SELECTED TOPICS IN THE THEORY OF ELEMENTARY PARTICLES

TOPICS ON ELEMENTARY THEORY OF SCATTERING

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I. SCATTERING BY A CENTRAL POTENTIAL

1. Phase shift analysis

In this section we give a short review of the scattering of a particle by a fixed centre of force described by a potential $V(r)$. (For a more detailed account see e.g.: L. Schiff, Quantum Mechanics).

The problem of elastic collision between two particles (with masses m_1, m_2) is easily reduced to the scattering of a single particle (with "reduced mass" $\mu = m_1 m_2/(m_1 + m_2)$) by a potential $V(r)$, by considering the relative motion of the two particles.

We shall use, in general, the coordinate system of the centre of mass of the two particles.

The incident particle (moving along the z axis in the positive direction) is described by a plane wave e^{ikz}; the scattered particle is described, at a great distance from the scattering centre, by an outgoing spherical wave

$$f(\vartheta)\,\frac{e^{ikr}}{r}\,,$$

where $f(\vartheta)$ is the scattering amplitude (ϑ is the angle between the z axis and the direction of motion of the scattered particle; r is the distance of the particle from the scattering centre); thus, the complete solution of the stationary Schrödinger equation

$$-\frac{\hbar^2}{2\mu}\nabla^2\psi + V(r)\psi = E\psi \qquad (1)$$

is represented, at large distances from the scattering centre, by the asymptotic form

$$\psi(r,\vartheta) \to e^{ikz} + f(\vartheta)\,\frac{e^{ikr}}{r}\,. \tag{2}$$

The solution of Eq. (1) can be expanded in terms of the Legendre polynomials $P_\ell(\cos\vartheta)$:

$$\psi(r,\vartheta) = \frac{1}{r}\sum_{\ell=0}^{\infty} u_\ell(r)\,P_\ell(\cos\vartheta) \tag{3}$$

where $u_\ell(r)$ is the solution of the radial Schrödinger equation

$$\frac{d^2}{dr^2}u_\ell + \left[k^2 - U(r) - \frac{\ell(\ell+1)}{r^2}\right]u_\ell = 0 \tag{4}$$

with

$$k^2 = \frac{2\mu E}{\hbar^2}\,,\quad U(r) = \frac{2\mu V(r)}{\hbar^2}\,.$$

In the case of a potential with a finite range R, the u_ℓ's have the asymptotic behaviour (for $r >> R$):

$$u_\ell(r) \to a_\ell\,\frac{1}{kr}\sin\left(kr - \frac{1}{2}\ell\pi + \delta_\ell\right) \tag{5}$$

where the quantity δ_ℓ (which depends, in general, on the energy E) is the <u>phase shift</u> for the scattering in the state of angular momentum ℓ. The general asymptotic form of the solution Eq. (4) can then be written as

$$\psi(r,\vartheta) \to \frac{1}{kr}\sum_{\ell=0}^{\infty} a_\ell\,\sin\left(kr - \frac{1}{2}\ell\pi + \delta_\ell\right)P_\ell(\cos\vartheta). \tag{6}$$

We have now to chose the quantities a_ℓ in such a way that the two expressions Eqs. (2) and (6) coincide. Using the expansion

$$e^{ikz} = \frac{1}{kr}\sum_{\ell=0}^{\infty} i^\ell(2\ell+1)\,\sin\left(kr - \frac{1}{2}\ell\pi\right)P_\ell(\cos\vartheta) \tag{7}$$

we can write

$$f(\vartheta)\,\frac{e^{ikr}}{r} = \psi(r,\vartheta) - e^{ikz} =$$

$$= \frac{1}{kr}\sum_{\ell=0}^{\infty}\left\{a_\ell \sin\left(kr - \tfrac{1}{2}\ell\pi + \delta_\ell\right) - i^\ell(2\ell+1)\sin\left(kr - \tfrac{1}{2}\ell\pi\right)\right\} P_\ell(\cos\vartheta)$$

from which the following relations are easily derived:

$$a_\ell = i^\ell(2\ell+1)\,e^{i\delta_\ell} \tag{8}$$

$$f(\vartheta) = \frac{1}{2ik}\sum_{\ell=0}^{\infty}(2\ell+1)\left(e^{2i\delta_\ell} - 1\right)P_\ell(\cos\vartheta)\ . \tag{9}$$

The formula Eq. (9) solves the problem of expressing the scattering amplitude in terms of the phase shifts δ_ℓ. We shall sometimes use the notation

$$f(\vartheta) = \sum_{\ell=0}^{\infty}(2\ell+1)\,f_\ell(k)\,P_\ell(\cos\vartheta) \tag{10}$$

where

$$f_\ell(k) = \frac{e^{2i\delta_\ell} - 1}{2ik} = \frac{e^{i\delta_\ell}\sin\delta_\ell}{k} \tag{11}$$

represents the scattering amplitude of the ℓ-th partial wave.

The differential cross-section is given by

$$\frac{d\sigma}{d\Omega} = |f(\vartheta)|^2 \tag{12}$$

and the total cross-section is obtained by integration over all the angles:

$$\sigma(k) = 2\pi\int_0^\pi |f(\vartheta)|^2 \sin\vartheta\, d\vartheta = \frac{4\pi}{k^2}\sum_{\ell=0}^{\infty}(2\ell+1)\sin^2\delta_\ell\ . \tag{13}$$

It is easy to derive from Eqs. (8) and (13) the relation

$$\mathrm{Im}\, f(0) = \frac{k}{4\pi}\,\sigma(k)\ . \tag{14}$$

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This relation between the total cross-section and the forward scattering amplitude is known as "optical theorem".

2. Determination of the phase shifts

The determination of the phase shifts depends on the specific form of the potential $V(r)$. We are interested here in the general properties of the phase shifts valid for a potential of finite range R; we refer again to textbooks, such as Shiff, for detailed analyses.

The radial solution of Eq. (4) for $r > R$ (in this region $V(r) = 0$) can be written as:

$$u_\ell(r > R) = \cos\delta_\ell \cdot j_\ell(kr) + \sin\delta_\ell \cdot \eta_\ell(kr) \qquad (15)$$

where j_ℓ and η_ℓ are the spherical Bessel and Neuman functions. The phase shifts are determined by joining at the boundary $r = R$ the wave function and its first derivative, i.e. by joining the logarithmic derivative of the wave function. We write:

$$g_\ell(k) = R\left(\frac{u'_\ell}{u_\ell}\right)_{r=R} \qquad (16)$$

where the quantity $g_\ell(k)$ is R times the logarithmic derivative of the wave function valid inside the potential region evaluated at $r = R$. Clearly g_ℓ depends on the form of the potential; but we shall consider it here as a given parameter.

Suppose now that $kR << 1$; in this case we can use the asymptotic expressions

$$j_\ell(x) \approx \frac{x^\ell}{(2\ell+1)!!} \; ; \quad \eta_\ell(x) \approx \frac{(2\ell-1)!!}{x^{\ell+1}} \qquad (x << 1) \qquad (17)$$

in the above relations Eqs. (15) and (16). We obtain in this way:

$$\mathrm{tg}\,\delta_\ell = \alpha_\ell \frac{(kR)^{2\ell+1}}{(2\ell+1)!!\,(2\ell-1)!!} \qquad (18)$$

with

$$\alpha_\ell = \frac{\ell - g_\ell}{\ell + 1 + g_\ell} \, . \qquad (19)$$

For a regular behaviour of α_ℓ, tg δ_ℓ decreases strongly with the increasing angular momentum ℓ. For very small values of kR, we may neglect all the terms except the first. In this approximation (equivalent to keep only the S wave) one gets:

$$\mathrm{tg}\ \delta_0 \approx \delta_0 \approx \alpha_0 k \tag{20}$$

and

$$f(\vartheta) \approx \alpha_0 \quad ; \quad \sigma \approx 4\pi \alpha_0^2 \ . \tag{21}$$

Then, for small velocity, the scattering is <u>isotropic</u>, and the cross-section is independent of the energy. The same result is obtained for the low-energy scattering by an impenetrable sphere of radius $R = \alpha_0$.

3. Resonance scattering

We have assumed, in the previous section, that for very small kR the quantities α_ℓ are also small. This is obviously not true in the special case

$$g_\ell(k_R) = -(\ell + 1) \ . \tag{22}$$

In this case tg δ_ℓ goes to infinity and $\delta_\ell = \pi/2$ at the value $k = k_R$.

We can expand $g_\ell(k)$ for values of k close to k_R (or equivalently for values of the energy E close to $E_R = \hbar^2 k_R^2 / 2\mu$):

$$g_\ell(E) \simeq -(\ell + 1) + (E_R - E) \left(\frac{dg_\ell}{dE}\right)_{E=E_R} \ . \tag{23}$$

We can then re-write Eq. (18) as

$$\mathrm{tg}\ \delta_\ell = \frac{\Gamma_\ell / 2}{E_R - E} \tag{24}$$

with

$$\Gamma_\ell = \frac{2(kR)^{2\ell+1}}{\left(\frac{dg_\ell}{dE}\right)_{E_R} \left[(2\ell - 1)!!\right]^2} \ . \tag{25}$$

This behaviour corresponds to a resonance which occurs in the scattering of the ℓ-th partial wave at the energy E_R.

If we write for the total cross-section

$$\sigma = \sum_{\ell=0}^{\infty} \sigma_\ell$$

we see that for E close to E_R the "partial wave cross-section" σ_ℓ becomes:

$$\sigma_\ell(E) = \frac{4\pi}{k^2}(2\ell+1)\frac{(\Gamma/2)^2}{(E-E_R)^2+(\Gamma/2)^2} . \qquad (26)$$

It is clear that at the resonance $E = E_R$, σ_ℓ reaches the maximum value

$$\sigma_\ell = \frac{4\pi}{k_R^2}(2\ell+1) \qquad (27)$$

which is called the geometrical value. The quantity Γ is the width of the resonance; at $E = E_R \pm \Gamma/2$ the cross-section decreases to ½ of the maximum value.

4. Graphical representation

We give here a graphical representation for the scattering amplitude given by Eq. (11), which can also be written as

$$f_\ell(k) = \frac{1}{k\cot\delta_\ell - ik} . \qquad (28)$$

We also give the following relations

$$\mathrm{Im}\, f_\ell^{-1} = -k \qquad (29)$$

$$\mathrm{Re}\, f_\ell^{-1} = k\cot\delta_\ell \qquad (30)$$

the first of which corresponds to the "unitarity" of the S matrix, (see following).

