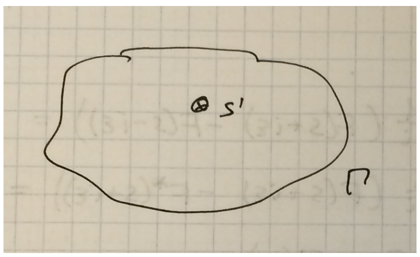
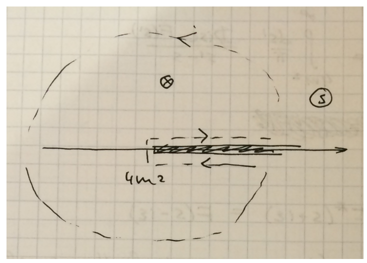
Indiana's school lecture (black board);
prepared by I.V.Danilkin (JLab, June, 2015)

Literature:

[1] M. E. Peskin and D. V. Schroeder. An Introduction to Quantum Field Theory. Westview Press, 1995.
[2] Bastian Kubis lectures https://www.jlab.org/conferences/asi2012/Kubis/Kubis_lectures123.pdf
[3] F. J. Yndurain. Low-energy pion physics. arXiv: hep-ph/0212282, 2002.

Dispersion relation (10 min)





Cauchy theorem

$$F(s) = \frac{1}{2\,i} \int_{\Gamma} \frac{d\,s'}{\pi} \frac{F(s')}{s'-s} \tag{1}$$

for holomorphic function $F(s)$ and Γ is a rectifiable path. We can always deform the contour until the closest singularity.
If $F(s \to \infty) \to 0$ on the large semi-circle then

$$\begin{aligned} F(s) &= \frac{1}{2\,i} \int_{\Gamma} \frac{d\,s'}{\pi} \frac{F(s')}{s'-s} \to \frac{1}{2\,i} \int_{4\,m^2}^{\infty} \frac{d\,s'}{\pi} \frac{F(s'+i\,\epsilon)}{s'-s} - \frac{1}{2\,i} \int_{4\,m^2}^{\infty} \frac{d\,s'}{\pi} \frac{F(s'-i\,\epsilon)}{s'-s} = \\ & \frac{1}{2\,i} \int_{4\,m^2}^{\infty} \frac{d\,s'}{\pi} \frac{F(s'+i\,\epsilon)-F(s'-i\,\epsilon)}{s'-s} = \frac{1}{2\,i} \int_{4\,m^2}^{\infty} \frac{d\,s'}{\pi} \frac{\text{Disc}\,F(s')}{s'-s} \\ & \text{Disc}\,F(s) = F(s+i\,\epsilon) - F(s-i\,\epsilon) \end{aligned} \tag{2}$$

Analyticity:

$$F(s) = \frac{1}{2\,i} \int_{4\,m^2}^{\infty} \frac{d\,s'}{\pi} \frac{\text{Disc}\,F(s')}{s'-s} \tag{3}$$

Reflection principle:

$$F^*(s+\,i\,\epsilon) = F(s-i\,\epsilon) \tag{4}$$

allows to relate imaginary part of the amplitude (unitarity) to analytical continuation of the amplitude (i.e. dispersion relation)

Exercise:

$$\text{Disc}\,F(s) = F(s+i\,\epsilon) - F(s-i\,\epsilon) = F(s+i\,\epsilon) - F^*(s+i\,\epsilon) = 2\,i\,\text{Im}\,F(s) \tag{5}$$

Subtractions: In general, one can always introduce subtractions

$$F(s) = \int_{4m^2}^{\infty} \frac{ds'}{\pi} \frac{\operatorname{Im} F(s')}{s'-s} = \int_{4m^2}^{\infty} \frac{ds'}{\pi} \left(\frac{s'-s+s-s_0}{s'-s_0}\right) \frac{\operatorname{Im} F(s')}{s'-s} = \int_{4m^2}^{\infty} \frac{ds'}{\pi} \frac{\operatorname{Im} F(s')}{s'-s_0} + \frac{(s-s_0)}{\pi} \int_{4m^2}^{\infty} \frac{ds'}{s'-s_0} \frac{\operatorname{Im} F(s')}{s'-s} \quad (6)$$

