scattering & physics in finite volume

Jozef Dudek

Old Dominion University

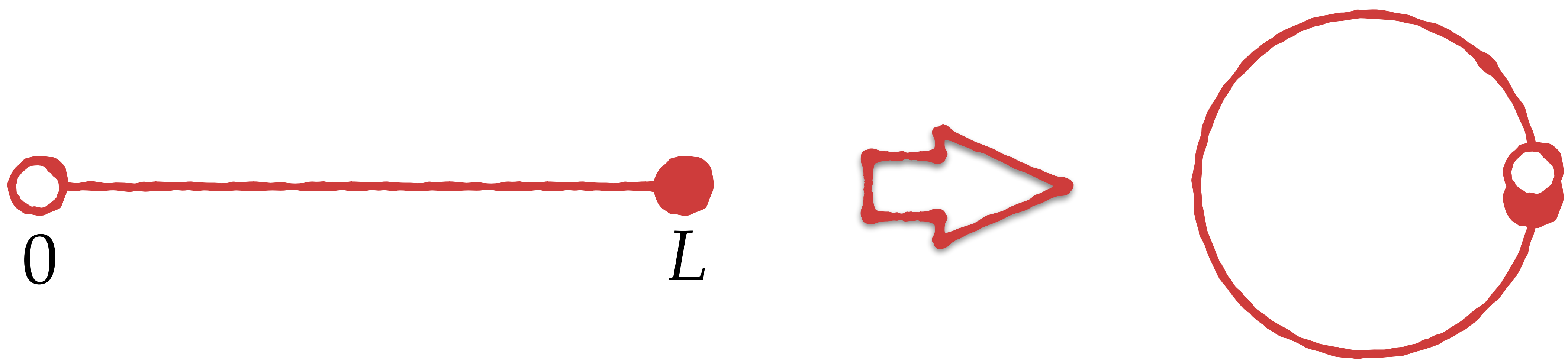
Jefferson Lab

one-dim quantum mechanics

free-particle momentum eigenstates $\psi_p(x) \sim e^{ipx}$

consider a system of length *L*

with periodic boundary conditions



$$\psi_p(0) = \psi_p(L)$$

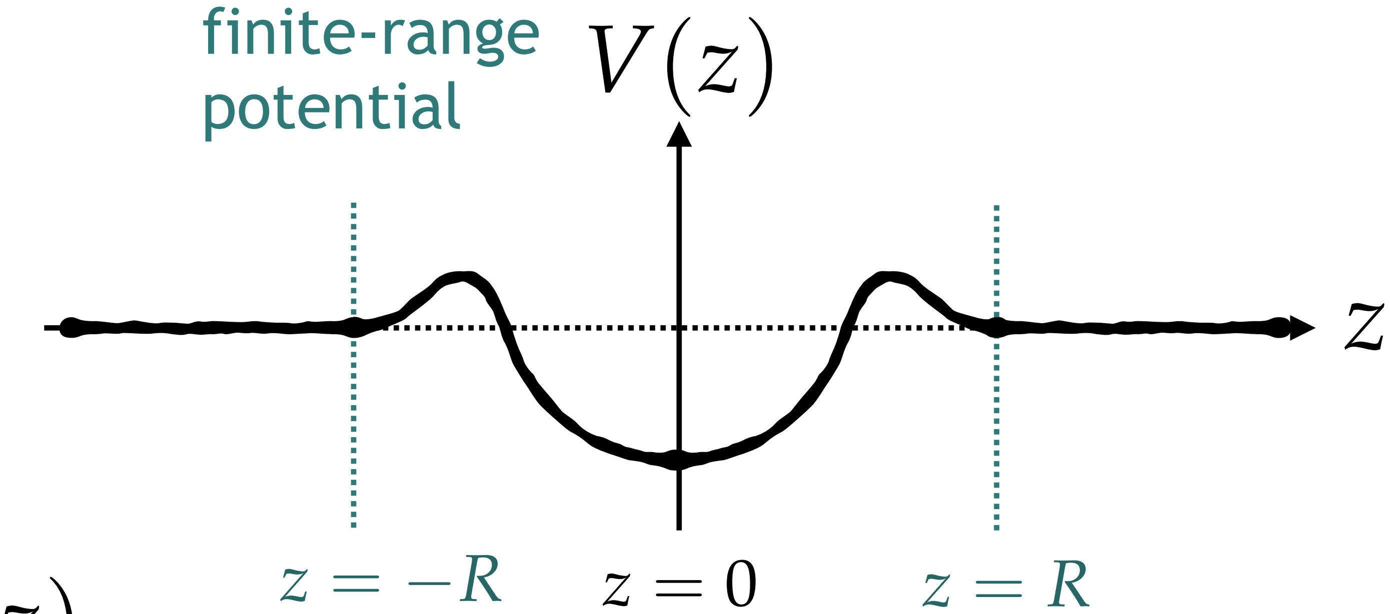
$$1 = e^{ipL}$$

$$p = \frac{2\pi}{L} n$$

quantized momentum

one-dim quantum mechanics

- consider scattering of two identical bosons separated by *z*



$$-\frac{1}{m}\frac{d^2\psi}{dz^2}+V(z)\psi(z)=E\psi(z)$$

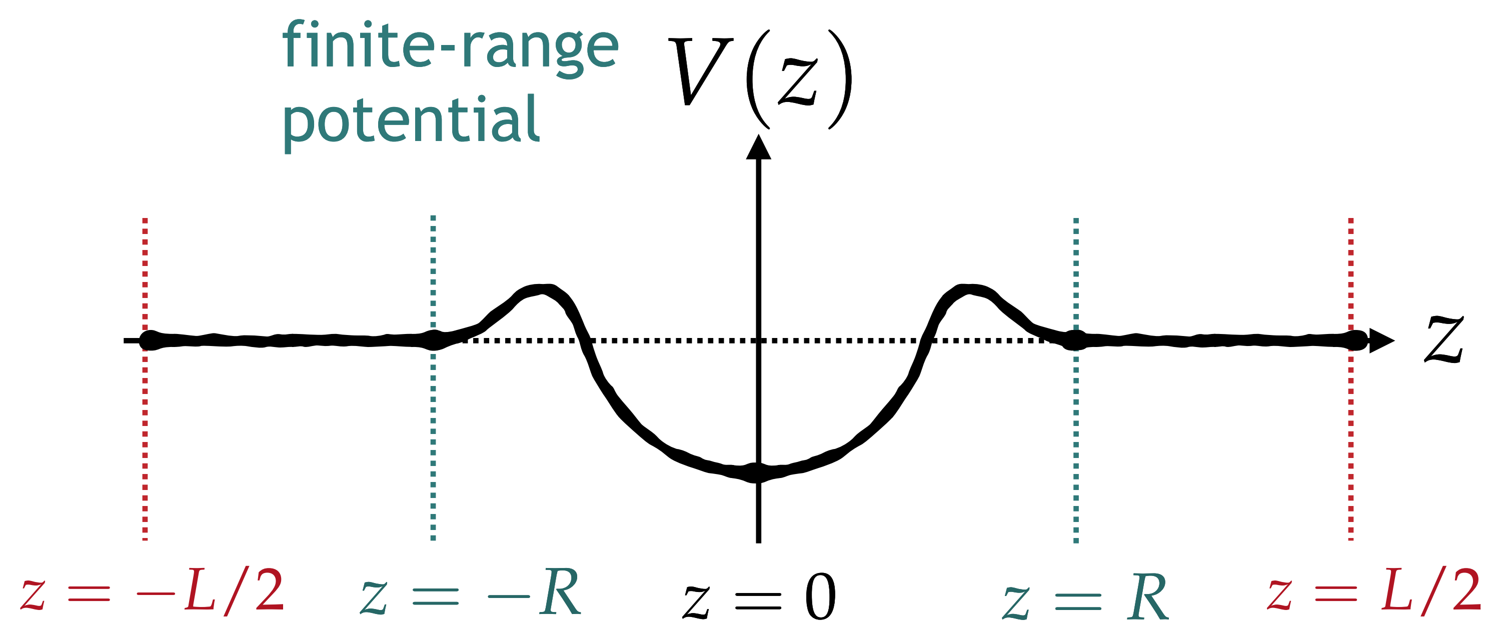
outside the well

$$\psi(|z|>R)\sim\cos\big(p|z|+\delta(p)\big)$$

$\delta(p)$ elastic scattering phase-shift

one-dim quantum mechanics

- consider scattering of two identical bosons separated by z



outside the well

$$\psi(|z|>R) \sim \cos\big(p|z| + \delta(p)\big)$$

- apply a periodic boundary condition

$$\left.\begin{aligned}\psi(-L/2) &= \psi(L/2)\\ \frac{d\psi}{dz}(-L/2) &= \frac{d\psi}{dz}(L/2)\end{aligned}\right\} \quad \frac{pL}{2} + \delta(p) = n\pi$$