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In the case of a resonance in the ℓ partial wave, we get from Eq. (24):

$$f_\ell(k) = \frac{1}{k}\,\frac{\Gamma/2}{(E_R - E) - i\Gamma/2} \quad . \tag{31}$$

It is easy to show that the quantity

$$F_\ell(k) = kf_\ell(k) = \frac{1}{\cot\,\delta_\ell - i} \tag{32}$$

is represented by a circle in the complex plane of F_ℓ (Im F_ℓ versus Re F_ℓ). In fact, by writing

$$\epsilon = \cot\,\delta_\ell$$
$$x = \mathrm{Re}\,F_\ell = \frac{\epsilon}{\epsilon^2 + 1}$$
$$y = \mathrm{Im}\,F_\ell = \frac{1}{\epsilon^2 + 1}$$

we get

$$x^2 = (1 - y)y$$

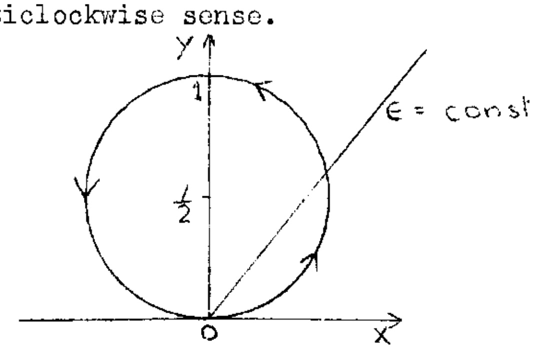
which is the equation of a circle with centre at the point $x = 0$, $y = 1/2$, and radius = $1/2$ (see Fig. 1). For ϵ = const, one gets straight lines of equation: $y = \frac{1}{\epsilon}\,x$.

In the case of a resonance ϵ is given by

$$\epsilon = \frac{2}{\Gamma}\,(E_R - E) \; . \tag{33}$$

At the resonance $E = E_R$: $\epsilon = 0$, $x = 0$, $y = 1$; for $E < E_R$: $\epsilon > 0$; for $E > E_R$: $\epsilon < 0$. The point $x = y = 0$ corresponds to $\delta_\ell = 0, \pi, \ldots$. (We assume $\delta_\ell \to 0$ as $E \to 0$). Thus, as the energy E passes through E_R, the representative point on the circle in Fig. 1 passes through the point $x = 0$, $y = 1$ in an anticlockwise sense.

Fig. 1



II. SCATTERING OF PARTICLES WITH SPIN

1. Scattering of a spin-0 particle by a spin-½ particle

We generalize now the previous results to the case in which the particles have spins different from zero, and the potential depends on the spin orientation, as e.g. in the case of the spin-orbit potential.

In this case the solution of the Schrödinger equation can be expanded in terms of the eigenstates $\mathcal{Y}^{M}_{J\ell}(\vartheta,\varphi)$ of J, M, ℓ (J is the total angular momentum and M is its component along the z axis):

$$\psi(r,\vartheta,\varphi) = \frac{1}{r}\sum_{J,\ell} u_{J\ell}(r)\mathcal{Y}^{M}_{J\ell}(\vartheta,\varphi) \ . \tag{34}$$

The $\mathcal{Y}^{M}_{J\ell}$ are expressed in terms of the usual spherical harmonics Y^{m}_{ℓ}, and the spin eigenfunctions $\chi^{m_s}_{S}$:

$$\mathcal{Y}^{M}_{J\ell}(\vartheta,\varphi) = \sum_{m,m_s} C_{\ell S}(J,M;m,m_s)\ \chi^{m_s}_{S}\ Y^{m}_{\ell}(\vartheta,\varphi) \tag{35}$$

where the $C_{\ell S}$'s are the Clebsch-Gordan coefficients (m, m_s are the z components of ℓ and S).

The radial Schrödinger equation is now

$$\frac{d^2}{dr^2}\ u_{J,\ell} + \left[k^2 - U_{J,\ell}(r) - \frac{\ell(\ell+1)}{r^2}\right]u_{J,\ell} = 0 \ . \tag{36}$$

We consider here the simple case in which the two particles have spin zero and spin ½; for a given value of ℓ, the total angular momentum J can have the two values $J = \ell \pm \frac{1}{2}$ and the potential splits into two terms (we assume $U_{\ell+\frac{1}{2},\ell} \neq U_{\ell-\frac{1}{2},\ell}$; clearly, if the potential does not depend on J, the results of the preceding sections remain unchanged).

Since for the incident wave m = 0 (there is symmetry around the z axis), the z component of the total angular momentum is $M = m_s = \pm\frac{1}{2}$ in the initial state, and then it will be always $M = \pm \frac{1}{2}$. On the other hand, if the incident particle has a given $m_s = + \frac{1}{2}$, the outgoing particle can have both values $m_s' = + \frac{1}{2}$, $m_s' = - \frac{1}{2}$.

The asymptotic wave function can then be written as follows:

$$\psi_{\pm\frac{1}{2}}(\vartheta\ \varphi) \rightarrow \begin{pmatrix}\alpha\\ \beta\end{pmatrix} e^{ikz} + \begin{pmatrix}\alpha\\ \beta\end{pmatrix} \frac{e^{ikr}}{r} f(\vartheta,\varphi) + \begin{pmatrix}\beta\\ \alpha\end{pmatrix} \frac{e^{ikr}}{r} g(\vartheta,\varphi) \qquad (37)$$

where we have used the notation $\alpha = \chi^{\frac{1}{2}}_{\frac{1}{2}}$, $\beta = \chi^{-\frac{1}{2}}_{\frac{1}{2}}$. The quantity g is called spin-flip amplitude, and f non spin-flip amplitude. These amplitudes determine the differential cross-section

$$\frac{d\sigma}{d\Omega} = |f\ (\vartheta,\varphi)|^2 + |g\ (\vartheta,\varphi)|^2 \ . \qquad (38)$$

The asymptotic form of the general solution is given by Eq. (34) with

$$u_{J,\ell}(r) \rightarrow \frac{1}{r}\, a_{J,\ell}\ \sin\ (kr - \frac{1}{2}\ell\pi + \delta_{J,\ell})\ ,\ (J = \ell \pm \frac{1}{2}) \qquad (39)$$

In this specific case we give the explicit expressions for the $\mathcal{Y}^M_{J,\ell}$:

$$\mathcal{Y}^{\frac{1}{2}}_{\ell+\frac{1}{2}} = \sqrt{\frac{\ell + 1}{2\ell + 1}}\ \alpha Y^0_\ell + \sqrt{\frac{\ell}{2\ell + 1}}\ \beta Y^{-1}_\ell$$

$$\mathcal{Y}^{\frac{1}{2}}_{\ell-\frac{1}{2}} = -\sqrt{\frac{\ell}{2\ell + 1}}\ \alpha Y^0_\ell + \sqrt{\frac{\ell + 1}{2\ell + 1}}\beta Y^1_\ell \ .$$

Using again the expansion Eq. (7), written in the form

$$e^{ikz} = \frac{1}{kr} \sum_\ell \sqrt{4\pi(2\ell + 1)}\ i^\ell\ \sin\ (kr - \frac{1}{2}\ell\pi)\ Y^0_\ell \qquad (40)$$

one obtains by comparison of Eqs. (37) and (34):

$$\left.\begin{aligned} a_{\ell+\frac{1}{2},\ell} &= \sqrt{\frac{4\pi}{\ell + 1}}\ i^\ell\ e^{i\delta_{\ell+\frac{1}{2},\ell}} \\ a_{\ell-\frac{1}{2},\ell} &= -\sqrt{4\pi\ell}\ i^\ell\ e^{i\delta_{\ell-\frac{1}{2},\ell}} \end{aligned}\right\} \qquad (41)$$

and

$$f(\vartheta,\varphi) = \frac{1}{k} \sum_{\ell} \sqrt{\frac{4\pi}{2\ell+1}} \left[(\ell+1)\, f_{\ell+1/2} + \ell\, f_{\ell-1/2} \right] Y_{\ell}^{0}\ (\vartheta)$$

(42)

$$g(\vartheta,\varphi) = \frac{1}{k} \sum_{\ell} \sqrt{\frac{4\pi}{2\ell+1}} \sqrt{\ell(\ell+1)} \left[f_{\ell+1/2} - f_{\ell-1/2} \right] Y_{\ell}^{1}\ (\vartheta,\varphi)$$

where

$$f_{\ell\pm 1/2} = \frac{e^{2i\delta_{\ell\pm 1/2}}-1}{2i} = e^{i\delta_{\ell\pm 1/2}} \sin \delta_{\ell\pm 1/2} \ . \qquad (43)$$

It is clear that the differential cross-section does not depend on the angle φ. By integration over all the angles one can obtain the total cross-section

$$\sigma = \frac{4\pi}{k^2} \sum_{\ell=0}^{\infty} \left[(\ell+1) \sin^2 \delta_{\ell+1/2} + \ell \sin^2 \delta_{\ell-1/2} \right] =$$

$$= \frac{4\pi}{k^2} \sum_{J=1/2}^{\infty} (J+1/2) \sin^2 \delta_{J,\ell} \ . \qquad (44)$$

This formula can be easily extended to the general case in which the two particles have spins s_1 and s_2. The result is the following

$$\sigma = \frac{4\pi}{k^2} \left[(2S_1+1)\ (2S_2+1) \right]^{-1} \sum_{J,S} (2J+1) \sin^2 \delta_{\ell,J,S} \qquad (45)$$

where S is the total spin ($\vec{S} = \vec{s}_1 + \vec{s}_2$).

These results can be applied to the case of a resonance occuring at a particular value of J.

2. Example: the π^+p scattering at low energy

We consider here as an example the π^+p scattering at low-energy (π^+ kinetic energy up to $\simeq$ 200 MeV).

