Unitarity, Analyticity and Crossing Symmetry in Two- and Three-Hadron Final State Interactions

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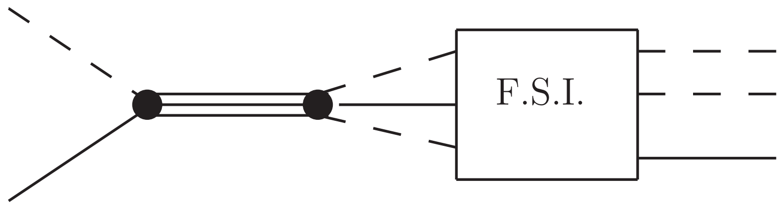
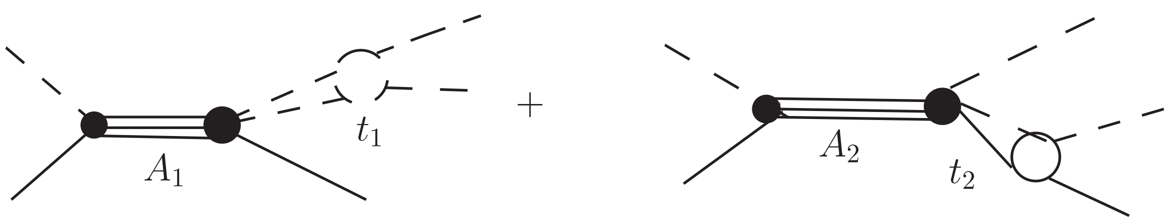
N.B. Fuller write-up available on Workshop Resources

Motivation



"Isobar Model"



$|A_1 t_1 + A_2 t_2|^2$ t_i two-hadron amplitudes

Interference term Re$[(A_1 A_2^*)(t_1 t_2^*)]$

Extract phase of $A_1 A_2^*$ IF know phase of $t_1 t_2^*$

But rescattering alters phases

Similarly for extraction of weak CP-violating phases

Three-body problem!

"Minimum Theory of FSI"

Lecture 1: **Two-hadron fsi**

Unitarity → K-matrix, P-vector

+ Analyticity → M-O solution, dispersion relations

Lectures 2, 3, 4: **Three-hadron fsi**

2: Kinematics, Dalitz plot, isobar model; sub-energy unitarity; violated by isobar model; U+analyticity → K-T type equations for isobar correction factors

3: Single variable integral equation for correction factors; RPE process; examples ($\pi\pi\pi$, $\pi\pi N$)

4: 3-body unitarity; particle-resonance scattering

1. Two-hadron fsi

1.1 Elastic 2→2 unitarity, s-waves

$$T(s) - T^*(s) = 2\mathrm{i}\rho(s)T^*(s)T(s)$$

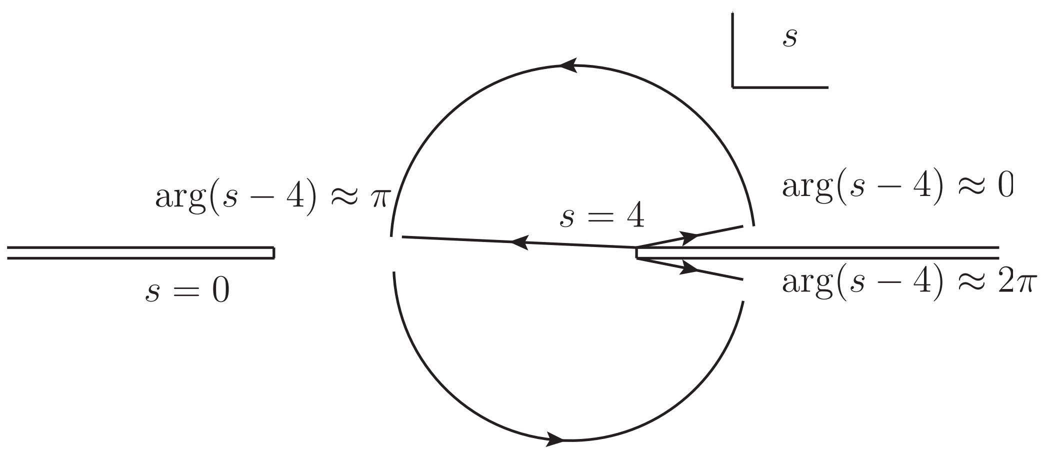
$$\rho(s) = \frac{1}{16\pi}\left(\frac{s-4}{s}\right)^{1/2}$$

$$T(s) = \mathrm{e}^{\mathrm{i}\delta}\sin\delta/\rho = (\rho\cot\delta - \mathrm{i}\rho)^{-1}$$