$$\boxed{p = \frac{2\pi}{L}n - \frac{2}{L}\delta(p)}$$

discrete energy spectrum

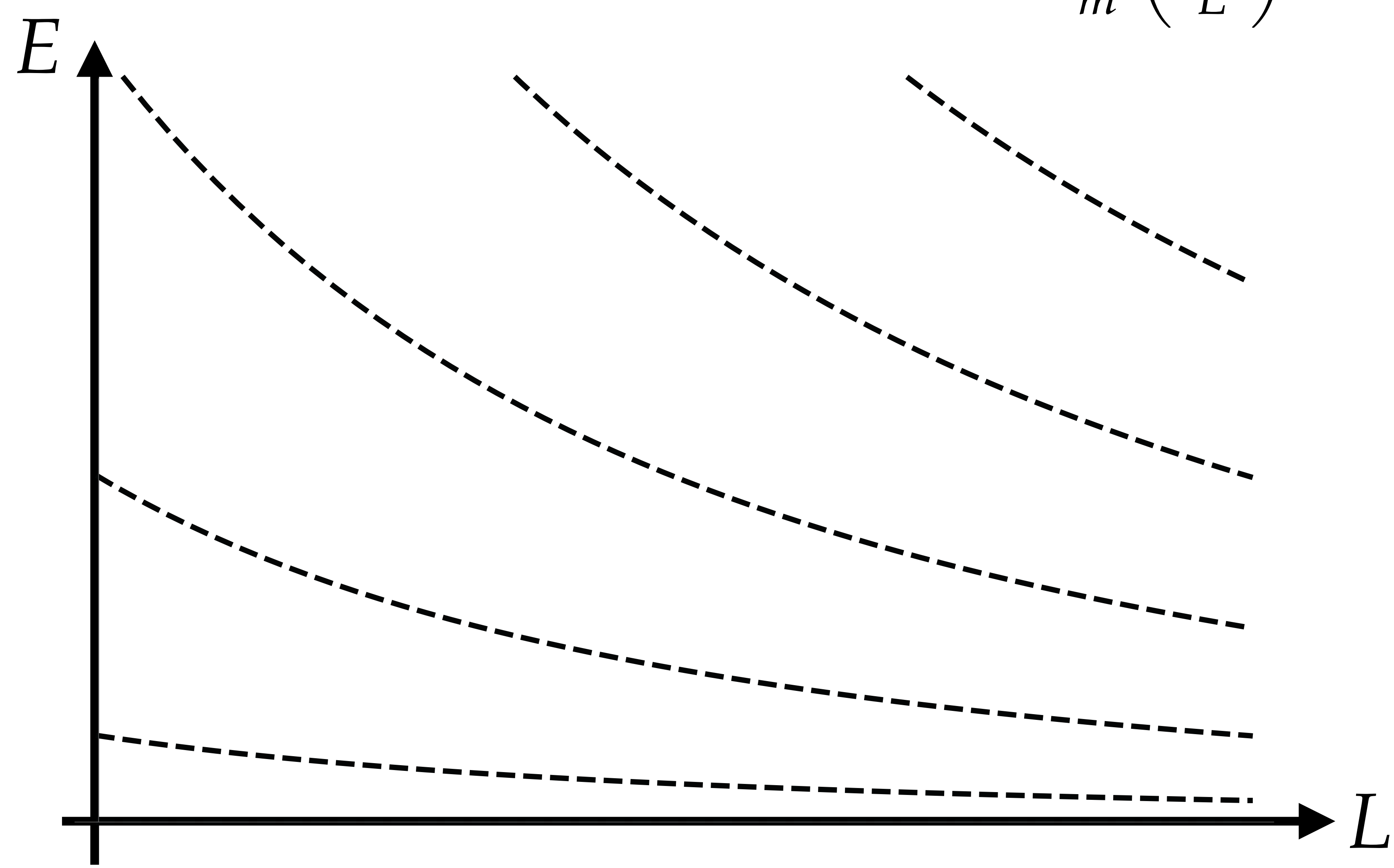
one-dim quantum mechanics

$$p = \frac{2\pi}{L} n - \frac{2}{L} \delta(p)$$

discrete energy spectrum

e.g. no interaction, $\delta(p) = 0$

$$E_n(L) = \frac{1}{m} \left(\frac{2\pi}{L} \right)^2 n^2$$

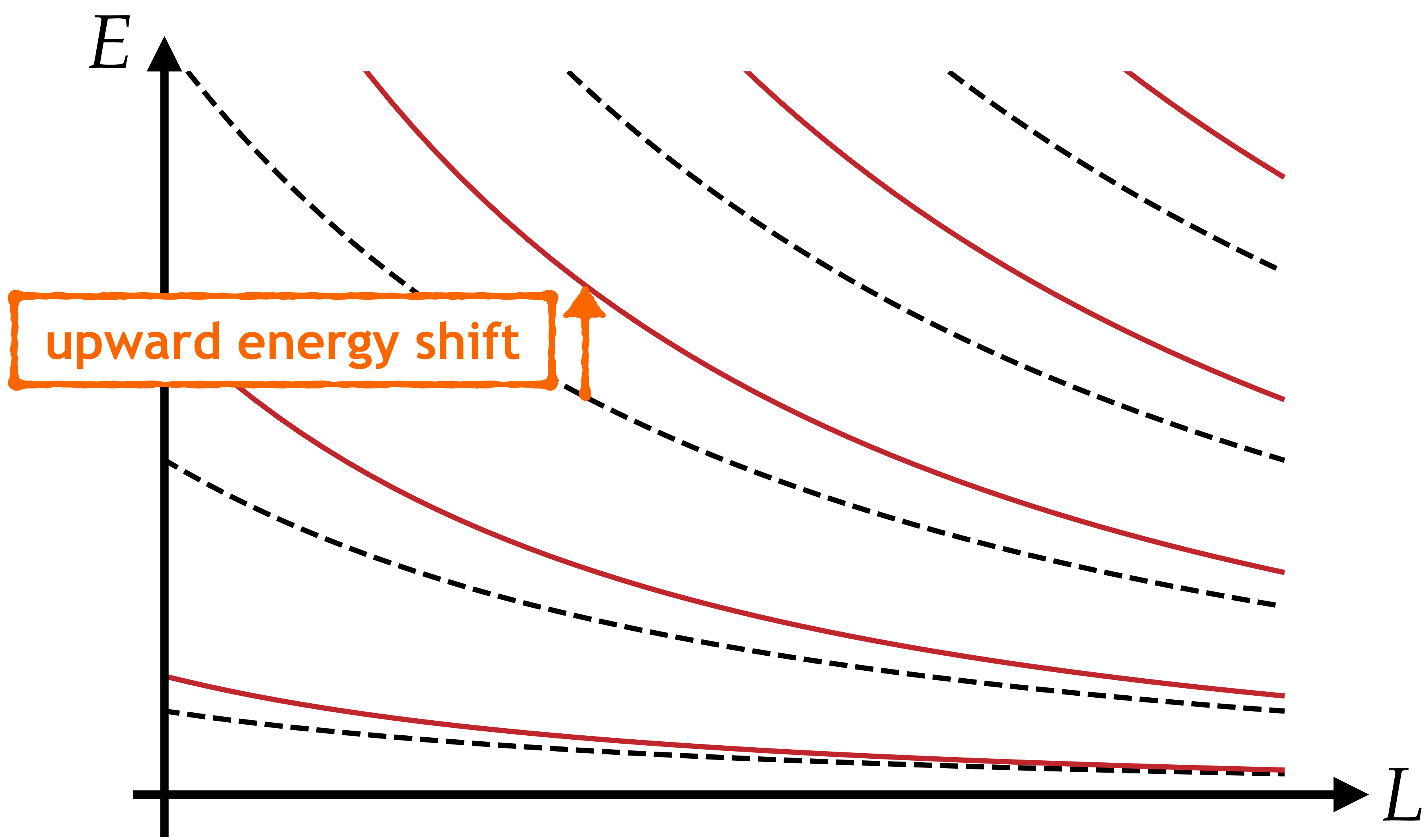


one-dim quantum mechanics

$$p = \frac{2\pi}{L}n - \frac{2}{L}\delta(p)$$

discrete energy spectrum

e.g. a weak repulsive interaction, $\delta(p) = -ap$

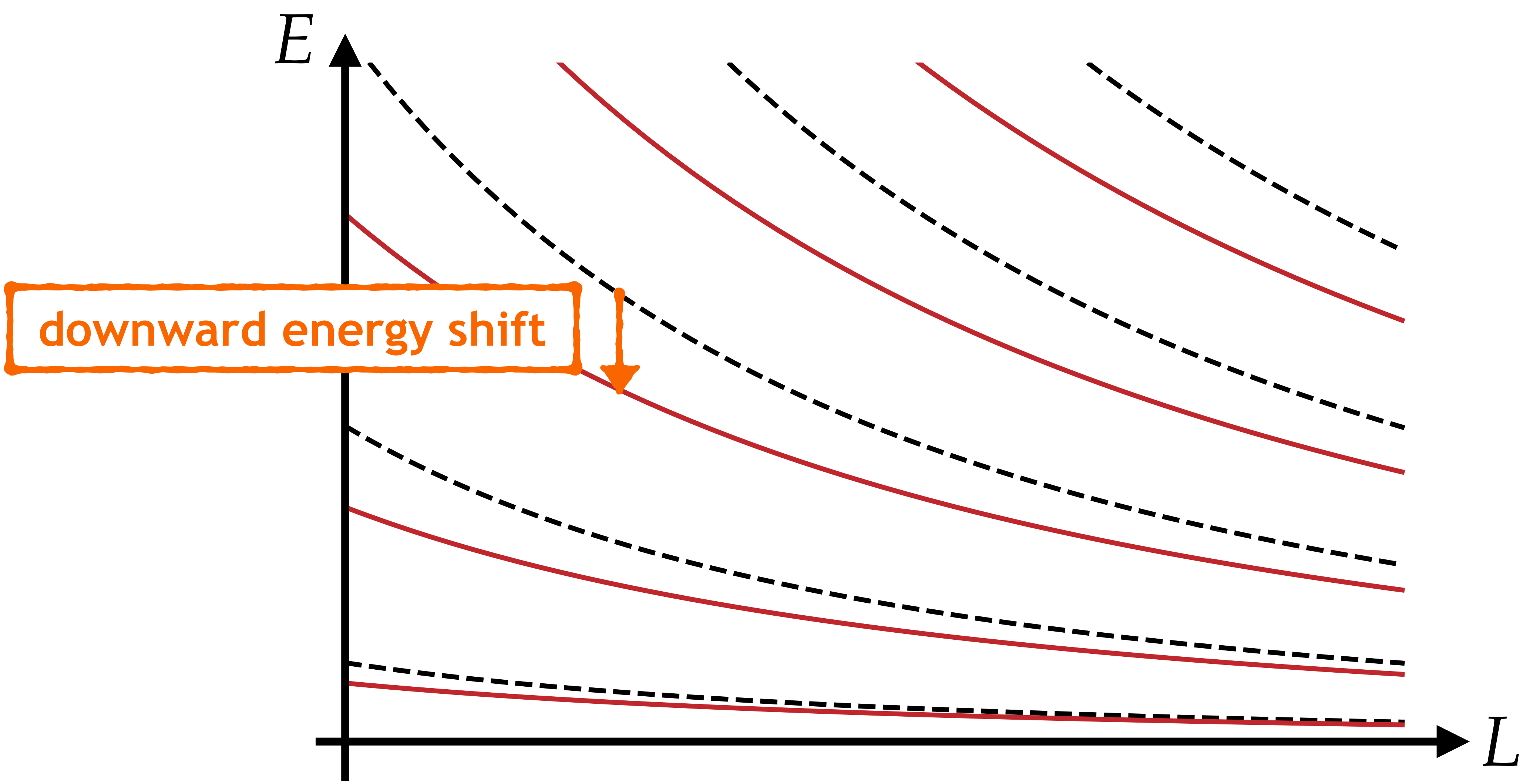


one-dim quantum mechanics

$$p = \frac{2\pi}{L} n - \frac{2}{L}\delta(p)$$

discrete energy spectrum

e.g. a weak attractive interaction, $\delta(p) = ap$

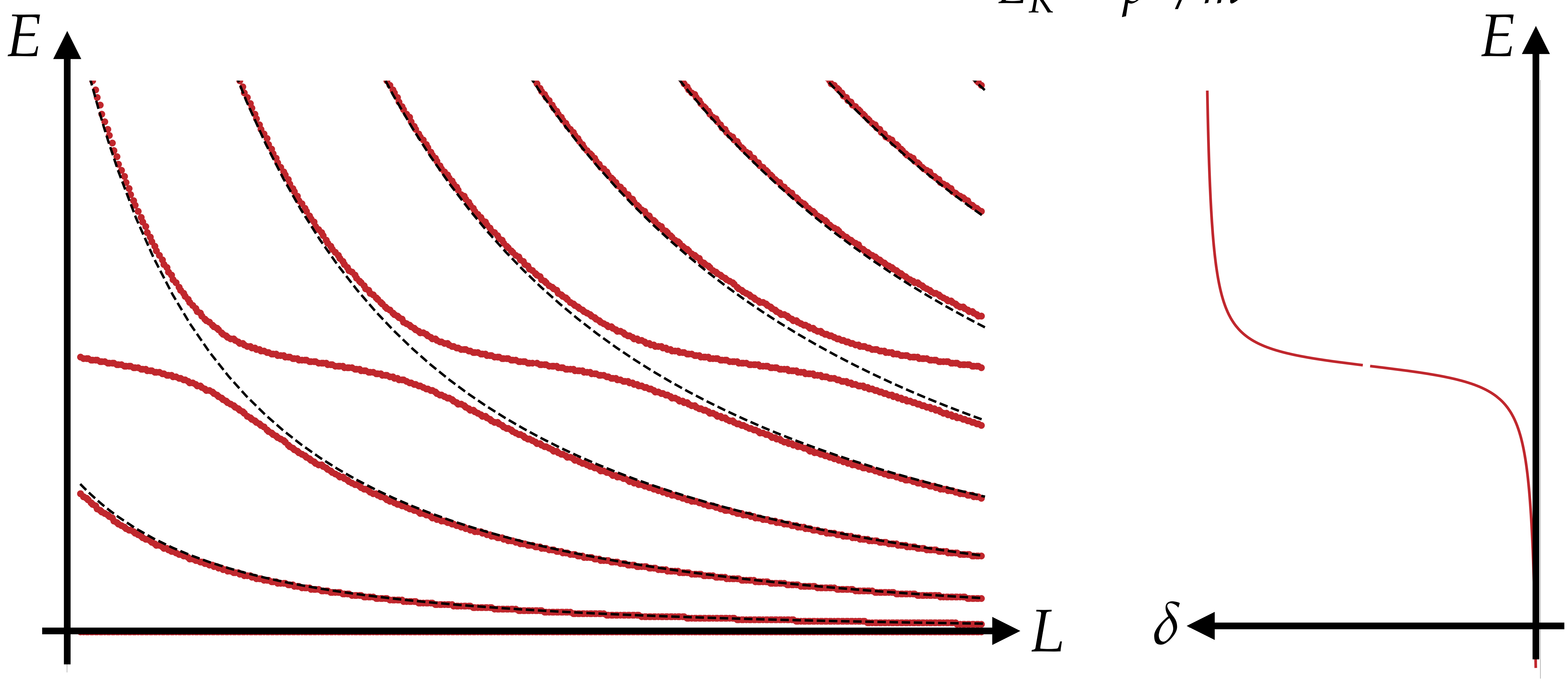


one-dim quantum mechanics

$$p = \frac{2\pi}{L} n - \frac{2}{L} \delta(p)$$

discrete energy spectrum

e.g. a non-rel Breit-Wigner resonance $\tan \delta(p) = \dfrac{\Gamma/2}{E_R - p^2/m}$

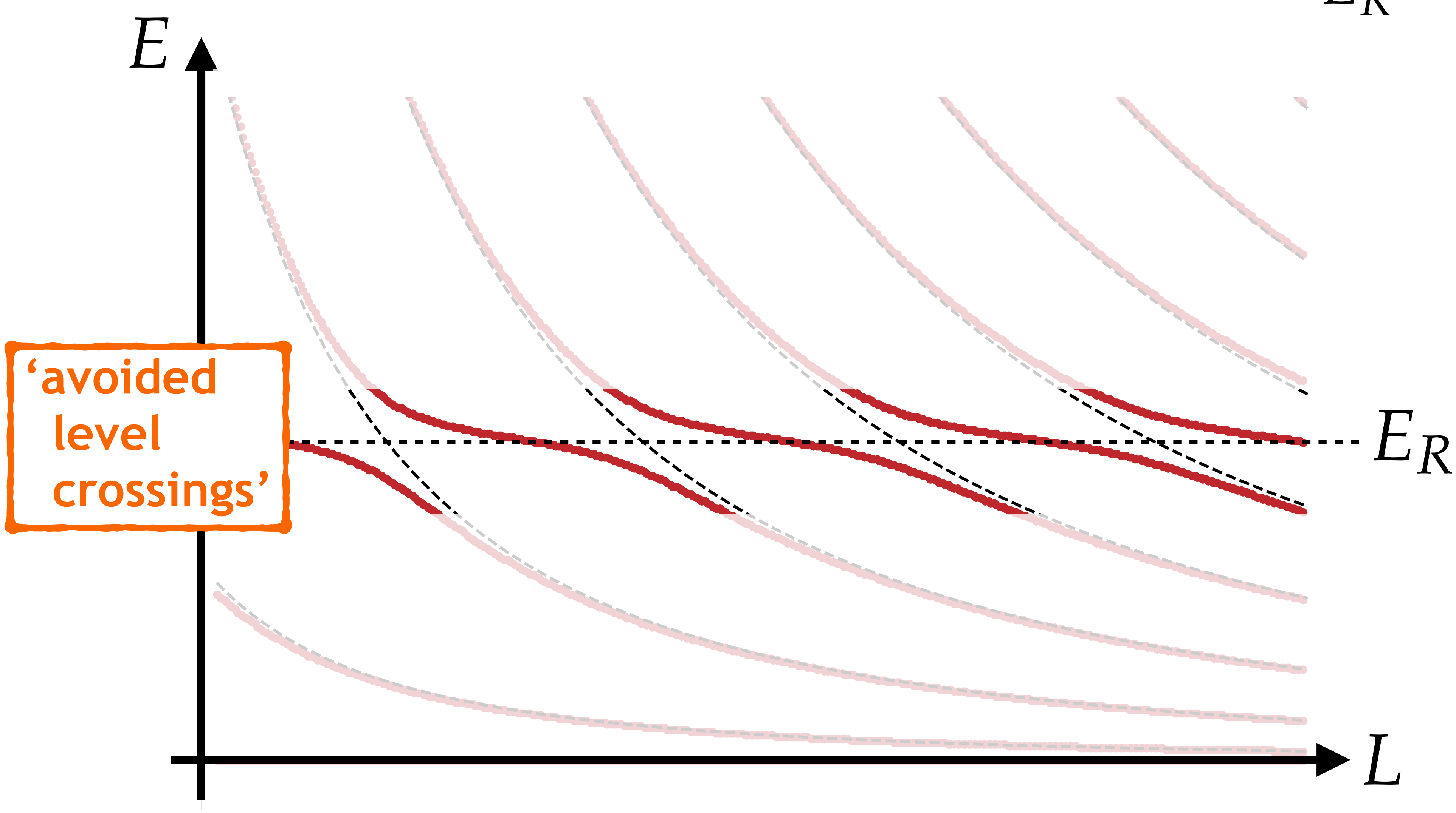


one-dim quantum mechanics

$$p = \frac{2\pi}{L} n - \frac{2}{L}\delta(p)$$

discrete energy spectrum

e.g. a non-rel Breit-Wigner resonance $\tan\delta(p) = \dfrac{\Gamma/2}{E_R - p^2/m}$



3+1 dim quantum field theory

periodic boundary conditions in (*x*,*y*,*z*) → periodic cube → hypertoroid

allowed free particle momenta $\vec{p} = \frac{2\pi}{L}\left[n_x, n_y, n_z\right]$

relationship between spectrum & elastic scattering phase-shift worked out by Lüscher

somewhat complicated by lack
of full rotational symmetry (cube ≠ sphere)

Lüscher NPB354 (1991) 531

ignoring the complications for a moment, we get

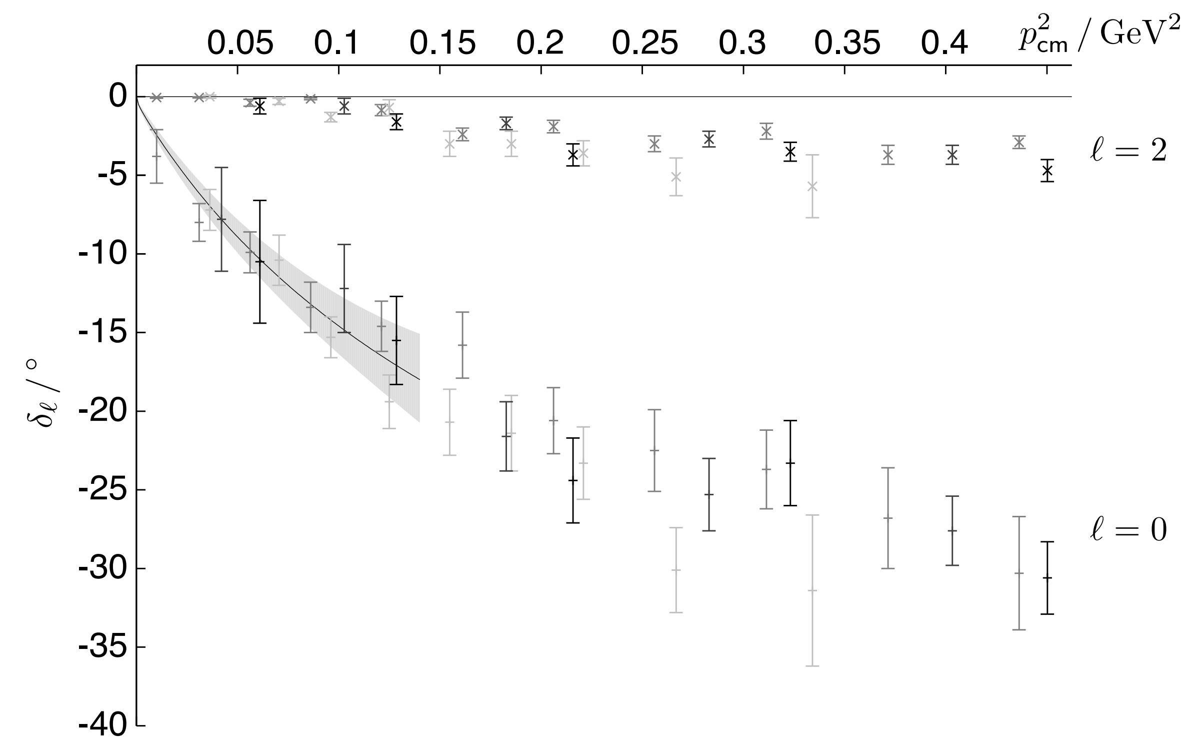
$$\delta_\ell(E) = \boxed{f_\ell(E, L)}$$

known function

ππ I=2 scattering

experimentally, weak and repulsive

ππ I=2 PHASE-SHIFT



$\pi\pi$ I=2 scattering from lattice QCD

compute the spectrum of I=2 eigenstates with $J^P = 0^+$

⇒ evaluate correlation functions with operators having these quantum numbers

what operators should we use ?

minimal quark content $\bar{u}\bar{u}dd$

since we expect the physics at low-energy to be $\pi\pi$ scattering,
how about **operators resembling a pair of pions** ?

$$\mathcal{O}_{\pi\pi}^{|\vec{p}|} = \sum_{\hat{p}} \pi^+(\vec{p})\pi^+(-\vec{p})$$

i.e. want large values of $\langle \mathfrak{n} | \mathcal{O} | 0 \rangle$

in $C(t) = \sum_{\mathfrak{n}} e^{-E_{\mathfrak{n}} t} \left| \langle \mathfrak{n} | \mathcal{O} | 0 \rangle \right|^2$

$\pi\pi$ I=2 scattering from lattice QCD

compute the spectrum of I=2 eigenstates with $J^P = 0^+$

⇒ evaluate correlation functions with operators having these quantum numbers

what operators should we use ?

minimal quark content $\bar{u}\bar{u}dd$

since we expect the physics at low-energy to be $\pi\pi$ scattering,
how about operators resembling a pair of pions ?

$$\mathcal{O}_{\pi\pi}^{|\vec{p}|} = \sum_{\hat{p}} \pi^+(\vec{p})\pi^+(-\vec{p})$$

for pion operators use the 'variationally optimal' combinations $\pi^+ = \sum_i v_i \left(\bar{u}\mathbf{\Gamma}_i d\right)$

to make a basis, consider different relative momentum

$\pi_{[000]}\pi_{[000]}$

$\pi_{[100]}\pi_{[-100]}$

$\pi_{[110]}\pi_{[-1-10]}$

$\pi_{[111]}\pi_{[-1-1-1]}$

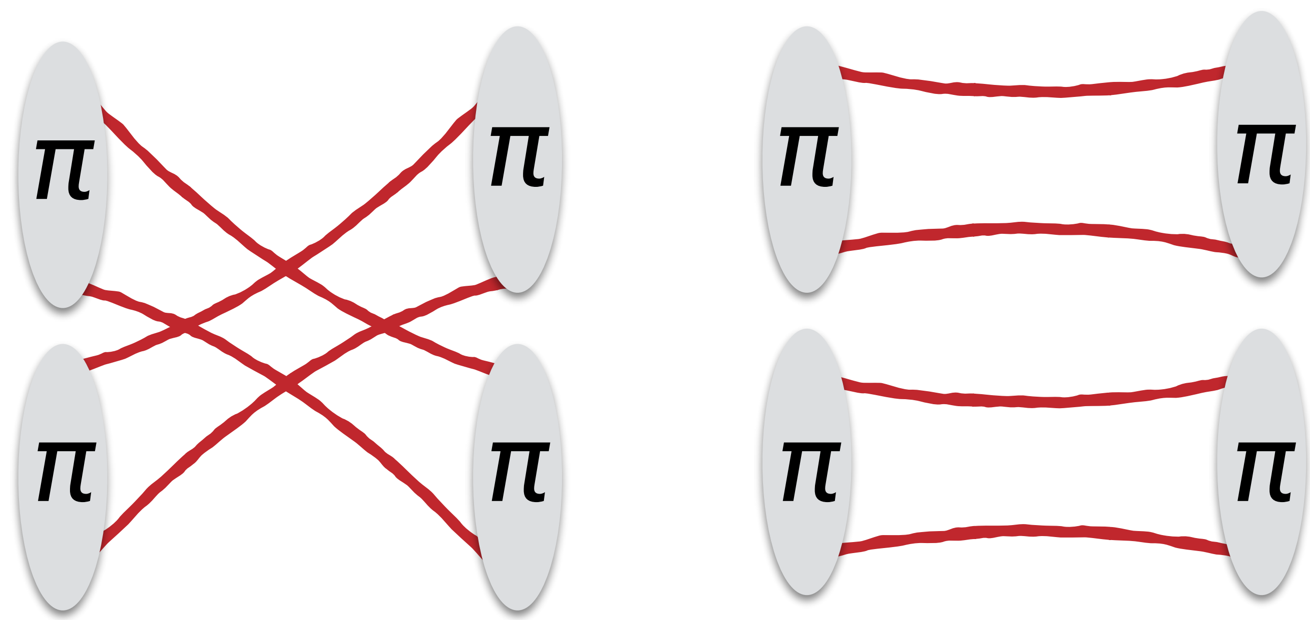
$\pi\pi$ I=2 scattering from lattice QCD

evaluate a matrix of correlation functions

$$C_{|\vec{p}|,|\vec{q}|} = \langle 0 | \mathcal{O}_{\pi\pi}^{|\vec{p}|}(t)\, \mathcal{O}_{\pi\pi}^{|\vec{q}|\dagger}(0) | 0 \rangle$$

formally integrate out the quark fields ...

need to compute the following quark propagation diagrams, averaged over an ensemble of gauge configurations



an aside on the lattices

I'm going to present some results from a particular lattice QCD set-up *PRD79 034502 (2009)*