Assuming that only the S and P waves are important and that the higher waves can be neglected, we get from Eq. (42):

$$f = \frac{1}{k} \left[f_{0,1/2} + (2f_{1,3/2} + f_{1,1/2}) \cos\vartheta \right]$$
$$g = \frac{1}{k} (f_{1,3/2} - f_{1,1/2}) \sin\vartheta \, e^{i\varphi} \, . \qquad (46)$$

The differential cross-section is

$$\frac{d\sigma}{d\Omega} = \frac{1}{k^2} (A + B \cos\vartheta + C \cos^2\vartheta) \qquad (47)$$

with

$$A = |f_{0,1/2}|^2 + |f_{1,1/2} - f_{1,3/2}|^2$$
$$B = 2\mathrm{Re}\, f^*_{0,1/2} (2f_{1,3/2} + f_{1,1/2}) \qquad (48)$$
$$C = 3|f_{1,3/2}|^2 + 6\mathrm{Re}\, f^*_{1,3/2} f_{1,1/2}$$

In the case of pure $P_{3/2}$ ($\ell = 1$, $J = 3/2$) state, which is the state of the well-known resonance in π^+p at $\simeq$ 200 MeV, one has

$$A = |f_{1,3/2}|^2 \, , \quad B = 0 \, , \quad C = 3|f_{1,3/2}|^2$$

and the differential cross-section is

$$\frac{d\sigma}{d\Omega} = |f_{1,3/2}|^2 (1 + 3\cos^2\vartheta) \, . \qquad (49)$$

The angular distribution is then symmetric around $\vartheta = \pi/2$ for the pure $P_{3/2}$ resonant state. The presence of the S wave and its interference with the $P_{3/2}$ wave gives rise to the asymmetric term

$$B \cos\vartheta = 4\mathrm{Re}\, f^*_{0,1/2} f_{1,3/2} \cos\vartheta \, .$$

This is, in fact, the experimental feature.

III. INELASTIC COLLISIONS

1. Elastic and inelastic cross-sections

We call *inelastic* a collision which produces a change in the nature or in the number of the colliding particles. Examples of inelastic collisions between elementary particles are the following: $K^- + p \rightarrow \Sigma^+ + \pi^+$, $\pi^- + p \rightarrow p + \pi^- + \pi^0$. In general, several final states are possible for a given initial state. For example, considering only the two particle states, the initial system $K^- + p$ can go into the final states: $K^- + p$, $\bar{K}^0 + n$, $\Lambda^0 + \pi^0$, $\Sigma^- + \pi^+$, $\Sigma^0 + \pi^0$. Each different state is called a different *channel*.

a) The asymptotic form of the wave function of the system of the two colliding particles will be a sum of different terms, each representing a possible channel. Amongst these terms there is always, in particular, a term corresponding to the elastic scattering. In addition there is, of course, a term describing the particles before the collisions.

For the moment we consider only the global effect of the inelastic processes, specifically on the elastic scattering, and devote Section IV to the study of separate inelastic channels. We consider, for the sake of simplicity, spin-zero particles.

The asymptotic expressions of the radial functions $u_\ell(r)$ are now modified with respect to the pure elastic case. The expression (5) can be considered as the sum of an incident and outgoing waves with the same amplitudes. In the present case, since several channels are present in the final states, the amplitude of the final outgoing wave must be less than that of the ingoing wave. We write then:

$$u_\ell(r) \simeq \frac{\alpha_\ell \, e^{i(kr - \frac{1}{2}\ell\pi)} - e^{-i(kr - \frac{1}{2}\ell\pi)}}{2ikr} \qquad (50)$$

where α_ℓ is in general a complex quantity, with modulus less than unity; it can be written as: $\alpha_\ell = \eta_\ell \, e^{i\delta_\ell}$ (δ_ℓ, η_ℓ real numbers).

The asymptotic expression of the wave function $\psi(r,\vartheta)$ is then given by:

$$\psi(r,\vartheta) \to \sum_{\ell} i^{\ell}\,(2\ell+1)\,\frac{\alpha_{\ell}\,e^{i(kr-\frac{1}{2}\ell\pi)} - e^{-i(kr-\frac{1}{2}\ell\pi)}}{2ikr}\,P_{\ell}(\cos\vartheta). \quad (51)$$

Using the same procedure of Section I.1, one can find by comparison of Eqs. (2) and (51):

$$f(\vartheta) = \sum_{\ell}\,(2\ell+1)\,f_{\ell}(k)\,P_{\ell}(\cos\vartheta)$$

with

$$f_{\ell}(k) = \frac{\alpha_{\ell}-1}{2ik} = \frac{\eta_{\ell}e^{i\delta_{\ell}}-1}{2ik}\,. \quad (52)$$

The total elastic cross-section is given by:

$$\sigma_{e\ell} = \frac{\pi}{k^2}\sum_{\ell}\,(2\ell+1)\,|1-\alpha_{\ell}|^2 = \frac{4\pi}{k^2}\sum_{\ell}\,(2\ell+1)\left|\frac{\eta_{\ell}e^{i\delta_{\ell}}-1}{2i}\right|^2. \quad (53)$$

The total inelastic cross-section (for all possible final states of the particles) can also be expressed in terms of the quantities α_{ℓ}. For each value of ℓ the intensity of the outgoing wave is reduced by the ratio $|\alpha_{\ell}|^2 < 1$ with respect to the intensity of the ingoing wave. This reduction is entirely due to inelastic scattering:

$$\sigma_{in} = \frac{\pi}{k^2}\sum_{\ell}\,(2\ell+1)\,(1-|\alpha_{\ell}|^2) = \frac{\pi}{k^2}\sum_{\ell}\,(2\ell+1)\,(1-\eta_{\ell}^2)\,. \quad (54)$$

Obviously, for $\eta_{\ell} = 1$, $\sigma_{e\ell}$ becomes identical to Eq. (13), and σ_{in} vanishes.

We note that only $\sigma_{e\ell}$ contains the complex term α_{ℓ} (while σ_{in} contains only the modulus $|\alpha_{\ell}|$): this corresponds to the fact that in the elastic scattering the incident and outgoing wave are coherent.

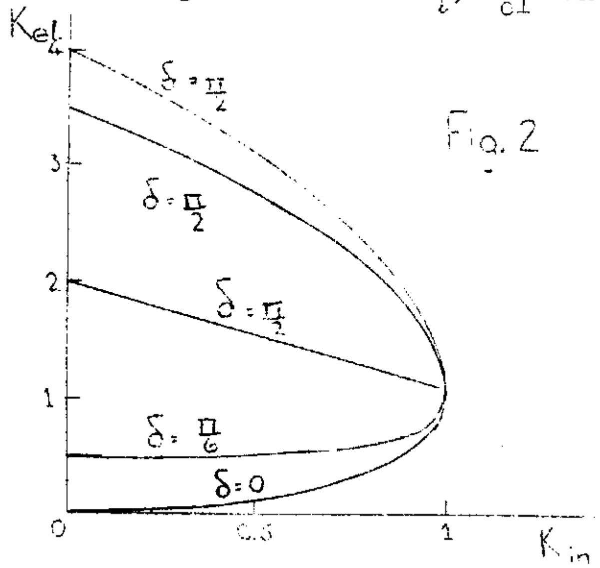
It is easy to check that the optical theorem expressed by Eq. (14) holds for the total cross section $\sigma_{tot} = \sigma_{e\ell} + \sigma_{in}$

Let us denote by $\sigma_\ell^{(el)}$, $\sigma_\ell^{(in)}$ the terms in Eqs. (53) and (54) and define:

$$K_{el} = k^2\sigma_\ell^{(el)}/(2\ell+1)\pi = |\eta_\ell e^{2i\delta_\ell} - 1|^2$$

$$K_{in} = k^2\sigma_\ell^{(in)}/(2\ell+1)\pi = (1 - \eta_\ell^2) \quad . \tag{56}$$

For a given value of δ_ℓ, K_{el} can be expressed in terms of K_{in}. We plot in Fig. 2 K_{el} versus K_{in} for a few typical values of δ_ℓ. The boundary ($\delta_\ell = 0$, $\delta_\ell = \pi/2$) corresponds to the minimum and maximum values of the elastic cross-section, as a function of the inelastic one. For $\eta_\ell = 1$, K_{el} reaches the maximum value $|e^{2i\delta_\ell} - 1|^2$, while $K_{in} = 0$ (pure elastic scattering). The maximum value of K_{in} is obtained for $\eta_\ell = 0$ (complete absorption): in this case $K_{el} = K_{in} = 1$.



The previous results can be generalized to the case of spin different from zero, (see e.g. Blatt and Weisskopf, Theoretical Nuclear Physics, Chapter VIII). Usually a channel is characterized also by a given value of the total spin $\vec{S} = \vec{s}_1 + \vec{s}_2$ (we note the scattering without a change of S is coherent); this channel has the statistical weight

$$g(S) = \frac{2S+1}{(2s_1+1)\,(2s_2+1)} \quad . \tag{57}$$

For given values of S and J the elastic and inelastic cross-sections are given by:

$$\sigma_{el}(S,J) = \frac{\pi}{k^2}\,\frac{2J+1}{2S+1}\sum_{\ell,\ell'=|J-S|}^{J+S} |\delta_{\ell\ell'} - \alpha_{\ell\ell'}(S,J)|^2 \tag{58}$$

$$\sigma_{in}(S,J) = \frac{\pi}{k^2}\,\frac{2J+1}{2S+1}\sum_{\ell,\ell'=|J-S|}^{J+S} (\delta_{\ell\ell'} - |\alpha_{\ell\ell'}(S,J)|^2) \tag{59}$$

where one has to take into account that the orbital angular momentum ℓ' in the final state can be different from the corresponding value ℓ in the initial state, for given values of J and S (the conservation of parity implies that ℓ and ℓ' differ by an even number).

The total cross-sections for a given S, $\sigma_{el}(S)$, $\sigma_{in}(S)$ are obtained by summing $\sigma_{el}(S,J)$, $\sigma_{in}(S,J)$ over all the values of J. For unpolarized beams one has to sum over all possible values of S, multiplying $\sigma_{el}(S)$, $\sigma_{in}(S)$ by the statistical weight g(S). The results obtained in this way are the complete generalization of the formulae Eqs. (53) and (54).

2. Scattering by a black sphere

We consider here, as an example, the scattering by a "black" sphere, i.e. a sphere which absorbs all the particles which strike on it. We consider the case in which the wave length of the incident particle 1/k is much smaller than the radius R of the sphere.

These assumptions can be expressed as follows:

$$\alpha_\ell = 0 \quad \text{for} \quad \ell \leq kR \quad \text{(no outgoing wave)}$$
$$\alpha_\ell = 1 \quad \text{for} \quad \ell > kR \quad \text{(no scattering)}$$

From Eqs. (53) and (54) one gets

$$\sigma_{el} = \sigma_{in} = \frac{\pi}{k^2} \sum_{\ell=0}^{\ell_{max}} (2\ell + 1) \simeq \frac{\pi}{k^2} \sum_{\ell=0}^{kR} (2\ell + 1) \simeq \pi R^2 . \qquad (60)$$

Both the elastic and inelastic cross-sections are equal to the geometrical cross-section. The elastic scattering, which is called shadow scattering, is due to the diffraction effects which occur at the edge of the target. The scattering angles of this diffraction scattering are rather small and of the order of $\ell_{max}^{-1} \simeq (kR)^{-1}$; the differential cross-section presents a very narrow peak in the forward direction.