BW: $\rho\cot\delta = (s_{\mathrm{R}} - s)/g^2$

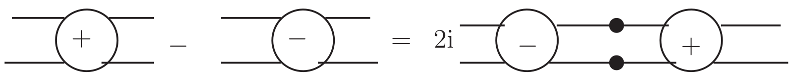
$$T_{\mathrm{res}} = g^2/(s_{\mathrm{R}} - s - \mathrm{i}\rho(s)g^2)$$

will allow s to be a complex variable



Physical limit for $s \geq 4$ is $s + \mathrm{i}\epsilon$ (s_+)
Hermitian analyticity: $f^*(s) = f(s^*)$

unitarity $T_+ - T_- = 2\mathrm{i}\rho T_+ T_-$



unitarity tells us the discontinuity of T across the $s \geq 4$ cut

U: $T_+^{-1} - T_-^{-1} = -2\mathrm{i}\rho$
Since $\rho_+ - \rho_- = 2\rho$, can satisfy U by

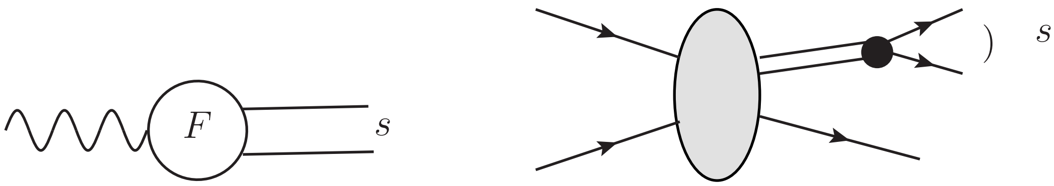
$$T^{-1} = K^{-1} - \mathrm{i}\rho$$

$K^{-1} \equiv \rho \cot \delta$; e.g. $K_{\text{res}} = g^2/(s_{\text{R}} - s)$

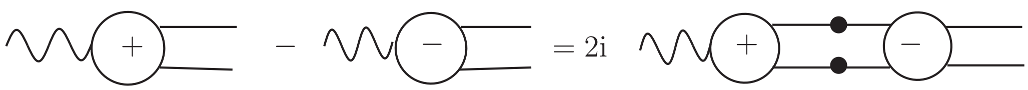
$$T = K(1 - \mathrm{i}\rho K)^{-1} = (1 - \mathrm{i}K\rho)^{-1}K$$

generalizes to matrices in space of channels
Plus can add resonances in K, add background .... still T is unitary!
e.g. $K_{ij} = \sum_a g_{ia} g_{aj}/(s_a - s) + B_{ij}$

1.2 Unitarity in 2-hadron f.s.i.



$$F_{+} - F_{-} = 2\mathrm{i}T_{+}\rho F_{-} = 2\mathrm{i}T_{-}\rho F_{+}$$



$$\mathrm{Im}F = T\rho F^{*} = T^{*}\rho F$$

So unitarity $\rightarrow$ F must have the phase of T

(Watson's Thm)

provided only one strong f.s.i.

$$F_+ - F_- = 2\mathrm{i}T_-\rho F_+ \Rightarrow (1 - 2\mathrm{i}T_-\rho)F_+ = F_-$$

But $T_- = (1 + \mathrm{i}K\rho)^{-1}K$. So substituting,

$$\frac{1}{1+\mathrm{i}K\rho}(1 - \mathrm{i}K\rho)F_+ = F_-$$

which by inspection is satisfied by

$$\boxed{F_+ = \frac{1}{1-\mathrm{i}K\rho}P}$$

where P has no branch point at $s = 4$.

Generalizes again to many channels.