"anisotropic Clover lattices"

lattice spacing in space directions: $a_s \sim 0.12\,\mathrm{fm}$

lattice spacing in time direction: $a_t \sim a_s/3.5$ $a_t^{-1} \sim 6\,\mathrm{GeV}$

three flavors of quark, two light & one strange

$$m_s \approx m_s^{\mathrm{phys}}$$

$$m_u = m_d > m_{u,d}^{\mathrm{phys}}$$

exact isospin symmetry

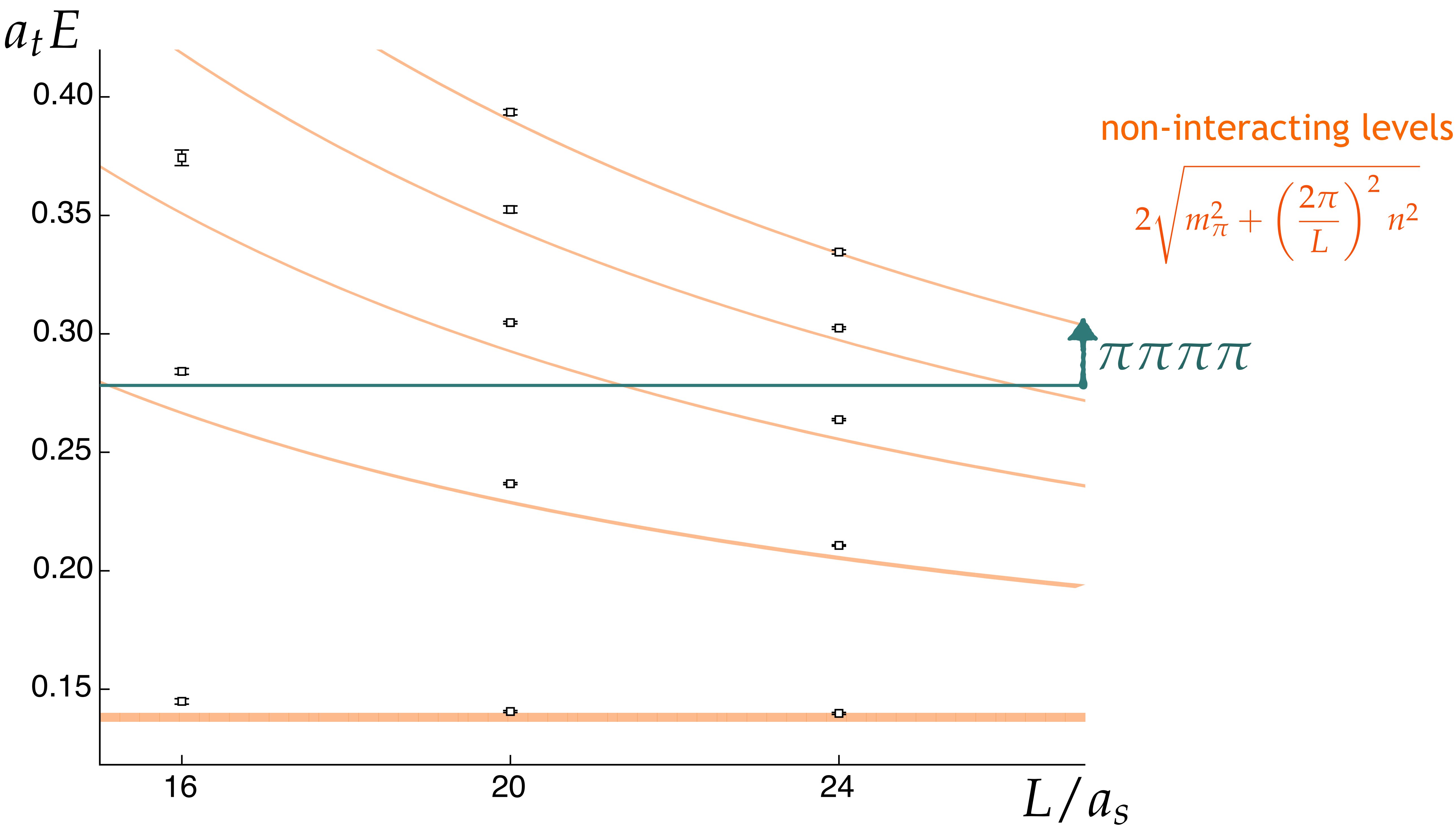
$m_\pi \sim 391\,\mathrm{MeV}$

multiple volumes: 16^3, 20^3, 24^3

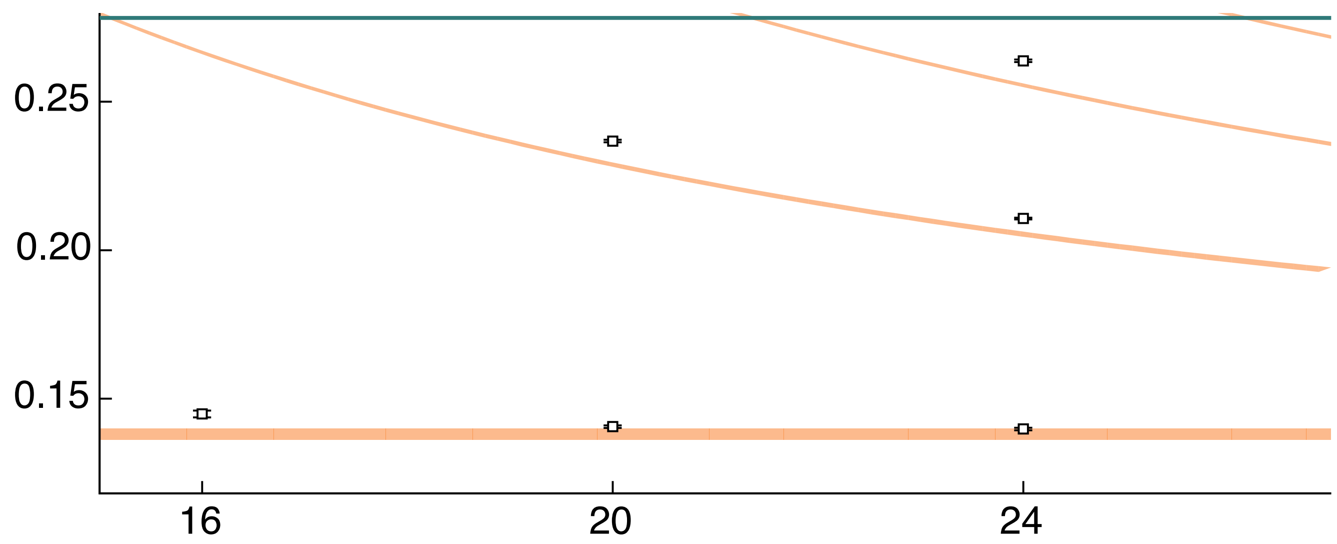
~ 2.0, 2.5, 3.0 fm

$m_\pi L$ ~ 4, 5, 6

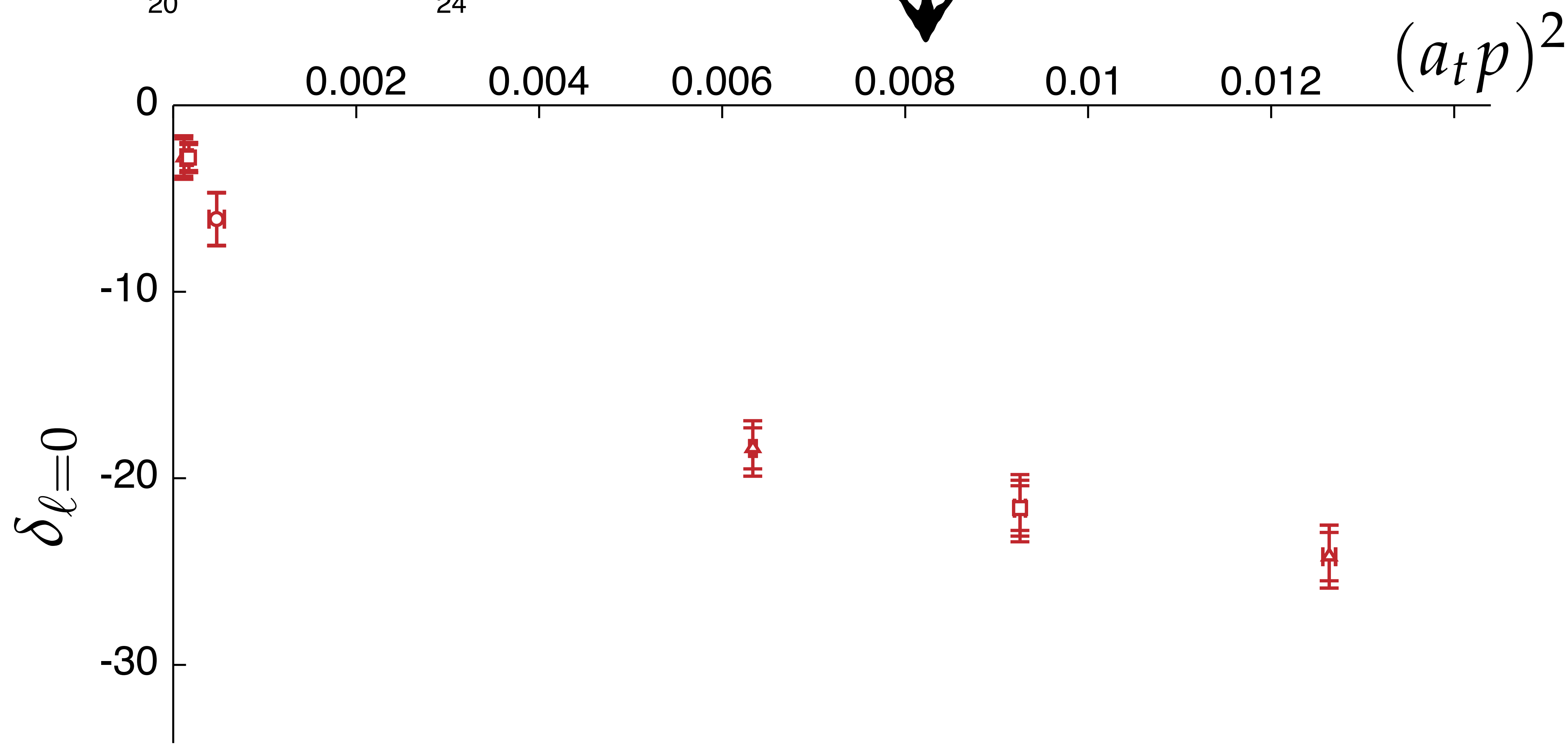
ππ *I*=2 scattering from lattice QCD



$\pi\pi$ *I*=2 scattering from lattice QCD



$$\delta_\ell(E) = f_\ell(E, L)$$



moving frames - more information

in a finite-volume considering a moving frame contains extra info

... surely some mistake ?

length contraction along the direction of motion changes the quantization condition

(and also reduces the symmetry group)

Gottlieb & Rumm. NPB450 (1995) 397
Kim et. al. NPB727 (2005) 218
& others

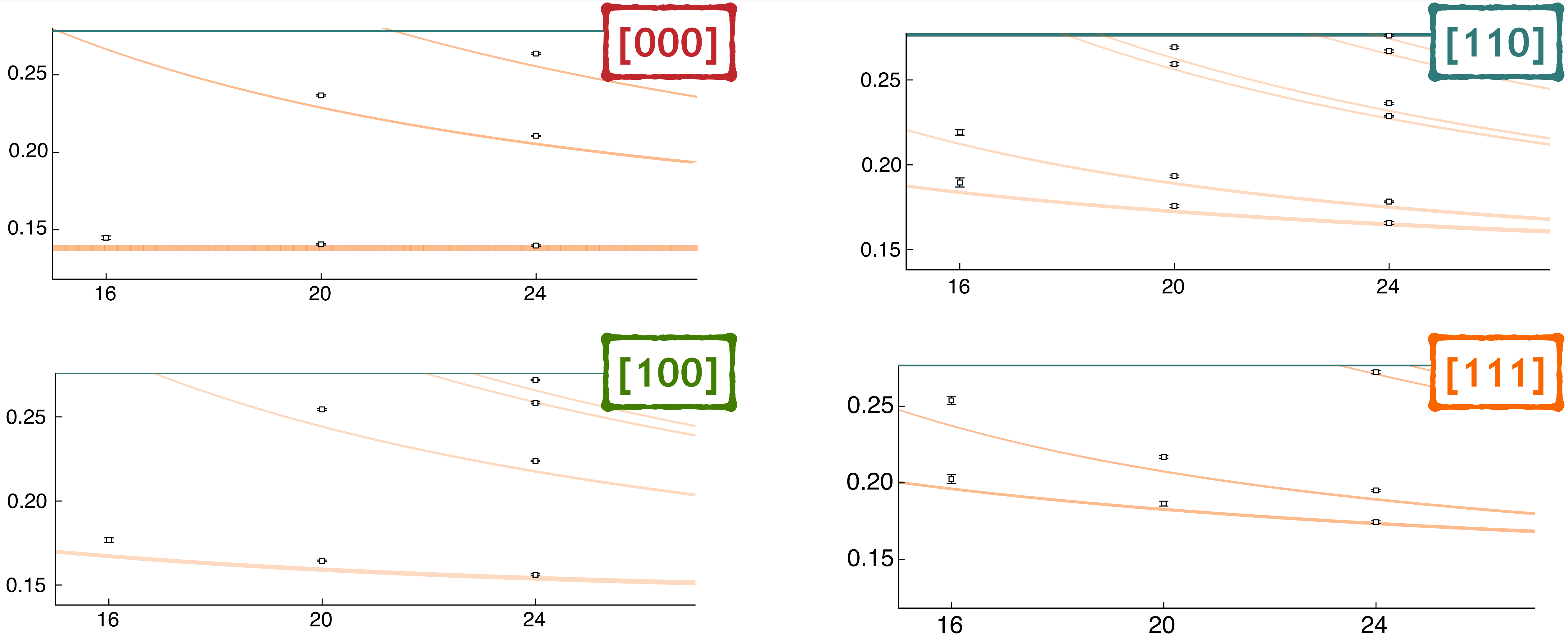
after a bit of group theory, it's quite easy to construct the relevant operators

$$\mathcal{O}_{\pi\pi}^{\vec{P},\Lambda;|\vec{p}|} = \sum_{\hat{p}} \boxed{C(\vec{P},\Lambda;\vec{p},\vec{P}-\vec{p})}\, \pi^{+}(\vec{p})\, \pi^{+}(\vec{P}-\vec{p})$$

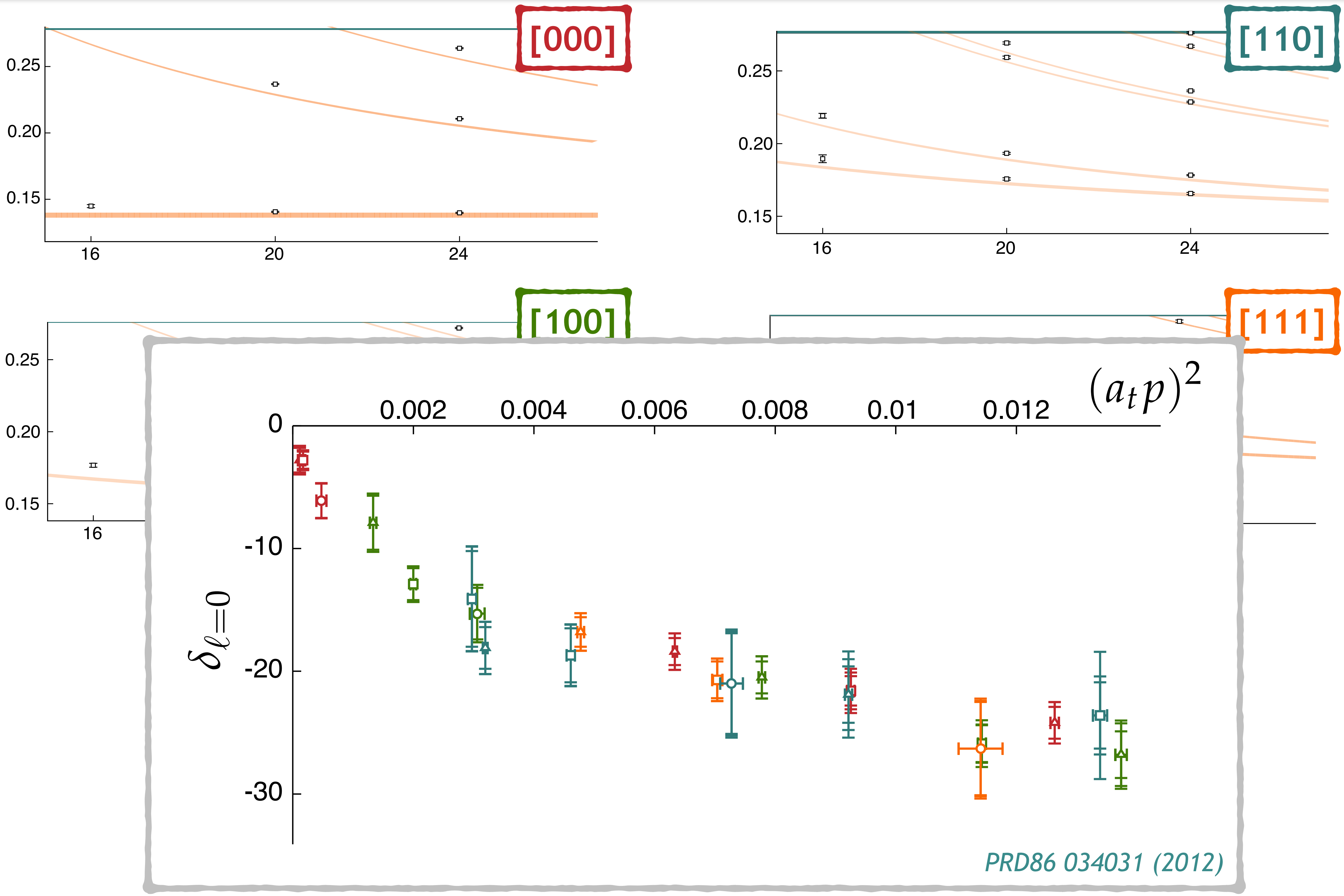
Clebsch-Gordan for irreducible representation Λ of the 'little-group'