We can give some estimate of the shape of the diffraction peak. The scattering amplitude $f(\vartheta)$ is pure imaginary, as one can see from Eq. (52)

$$f(\vartheta) = \frac{i}{2k} \sum_{\ell=0}^{\ell_{max}} (2\ell+1)\, P_\ell(\cos\vartheta)\ . \qquad (61)$$

For small angles and large values of ℓ one can write

$$P_\ell(\cos\vartheta) \simeq J_0(\ell \sin\vartheta)$$

and the scattering amplitude, by replacing the sum over ℓ by an integral, $f(\vartheta)$ becomes

$$\mathrm{Im}\, f(\vartheta) \simeq \frac{1}{2k} \int_0^{kR} 2\ell J_0(\ell \sin\vartheta)\, d\ell = \frac{R}{\sin\vartheta} J_1(kR \sin\vartheta)\ . \qquad (62)$$

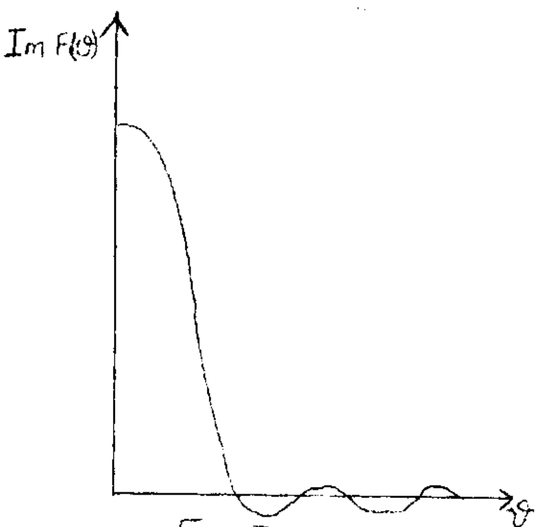


Fig. 3

The behaviour of $\mathrm{Im}\, f(\vartheta)$ versus ϑ is represented in Fig. 3. The value at $\vartheta = 0$ is given by

$$\mathrm{Im}\, f(0) = \lim_{\vartheta\to 0} \left(\frac{R}{\sin\vartheta} \cdot \frac{kR \sin\vartheta}{2} \right) = \frac{kR^2}{2}$$

as given also by the optical theorem Eq. (55). The first zero of $J_1(kR \sin\vartheta)$ is at $\vartheta \simeq 3.8/kR$.

IV. MULTI-CHANNEL FORMALISM

1. The scattering matrix

We have considered in the previous chapter the case of multi-channel processes, and we have examined in particular the elastic channel. We present here a more general approach, which allows us to obtain the cross-sections for the transition from a given initial channel to a different final channel.

We consider, for the sake of simplicity, only two spin-zero particles in each channel.

The asymptotic wave function describing the relative motion of the two particles in a given channel i, with orbital angular momentum ℓ_i, is represented by

$$\Psi_{\ell_i}^{(i\to i)}(r) = \frac{-1}{2_i k_i r}\left[e^{-i(k_i r - \frac{1}{2}\ell_i \pi)} - S_{ii}\, e^{i(k_i r - \frac{1}{2}\ell_i \pi)}\right]. \quad (63)$$

This expression represents the ingoing and outgoing spherical waves in the same channel i. It corresponds to the ℓ-th term of Eq. (51) with α_ℓ replaced by S_{ii}.

However, with an ingoing wave in the channel i, there will be outgoing waves also in all the other different channels which are open (a channel is said to be open when the available energy is greater than its threshold). The outgoing wave in the f channel is written as

$$\Psi_{\ell_f}^{(i\to f)}(r) = \frac{1}{2i\sqrt{k_f k_i}\, r}\, S_{fi}\, e^{i(k_f r - \frac{1}{2}\ell_f \pi)} . \quad (64)$$

The ratio of the flux of the outgoing wave in the channel f to the incident plane wave flux gives the cross-section

$$\sigma_{\ell_i}^{(i\to f)} = (2\ell_i + 1)\, \frac{\pi}{k_i^2}\, |S_{fi}|^2 . \quad (65)$$

This formula is valid for i ≠ f; in order to obtain the scattered wave in the elastic channel, we have to subtract from the outgoing wave in Eq. (63) the ℓ-th term of the outgoing part of the plane wave Eq. (6):

$$\sigma_{\ell_i}^{(i\to i)} = (2\ell_i + 1)\, \frac{\pi}{k_i^2}\, |1 - S_{ii}|^2 . \quad (66)$$

The expressions Eqs. (65) and (66) can be written in a single formula:

$$\sigma_{\ell_i}^{(i\to f)} = (2\ell_i + 1)\, \frac{\pi}{k_i^2}\, |\delta_{fi} - S_{fi}|^2 . \quad (67)$$

The complex quantities S_{fi} ($i,f = 1, 2, \ldots, n$; n = number of channels) can be ordered in an $n \times n$ matrix S, which is called the <u>scattering matrix</u>:

$$S = \begin{pmatrix} S_{11} & S_{12} & S_{13} & \ldots \\ S_{21} & S_{22} & S_{23} & \ldots \\ S_{31} & S_{32} & S_{33} & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{pmatrix}$$

We quote here two important properties of the S matrix (for the proof see e.g.: Blatt and Weisskopf):

1) S is a <u>unitary matrix</u>: $SS^{+} = 1$ (S^{+} is the hermitian conjugate of S: $S^{+}_{ij} = S^{*}_{ji}$).
2) S is a <u>symmetrical matrix</u>: $\tilde{S} = S$ ($\tilde{S}$ is the transposed matrix of S: $\tilde{S}_{ij} = S_{ji}$).

The first property is connected with the <u>conservation of flux</u>, the second with <u>time reversal invariance</u>.

In the case of one channel, the S matrix, which reduces to one element, can be written, using the first property:

$$S_{\ell} = e^{2i\delta_{\ell}} \tag{68}$$

where δ_{ℓ} is the usual phase shift. In fact, the cross-section Eq. (67) with Eq. (68) becomes:

$$\sigma_{\ell} = \frac{4\pi}{k^2}(2\ell + 1)\sin^2\delta_{\ell}$$

which agrees with the formula Eq. (13).

In the case of two channels, using both unitarity and symmetry, the S matrix can be written as follows

$$S_{\ell} = \begin{pmatrix} \eta_{\ell}\, e^{2i\delta_{\ell}^{(1)}} & i\sqrt{1-\eta_{\ell}^2}\, e^{i(\delta_{\ell}^{(1)}+\delta_{\ell}^{(2)})} \\ i\sqrt{1-\eta_{\ell}^2}\, e^{i(\delta_{\ell}^{(1)}+\delta_{\ell}^{(2)})} & \eta_{\ell}\, e^{2i\delta_{\ell}^{(2)}} \end{pmatrix} \tag{69}$$

where the quantities $\delta_\ell^{(1)}$, $\delta_\ell^{(2)}$, η_ℓ are real. In this case Eq. (67) gives:

$$\sigma_\ell^{(i\to i)} = (2\ell+1)\,\frac{\pi}{k_i^2}\left|1-\eta_\ell\, e^{2i\delta_\ell^{(i)}}\right|^2$$

$$\sigma_\ell^{(i\to f)} = (2\ell+1)\,\frac{\pi}{k_i^2}\left(1-\eta_\ell^2\right)$$

which agree with the results obtained in Eqs. (53) and (54).

For more than two channels, the situation is much more complicated. Since the S matrix is unitary and symmetrical, the number of independent real parameters in the case of n channels is $\frac{1}{2}n\,(n+1)$.

We close this section by giving, without proof, a general expression for the differential cross-section, by which one can evaluate the angular distribution for a reaction of particles with spin:

$$\frac{d\sigma^{(i\to f)}}{d\Omega} = \frac{(2k_i)^{-1}}{(2S_{1i}+1)(2S_{2i}+1)}\sum_{S_i,S_f}\left|\sum_{J,\ell_i,\ell_f}\sqrt{4\pi(2\ell_i+1)}\; C_{\ell_i S_i}(J,m_{s_i};0,m_{s_i})\,\cdot\right.$$

$$\left.\cdot\; C_{\ell_f S_f}(J,m_{s_i};m_f,m_{sf})\left[\delta_{fi}-S_{fi}(J,S_i,S_f,\ell_i,\ell_f)\right]Y_{\ell_f}^{m_f}(\vartheta\,\varphi)\right|^2 . \qquad (70)$$

The S matrix depends, in this case, for a given total angular momentum J, on the total spin $\vec{S} = \vec{S}_1+\vec{S}_2$ and angular momenta ℓ_i, ℓ_f of the initial and final channels.

2. The scattering amplitude and the reaction matrix

It is useful to introduce two other matrices, besides the S matrix. We define a matrix T, which is the generalization of the scattering amplitude for a multi-channel system, by:

$$S_{fi} = \delta_{fi} + 2i\, k_i^{1/2}\, k_f^{1/2}\, T_{fi} . \qquad (71)$$

In the matrix notation:

$$S = 1 + 2i\, k^{1/2}\, T\, k^{1/2} \qquad (72)$$

or

$$T = k^{-1/2}\, \frac{S-1}{2i}\, k^{-1/2} \qquad (73)$$

where k is a diagonal matrix.

In terms of the elements T_{fi}, the cross-section Eq. (67) becomes

$$\sigma_{\ell_i}^{(i\to f)} = 4\pi\, (2\ell_i + 1)\, \frac{k_f}{k_i}\, |T_{fi}|^2 . \qquad (74)$$

One sees immediately that, in the case of one channel, T coincides with the scattering amplitude

$$T_\ell = \frac{e^{2i\delta_\ell} - 1}{2ik} . \qquad (75)$$

The expressions Eqs. (63) and (64), by use of Eq. (71), can be replaced by

$$\psi_{\ell_f}^{(i\to f)}(r) = \delta_{fi}\, \frac{\sin (k_f r - \frac{1}{2}\ell_f \pi)}{k_f r} + T_{fi}\, \frac{e^{i(k_f r - \frac{1}{2}\ell_f \pi)}}{r} \qquad (76)$$

which is a generalization of the ℓ-th term of Eq. (2). The expression Eq. (76) represents then an incident wave of unit amplitude in the i channel, together with an outgoing wave of amplitude T_{fi} in the f channel.