As a result we got an improved convergence at ∞ (due to additional power of s' in the denominator); sum rule (when the integral converges)

$$F(s_0) = \int_{4m^2}^{\infty} \frac{ds'}{\pi} \frac{\operatorname{Im} F(s')}{s'-s_0} \quad (7)$$

Exercise: If the integral does not converge sufficiently fast on the large semi-circle, derive dispersive relation for (assuming $F(s_0)$ is Real)

$$\frac{F(s) - F(s_0)}{s - s_0} \quad (8)$$

General formula. Note that by introducing subtractions you improve the convergence at $s \to \infty$, however there are additional parameters to determine.

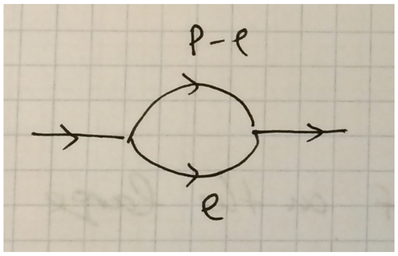
$$F(s) = \sum_{i=0}^{n-1} \frac{1}{i!} F^{(i)}(s_0)\,(s-s_0)^i + \frac{(s-s_0)^n}{\pi} \int_{4m^2}^{\infty} \frac{ds'}{(s'-s_0)^n} \frac{\operatorname{Im} F(s')}{s'-s} \quad (9)$$

Calculation of the Disc (10 min)

Instead of making full loop calculations one can use **Cutkosky (cutting) rule:**

$$\frac{1}{p^2 - m^2 + i\,\epsilon} \to -2\,\pi\,i\,\delta\left(p^2 - m^2\right) \quad (10)$$

Example:



$$i\,M = \int \frac{d^4 l}{(2\pi)^4} \frac{1}{\left(l^2 - m^2 + i\,\epsilon\right)\left((p-l)^2 - m^2 + i\,\epsilon\right)}$$

$$\operatorname{Disc} M = (-i) \int \frac{d^4 l}{(2\pi)^4} (-2\,\pi\,i)\,\delta\left(l^2 - m^2\right)(-2\,\pi\,i)\,\delta\left((p-l)^2 - m^2\right)$$

$$d^4 l = d\,l_0\,\boldsymbol{l}^2\,d\,\boldsymbol{l}\,d\Omega,\;\; \delta\left(l^2 - m^2\right) = \delta\left(l_0 - \boldsymbol{l}^2 - m^2\right) = \frac{\delta\left(l_0 - \sqrt{\boldsymbol{l}^2 + m^2}\right)}{2\,l_0}$$

$$\delta\left((p-l)^2 - m^2\right) = \delta\left(p^2 + l^2 - 2\,p\,l - m^2\right) = \delta\left(s - 2\sqrt{s}\;l_0\right),$$

$$p\,l = p_0\,l_0 - \boldsymbol{p}\,\boldsymbol{l} = \sqrt{s}\;l_0,\;\; \text{cm}:\; p = \left(\sqrt{s}\,,0\right),\;\; l = (l_0, \boldsymbol{l})$$

$$\text{Disc}\, M = \frac{i}{4\pi^2}\int \frac{\boldsymbol{l}^2\, d\boldsymbol{l}}{2\, l_0}\, d\Omega\, \delta\left(s - 2\sqrt{s}\; l_0\right)$$

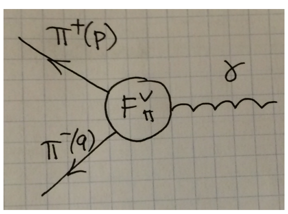
$$\boldsymbol{l}^2\, d\boldsymbol{l} = \sqrt{l_0^2 - m^2}\; l_0\, d\, l_0,\;\; \delta\left(s - 2\sqrt{s}\; l_0\right) = \frac{\delta\left(l_0 - \frac{\sqrt{s}}{2}\right)}{2\sqrt{s}}$$

$$\text{Disc}\, M = \frac{i}{8\pi}\sqrt{1 - \frac{4\, m^2}{s}} = 2\, i\, \text{Im}\, M$$