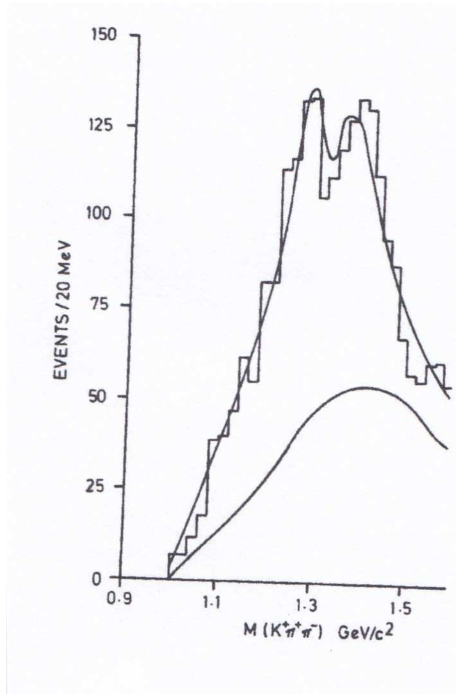
P is a vector (which f.s. channel).

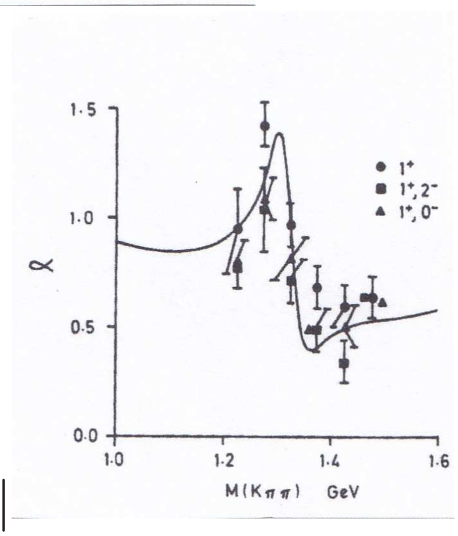
e.g. $P_i = \sum_a g_{ia}\frac{1}{s_a - s}f_{ap} + B_i$.

Note P also has the resonance poles

IJRA Nucl. Phys. A **189** 417 (1972)

$K_1(1270), K_2(1400)$ $P =$ two resonances + Deck B M. G. Bowler *et al.* Nucl. Phys. B97, 227 (1975)





$\alpha = |F_{\rho K}/F_{K^*\pi}|$

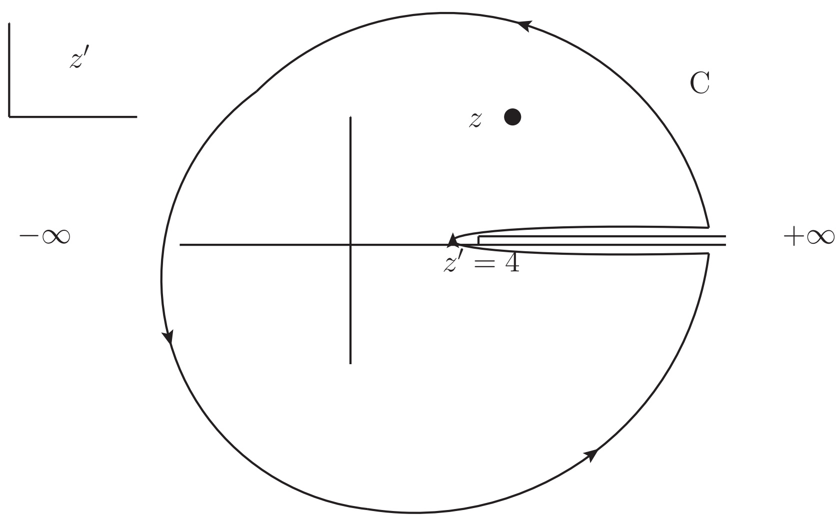
See lectures by Ed Berger!

