific $\{\vec{Y}_{j\ell m}$'s$\}$ for small j and ℓ are

$$\vec{Y}_{10m} = \frac{1}{\sqrt{4\pi}}\,\vec{e}_m \qquad \text{(A1.11)}$$

$$\vec{Y}_{11m} = -i\sqrt{\frac{3}{8\pi}}\;\hat{r}\times\vec{e}_m \qquad \text{(A1.12)}$$

$$\vec{Y}_{12m} = \frac{1}{\sqrt{8\pi}}\left\{\vec{e}_m - 3(\hat{r}\cdot\vec{e}_m)\hat{r}\right\} \qquad \text{(A1.13)}$$

A final useful result is the cross product of two vecto spherical harmonics, which is implicit in the relation

$$\int d\Omega\, \vec{Y}_{j_1\ell_1m_1}\cdot\left(\vec{Y}_{j_2\ell_2m_2}\times\vec{Y}_{j_3\ell_3m_3}\right)$$
$$= i\sqrt{\frac{3}{2\pi}}\left[\prod_{i=1}^{3}\sqrt{(2j_i+1)(2\ell_i+1)}\;(-)^{j_i+1}\right.$$
$$\cdot\begin{pmatrix}\ell_1 & \ell_2 & \ell_3\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}j_1 & j_2 & j_3\\ m_1 & m_2 & m_3\end{pmatrix}\begin{Bmatrix}j_1 & j_2 & j_3\\ \ell_1 & \ell_2 & \ell_3\\ 1 & 1 & 1\end{Bmatrix} \qquad \text{(A1.14)}$$

where the curly bracket is a 9-j symbol and the 3-j symbols are related to CG coefficients by

$$\langle j_1m_1, j_2m_2 | j_3m_3\rangle = (-)^{j_1-j_2-m_3}\sqrt{2j_3+1}\begin{pmatrix}j_1 & j_2 & j_3\\ m_1 & m_2 & -m_3\end{pmatrix} \qquad \text{(A1.15)}$$

... Dirac on-shell spinors are as follows:

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Particle at rest with spin projection referred to an arbitrary direction (θ, ϕ):

$$\underset{\sim}{\varepsilon}_R(\pm 1) = \frac{e^{\pm i\phi}}{\sqrt{2}}\left(\mp\cos\theta\cos\phi + i\sin\phi, \mp\cos\theta\sin\phi - i\cos\phi, \pm\sin\theta\right)^{*}$$

$$\underset{\sim}{\varepsilon}_R(0) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$

A general polarization vector ε^{μ} can be obtained by boosting the polarization vector at rest ε_R^{μ}:

$$\varepsilon^{\mu}(p\lambda) = \left(\frac{\underset{\sim}{p}\cdot\underset{\sim}{\varepsilon}_R}{m}, \underset{\sim}{\varepsilon}_R + \left(\frac{E-m}{m}\right)\hat{\underset{\sim}{p}}\cdot\underset{\sim}{\varepsilon}_R\,\hat{\underset{\sim}{p}}\right)$$

where m is the mass, $E = \sqrt{\underset{\sim}{p}^2 + m^2}$ and $\underset{\sim}{p} = |p|\hat{\underset{\sim}{p}}$.

The HELICITY polarization vectors are obtained by boosting along the spin polarization axis. One obtains:

$\varepsilon_H^{\mu}(p\lambda) \Rightarrow$ HELICITY pol. vectors

$$\varepsilon_H^{\mu}(p\pm 1) = \frac{e^{\pm i\phi}}{\sqrt{2}}\left(0, \mp\cos\theta\cos\phi + i\sin\phi, \mp\cos\theta\sin\phi - i\cos\phi, \pm\sin\theta\right) = (0, \underset{\sim}{\varepsilon}_R)$$

$$\varepsilon_H^{\mu}(p0) = \left(\frac{p}{m}, \frac{E}{m}\hat{\underset{\sim}{p}}\right) = \frac{1}{m}\left(p, E\sin\theta\cos\phi, E\sin\theta\sin\phi, E\cos\theta\right)$$