ππ I=2 scattering from lattice QCD

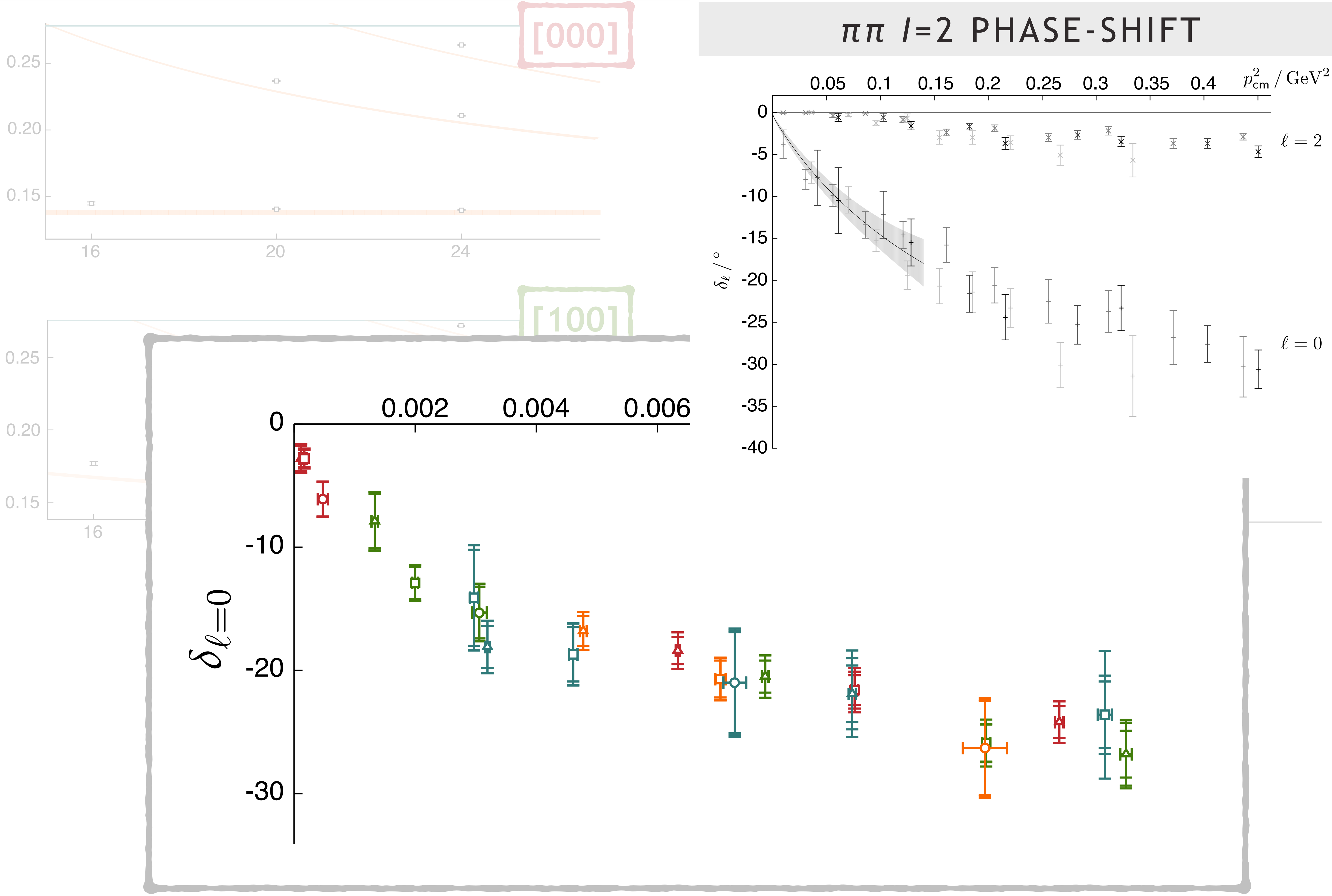
$m_\pi \sim 391\,\mathrm{MeV}$



ππ I=2 scattering from lattice QCD



ππ I=2 scattering from lattice QCD



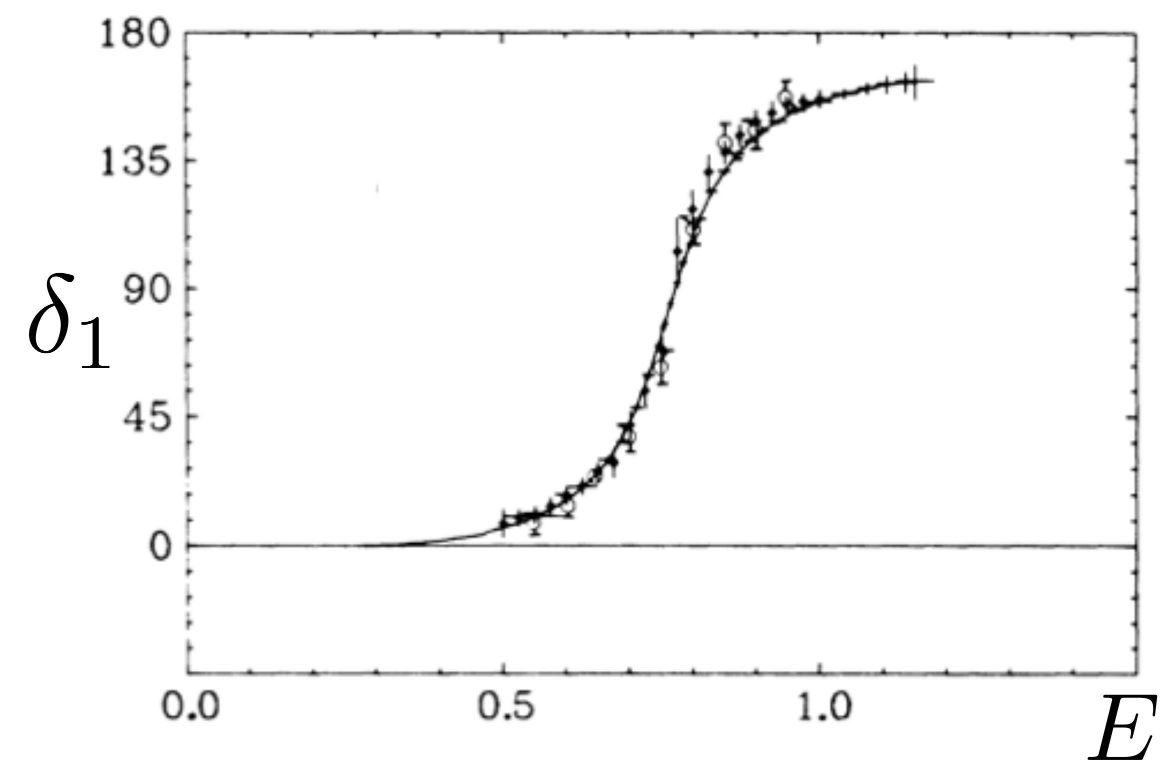
$\pi\pi$ I=1 scattering from lattice QCD

experimentally the I=1 P-wave is qualitatively different - there is a resonance

ρ(770) [h] $I^G(J^{PC}) = 1^+(1^{--})$

Mass $m = 775.26 \pm 0.25$ MeV
Full width $\Gamma = 149.1 \pm 0.8$ MeV

RESONANT PHASE SHIFT



but we'll follow the same approach - first, compute the spectrum in finite volume …

$\pi\pi$ *I*=1 scattering from lattice QCD

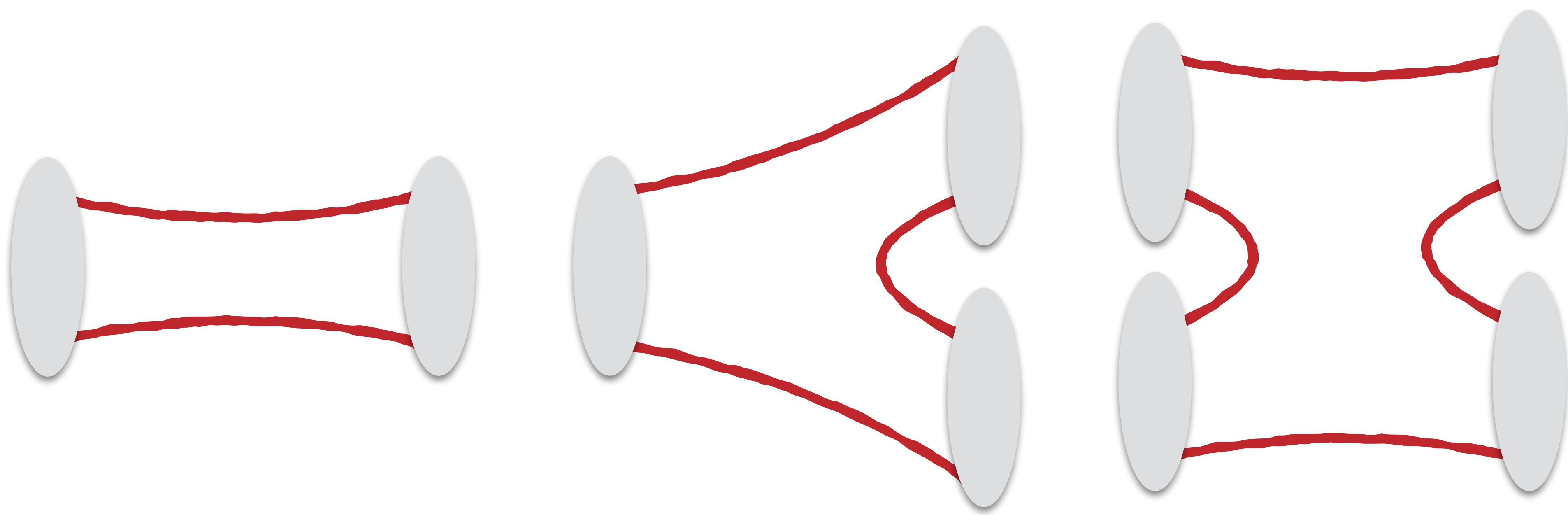
we can again consider a basis of $\pi\pi$-like operators

but in isospin=1, we can also have a smaller quark content, $\bar{u}d$

⇒ why not supplement with a basis of $\bar{u}\Gamma D \ldots Dd$ constructions

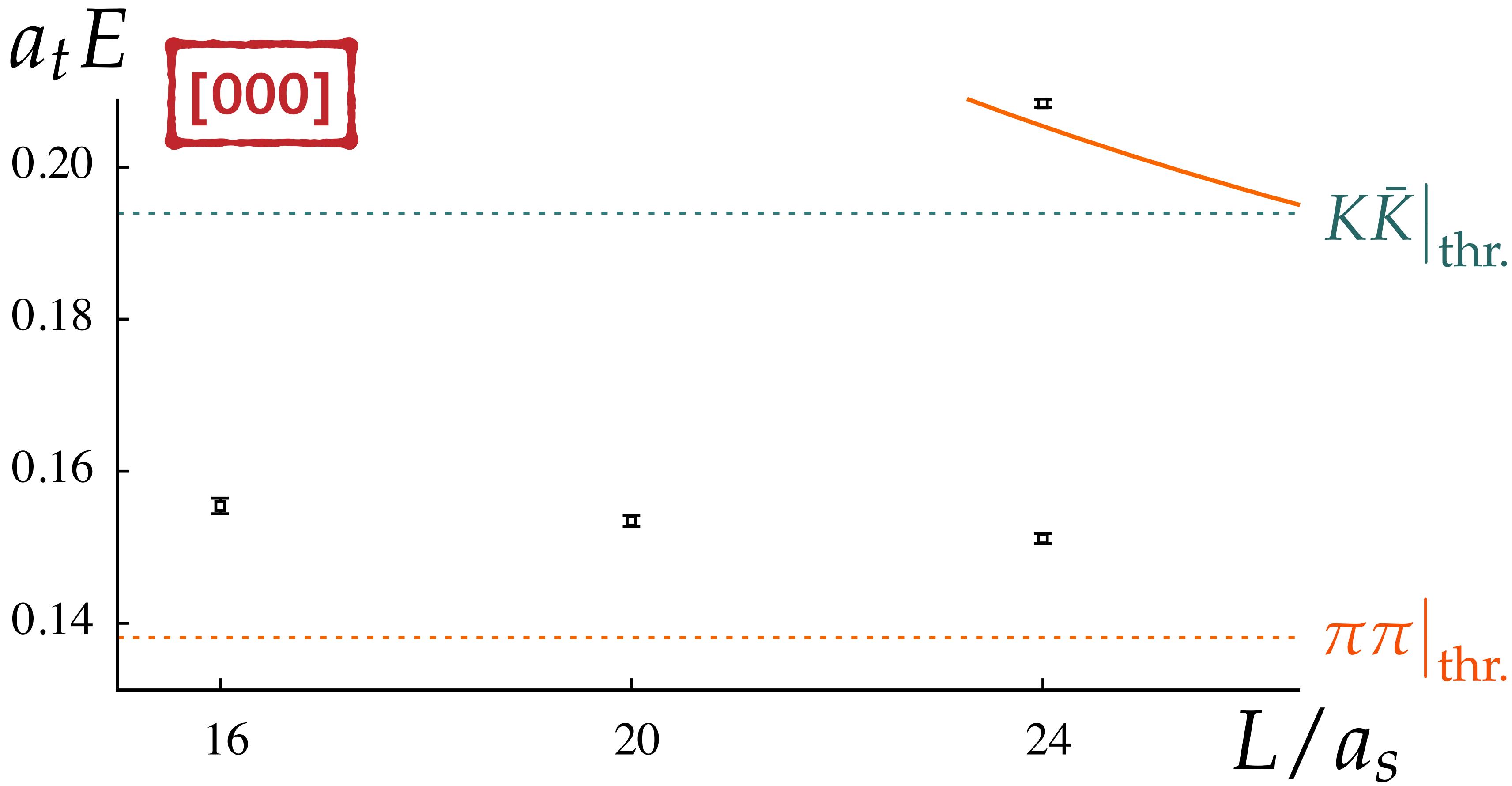
formally integrate out the quark fields …

more involved set of quark propagations:

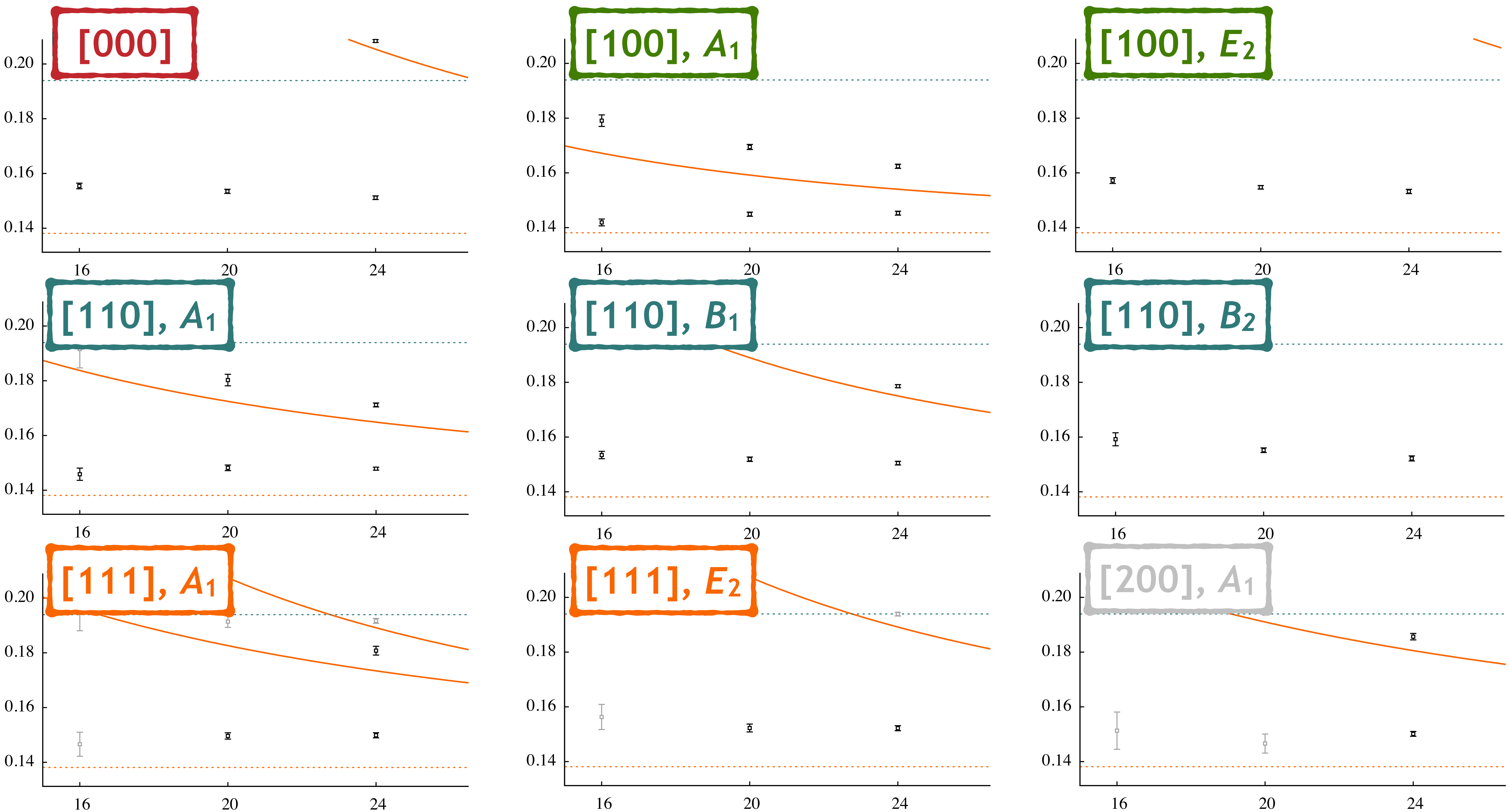


but we've got the technology to accomplish this …

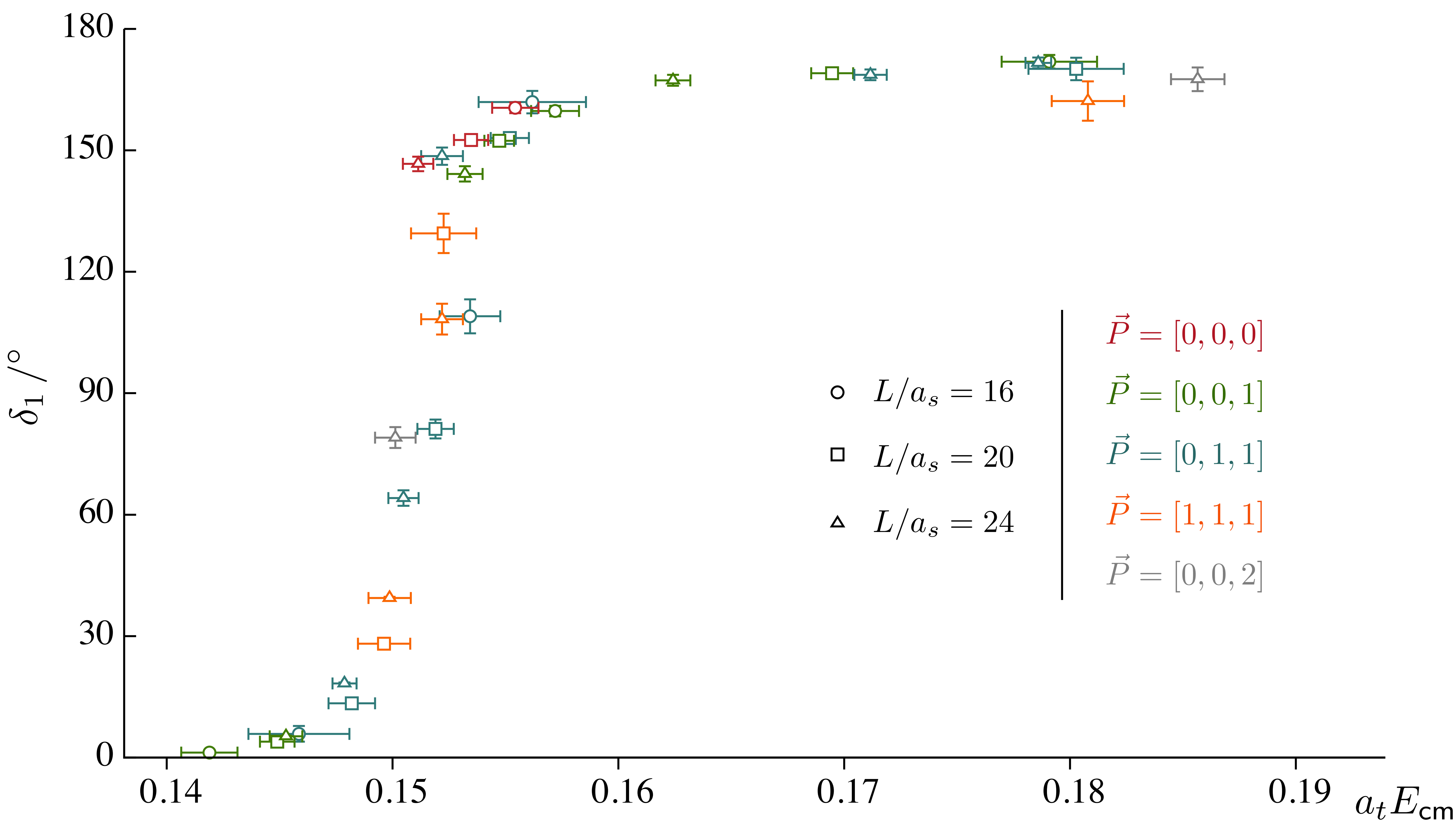
rest frame spectrum



in-flight spectra

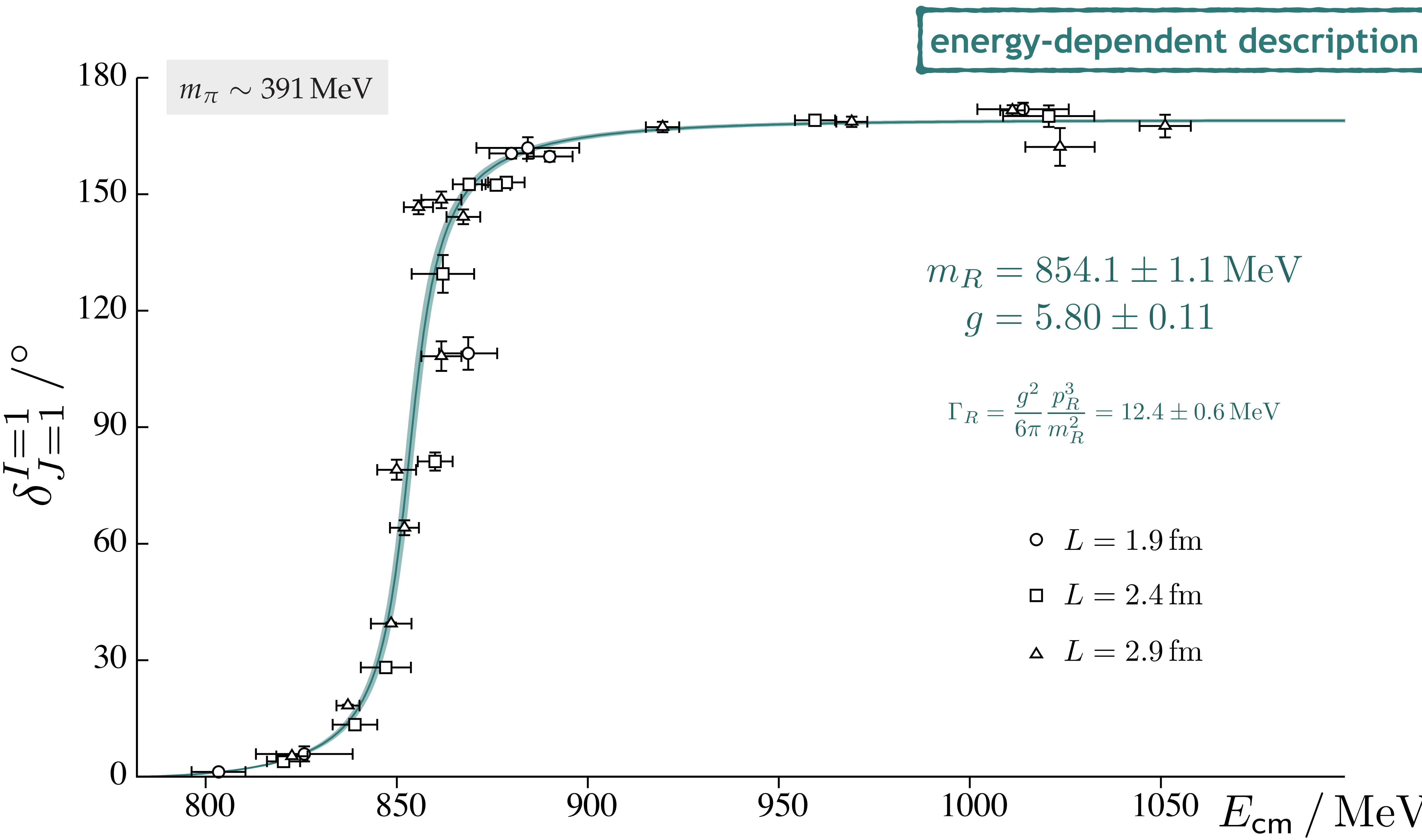


$\pi\pi$ I=1 scattering from lattice QCD



PRD87 034505 (2013)

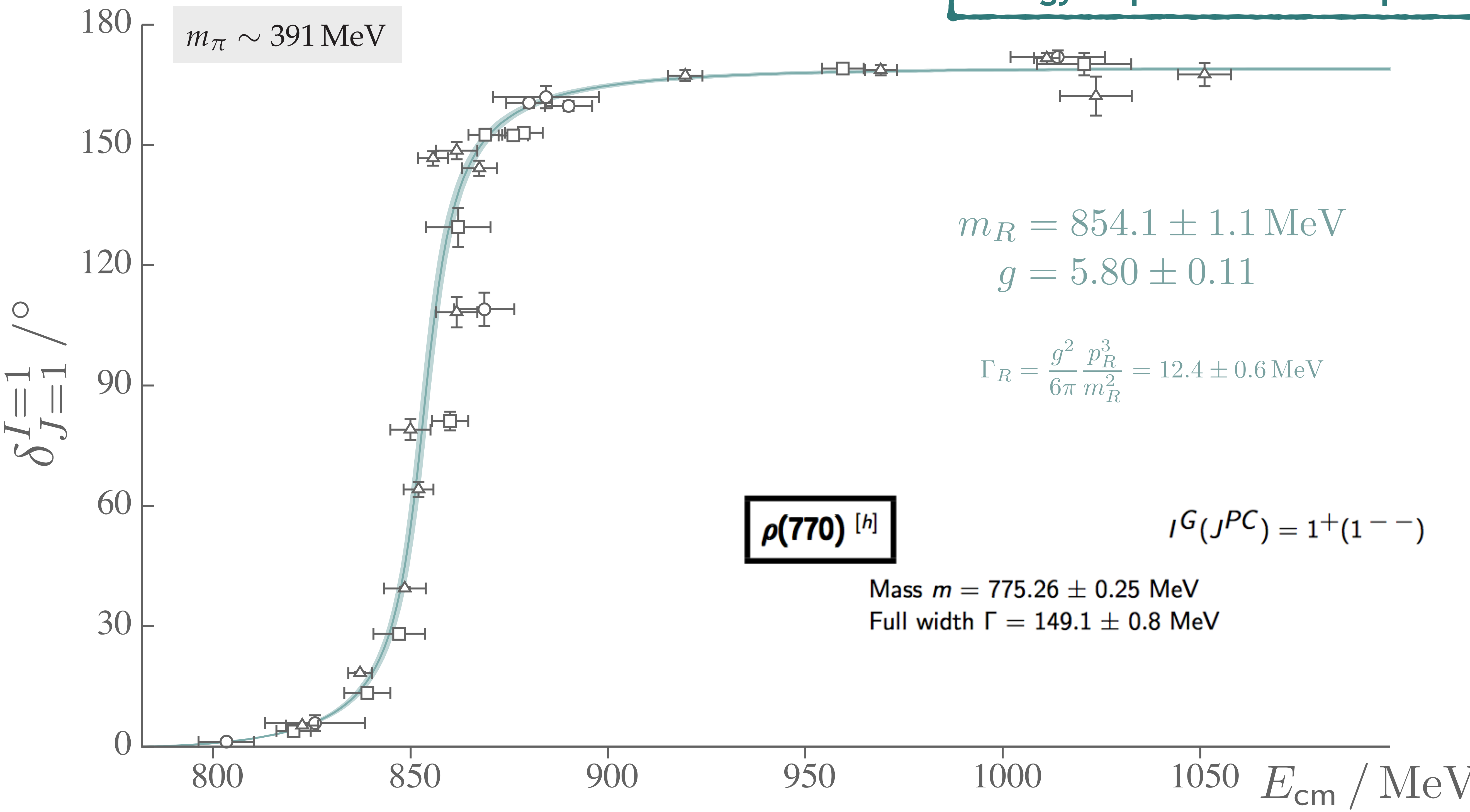
ππ *P*-wave phase-shift



PRD87 034505 (2013)

$\pi\pi$ *P*-wave phase-shift

energy-dependent description



$\rho(770)$ [h]

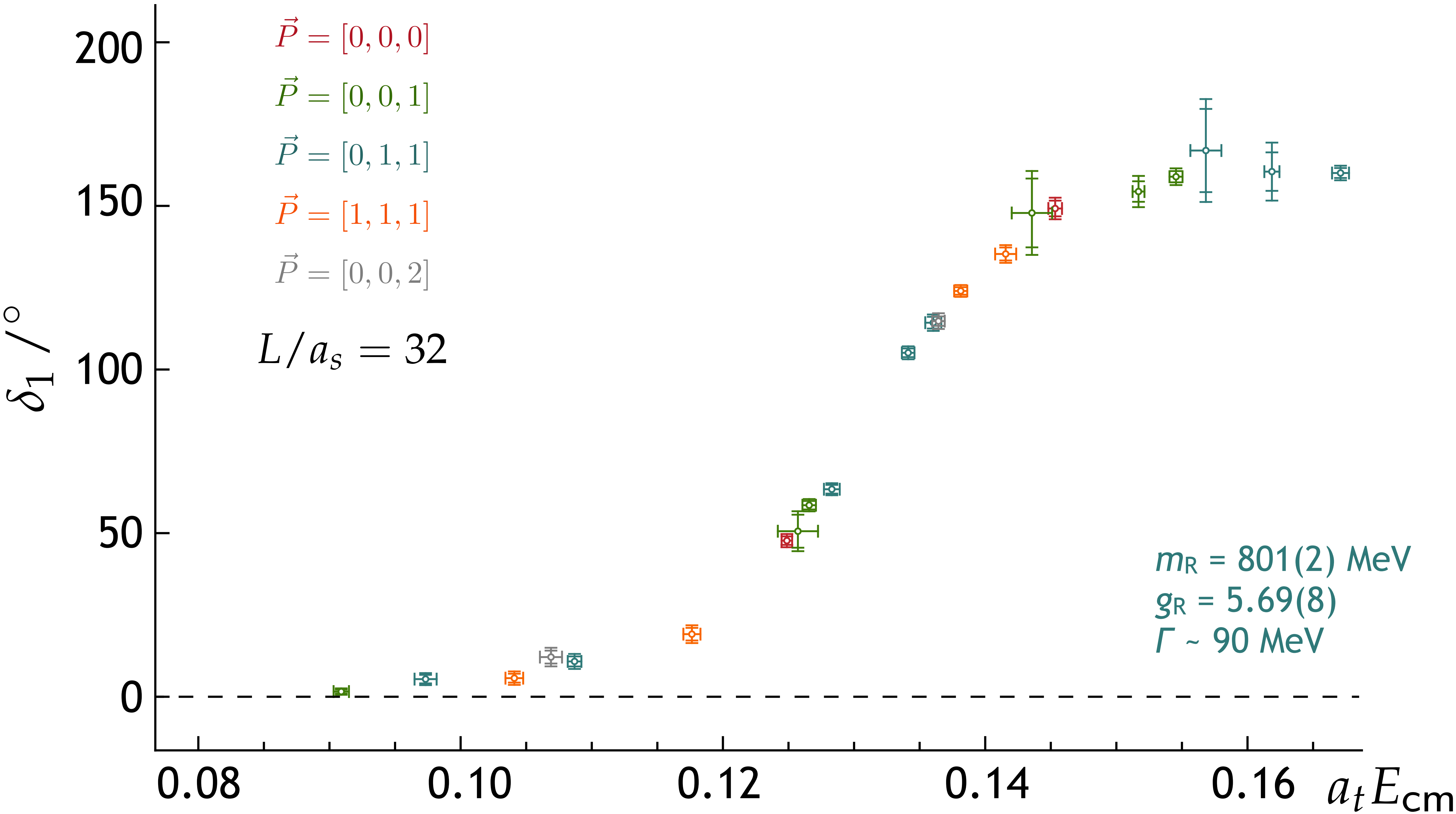
$I^G(J^{PC}) = 1^+(1^{--})$

Mass $m = 775.26 \pm 0.25$ MeV
Full width $\Gamma = 149.1 \pm 0.8$ MeV

PRD87 034505 (2013)

at a smaller quark mass ...

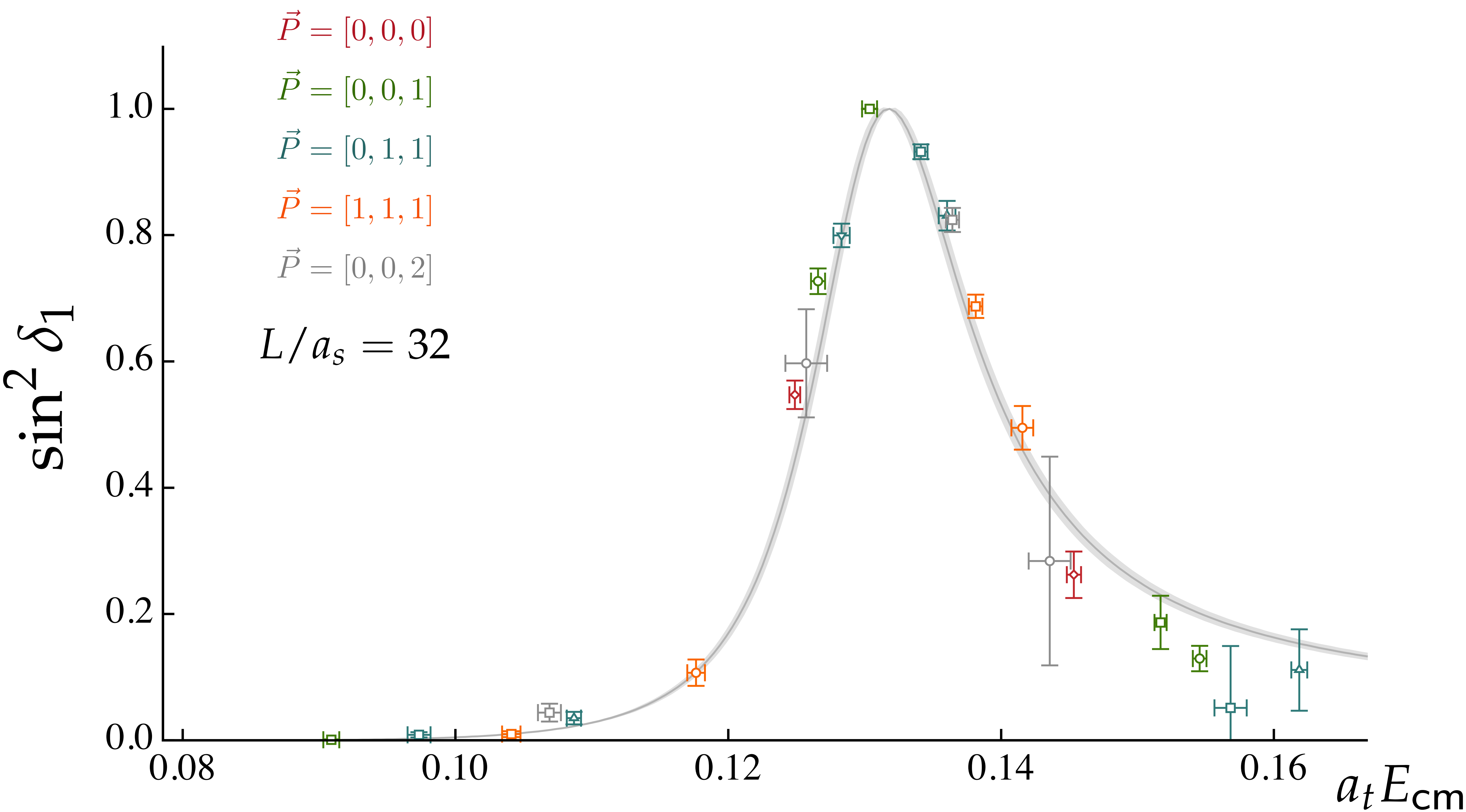
$m_\pi \sim 230\,\mathrm{MeV}$



PRELIMINARY ... to appear soon

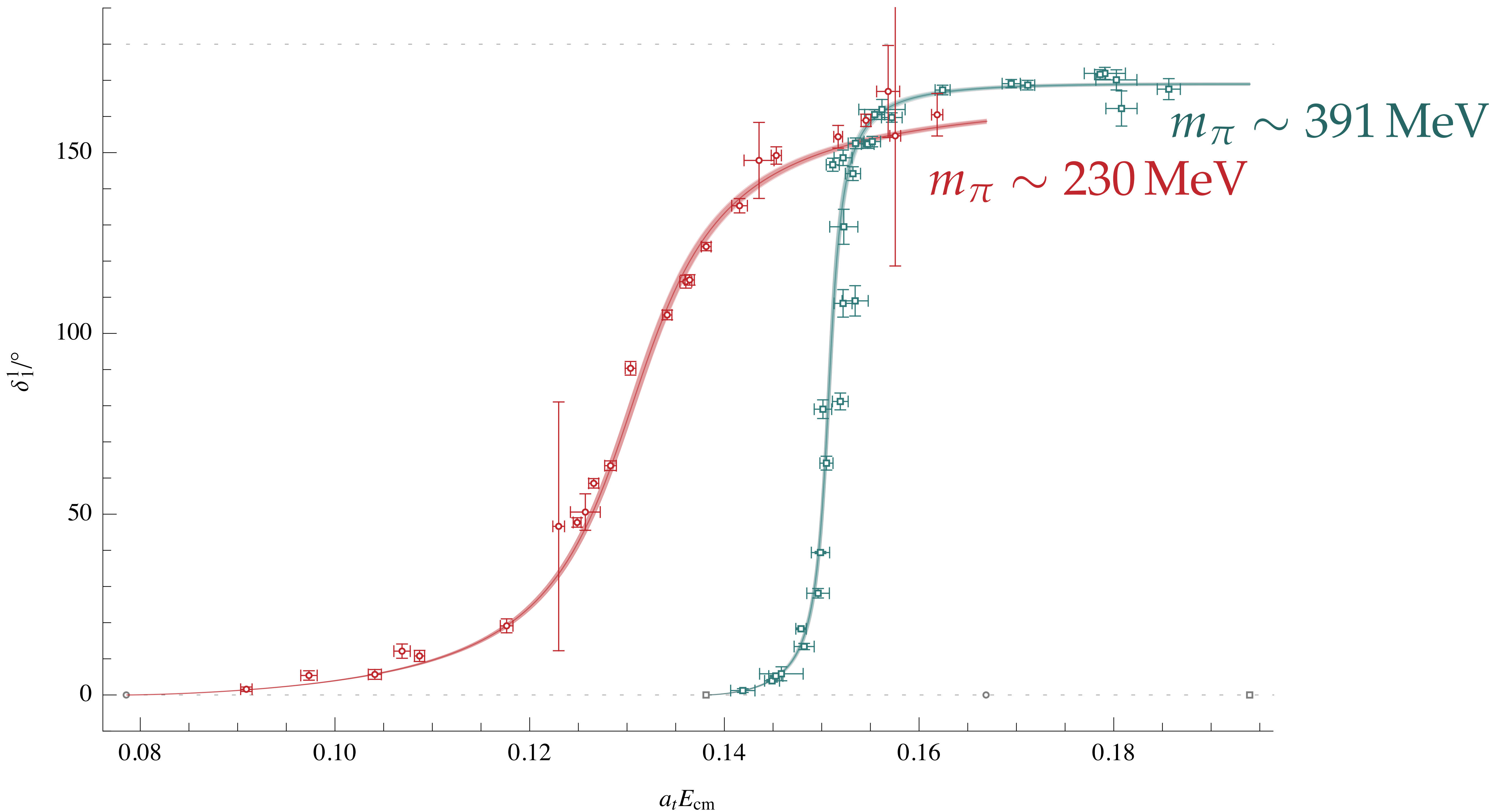
at a smaller quark mass ...