Therefore

$$\text{Im}\, M = \frac{1}{16\pi}\sqrt{1 - \frac{4\, m^2}{s}} = \frac{1}{16\pi}\rho(s) \tag{12}$$

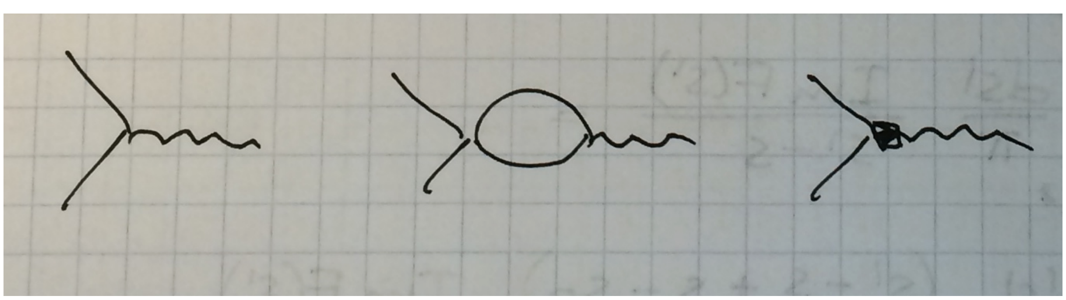
Pion vector form factor (15 min)



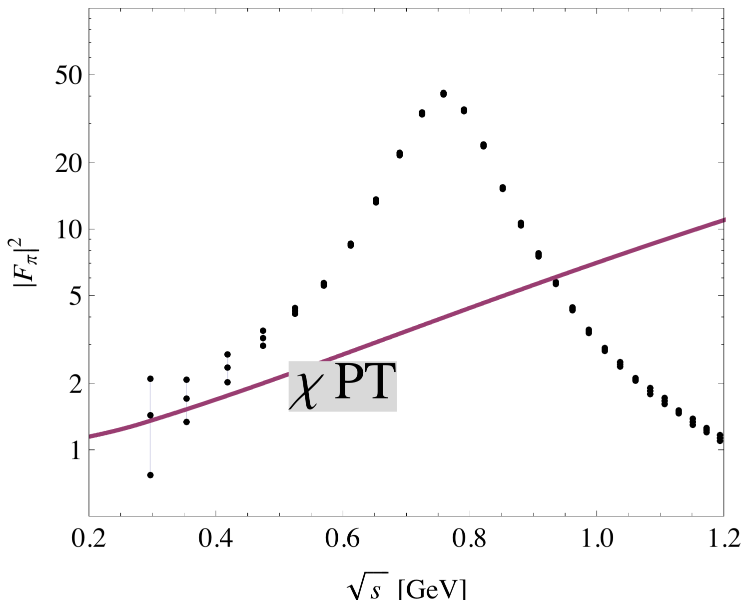
We consider the process of a transition of a photon into a pair of pions. This is an important building block: resposible for a hadronic part of $e^+ e^- \to \pi^+ \pi^-$, $\tau^- \to \pi^- \pi^0 \nu_\tau$, ... The matrix element:

$$\left\langle \pi^+(p)\, \pi^-(q) \,\middle|\, J_\mu(0) \,\middle|\, 0 \right\rangle = (p - q)_\mu\, F_\pi^V(s) \tag{13}$$

where J_μ is the EM current and $F_\pi^V(s)$ is the pion vector form factor (normalized $F_\pi^V(0) = 1$). This process does not have any crossed channel exchanges and therefore no left-hand cuts. Note also, that the factor $(p - q)_\mu$ insures gauge invariance when the two pions are on-shell
(i.e. $(p - q)_\mu\, (p + q)^\mu = p^2 - q^2 = m^2 - m^2 = 0$). At very low energy the pion vector form factor can be calculated in Chiral Perturbation Theory (χPT). At NLO it reads