1.3 Unitarity + analyticity

1.3.1 Elastic 2 → 2 reactions

If $f(s)$ has ONLY the $s = 4$ branch point,

$$f(s) = \frac{1}{2\pi \mathrm{i}} \int_4^\infty \frac{f(s'_+) - f(s'_-)}{s' - s} \mathrm{d}s'$$



disc$f$ i.e. unitarity + analyticity will determine f (assuming convergence)

Indeed, disc$T^{-1} = -2\mathrm{i}\rho$. So maybe

$$T^{-1}(s) \stackrel{?}{=} -\frac{1}{\pi}\int_4^\infty \frac{1}{16\pi}\sqrt{\frac{s'-4}{s'}}\frac{\mathrm{d}s'}{s'-s} \equiv I(s)$$

But the integral diverges. So add one parameter, value of $T^{-1}(s)$ at some point s_0. Then $T^{-1}(s) = I(s_0) + [I(s) - I(s_0)]$ and the "subtracted" integral converges. Convenient to take $s_0 = 4$. Then $T^{-1}(s) = \text{constant} + L(s)$ where

$$L(s) = \frac{1}{16\pi^2}\sqrt{\frac{s-4}{s}}\ln\left(\frac{\sqrt{s-4}+\sqrt{s}}{-\sqrt{s-4}+\sqrt{s}}\right)$$

and $\mathrm{Im}\ln = -\pi$ for s real, > 4

(Chew-Madelstam function) Could still satisfy U if replace "constant" by function with no RH cut e.g. K^{-1}. K-matrix with C-M phase space.

Actually $T(s)$ has "LH" cut $s \leq s_{\mathsf{L}}$ as well

$$T(s) = \frac{\mathrm{e}^{\mathrm{i}\delta}\sin\delta}{\rho} = \frac{N(s)(\mathsf{L})}{D(s)(\mathsf{R})}$$

$$D_{+} - D_{-} = N(T_{+}^{-1} - T_{-}^{-1}) = -2\mathrm{i}\rho N$$

$$= -2\mathrm{i}D_{+}\mathrm{e}^{\mathrm{i}\delta}\sin\delta$$

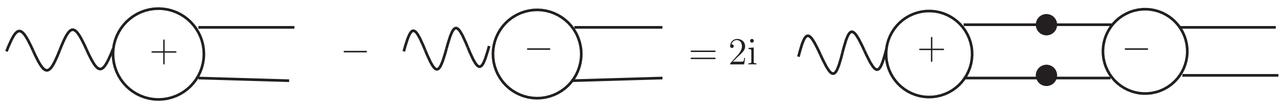
$$\Rightarrow D_{+} = D_{-}\mathrm{e}^{-2\mathrm{i}\delta}$$

Take log of both sides

$$\boxed{D(s_{+}) = \exp\{-\textstyle\int_{4}^{\infty} \frac{\delta(s')}{s'-s_{+}}\,\mathrm{d}s'\}}$$

N.B. $\frac{1}{s'-s-\mathrm{i}\epsilon} = \frac{P.V.}{s'-s} + \mathrm{i}\pi\delta(s'-s)$

1.3.2 Two-hadron fsi



$$D_+ = D_- \mathrm{e}^{-2\mathrm{i}\delta}$$

$$F_+ = (1 + 2\mathrm{i}T_+\rho)F_- = \mathrm{e}^{2\mathrm{i}\delta}F_-$$

Hence $F_+D_+ = F_-D_- \Rightarrow F(s) = C(s)/D(s)$ where $C(s)$ is regular at $s = 4$.

Now suppose we want to include "background" term (e.g. Deck) with L cut In this case, can write

$$F(s) = B(s) + \frac{1}{\pi}\int_4^\infty \frac{T^*(s')\rho(s')F(s')}{s'-s}\,\mathrm{d}s'$$

which is an *integral equation* for F. Remarkably, exact solution exists (Muskhelishvili-Omnès)

M-O solution:

$$\mathrm{disc_R}(DF-DB) = D_+F_+ - D_-F_- - (D_+ - D_-)B$$
$$= -(D_+ - D_-)B = 2\mathrm{i}\rho NB.$$

Hence

$D_+(F_+ - B) = \frac{1}{\pi}\int_4^\infty \frac{\rho(s')N(s')B(s')}{s'-s-\mathrm{i}\epsilon}\,\mathrm{d}s'$

or

$$\boxed{F(s) = B(s) + \frac{1}{\pi D(s)}\int_4^\infty \frac{\rho' N' B'}{s'-s-\mathrm{i}\epsilon}\,\mathrm{d}s'} + C(s)/D(s)$$

Muskhelishvili-Omnès solution

M.G.Bowler *et al.* N.P. B97, 227 (1975); Deck + direct + rescattering for $a_1(1260)$