$m_\pi \sim 230\,\mathrm{MeV}$



PRELIMINARY ... to appear soon

at a smaller quark mass …

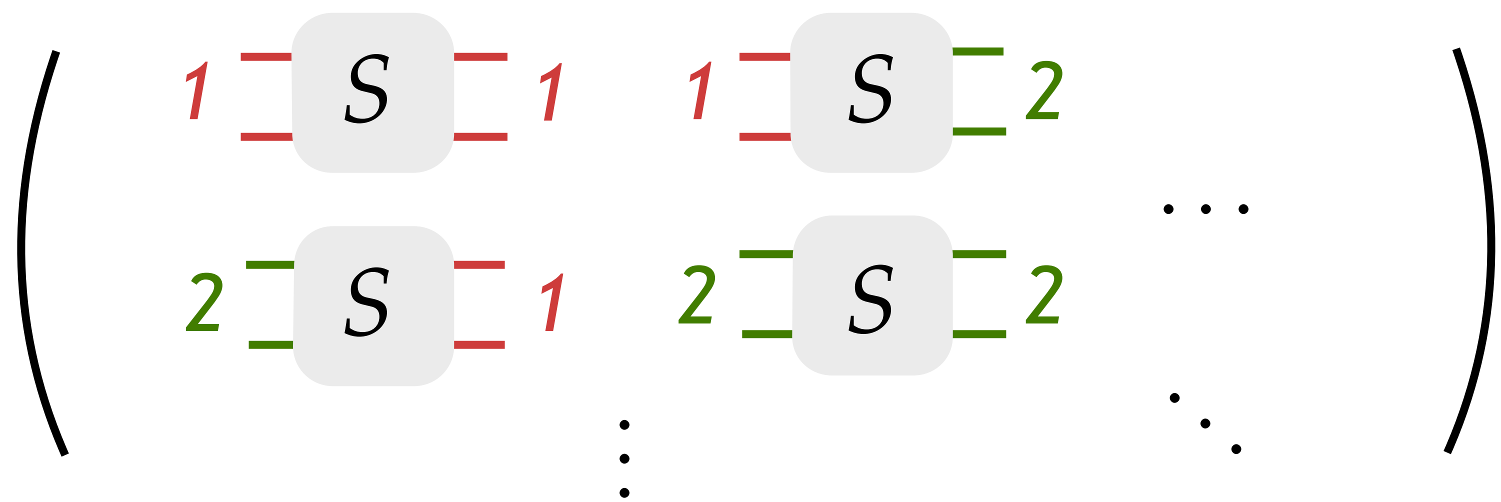


extracting resonances from finite-volume spectra computed in lattice QCD looks promising …

coupled-channel scattering

scattering now described by an S-matrix

e.g. $\mathbf{S} =$



or equivalently a t-matrix $$\mathbf{S} = \mathbf{1} + 2i\sqrt{\boldsymbol{\rho}}\,\mathbf{t}\sqrt{\boldsymbol{\rho}}$$

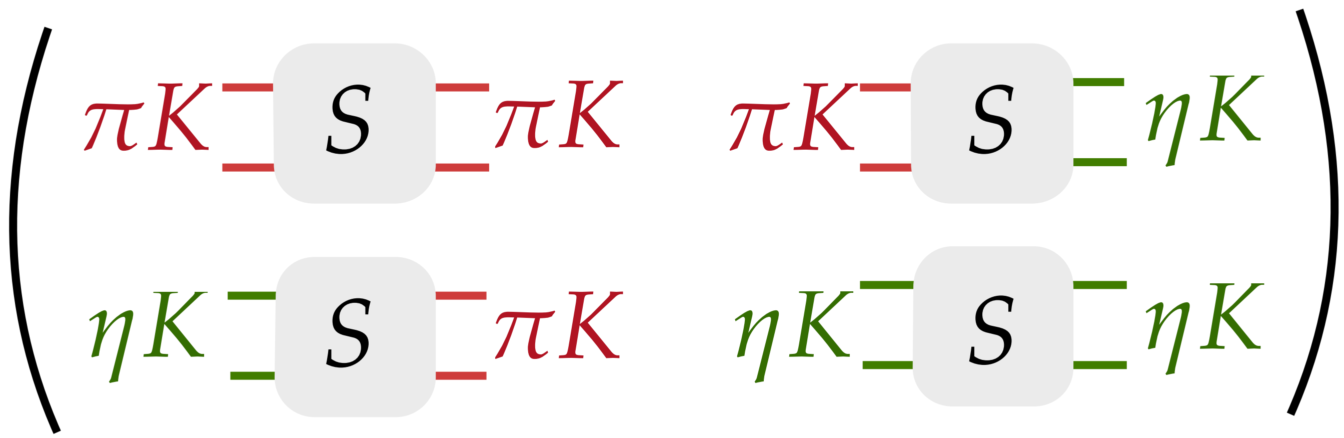
phase space $$\rho_{ij} = \delta_{ij}\frac{2k_i}{\sqrt{s}}$$

unitarity condition $$\mathrm{Im}\big[t^{-1}(s)\big]_{ij} = -\delta_{ij}\,\rho_i(s)\,\Theta(s - s_i^{\mathrm{thr.}})$$

coupled πK, ηK scattering

below η′K threshold, a two-channel system

$$\mathbf{S} =$$



$$S_{\pi K,\pi K} = \eta\, e^{2i\delta_{\pi K}}$$
$$S_{\eta K,\eta K} = \eta\, e^{2i\delta_{\eta K}}$$
$$S_{\pi K,\eta K} = i\sqrt{1-\eta^2}\, e^{i(\delta_{\eta K}+\delta_{\pi K})}$$

phase-shifts & inelasticity parameterization

experimentally:

S-wave: broad resonances ? κ(700), K^*_0(1430)
P-wave: narrow resonance *K**(892)
D-wave: narrow resonance K^*_2(1430)

all essentially decoupled from *ηK*

coupled-channels in a finite-volume

there is again a discrete spectrum determined by the scattering amplitudes

He, JHEP 0507 011
Hansen, PRD86 016007
Briceno, PRD88 094507
Guo, PRD88 014051

$$\det\left[\mathbf{t}^{-1}(E) + i\boldsymbol{\rho}(E) - \mathbf{M}(E,L)\right] = 0$$

known 'kinematic' functions

spectrum given by the values of *E* which solve this equation

computing the $\pi K, \eta K$ spectrum in LQCD

operator basis :

$q\bar{q}$ -like

$$\bar{u}\boldsymbol{\Gamma}s = \bar{u}\,\Gamma D \ldots D\, s$$

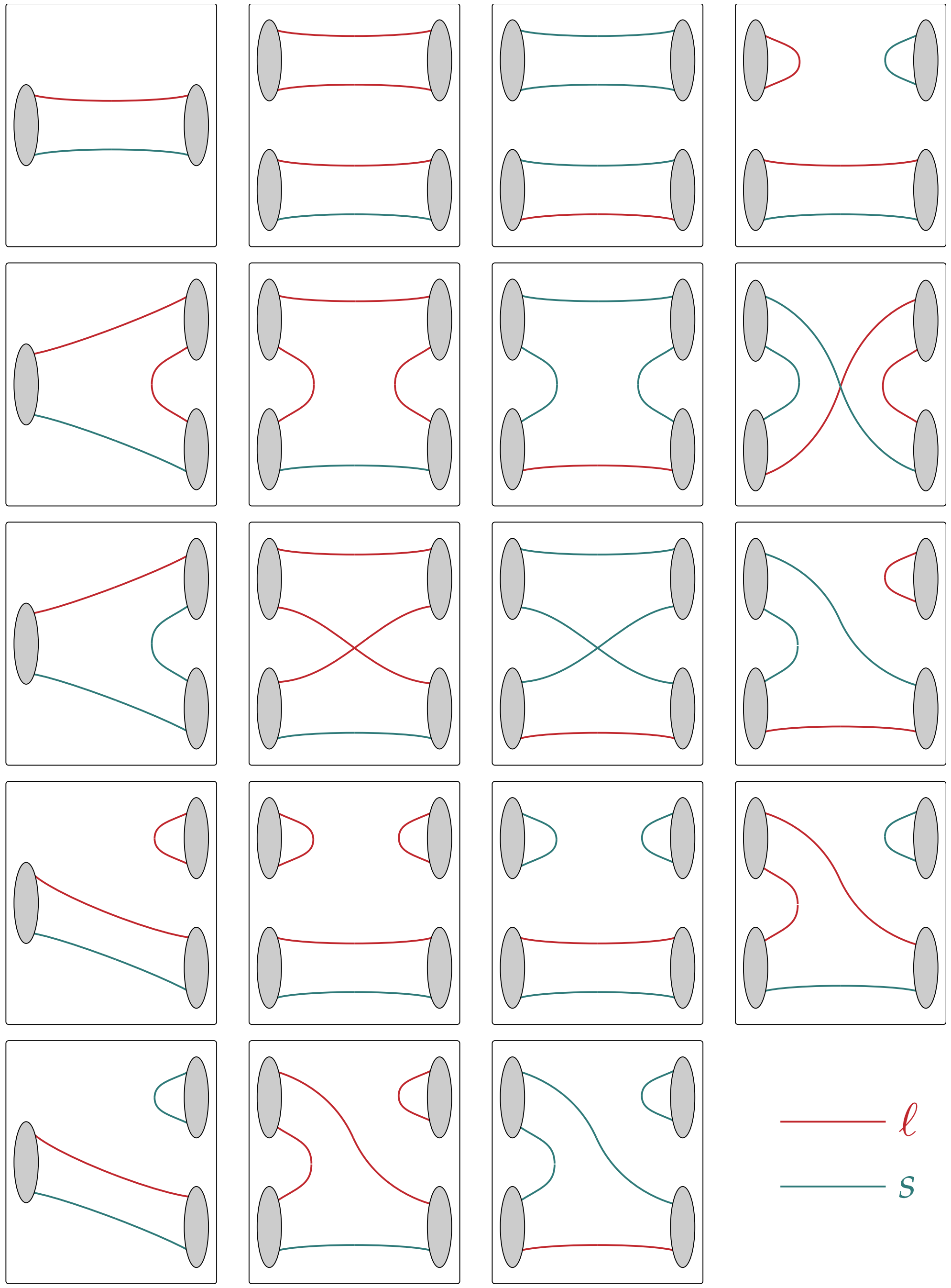
πK -like

$$\sum_{\hat{p}_1,\hat{p}_2} C(\Lambda,\vec{P};\vec{p}_1,\vec{p}_2)\,\pi(\vec{p}_1)K(\vec{p}_2)$$

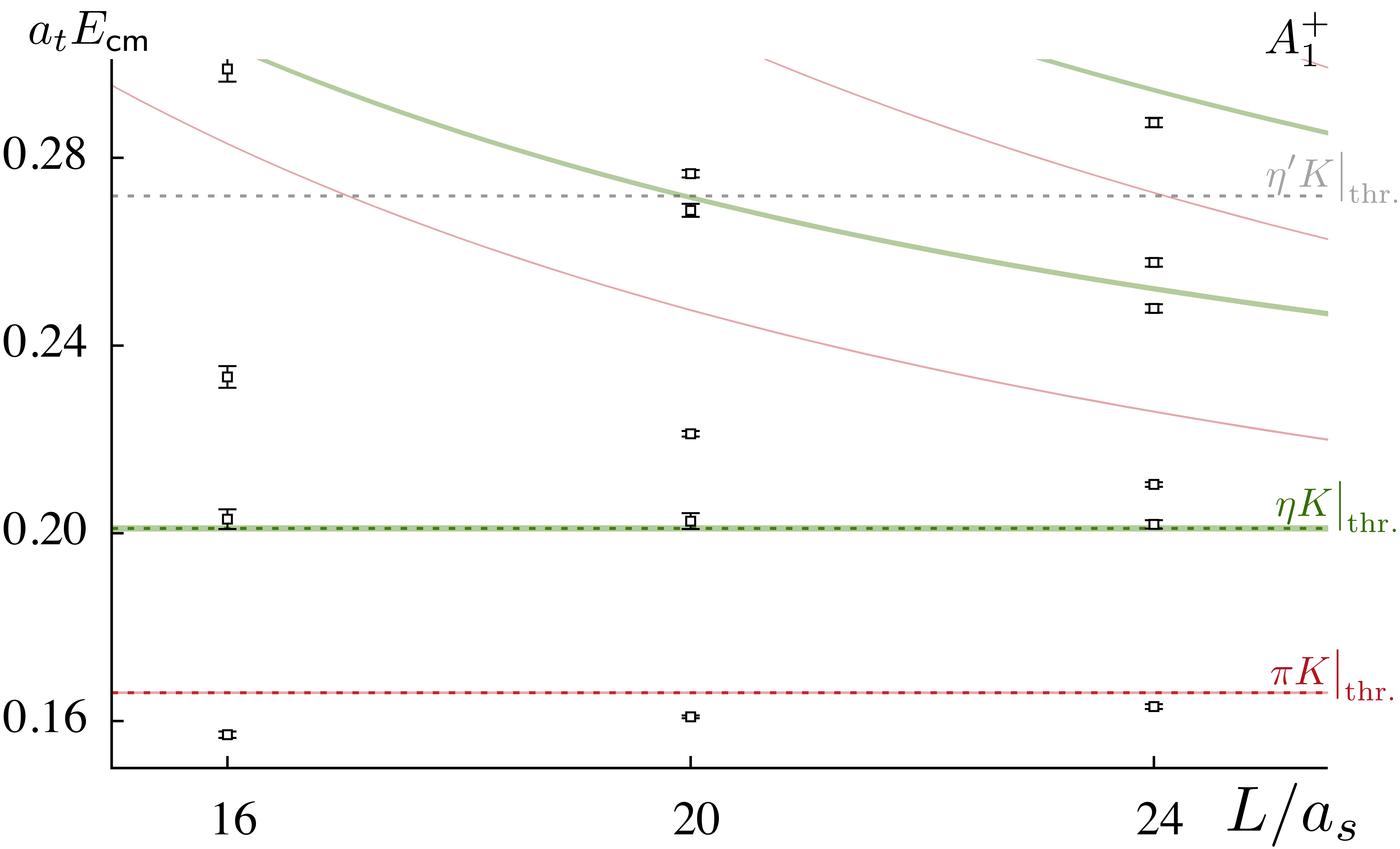
ηK -like

$$\sum_{\hat{p}_1,\hat{p}_2} C(\Lambda,\vec{P};\vec{p}_1,\vec{p}_2)\,\eta(\vec{p}_1)K(\vec{p}_2)$$

WICK CONTRACTIONS

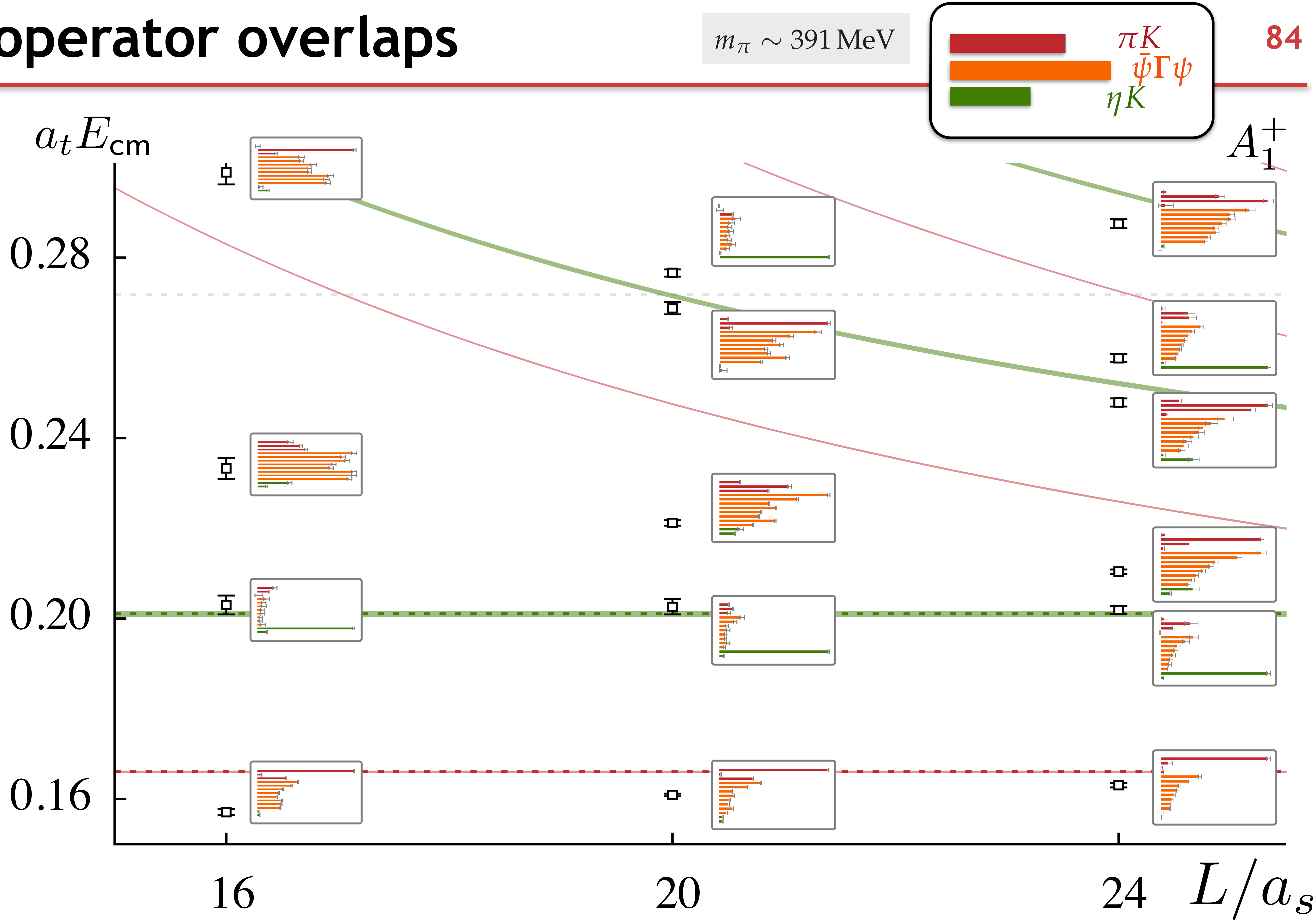


the calculated $\pi K, \eta K$ spectrum



operator overlaps

$m_\pi \sim 391\,\mathrm{MeV}$



coupled-channels in a finite-volume

there is again a discrete spectrum determined by the scattering amplitudes

$$\det\left[\mathbf{t}^{-1}(E) + i\boldsymbol{\rho}(E) - \mathbf{M}(E,L)\right] = 0$$

known 'kinematic' functions

spectrum given by the values of *E* which solve this equation

if we have the energy levels – how do we find ***t***(*E*) ?

for each energy level, $\frac{1}{2}N(N+1)$ unknowns = 3 in the two-channel case

so can't just solve level-by-level like we did in the elastic case

how about parameterizing the energy dependence ?