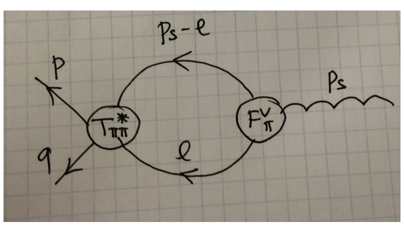


$$L \sim i\, A_\mu (\pi^+ \partial^\mu \pi^- - \pi^- \partial^\mu \pi^+ + \ldots)$$
$$F_\pi^V(s) = 1 + \frac{1}{6} \frac{1}{(4\pi f_\pi)^2} (L_6 - 1)\, s + \frac{1}{6 f_\pi^2} \left(s - 4 m_\pi^2\right) \bar{J}(s) + O\!\left(s^2\right),\ L_6 \approx 16 \tag{14}$$



Improvement: dispersion relation

→ need to calculate the Discontinuity (Cutkosky rules again)



$$(p-q)_\mu \operatorname{Disc} F_\pi^V(s) =$$
$$\frac{1}{2}(-i) \int \frac{d^4 l}{(2\pi)^4} (-2\pi i)\, \delta\!\left(l^2 - m_\pi^2\right) (-2\pi i)\, \delta\!\left((p_s - l)^2 - m_\pi^2\right) T_I^*(s, z)\, (p_s - 2 l)_\mu F_\pi^V(s)$$
$$p_s = p + q,\ s = (p+q)^2, \tag{15}$$
$$z = \cos\theta \text{ is } c.m. \text{ angle } \left(\text{note also that } (p+q) = \left\{\sqrt{s}\,, 0\right\},\ p_0 = q_0 = \frac{\sqrt{s}}{2} = l_0,\ \ |\boldsymbol{p}| = |\boldsymbol{q}| = |\boldsymbol{l}| = \frac{\sqrt{s}}{2} \rho(s)\right)$$

We obtain

$$(p-q)_\mu \operatorname{Disc} F_\pi^V(s) = \frac{i}{64\pi^2} \rho(s) F_\pi^V(s) \int d\Omega\, T_I^*(s, z)\, (p_s - 2 l)_\mu$$
$$\int d\Omega\, T_I^*(s, z)\, (p_s - 2 l)_\mu = L_1 (p+q)_\mu + L_2 (p-q)_\mu \quad \text{... then contract with } (p+q), \tag{16}$$
$$(p-q)\ \ldots$$

Exercise: Show that

$$\int d\Omega\, T_I^*(s,z)\,(p_s - 2\,l)_\mu = 2\,\pi\,(p-q)_\mu \int_{-1}^{1} dz\, z\, T_I^*(s,z) \tag{17}$$

where $T_I(s, z)$ is the $\pi\pi$ amplitude

$$\begin{aligned} T_I(s,z) &= 32\,\pi \sum_{l=0}^{\infty} (2\,l+1)\, t_l^I(s)\, P_l(z) \\ \int_{-1}^{1} dz\, P_l(z)\, P_{l'}(z) &= \frac{2\,\delta_{l\,l'}}{2\,l+1} \end{aligned} \tag{18}$$

Since Disc $F_\pi^V(s) = 2\,i\,\mathrm{Im}\,F_\pi^V(s)$, we get

$$\mathrm{Im}\,F_\pi^V(s) = \rho(s)\,F_\pi^V(s)\,t_1^{1*}(s)\,\theta\left(s > 4\,m_\pi^2\right) \tag{19}$$

If we consider only elastic scattering:

$$t_1^1(s) = \frac{\sin\delta_1^1(s)\,e^{i\,\delta_1^1(s)}}{\rho(s)}, \quad \rho(s)\,t_1^{1*}(s) = \sin\delta_1^1(s)\,e^{-i\,\delta_1^1(s)} \tag{20}$$