$L = J \pm \;=\; 0, 2$

$A \cdot k \;\Rightarrow\; A_{\perp}^{\dagger \text{ vol}}$

and hence

$$\int d\vec{x}\; \vec{A}_{n'}^{*} \cdot \vec{A}_{n} = \frac{1}{2\omega_n}\delta_{nn'} \qquad (2.7)$$

$$\int d\vec{x}\; \psi_{n'}^{\dagger}\psi_{n} = \delta_{nn'}$$

b) <u>Gluon Modes</u>

The normalized classical gluon modes are of TE ("magnetic") and TM ("electric") type

<u>TE</u>

$B = A \times r$

$$\vec{A}_{jm} = \alpha_j^{TE}\, j_j(\omega r)\, \vec{Y}_{jjm} \qquad (2.8)$$

$$\vec{A}_{jm}^{*} = (-)^{m+1}\vec{A}_{j,-m} \qquad (2.9)$$

$$\vec{H}_{jm} = \frac{i\alpha_j^{TE}}{\sqrt{2j+1}}\left\{\sqrt{j+1}\, j_{j-1}(\omega r)\vec{Y}_{jj-1m} - \sqrt{j}\, j_{j+1}(\omega r)\vec{Y}_{jj+1m}\right\} \qquad (2.10)$$

$$\vec{H}_{jm}^{*} = (-)^{m+1}\vec{H}_{j,-m} \qquad (2.11)$$

<u>TM</u>

$$\vec{A}_{jm} = \frac{\alpha_j^{TM}}{\sqrt{2j+1}}\left\{\sqrt{j+1}\, j_{j-1}(\omega r)\vec{Y}_{jj-1m} - \sqrt{j}\, j_{j+1}(\omega r)\vec{Y}_{jj+1m}\right\} \qquad (2.12)$$

$$\vec{A}_{jm}^{*} = (-)^{m}\vec{A}_{j,-m} \qquad (2.13)$$

$$\vec{H}_{jm} = i\alpha_j^{TM}\,\omega\, j_j(\omega r)\vec{Y}_{jjm} \qquad (2.14)$$

$$\vec{H}_{jm}^{*} = (-)^{m}\vec{H}_{j,-m} \qquad (2.15)$$

with the normalization

$$\alpha_j^{TE} = \left|\left\{x_j\left(1 - \frac{j(j+1)}{x_j^2}\right)\right\}^{-1/2} j_j(x_j)^{-1} a^{-1}\right|$$

$$\alpha_j^{TM} = \left|x_j^{-1/2}\, j_{j+1}(x_j)^{-1} a^{-1}\right|$$

The mode numbers $\chi_j \equiv \omega_j a$ and modes are those giv

Barnes [11], although his gluon normalizations $\{\alpha_j\}$ are

large.

The mode numbers for the first few modes are

<u>TE</u> $\quad j_{j-1}(x) = \frac{j}{j+1}\, j_{j+1}(x)$

J^P	x	
1^+	2.744	6.117
2^-	3.870	7.443

<u>TM</u> $\quad j_j(x) = 0$

J^P	x	
1^-	4.493	7.725
2^+	5.763	9.905

As we shall repeatedly consider the lowest TE a

modes, their properties are of special interest. Th

explicitly

l that appears in the functional integral for the expression of the ting functional $Z[J]$. In this connection the lattice regularization has equently used in the recent works for evaluating physical quantities of ns (see, e.g. [Cre 83]).

mensional regularization

rgent multiple integral may be made convergent by reducing the r of multiple integrals. For example, the linearly divergent 4- ...als in Eq. (2.4.9) would be finite if the space-time were 2- .. This fact is the basic idea of dimensional regularization [Bol 72, Ash 72, Cic 72]. In dimensional regularization we keep the space- nension D lower than four and replace the divergent 4-dimensional by a convergent D-dimensional one. By making momentum ions explicitly we obtain an analytic expression as a function of the on D. We make the analytic continuation in D in this expression. Then nal divergence will show up as a pole at $D = 4$ in the above analytic on. For a review, see [Lei 75].

in the dimensional regularization nothing has been violated except ace-time is not 4-dimensional, all the physical requirements are ed. Hence this regularized theory is Lorentz invariant, gauge invariant, ... [Ho 72, Spe 74]. In this sense dimensional regularization is the e for gauge theories. We shall give a more detailed account of this ext subsection.