parameterizing *t(E)*

must be a unitarity-preserving parameterization

$$\det\left[\mathbf{t}^{-1}(E) + i\boldsymbol{\rho}(E) - \mathbf{M}(E,L)\right] = 0$$

$$\det\left[\mathrm{Re}(\mathbf{t}^{-1}) + i\,\mathrm{Im}(\mathbf{t}^{-1}) + i\boldsymbol{\rho} - \mathbf{M}\right] = 0$$

must vanish to have solutions

real above threshold

***S*-matrix constraints are entering the game …**

e.g. *K*-matrix form

$$\mathbf{t}^{-1}(E) = \mathbf{K}^{-1}(E) + \mathbf{I}(E)$$

real function

$$\mathrm{Im}\, I_{ij}(E) = -\delta_{ij}\,\rho_i(E)$$

e.g. Chew-Mandelstam form

e.g. 6 parameter "pole plus constant" form

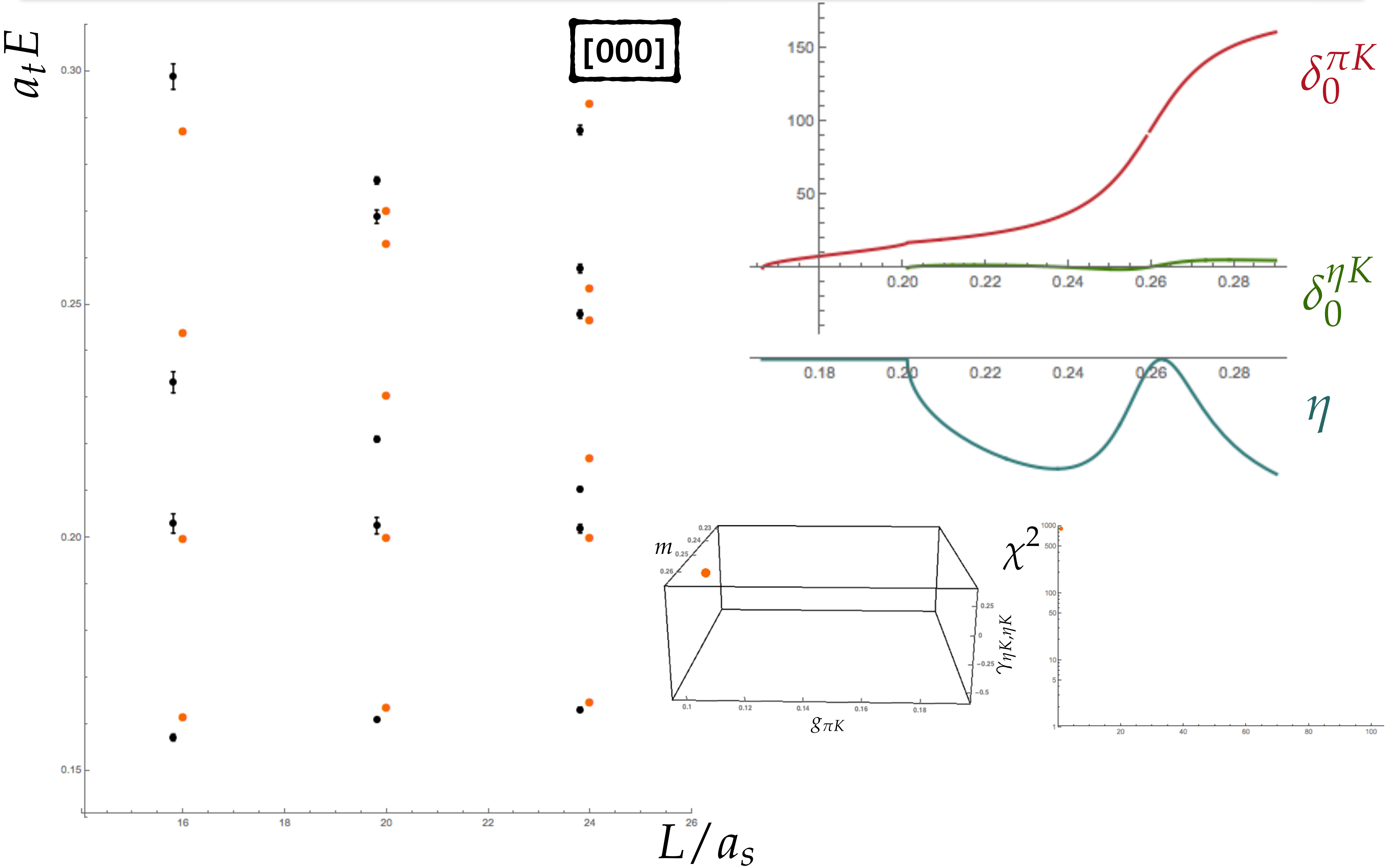
$$K_{ij}(E) = \frac{g_i\, g_j}{m^2 - E^2} + \gamma_{ij}$$

with variables

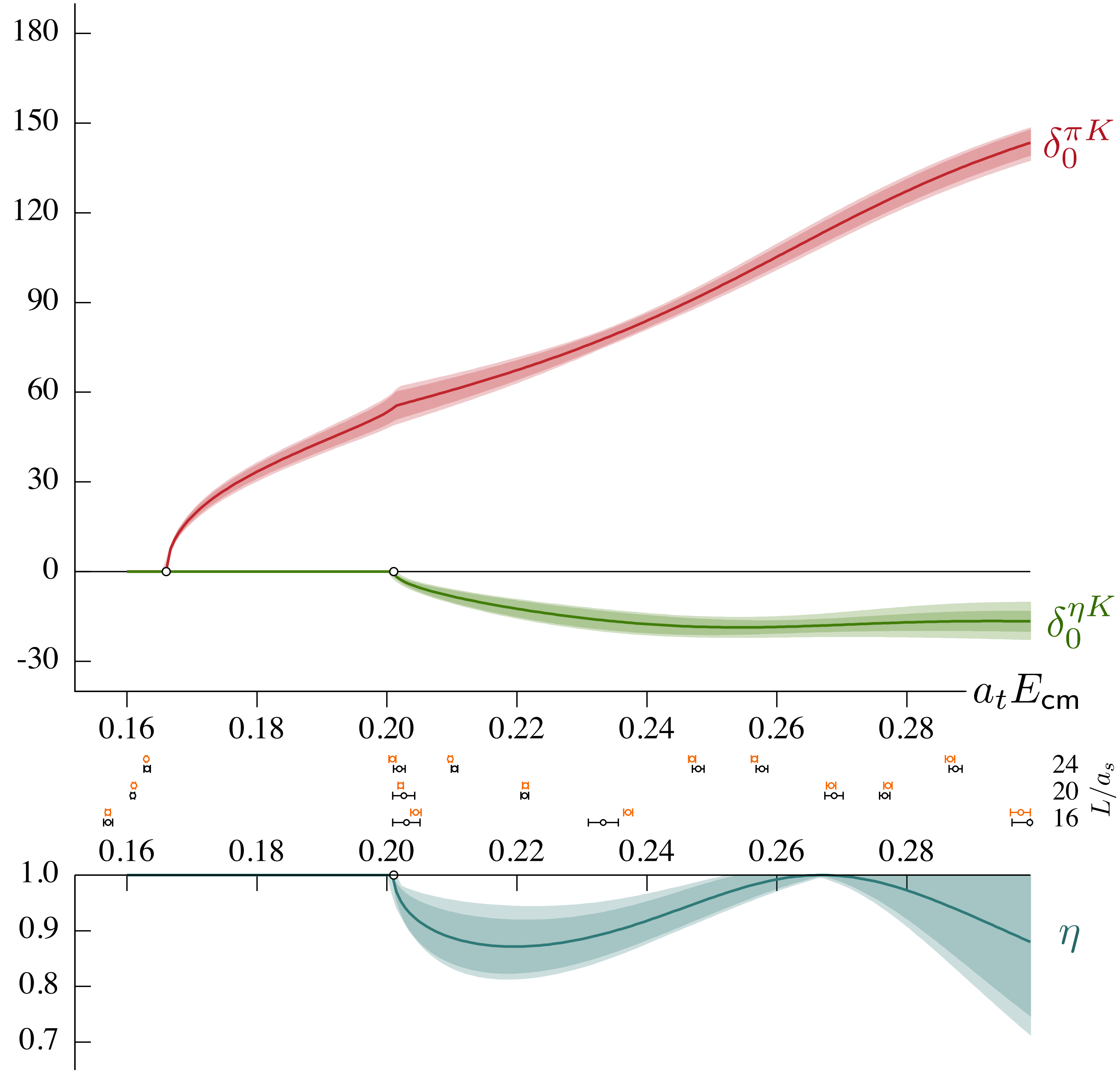
$m,\ g_1,\ g_2,\ \gamma_{11},\ \gamma_{12},\ \gamma_{22}$

a cartoon of the minimization

$m_\pi \sim 391\,\mathrm{MeV}$



S-wave amplitudes



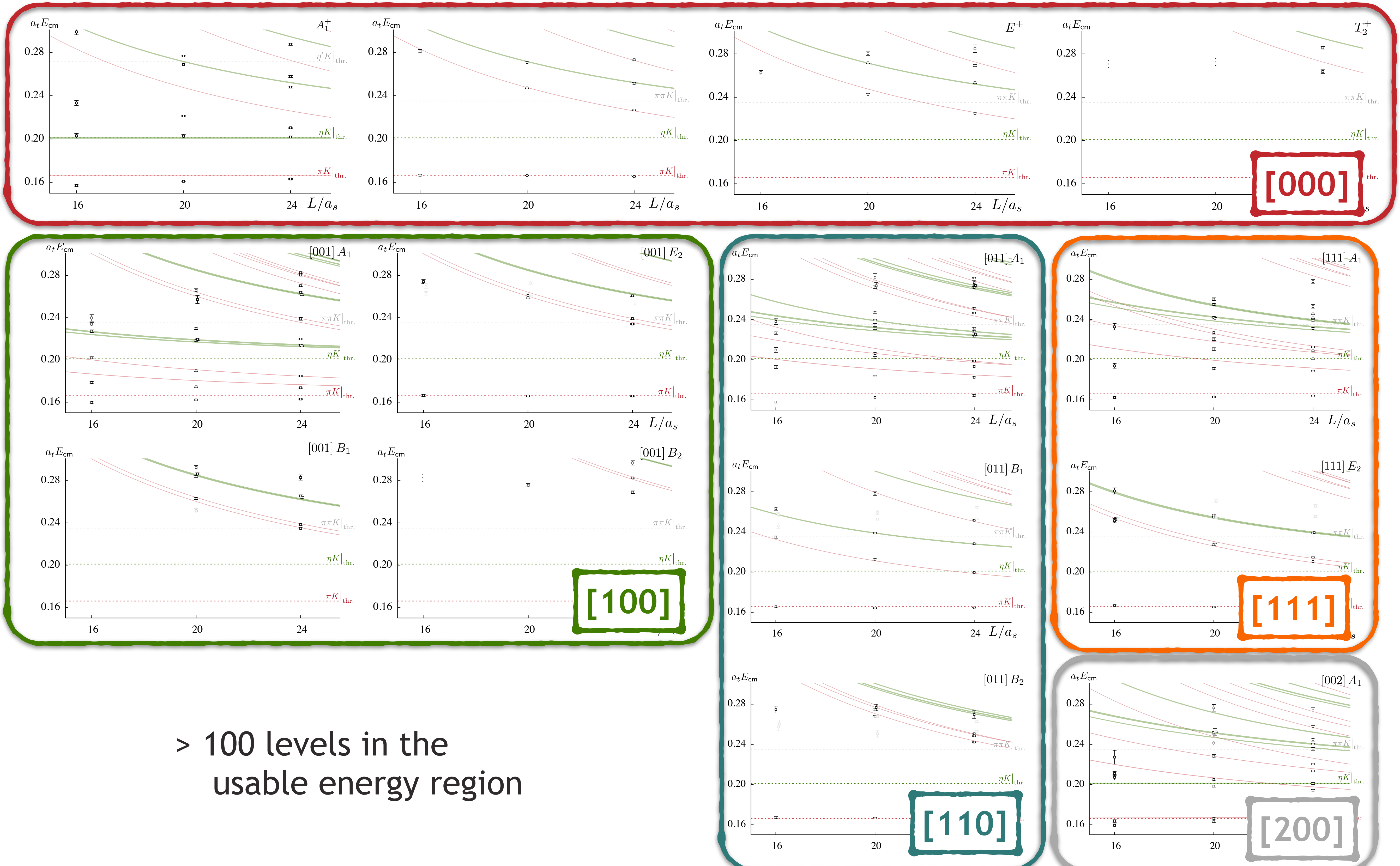
resonant (?) πK scattering

weak ηK scattering

weakly coupled

$\pi K/\eta K$ coupled-channel scattering

$m_\pi \sim 391\,\text{MeV}$



> 100 levels in the usable energy region

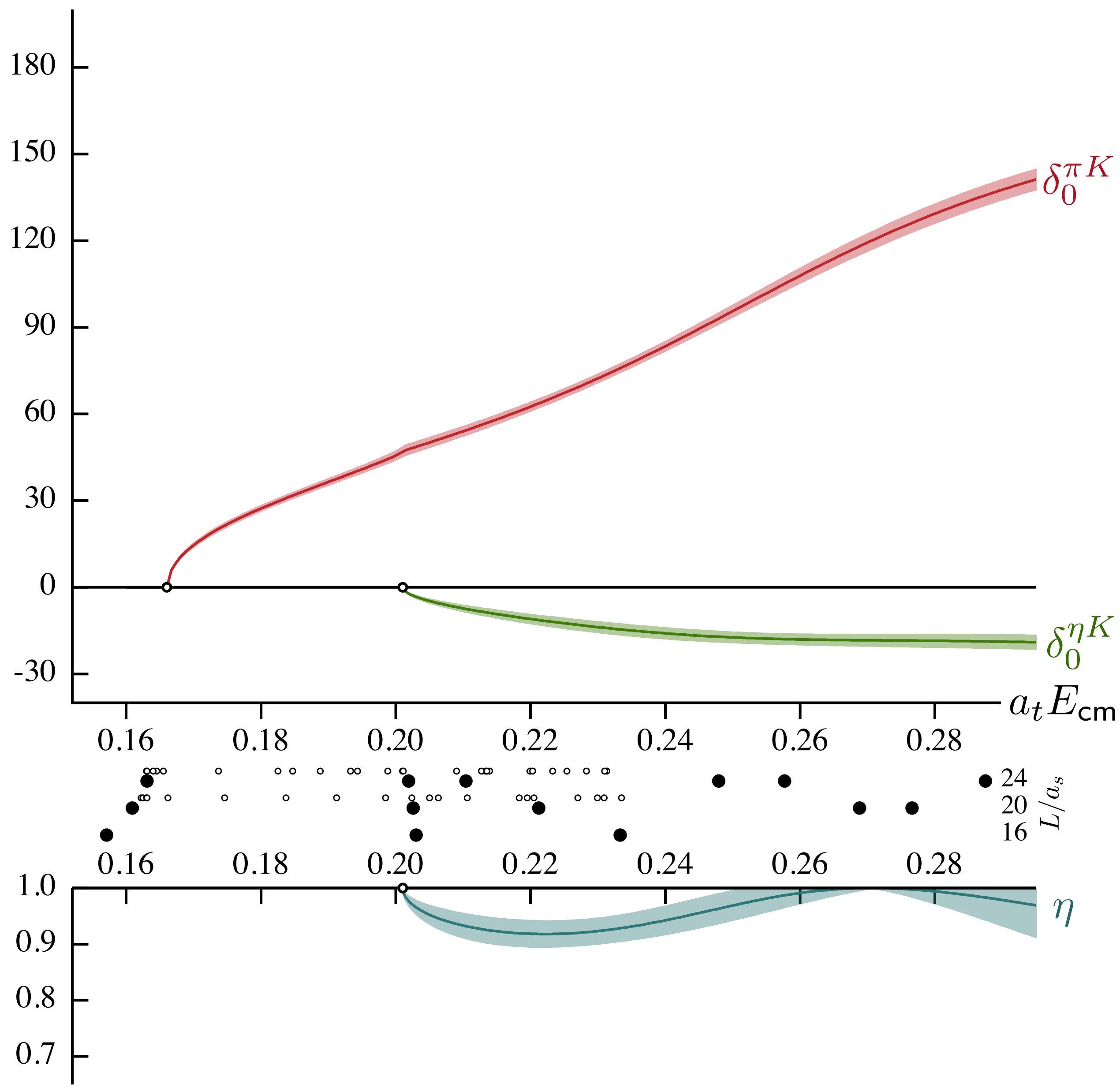
$\pi K/\eta K$ coupled-channel scattering