Watson final state theorem: the phase of the form factor is determined by the two-particle scattering phase shift: $F_\pi^V(s) = \left|F_\pi^V(s)\right| e^{i\,\delta_1^1(s)}$

$$\begin{aligned} F_\pi^V(s) &= \left|F_\pi^V(s)\right| e^{i\,\delta_1^1(s)} \\ \mathrm{Arg}\left(F_\pi^V(s)\right) &= \delta_1^1(s) \end{aligned} \tag{21}$$

The full Omnes–Muskhelishvili problem (20 min)

We want to find the most general representation for a function, $F(s)$, which is the analytic in the complex s plane with the cut from s= $\left[4\,m^2, \infty\right]$, assuming that we know its phase on the cut,

$$\mathrm{Arg}\,(F(s)) = \delta(s), \quad s > 4\,m^2 \tag{22}$$

Solution is not unique, if $F_0(s)$ is a solution, then $e^{\alpha\,s}\,F_0(s)$ is a solution too. Need to know the **asymptotic information**! We look for a solution in the from

$$\begin{aligned} &F(s) = P(s)\,\Omega(s) \\ &\frac{1}{2\,i}\,(\Omega(s+i\,\epsilon) - \Omega(s-i\,\epsilon)) = \Omega(s+i\,\epsilon)\,\sin\delta(s)\,e^{-i\,\delta(s)}, \\ &\Omega(s+i\,\epsilon)\left(\frac{1}{2\,i} - \left(\frac{e^{i\,\delta(s)} - e^{-i\,\delta(s)}}{2\,i}\right)e^{-i\,\delta(s)}\right) = \Omega(s+i\,\epsilon)\,e^{-2\,i\,\delta(s)}\,\frac{1}{2\,i} = \Omega(s-i\,\epsilon)\,\frac{1}{2\,i} \\ &\Omega\,(s+i\,\epsilon)\,e^{-2\,i\,\delta(s)} = \Omega(s-\,i\,\epsilon),\;\; \ln\Omega\,(s+i\,\epsilon) - 2\,i\,\delta = \ln\Omega\,(s+i\,\epsilon) \\ &\mathrm{Disc}(\ln\Omega(s)) = 2\,i\,\delta(s) \end{aligned} \tag{23}$$

Dispersion relation

$$\ln\Omega(s) = \frac{1}{2i}\int_{4m^2}^{\infty}\frac{ds'}{\pi}\frac{\text{Disc}(\ln\Omega(s'))}{s'-s} = \int_{4m^2}^{\infty}\frac{ds'}{\pi}\frac{\delta(s')}{s'-s},$$
$$\Omega(s) = \text{Exp}\left(\int_{4m^2}^{\infty}\frac{ds'}{\pi}\frac{\delta(s')}{s'-s}\right), \quad \Omega(s) = \text{Exp}\left(a + \frac{s}{\pi}\int_{4m^2}^{\infty}\frac{ds'}{s'}\frac{\delta(s')}{s'-s}\right) \tag{24}$$

$$\Omega(s) = \text{Exp}\left(\frac{s}{\pi}\int_{4m^2}^{\infty}\frac{ds'}{s'}\frac{\delta(s')}{s'-s}\right), \quad \Omega(0) = 1 \tag{25}$$

One subtraction: normalization $\Omega(0) = 1$. The function $\Omega(s)$ is known as the **Omnes function**. Many applications!