Digression

SEFUL FORMULAS FOR SU(N): In the course of QCD calculations we often quantities like T^aT^a, $f^{acd}f^{bcd}$, $T^aT^bT^a$, etc, where T^a ($a=1, 2, \ldots, N^2-1$) are SU(N) s and f^{abc} the structure constants. We shall present here a derivation of formulas for the ntities.

ebra of SU(N) is defined by the commutation relations

$$[T^a, T^b] = i f^{abc} T^c \quad . \tag{2.4.13}$$

alization condition we adopt

$$\mathrm{Tr}[T^a T^b] = \frac{1}{2}\delta^{ab} \quad , \tag{2.4.14}$$

trace refers to the $N \times N$ matrix in the fundamental representation. derive the useful formula

$$T^a_{ij} T^a_{kl} = \frac{1}{2}(\delta_{il}\delta_{jk} - \frac{1}{N}\delta_{ij}\delta_{kl}) \quad . \tag{2.4.15}$$

this formula we note that the identity 1 and T^a ($a=1, 2, \ldots, N^2-1$) form the basis of a ... $N \times N$ matrices A, and ...

$$\ldots c^a T^a \quad , \tag{2.4.16}$$

where c^0, c^1, ... are real numbers. The constants c^0 and c^a are determined by using the normalization condition (2.4.14):

$$c^0 = \mathrm{Tr}[A]/N \quad , \tag{2.4.17}$$

$$c^a = 2\mathrm{Tr}[T^a A] \quad . \tag{2.4.18}$$

Inserting Eqs. (2.4.17–18) into Eq. (2.4.16) we have

$$A_{lk}(2T^a_{ij}T^a_{kl} + \frac{1}{N}\delta_{kl}\delta_{ij} - \delta_{il}\delta_{jk}) = 0 \quad . \tag{2.4.19}$$

Since Eq. (2.4.19) holds for arbitrary hermitean matrix A, we obtain Eq. (2.4.15).
An immediate consequence of Eq. (2.4.15) is that

$$(T^a T^a)_{ij} = \frac{N^2-1}{2N}\delta_{ij} \quad , \tag{2.4.20}$$

$$(T^a T^b T^a)_{ij} = -\frac{1}{2N}T^b_{ij} \quad . \tag{2.4.21}$$

We can also calculate $f^{acd}f^{bcd}$ as follows. Using Eqs. (2.4.13) and (2.4.14) we have

$$\begin{aligned} f^{acd}f^{bcd} &= -2\mathrm{Tr}[[T^a, T^c][T^b, T^c]] \quad , \\ &= -2\mathrm{Tr}[2T^aT^cT^bT^c - (T^aT^b + T^bT^a)T^cT^c] \quad . \end{aligned} \tag{2.4.22}$$

Applying Eqs. (2.4.20) and (2.4.21) to Eq. (2.4.22), we obtain

$$f^{acd}f^{bcd} = N\delta^{ab} \quad . \tag{2.4.23}$$

$f^{cda} f^{cdb} = N\delta^{ab}$

It should be remarked here that the T^aT^a appearing in Eq. (2.4.20) is the Casimir operator of the group SU(N) and hence commutes with all the operators in SU(N). By Schur's lemma such an operator is a multiple of the identity operator. In general the number of independent Casimir operators is equal to the rank of the group.

2.4.2. Dimensional regularization

We carry on our discussion by using our example of the one-loop quark self energy part in the F...n gauge (2.4.9). We now move off from 4-dimensional space-time ...eplace the 4-dimensional integral d^4k by the D-dimensional one d^Dk. ... performing the D-dimensional integration, we mention that some care ...o be given to the algebra in D dimensions.

First of all the space-time index μ now runs from 0 to $D-1$ so that components of a momentum vector are

$$p^\mu = (p^0, p^1, \ldots, p^{D-1}) \quad , \tag{2.4.24}$$

and the contracted metric tensor is

$$g^\mu_\mu = g_{\mu\nu}g^{\mu\nu} = D \quad .$$

The Dirac algebra in D dimensions is unchanged except that Eq. (2.4.25) whenev... ...actions of ... ...ices. Th