The situation in which the outgoing waves are replaced by standing waves (for all channels) is described by

$$\Phi_{\ell_f}^{(i\to f)}(r) = \delta_{fi}\, \frac{\sin (k_f r - \frac{1}{2}\ell_f \pi)}{k_f r} + K_{fi}\, \frac{\cos (k_f r - \frac{1}{2}\ell_f \pi)}{k_f r} . \qquad (77)$$

The quantities K_{fi} are the elements of a matrix K, called the <u>reaction matrix</u>.

It is possible to show that the following relation holds between the matrices T and K:

$$T = K\,(1 - ikK)^{-1} = (1 - ikK)^{-1}\,K \tag{78}$$

or, equivalently:

$$T^{-1} = K^{-1} - ik\,. \tag{79}$$

We will show that the definition Eq. (77) leads to the relation Eq. (78) in the particular case in which only the S waves are present in all channels, (see also: R.H. Dalitz: Strange particles and strong interactions; Oxford University Press, 1962). For S waves, Eq. (76) and (77) become:

$$\psi^{(i\to f)}(r) = \delta_{fi}\,\frac{\sin k_f r}{k_f r} + T_{fi}\,\frac{e^{ik_f r}}{r} \tag{80}$$

$$\Phi^{(i\to f)}(r) = \delta_{fi}\,\frac{\sin k_f r}{k_f r} + K_{fi}\,\frac{\cos k_f r}{k_f r}\,. \tag{81}$$

The wave function $\Phi(r)$ satisfies the following condition at $r = 0$:

$$\left(r\Phi^{(i\to f)}\right)_{r=0} = K_{fi} = \sum_\ell K_{f\ell}\,\delta_{\ell i} = \sum_\ell K_{f\ell}\left[\frac{d}{dr}\,r\Phi^{(i\to \ell)}\right]_{r=0}\,. \tag{82}$$

This is a linear condition which is valid for all channels i, and therefore it is valid for any linear combination of $\Phi^{(i\to f)}$. Since the Φ's form a a complete set, we can write in general the relation Eq. (82) for any wave function, in particular for the $\psi^{(i\to f)}$ given in Eq. (80). In this way, one gets:

$$T_{fi} = \sum_\ell K_{f\ell}\,(\delta_{\ell i} + ik_\ell\,T_{\ell i}) \tag{83}$$

or in matrix notation

$$T = K\,(1 + ikT)$$

which is equivalent to Eq. (78).

By means of Eqs. (72) and (78) we can also express the K matrix in terms of the S matrix:

$$K = ik^{-1/2}\ (1-S)\ (1+S)^{-1}\ k^{-1/2}\ . \tag{84}$$

It is easy to show that, since the S matrix is unitary and symmetric, the K matrix is hermitian ($K = K^+$) and symmetric, i.e. it is a <u>real</u> matrix. Of course, the number of parameters in both matrices is the same, and given by $\frac{1}{2}\,n(n+1)$ for an n channel system.

In the one channel case, using Eq. (48) we get:

$$K_\ell = \frac{1}{k\ \cot\ \delta_\ell}\ . \tag{85}$$

We consider now a two channel system, limiting ourselves to S waves. Following Dalitz (see above reference), we write:

$$K = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} \tag{86}$$

where α,β,γ are real quantities.

For an incident S wave in channel <u>1</u> we have

$$\begin{aligned} \psi^{(1\to1)}(r) &= \frac{\sin\ k_1 r}{k_1 r} + T_{11}\ \frac{e^{ik_1 r}}{r} \\ \psi^{(1\to2)}(r) &= T_{21}\ \frac{e^{ik_2 r}}{r}\ . \end{aligned} \tag{87}$$

The application of the condition Eq. (83) to (86) and (87) gives:

$$\begin{aligned} T_{11} &= \alpha\ (1+ik_1\ T_{11}) + i\beta k_2\ T_{21} \\ T_{21} &= \beta\ (1+ik_1\ T_{11}) + i\gamma k_2\ T_{21} \end{aligned} \tag{88}$$

from which one gets

$$T_{11} = \frac{A}{1-ik_1 A} \tag{89}$$

with

$$A = \alpha + \frac{ik_2\beta^2}{1 - ik_2\gamma} \tag{90}$$

and

$$T_{21} = \frac{\beta}{(1 - ik_2\gamma)\ (1 - ik_1 A)} \ . \tag{91}$$

From Eqs. (69) and (73), using a complex phase shift $e^{i\alpha} = \eta e^{i\delta}$, one can write

$$T_{11} = \frac{e^{i\alpha_1} \sin \alpha_1}{k_1}$$

from which it follows

$$k_1 \cot \delta_1 = \frac{1}{A} \ . \tag{92}$$

The complex quantity A is called the scattering length for channel 1. At low energy A can be considered independent of the energy. This approximation applied by Dalitz (see Rev.Mod.Phys. 33, 471 (1961)) to the study of the K^-p interactions at low energy, corresponds to a zero effective range. This nomenclature is taken from the effective range theory for one channel scattering; which is well known in Nuclear Physics (see e.g. Blatt and Weisskopf, Chapter II). In this case one expands k cot δ in powers of k^2:

$$k \cot \delta = \frac{1}{a} + \frac{1}{2} r_0^2 k^2 + \dots \tag{92'}$$

The first term "a" is a constant which gives the cross-section at zero energy ($\sigma_0 \to 4\pi a^2$): it is called scattering length. The quantity r_0 which depends only on the form of the potential, is called effective range.

An expansion similar to Eq. (92) is performed for the matrix K^{-1} (which for one channel coincides with k cot δ) in the case of multi-channel processes. This approximation, with effective range different from zero, has been applied, for instance, to the K^-p interactions (see Ross and Shaw, Ann.Phys. 9, 391 (1960)).

We go back now to the two channel system, whose K matrix is given in Eq. (86). For an incident S wave in channel 2, we have, in analogy with Eq. (87):

$$\psi^{(2\to 1)}(r) = T_{12}\, \frac{e^{ik_1 r}}{r}$$

$$\psi^{(2\to 2)}(r) = \frac{\sin k_2 r}{k_2 r} + T_{22}\, \frac{e^{ik_2 r}}{r} \quad . \tag{93}$$

In a similar way, one gets:

$$T_{22} = \frac{B}{1 - ik_2 B} \quad ; \quad k_2 \cot \alpha_2 = \frac{1}{B} \tag{94}$$

with

$$B = \gamma + \frac{ik_1 \beta^2}{1 - ik_1 \alpha} \tag{95}$$

and

$$T_{12} = \frac{\beta}{(1 - ik_1\alpha)\,(1 - ik_2 B)} \quad . \tag{96}$$

One can prove, by comparison, that the relation $T_{12} = T_{21}$ is satisfied.

The corresponding cross-sections are evaluated by means of Eq. (74).

3. The reduced reaction matrix

We have considered in the previous section only open channels. It is instructive to include in our consideration the case of a closed channel.

Going back to the two channel example, suppose that the energy at which we consider the process is below the threshold for channel 1. We can still use the expressions Eqs. (94) and (96) for the amplitudes in channel 2, provided we now take k_1 pure imaginary: we take $k_1 = i|k_1|$, so that the wave function $\psi^{(2\to 1)}(r)$ behaves like $e^{-|k_1|r}$ and is always finite. The amplitude T_{22} is again given by:

$$T_{22}^{-1} = B^{-1} - ik_2 \tag{97}$$

with

$$B = \gamma - \frac{|k_1|\beta^2}{1 + |k_1|\alpha} \quad . \tag{98}$$

These considerations can be extended to the case of n channels, by writing

$$K = \begin{pmatrix} \alpha & \beta \\ \tilde{\beta} & \gamma \end{pmatrix} \tag{99}$$

with

$$\alpha = K_{11} \qquad \tilde{\beta} = \begin{pmatrix} K_{12} \\ K_{13} \\ \vdots \\ K_{1n} \end{pmatrix} \qquad \gamma = \begin{pmatrix} K_{22} & K_{23} & \dots & K_{2n} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ K_{n2} & K_{n3} & & K_{nn} \end{pmatrix}$$

$$\beta = (K_{12} \quad K_{13} \; \dots \; K_{1n})$$

Suppose again that the channel 1 is closed (one can extend these considerations to the case of more channels closed, replacing α by an $m \times m$ matrix, and β by an $m \times n$ matrix). Then, for the T matrix which contains only the open channels, one gets the same formal relation Eq. (97):

$$T_r^{-1} = K_r^{-1} - ik \tag{100}$$

with

$$K_r = \gamma - \tilde{\beta}\ |k|\ \frac{1}{1 + |k|\alpha}\ \beta\ . \tag{101}$$

The matrix K_r, which is called <u>reduced reaction matrix</u>, plays the role of a K matrix for the open channels, since the expression (100) has exactly the same form as Eq. (79). The expression (101) relates the elements of K_r to all the elements of K, connecting in this way the open with the closed channels.

4. Resonances in a multi-channel system

We extend now the formula Eq. (31) obtained for the resonant one-channel scattering amplitude to the case of many channels.

For the sake of simplicity, we consider a system of n channels all open, and replace the K and T matrices by :

$$\begin{aligned} K' &= k^{1/2}\ K\ k^{1/2} \\ T' &= k^{1/2}\ T\ k^{1/2}\ . \end{aligned} \tag{102}$$

The following considerations hold also if there are some closed channels: one has simply to replace the K and T matrices, by the reduced K_r and T_r matrices (see also: Dalitz, Strange particles, etc.).