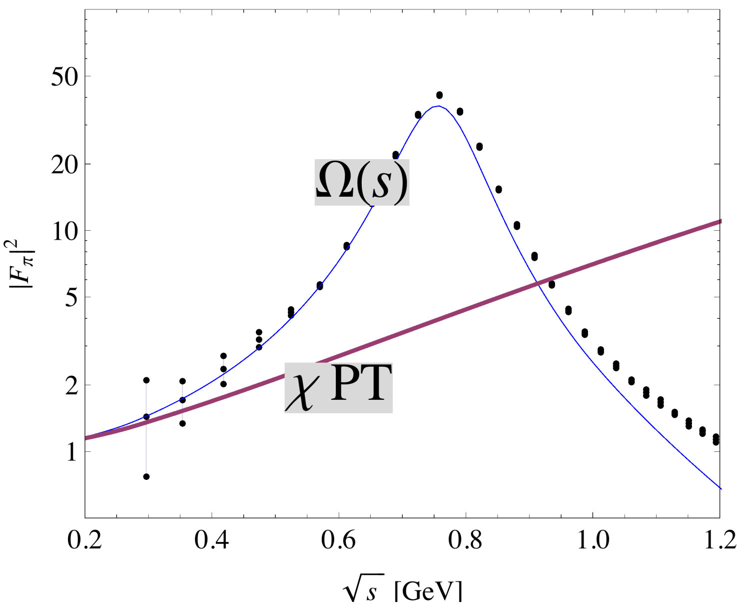
Exercise: Show that

$$\text{Arg}\,\Omega(s) = \delta(s)$$
$$\text{if } \delta(s) \to \alpha\pi, \text{ show } \Omega(s\to\infty) \to \frac{1}{s^{\alpha}} \tag{26}$$

Use

$$\int\frac{f(s')\,ds}{s'-s\mp i\epsilon} = p.v.\int\frac{f(s')\,ds}{s'-s} \pm i\pi f(s) \tag{27}$$

If one assume asymptotic: $F(\infty) \to 1/s$, then $F_\pi^V(s) = \Omega(s)$



Numerical implementation (25 min)

$$\Omega(s) = \text{Exp}\left(\frac{s}{\pi}\int_{4m^2}^{\infty}\frac{ds'}{s'}\frac{\delta(s')}{s'-s}\right) \tag{28}$$

Tangent stretching
Let’s now consider the integral

$$\int_a^\infty f(y)\, dy$$

In order to account the whole region, the replacement is needed $y \to y(x)$, which changes the integrating range (a,∞) into (0,1)

$$\int_a^\infty f(y)\, dy = \int_0^1 f(y(x)) \frac{dy}{dx}\, dx, \quad y(x) = a + C_{\text{ext}}\ \text{tg}\left(\frac{\pi}{2}\, x\right);\ \frac{dy}{dx} = \frac{C_{\text{ext}}\, \pi/2}{\cos\left(\frac{\pi}{2}\, x\right)^2}$$

$$\int_0^\infty f(y)\, dy = \sum_{i=1}^n f(y_i)\ \tilde{w}_i\ .$$

If the integrand is smooth, it is convenient to use Gaussian weights for integration. Few words about C_{ext}: As you can see $y\,(x = 1/2) = a + C_{\text{ext}}$. It means that n/2 points will be accounted before $a + C_{\text{ext}}$ and n/2 after. It is very useful when you know the general behavior of your function. If you do not, just put it equal to unity, $C_{\text{ext}} = 1$.

Example 2: $\int_2^\infty e^{-y^2}\, dy$

```
<< NumericalDifferentialEquationAnalysis`

n = 10; Cext = 1;
wg = GaussianQuadratureWeights[n, 0, 1];
yn = 2 + Table[Cext * Tan[π / 2 * wg[[i, 1]]], {i, n}];
wgn = Table[wg[[i, 2]] (Cext π / 2)/(Cos[π / 2 wg[[i, 1]]] ^ 2), {i, n}];
Sum[Exp[-yn[[i]] ^ 2] wgn[[i]], {i, 1, n}]
Integrate[Exp[-y^2], {y, 2., Infinity}]
```

0.00414553

0.00414553

If the integral contains a square root singularity $\left(\text{something like } \int_a^\infty f\left(y,\ \sqrt{y-a}\,\right) dy\right)$ it is useful to introduce another replacement

$$y(x) = a + C_{\text{ext}}\ \text{tg}\left(\frac{\pi}{2}\, x\right)^2;\ \frac{dy}{dx} = \frac{C_{\text{ext}}\, \pi/2}{\cos\left(\frac{\pi}{2}\, x\right)^2}\ 2\ \text{tg}\left(\frac{\pi}{2}\, x\right)$$

Principle values integrals

Very often one has to take integral of the following form

$$F(s) = \frac{s}{\pi} \int_{4m^2}^\infty \frac{ds'}{s'} \frac{f(s')}{s' - s - i\epsilon} = \theta\left(s < 4\,m^2\right) \frac{s}{\pi} \int_{4m^2}^\infty \frac{ds'}{s'} \frac{f(s')}{s' - s} + \theta\left(s > 4\,m^2\right) \frac{s}{\pi} \int_{4m^2}^\infty \frac{ds'}{s'} \frac{f(s')}{s' - s - \boldsymbol{i}\,\boldsymbol{\epsilon}} \qquad (29)$$

The latter integral can be written as

$$\frac{s}{\pi} \int_{4m^2}^\infty \frac{ds'}{s'} \frac{f(s')}{s' - s - \boldsymbol{i}\,\boldsymbol{\epsilon}} = \frac{s}{\pi} \left(p.v. \int_{4m^2}^\infty \frac{ds'}{s'} \frac{f(s')}{s' - s} + i\,\pi\, \frac{f(s)}{s}\right) \qquad (30)$$

For the p.v. integral we use the following trick:

$$p.v. \int_{4m^2}^\infty \frac{ds'}{s'} \frac{f(s')}{s' - s} = p.v. \int_{4m^2}^\infty \frac{ds'}{s'} \frac{f(s') - f(s) + f(s)}{s' - s} = \int_{4m^2}^\infty \frac{ds'}{s'} \frac{f(s') - f(s)}{s' - s} + \frac{f(s)}{s} \ln\left(\frac{4\,m^2}{s - 4\,m^2}\right) \qquad (31)$$

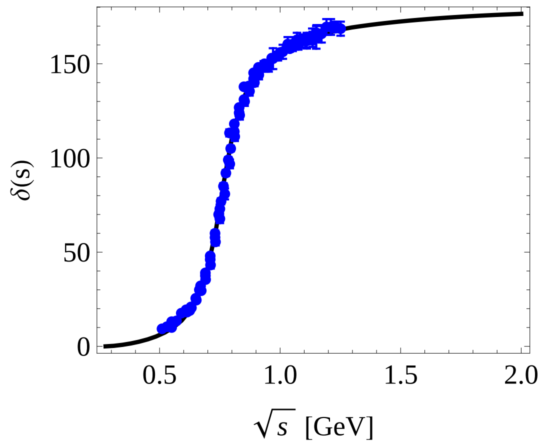
$$p.v.\int_{4m^2}^{\infty} ds' \frac{1}{s'(s'-s)} = \frac{1}{s}\ln\left(\frac{4m^2}{s-4m^2}\right)$$

All together (**Omnes function**)