The K′ matrix can be diagonalized by the eigenvalue equation

$$K'|\alpha\rangle = (\cot\delta_\alpha)^{-1}|\alpha\rangle \quad . \tag{103}$$

The eigenvalues $(\cot\delta_\alpha)^{-1}$ are all real, since K′ is real and symmetric. The same set of eigenstates $|\alpha\rangle$ diagonalizes also the T matrix, as one can see from Eq. (78):

$$T'|\alpha\rangle = (\cot\delta_\alpha - i)^{-1}|\alpha\rangle = e^{i\delta_\alpha}\sin\delta_\alpha|\alpha\rangle \quad . \tag{104}$$

The matrix elements T_{fi} given in Eqs. (74) and (79) correspond, however, to the representation $\langle f|T|i\rangle$ for the T matrix, where $|i\rangle$, $|f\rangle$ are the momentum eigenstates in the i, f channels. Since the eigenstates $|\alpha\rangle$ form a complete orthogonal set, one can write for each channel

$$|i\rangle = \sum_\alpha C_{i\alpha}|\alpha\rangle \tag{105}$$

where the coefficients $C_{i\alpha}$ are also real and satisfy the condition

$$\sum_i C_{i\alpha}\, C_{i\beta} = \delta_{\alpha\beta} \quad . \tag{106}$$

By means of Eqs. (104), and (105) one gets

$$\begin{aligned} T'_{fi} &= \langle f|T'|i\rangle = \sum_{\alpha,\beta} C_{f\beta}\, C_{i\alpha}\langle\beta|T'|\alpha\rangle = \\ &= \sum_\alpha C_{f\alpha}\, C_{i\alpha}\, e^{i\delta_\alpha}\sin\delta_\alpha = \sum_\alpha \frac{C_{f\alpha}\, C_{i\alpha}}{\cot\delta_\alpha - i} \quad . \end{aligned} \tag{107}$$

Suppose now that one of the eigenvalues of K′ becomes infinite at the energe $E = E_R$, i.e. that δ_R passes through $\pi/2$ at E_R. In analogy with the situation examined in the one channel case, we say that the amplitude T'_{fi} has a resonance at the energy E_R. If this resonance is isolated, or in other words if the other phase shifts δ_α $(\alpha \neq R)$ are small around E_R, one can write approximately in this region

$$T'_{fi} \simeq \frac{C_{fR}\, C_{iR}}{\cot \delta_R - i} \quad . \tag{108}$$

In the same region we can make a linear approximation for cot δ_R and write, in analogy with Eq. (24):

$$\cot \delta_R \simeq \frac{2}{\Gamma} (E_R - E) \; . \tag{109}$$

Then, we get

$$T'_{fi} = \frac{C_{fR}\, C_{iR}\, \Gamma/2}{E_R - E - i\, \Gamma/2} \; . \tag{110}$$

If we define:

$$\Gamma_i = C^2_{iR}\, \Gamma$$

since by Eq. (106):

$$\sum_i \Gamma_i = \Gamma$$

the resonant amplitude can also be written as:

$$T'_{fi} = \frac{\tfrac{1}{2}\, \Gamma_f^{1/2}\, \Gamma_i^{1/2}}{E_R - E - i\, \Gamma/2} \quad . \tag{111}$$

The cross-section is obtained by means of Eqs. (74) and (102):

$$\sigma^{(i\to f)}_{\ell_i} = \frac{\pi}{k_i^2} (2\ell_i + 1) \frac{\Gamma_i\, \Gamma_f}{(E - E_R)^2 + \Gamma^2/4} \quad . \tag{112}$$

This is the Breit-Wigner formula, valid for isolated resonances. The quantities Γ_i, Γ_f are the partial widths for the i,f channels: they are in general functions of the energy.

The graphical representation described for the amplitude Eq. (32) can be extended to the present multi-channel case. Defining:

$$\chi_{fi} = \frac{\sqrt{\Gamma_f \Gamma_i}}{\Gamma} \quad , \quad \epsilon = \frac{2}{\Gamma}(E_p - E)$$

the amplitude (111) can be re-written as

$$T'_{fi} = \frac{\chi_{fi}}{\epsilon - i} . \tag{113}$$

The real and imaginary parts of T'_{fi} are given by:

$$x = \text{Re } T'_{fi} = \chi_{fi} \frac{\epsilon}{\epsilon^2 + 1} \tag{114}$$
$$y = \text{Im } T'_{fi} = \chi_{fi} \frac{1}{\epsilon^2 + 1}$$

from which we get

$$x^2 = (\chi_{fi} - y)\, y \tag{115}$$

which is the equation of a circle of radius $R = \frac{1}{2} \chi_{fi}$ and centre at $x = 0$, $y = \frac{1}{2} \chi_{fi}$. The situation is described graphically in Fig. 4 where we have considered a few different values for χ_{fi}; clearly for $\chi_{fi} = 1$ one gets the limiting case valid for one channel. The lines ϵ = const are also here the straight lines $y = 1/\epsilon\, x$.

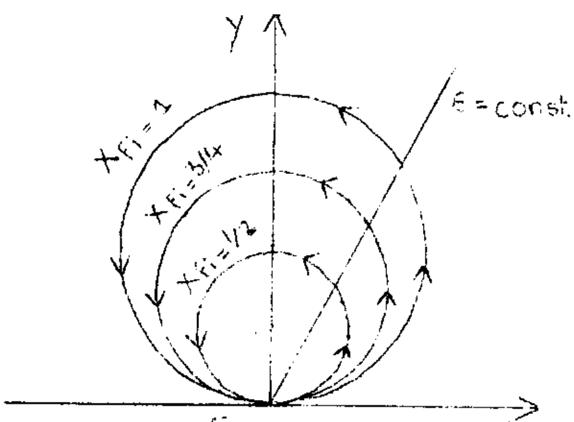


Fig. 4

Before closing this section, we want to point out that the resonances in a multi-channel system can be originated in two different ways:

1) The complete reaction matrix K can have a pole at a real value E_R of the energy, in the sense that each element of K has a pole at E_R. In this case the resonance is present in all the channels and it appears in all the reactions.

An example is given by the $P_{3/2}$ resonance of the $\pi - N$ system, which appears also in the photoproduction process $\gamma + p \to \pi^+ + n$ ($\pi^0 + p$) and in the

Compton scattering $\gamma + p \rightarrow \gamma + p$ (these processes can, in fact, be easily related by multi-channel formalism; (see e.g. Gell-Mann and Watson, Ann. Rev. of Nucl.Sci. 4, 218 (1954)).

2) A pole can be generated at E_R' in the reduced reaction matrix K_r by its connection with a closed channel. This pole does not appear, in general, in the complete K matrix. One example is given by Eq. (94), with B given by Eq. (98). B goes to infinity for

$$1 + \alpha |k_1| = 0 \tag{116}$$

$k_2 \cot \alpha_2$ passes through zero and a resonance appears in channel 2.

Suppose now that the (closed) channel 1 is weakly coupled with channel 2 ($\beta \simeq 0$). The condition Eq. (82) gives in this case

$$(r\psi^{(1\to1)})_{r=0} = \alpha \left[\frac{d}{dr} (r\psi^{(1\to1)}) \right]_{r=0} \tag{117}$$

which coincides with Eq. (116) in the case of a bound state.

We can interpret such a resonance in channel 2, as due to the effect of a strong interaction below threshold in channel 1. If the coupling between the two channels were rigorously zero, there would be a bound state in channel 1; in fact, it is a virtual bound state, since it can go into the open channel 2.

V. REGGE POLES

1. Regge poles in potential scattering

We again consider scattering by a central potential and go back to expression Eq. (10), which we re-write here giving explicitly the dependence on the energy E and on $z = \cos \vartheta$:

$$f(z,E) = \sum_{\ell=0}^{\infty} (2\ell + 1)\; f_\ell(E)\; P_\ell(z) \; . \tag{118}$$

It is possible to transform the sum over ℓ into an integral, considering ℓ as a continuous complex variable (Sommerfeld-Watson transformation):

$$f(z,E) = \frac{i}{2} \int_C \frac{(2\lambda+1)}{\sin \pi\lambda} f(\lambda,E)\; P_\lambda(-z)\; d\lambda \quad . \tag{119}$$

The partial wave amplitude $f_\ell(E)$ has been replaced by $f(\lambda,E)$ and λ replaces the discrete variable ℓ. The integral is performed in the complex λ-plane along the path C taken around the real axis (closed at infinity), as shown in Fig. 5.

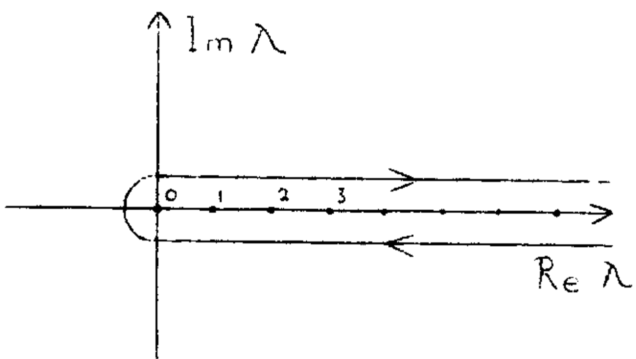


Fig. 5

In order to prove that the two expressions Eqs. (118) and (119) are indeed the same, we use the Chauchy theorem, which states that the integral of a meromorphic function along a closed path (in the clockwise sense) is equal to $(2\pi i)$ times the sum of the residues of the poles which are included in the path. For instance, in the case of only one pole at $z = z_0$ inside the closed region, one has for the analytic function $f(z)$:

$$\oint \frac{f(z)}{z - z_0}\, dz = 2\pi i\; f(z_0) \; .$$

One sees that the expression Eq. (119) has poles at all the integral values of $\lambda = \ell = 0, 1, \ldots$ $(\sin \pi\ell = 0)$, (other possible poles of $f(\lambda,E)$ on the real axis have to be excluded by the path). Around one of these poles, the integrand can be written as

$$\frac{i(2\lambda+1)\; f(\lambda,E)\; P_\lambda(-z)}{(-1)^\ell\; 2\pi\; (\lambda - \ell)}$$

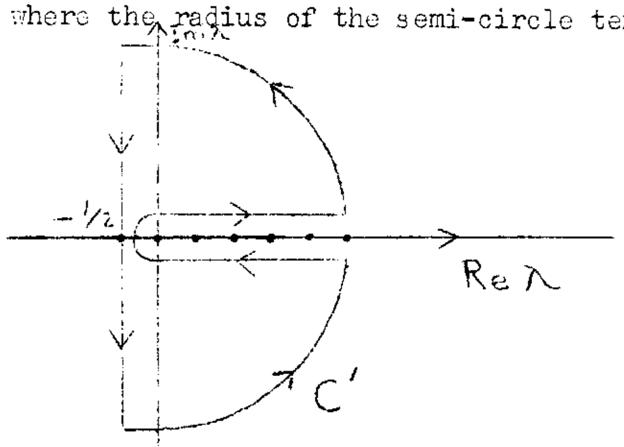
and the residue is

$$\frac{i}{2\pi}\,(2\ell+1)\; f_\ell(E)\; P_\ell(z)$$

which gives back the formula Eq. (118).

We consider the expression Eq. (119), but we take now the path C′ represented in Fig. 6, where the radius of the semi-circle tends to infinity.

Fig. 6



The poles on the real axis for integer λ are now outside the path, so that the integral along the new path C′ will be equal to $(2\pi i)$ times the residues of the eventual poles of $f(\lambda,E)$. Assuming that there are n such poles at λ_i $(i = 1, \dots n)$, we can write in general:

$$\frac{i}{2}\int_{C'} \frac{(2\lambda+1)}{\sin \pi\lambda} f(\lambda,E)\ P_\lambda(-z)\ d\lambda = \sum_{i=1}^{n} \frac{(2\lambda_i + 1)\ \varphi_i(E)}{\sin \pi\lambda_i} P_{\lambda_i}\ (-z) \qquad (120)$$

where the φ_i are the residues of $f(\lambda,E)$. It has been proven (see e.g. T. Regge, in the Proc. of the 1961 Herceg Novi Summer School) that for a rather general class of potentials (superposition of Yukawa potentials), the integral over the semi-circle at infinity vanishes, so that Eq. (120) can be re-written as:

$$\frac{i}{2}\int_{C} \frac{(2\lambda+1)}{\sin \pi\lambda} f(\lambda,E)\ P_\lambda(-z)\ dz - \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \frac{(2\lambda+1)}{\sin \pi\lambda} f(\lambda,E)\ P_\lambda(-z) d\lambda =$$

$$= \sum_{i} \frac{(2\lambda_i + 1)\ \varphi_i(e)}{\sin \pi\lambda_i} P_{\lambda_i}\ (-z) \qquad (121)$$

and by use of Eq. (119):

$$f(z,E) = \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \frac{(2\lambda+1)}{\sin \pi\lambda} f(\lambda,E)\ P_\lambda(-z)\ d\lambda = \sum_{i} \frac{(2\lambda_i + 1)\ \varphi_i(E)}{\sin \pi\lambda_i} P_{\lambda_i}\ (-z)\ .$$

It has also been proved that the poles of $f(\lambda,E)$ lie in the upper half plane of λ, and their number is limited. The position of such poles, which are called Regge poles, depends on the energy: $\lambda_i(E)$.

In order to clarify the meaning of such poles, we consider the contribution of a single Regge pole on the scattering amplitude, by writing simply:

$$f(z,E) \simeq (2\lambda+1)\,\frac{\varphi(E)}{\sin \pi\lambda(E)}\,P_{\lambda(E)}\,(-z)\ . \qquad (123)$$

If one projects out from $f(z,E)$ the partial wave amplitude $f_\ell(E)$, one obtains

$$f_\ell(E) = \frac{1}{2}\int_{-1}^{1} f(z,E)\,P_\ell(z)\,dz = \frac{1}{\pi}\,\frac{(2\lambda(E)+1)\ \varphi(E)}{(\lambda(E)+\ell+1)(\lambda(E)-1)} \qquad (124)$$

Suppose now that at $E = E_R$: $\mathrm{Re}\,\lambda(E_R) = \ell$ and $\mathrm{Im}\,\lambda(E_R)$ is small; around E_R we can then expand $\mathrm{Re}\,\lambda(E)$ as follows:

$$\mathrm{Re}\,\lambda(E) \simeq \ell + \left(\frac{d\,\mathrm{Re}\lambda}{dE}\right)_{E=E_R} (E-E_R) \qquad (125)$$

and Eq. (124) becomes

$$f_\ell(E) \simeq \frac{1}{\pi}\,\frac{\varphi(E_R)}{\left(\frac{d\,\mathrm{Re}\lambda}{dE}\right)_{E_R}(E-E_R) + i\ \mathrm{Im}\lambda(E_R)}\ . \qquad (126)$$

By comparison of this relation with the formula Eq. (31) for a resonant amplitude, we see that Eq. (126) can be interpreted as describing a resonance in the ℓ-th partial wave at the energy E_R and with the width

$$\Gamma = 2\ \mathrm{Im}\,\lambda(E_R) \Big/ \left(\frac{d\,\mathrm{Re}\,\lambda}{dE}\right)_{E=E_R}\ . \qquad (127)$$

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It is interesting to note that in Eq. (31) where ℓ is a real integer, the resonance appears as a pole in the scattering amplitude at the complex energy

$$E = E_R - i\,\frac{\Gamma}{2} \quad .$$

On the other hand, in Eq. (126) the resonance appears, for a real value of the energy, as a pole at the complex value of λ:

$$\lambda(E_R) = \ell + i \;\mathrm{Im}\; \lambda(E_R) \quad .$$

In general, the quantity $\lambda(E)$ moves with the energy on the complex λ-plane, describing a trajectory (<u>Regge trajectory</u>). We can say that a Regge pole represents a resonance whenever $\lambda(E)$ goes close to real integral values ℓ. One sees that a Regge pole can then connect different resonances occurring at different values of ℓ. When the energy becomes negative, one can show that Im λ = 0, and to an integral value of λ there corresponds a bound state.

2. Classification of the elementary particles on Regge trajectories

The concepts related to the Regge poles, derived in potential theory, have been extended to the strong interactions of the elementary particles by Chew and Frautschi (Phys.Rev.Lett. 8, 41 (1962)). Their conjecture is that all baryons and mesons (stable and unstable) are associated with Regge poles (poles of an S matrix which describes the strong interactions) which move on the complex angular momentum plane as functions of the energy (λ represents now the complex values of the spin J of the particle). The trajectory of a particular pole is characterized by a set of quantum numbers (isotopic spin, hypercharge, etc.) and by the evenness or oddness of the physical values of J for mesons and J- ½ for baryons. Below the threshold for the lowest channel with the given set of quantum numbers, one has: Im λ = 0. The stable and unstable particles occur when Re (E_R) = J, where E_R is the rest energy of the particle. The quantity Im $\lambda \neq 0$ gives the width for an unstable particle, according to Eq. (117).

Using these conjectures, all the strongly interacting particles can be classified on Regge trajectories. We report in Figs. 7 and 8 the Chew-Frautschi plot (Re λ is drawn versus the squared mass) as given by Rosenfeld (Proc. of the 1962 Int.Conf. on High-Energy Physics at CERN).

All the trajectories corresponding to the known particles lie below a trajectory which passes through J = 0, μ^2 = 0: this corresponds to the limiting trajectory consistent with the unitarity and analyticity of the S matrix. The physical state on this trajectory at J = 2 has been attributed to the f° meson (π-π resonance at ~ 1250 MeV).