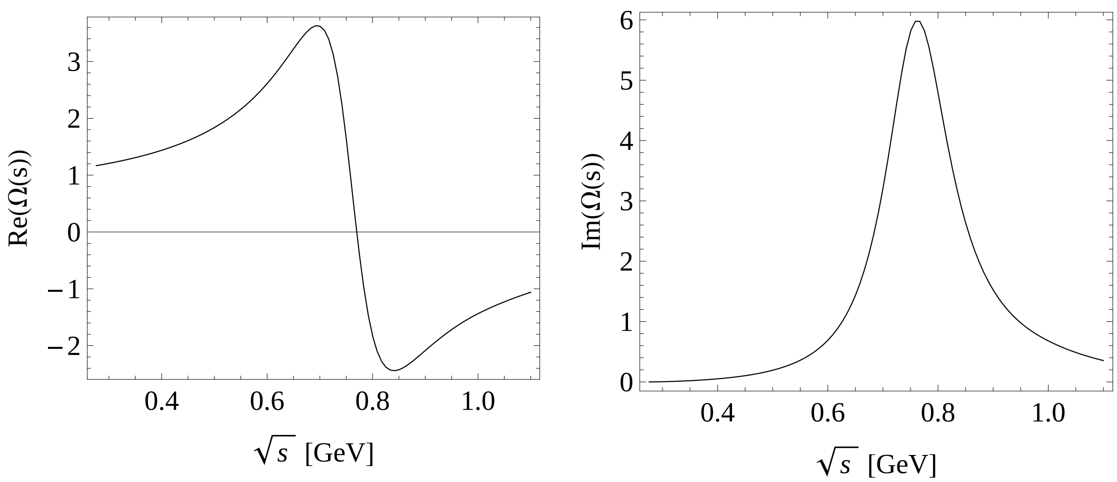
$$\Omega(s) = \text{Exp}\left(\frac{s}{\pi}\int_{4m^2}^{\infty}\frac{ds'}{s'}\frac{\delta(s')}{s'-s}\right) \tag{32}$$

```
Exit[];

<< NumericalDifferentialEquationAnalysis`
SetDirectory[NotebookDirectory[]];
StyleList = {AbsoluteThickness[1.0], AbsolutePointSize[5], #} & /@ {Blue, Red, Green, Purple};
SetOptions[Plot, Frame -> True, Axes -> {True, False},
  PlotStyle -> StyleList, AspectRatio -> 0.8, FrameStyle -> Directive[13]];
<< PiPi_Madrid.m;

mpi = 0.138;
mK = 0.4956745;
Mrho = 0.770;
ϵ = 0.00001;
LamPhShift = 1.3;

OmnesNInt[sig_][s_] := Exp[s / Pi * NIntegrate[deltaFinal[1][sb] / (sb (sb - s - I sig ϵ)),
      {sb, 4 mpi^2, 4 mK^2, Infinity}, AccuracyGoal → 5, MaxRecursion → 200]];

nOmn = 120;
Cx = 1.0;
wg = GaussianQuadratureWeights[nOmn, 0, 1];
s0 = 4 mpi^2;
sn = s0 + Table[Cx * Tan[Pi / 2 * wg[[i, 1]]]^2, {i, nOmn}];
wgn = Table[Cx * 2 * Tan[Pi / 2 * wg[[i, 1]]] * wg[[i, 2]] * Pi / 2 / (Cos[Pi / 2 * wg[[i, 1]]]^2), {i, nOmn}];
Deltan = Table[deltaFinal[1][sn[[i]]], {i, nOmn}];

Clear[deltas];
OmnesTemp0[s_] = Exp[s / π * Sum[Deltan[[i]] / (sn[[i]] (sn[[i]] - s)) * wgn[[i]], {i, 1, nOmn}]];
OmnesTemp[s_] = Exp[s / π * Sum[(Deltan[[i]] - deltas) / (sn[[i]] (sn[[i]] - s)) * wgn[[i]], {i, 1, nOmn}]];

Omnes[sig_][s_] := Which[sig == 0, OmnesTemp0[s],
sig == 1 || sig == -1, deltas = delta[1][s]; Which[s ≤ 4 mpi^2, OmnesTemp0[s],
s > 4 mpi^2, Exp[I * sig * delta[1][s]] * Exp[deltas / π Log[s0 / (s - s0)]] * OmnesTemp[s]]];
```

```
Show[Plot[deltaFinal[1][par^2] * 180 / Pi, {par, 2 mpi, 2.0}, PlotPoints → 50, MaxRecursion → 0,
  PlotStyle → {Black, Thick}, FrameLabel → {"√s  [GeV]", "δ(s)"}], DeltaExp[1], ImageSize → {250, 250}]
```




```
GraphicsRow[{Plot[Re[Omnes[1][par^2]], {par, 2 mpi, 1.1}, PlotStyle → {Black}, MaxRecursion → 0,
    PlotPoints → 100, PlotRange → All, ImageSize → {250, 250}, FrameLabel → {"√s  [GeV]", "Re(Ω(s))"}],
  Plot[{Im[Omnes[1][par^2]]}, {par, 2 mpi, 1.1}, PlotStyle → {Black}, MaxRecursion → 0,
    PlotPoints → 100, PlotRange → All, ImageSize → {250, 250}, FrameLabel → {"√s  [GeV]", "Im(Ω(s))"}]}]
```




```
Omnes[1][1]
OmnesNInt[1][1]
```

```
-1.43762 + 0.680652 i
```

```
-1.43777 + 0.680767 i
```