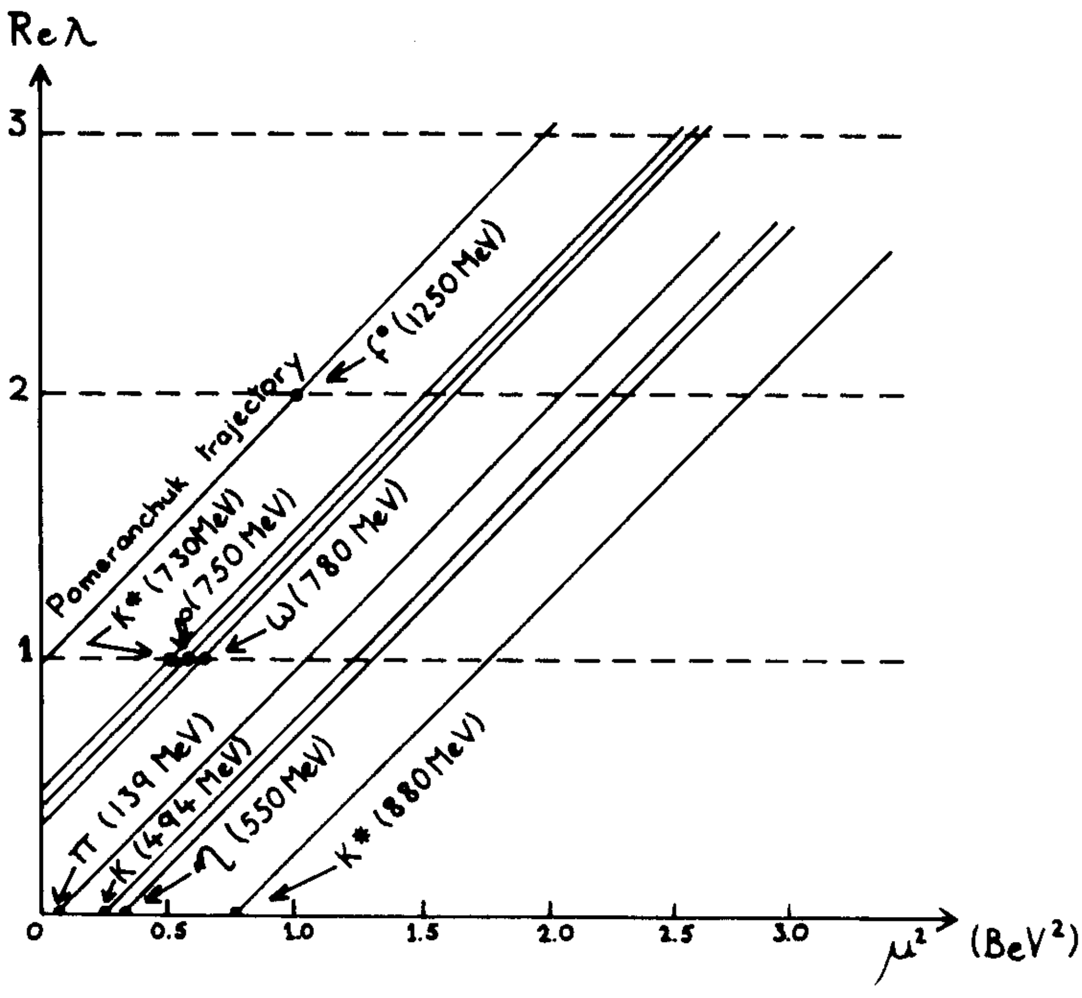

Fig. 7

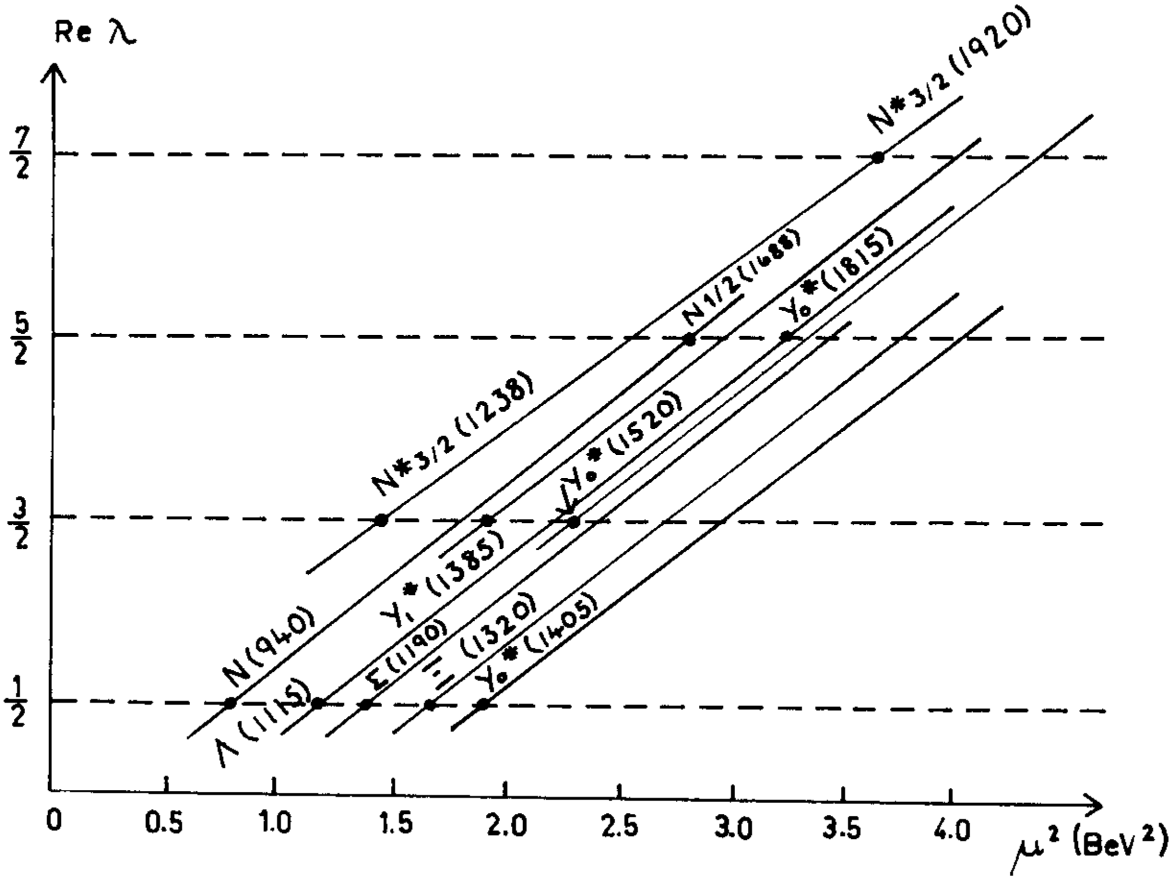
Re λ
7/2
5/2
3/2
1/2
0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
μ² (BeV²)
N*3/2 (1920)
N1/2 (1688)
Y₀* (1815)
N*3/2 (1238)
Y₀* (1520)
Y₁* (1385)
N (940)
Λ (1115)
Σ (1190)
Ξ (1320)
Y₀* (1405)

Fig. 8

3. High energy behaviour of the cross-sections

We will briefly consider here the implications of the Regge conjecture to the high-energy behaviour of the cross-sections.

We use here an invariant notation for the scattering amplitudes and the related variables (see also: G. Chew, in "Dispersion Relations", 1960 Scottish University Summer School). It is clear that two independent kinematical variables are sufficient for the scattering process

$$A + B \rightarrow A' + B' \qquad (I)$$

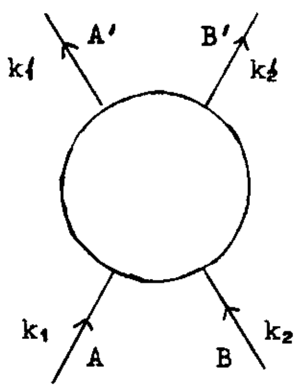

Fig. 9

which we represent graphically in Fig. 9. The two variables can be taken, for example, as the momentum k and the scattering angle ϑ in the c.m. system. The following set of variables is useful (we use the metric $k \equiv (k_0, i\vec{k})$):

$$\begin{aligned} s &= (k_1 + k_2)^2 = (k_1' + k_2')^2 \\ t &= (k_1 - k_1')^2 = (k_2 - k_2')^2 \\ u &= (k_1 - k_2')^2 = (k_2 - k_1')^2 \end{aligned} \qquad (128)$$

In the case of identical particles (with mass μ) they can be written as:

$$\begin{aligned} s &= 4(k^2 + \mu^2) \\ t &= -2k^2 \; (1 - \cos \vartheta) \\ u &= -2k^2 \; (1 + \cos \vartheta) \end{aligned} \qquad (129)$$

Of course only two of these variables are independent; in fact, the following relation holds

$$s + t + u = 4\mu^2 \; . \qquad (130)$$

For the reaction (I), s represents the squared energy in the c.m. system ($s > 0$), and t the squared momentum transfer ($t \leq 0$). The invariant scattering amplitude is defined by a function $A(s,t)$, which is related to the usual scattering amplitude $f(k,\vartheta)$ by

$$A(s,t) = \frac{\sqrt{s}}{2} f(k,\vartheta) \; . \tag{131}$$

The process

$$A + \bar{A}' \rightarrow \bar{B} + B' \tag{II}$$

where $\bar{A}'$, $\bar{B}$ stand for the antiparticles of A', B, is represented by the same graph of Fig. 9, if one considers k_1, $\bar{k}_1' = -k_1'$ as ingoing, and k_2', $\bar{k}_2 = -k_2$ as outgoing four-momenta. In this case one has:

$$\begin{aligned} t &= (p_1 + \bar{p}_1')^2 = 4(q^2 + \mu^2) \\ s &= (p_1 - \bar{p}_2)^2 = -2q^2(1 - \cos\bar{\vartheta}) \; . \end{aligned} \tag{132}$$

The variable t has now the role of the squared energy ($t > 0$), and s of the squared momentum transfer ($s \leq 0$) in the c.m. system for the reaction (II); q and $\bar{\vartheta}$ are the corresponding momentum and scattering angle.

The invariant scattering amplitude for the process (II) is again given by the function $A(s,t)$ in which the roles of the variables s,t have been interchanged with respect to the reaction (I). We can say that the same amplitude $A(s,t)$ represents the two reactions (I) and (II) when the variables s,t are defined in the two different physical regions Eqs. (129) and (132).

We assume now that the scattering amplitude has a Regge pole behaviour in the physical region for the reaction (II). We write, in analogy with Eq. (123):

$$A(s,t) = \frac{\beta(t)}{\sin \pi\lambda(t)} P_{\lambda(t)}(-\cos\bar{\vartheta}) \tag{133}$$

with

$$\cos\bar{\vartheta} = 1 - \frac{2s}{4\mu^2 - t} \tag{134}$$

and including in $\beta(t)$ terms which it is not necessary to specify here (for more details and other references see: Drell, Proc. of the 1962 Int.Conf. on High-Energy Physics at CERN, page 897).

The same amplitude describes the reaction (I) for $s > 0$, $t \leq 0$. For large values of s (high-energy for the reaction (I)), we can use the asymptotic expansion for the Legendre polynomials ($P_\ell(z) \sim z^\ell$ for large z) and write:

$$P_{\lambda(t)}(-\cos\bar{\vartheta}) \simeq P_{\lambda(t)}\left(\frac{2s}{4\mu^2 - t}\right) \simeq g(t)\ s^{\lambda(t)} \quad . \tag{135}$$

The scattering amplitude becomes

$$A(s,t) \simeq \frac{\beta(t)g(t)}{\sin\pi\lambda(t)}\ s^{\lambda(t)} \simeq a(t)\ s^{\lambda(t)} \tag{136}$$

and the differential cross-section for the reaction (I) can be written as:

$$\frac{d\sigma}{d\Omega} \simeq 4|a(t)|^2\ s^{2\lambda(t)-1} \quad . \tag{137}$$

By use of

$$\cos\vartheta = 1 - \frac{2t}{4\mu^2 - s} \simeq 1 + \frac{2t}{s} \tag{138}$$

one can also write

$$\frac{d\sigma}{dt} \simeq 16\pi\ |a(t)|^2\ s^{2(\lambda(t)-1)} \quad . \tag{139}$$

The optical theorem allows us to evaluate the total cross-section (t = 0 in the forward direction):

$$\sigma_{tot} \simeq \frac{16\pi}{s}\ \mathrm{Im}\ A(s,0) \simeq 16\pi\ \mathrm{Im}\ a(0)\ s^{\lambda(0)-1} \quad . \tag{140}$$

The experimental indication that the total cross-sections become constant at very high-energy implies:

$$\lambda(0) = 1 \quad . \tag{141}$$

This is equivalent to assuming the existence of a Regge pole which dominates all the amplitudes at very high energies; this pole corresponds to the higher trajectory (Pomeranchuk trajectory) considered in the previous section. Near $t = 0$ we can expand $\lambda(t)$ and, assuming that Eq. (141) holds, we get

$$\lambda(t) \simeq 1 + \lambda'(0)t \ . \qquad (142)$$

The differential cross-section Eq. (139) for small t ($t \leq 0$) can be written:

$$\frac{d\sigma}{dt} \simeq 16\pi \ |a(t)|^2 \ e^{2\lambda'(0)t\log s} \ . \qquad (143)$$

Theoretical conjectures lead to $\lambda'(0) > 0$. Then, since t is negative, one sees that the differential cross-section presents a forward peak which decreases exponentially with t and shrinks with increasing energy.

It is interesting to compare this behaviour with the diffraction peak obtained in the case of scattering by a black sphere. From Eq. (62) we get:

$$\frac{d\sigma}{d\Omega} \simeq \left| \frac{RJ_1(kR \sin \vartheta)}{\sin \vartheta} \right|^2 \ . \qquad (144)$$

For small angles one gets from Eq. (138):

$$2k \sin \vartheta/2 \approx k\vartheta \simeq \sqrt{-t}$$

and Eq. (144) can be re-written for large s, in the form

$$\frac{d\sigma}{dt} \simeq \pi \left| \frac{J_1(R\sqrt{-t})}{\sqrt{-t}} \right|^2 \ . \qquad (145)$$

For reasonable values of R($R \simeq 1$ fermi) and small values of t, this formula can be approximated by

$$\frac{d\sigma}{dt} \simeq \frac{\pi R^2}{4} \ \exp\left[\left(\frac{R}{2}\right)^2 t\right] \ . \qquad (146)$$

We see that in this case the forward peak decreases exponentially, but it is independent of the energy.

The behaviour given by Eq. (143) is a particular feature of the Regge pole hypothesis which, however, does not seem to be exhibited by the experimental results, at least at the available energies ($E \simeq 20$ GeV). Maybe the asymptotic behaviour given by the leading Regge pole will appear in the experiments only at higher energies.