$m_\pi \sim 391\,\mathrm{MeV}$ 90

describe all the finite-volume spectra

$$\chi^2/N_{\text{dof}} = \frac{49.1}{61-6} = 0.89$$

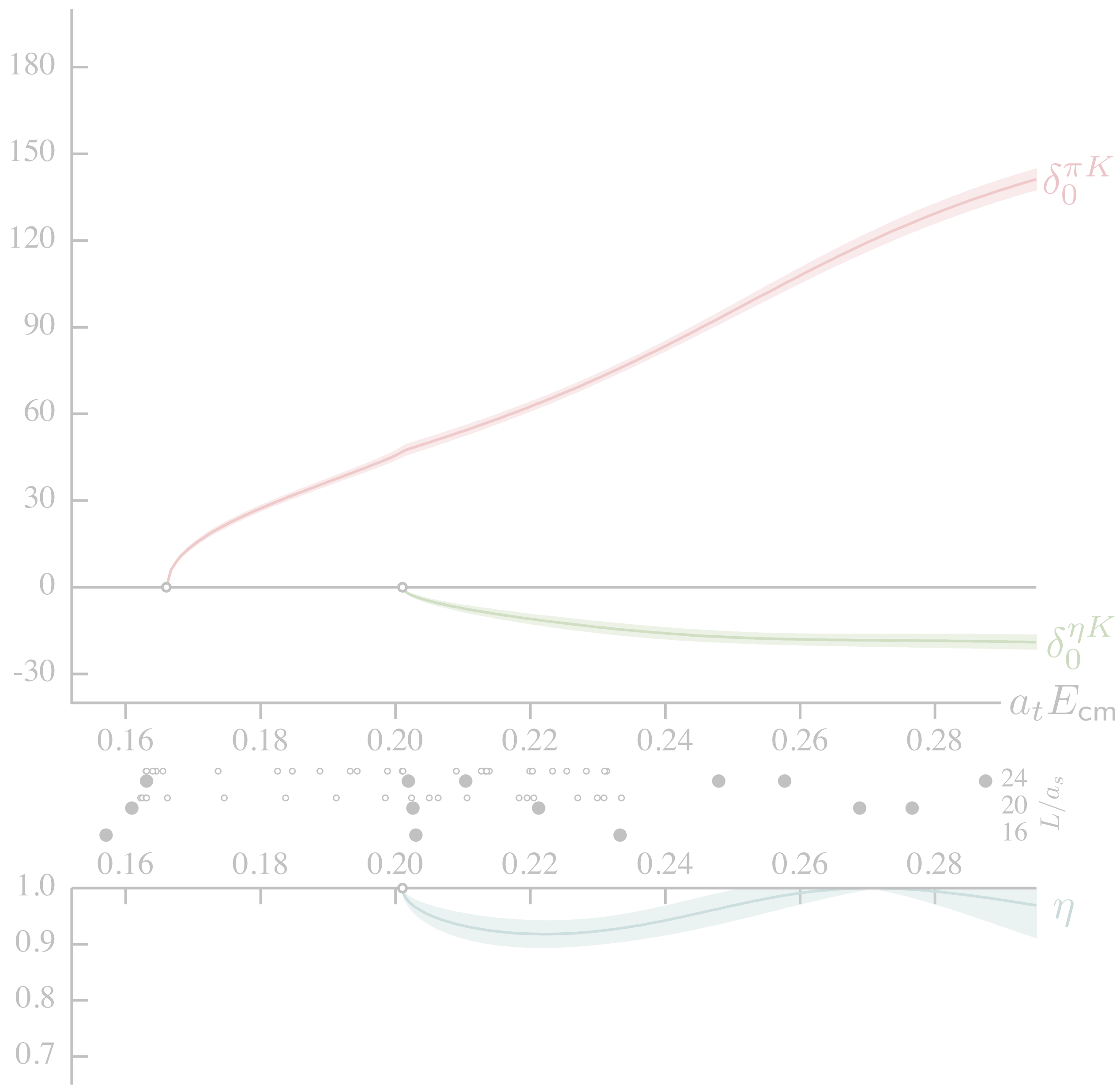
S-WAVE $\pi K/\eta K$ SCATTERING



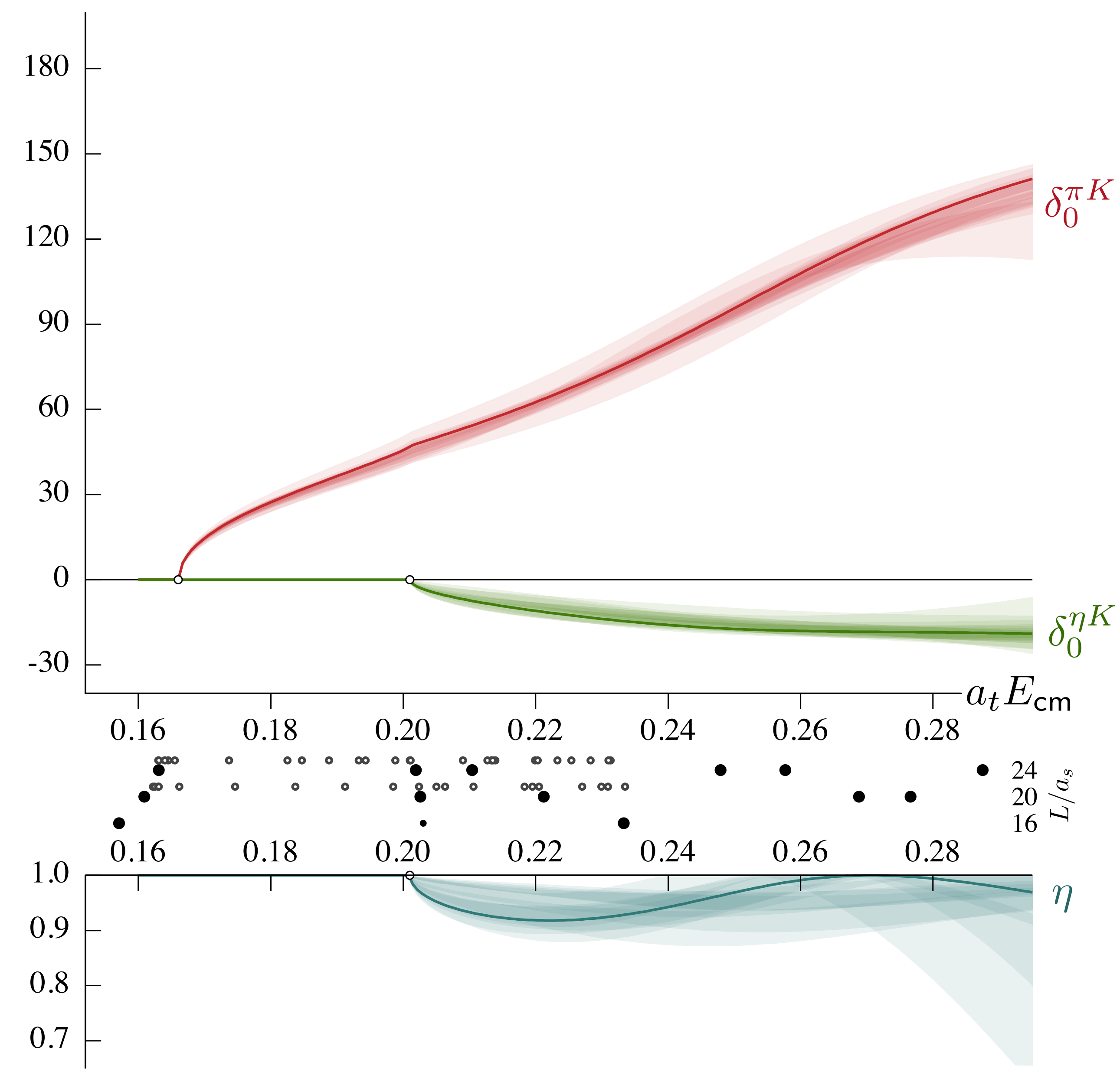
$\pi K/\eta K$ coupled-channel scattering

are the result parameterization dependent ?

– try a range of parameterizations ...



S-WAVE $\pi K/\eta K$ SCATTERING



– gross features are robust

$\pi K/\eta K$ coupled-channel scattering

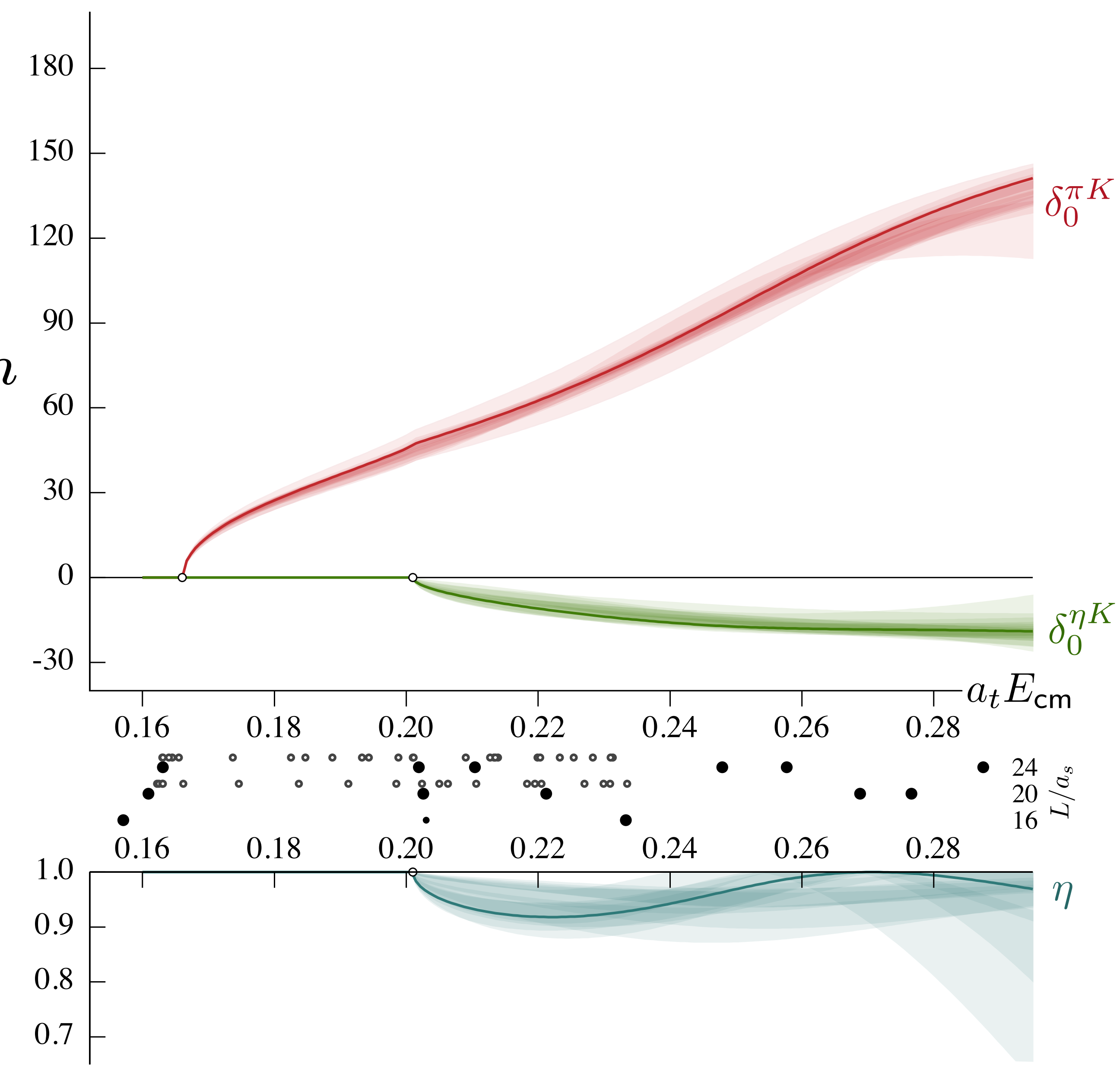
are the result parameterization dependent ?

– try a range of parameterizations ...

$$K^{-1}_{ij}(s) = \sum_{n=0}^{N_{ij}} c^{(n)}_{ij}\, s^n$$

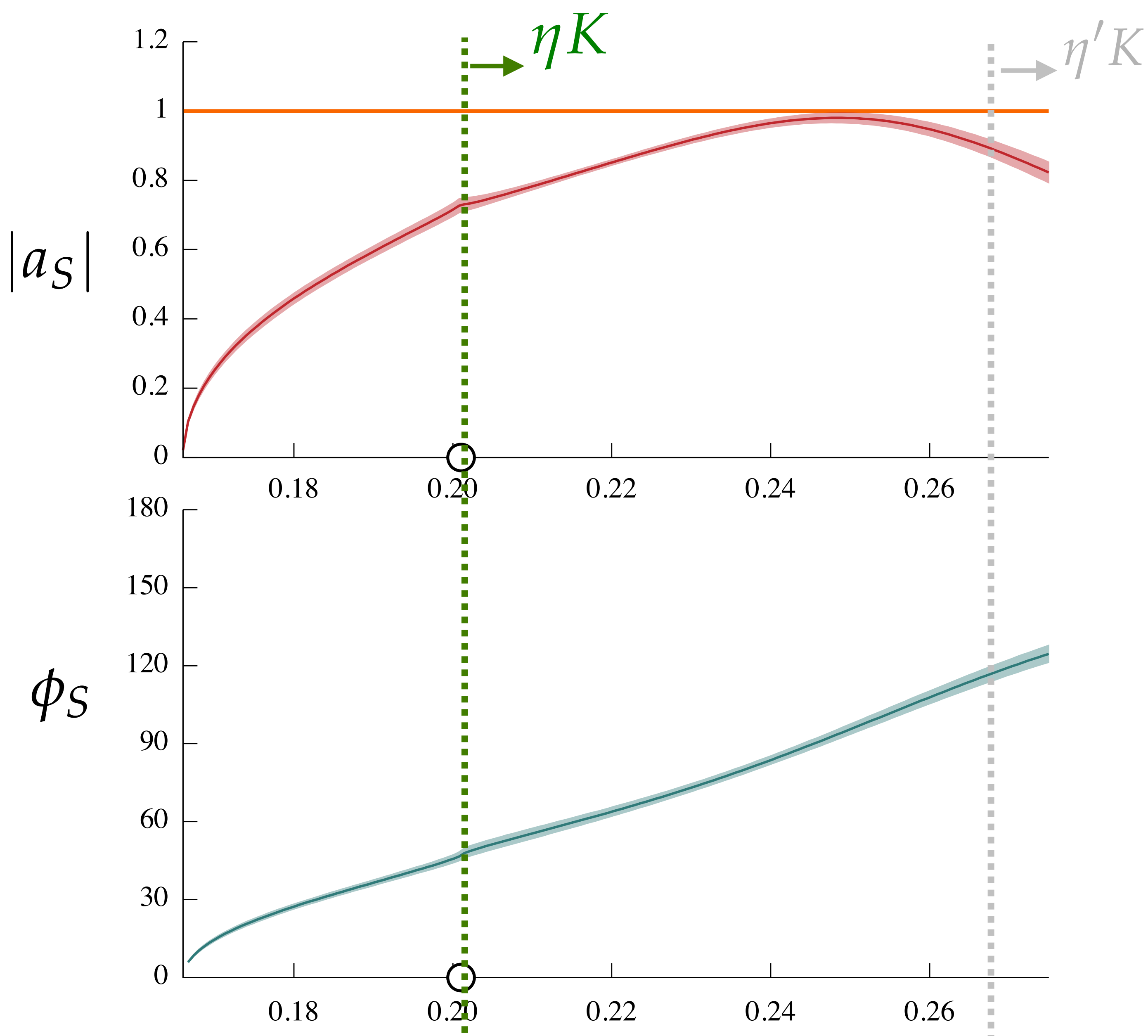
$$K_{ij}(s) = \sum_p \frac{g_i^{(p)} g_j^{(p)}}{m_p^2 - s} + \sum_n \gamma^{(n)}_{ij}\, s^n$$

S-WAVE $\pi K/\eta K$ SCATTERING

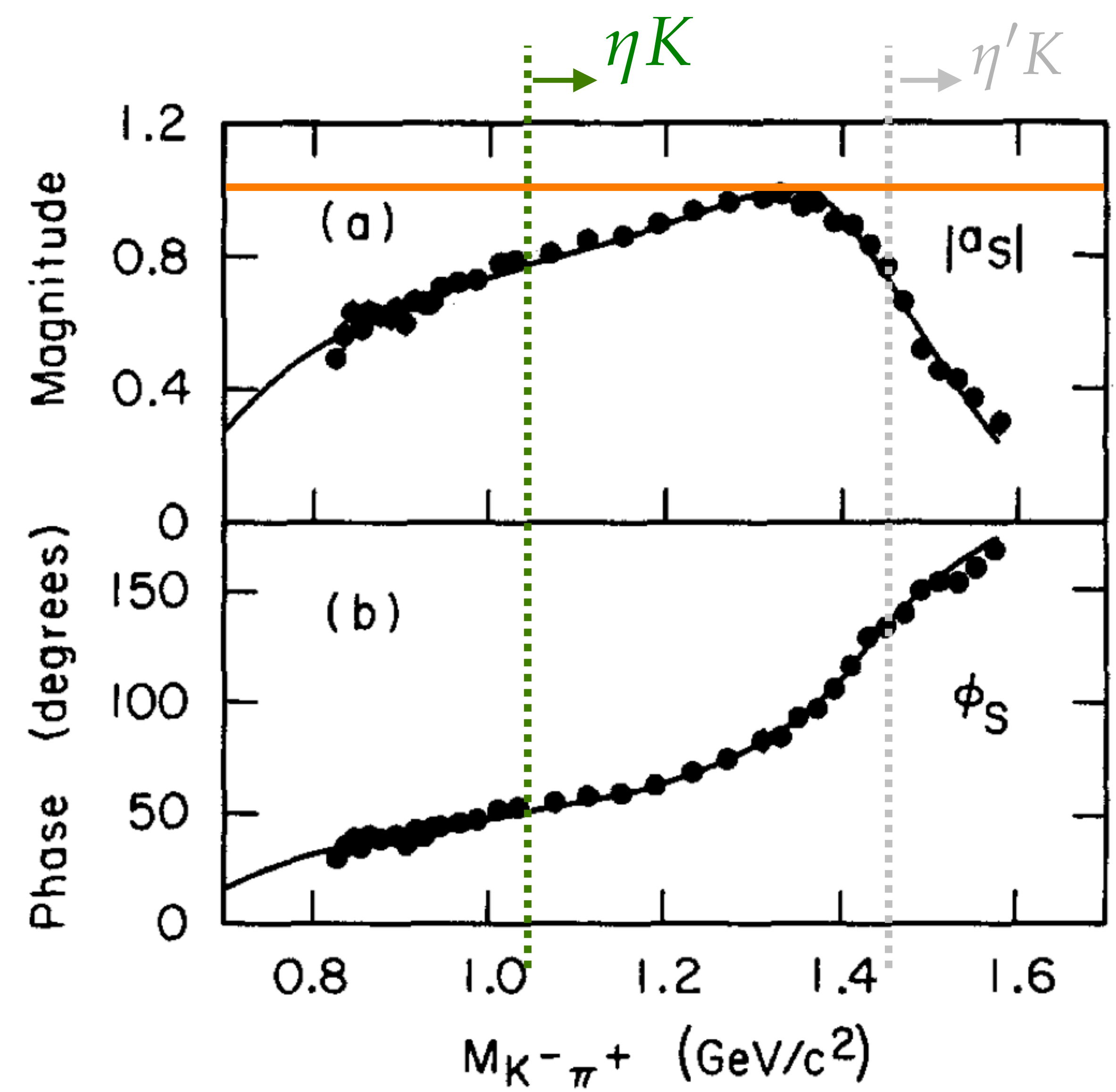


– gross features are robust

S-WAVE $\pi K \to \pi K$ AMPLITUDE



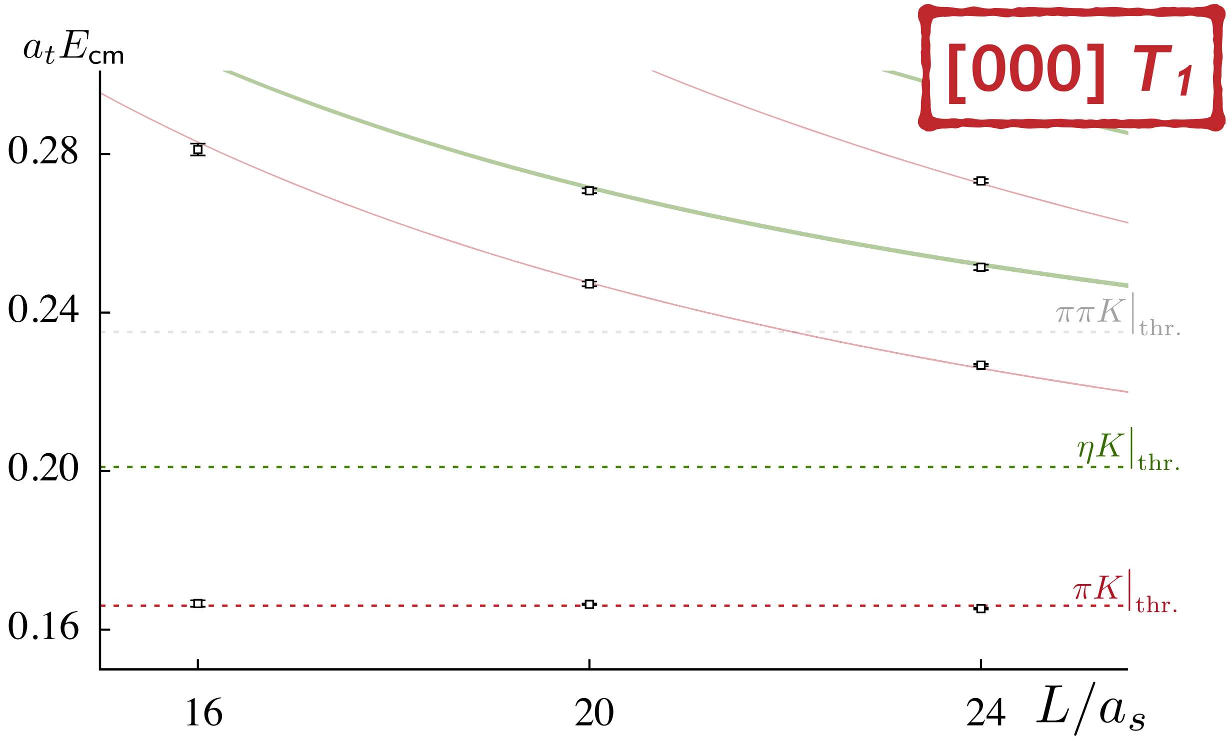
$m_\pi \sim 391$ MeV

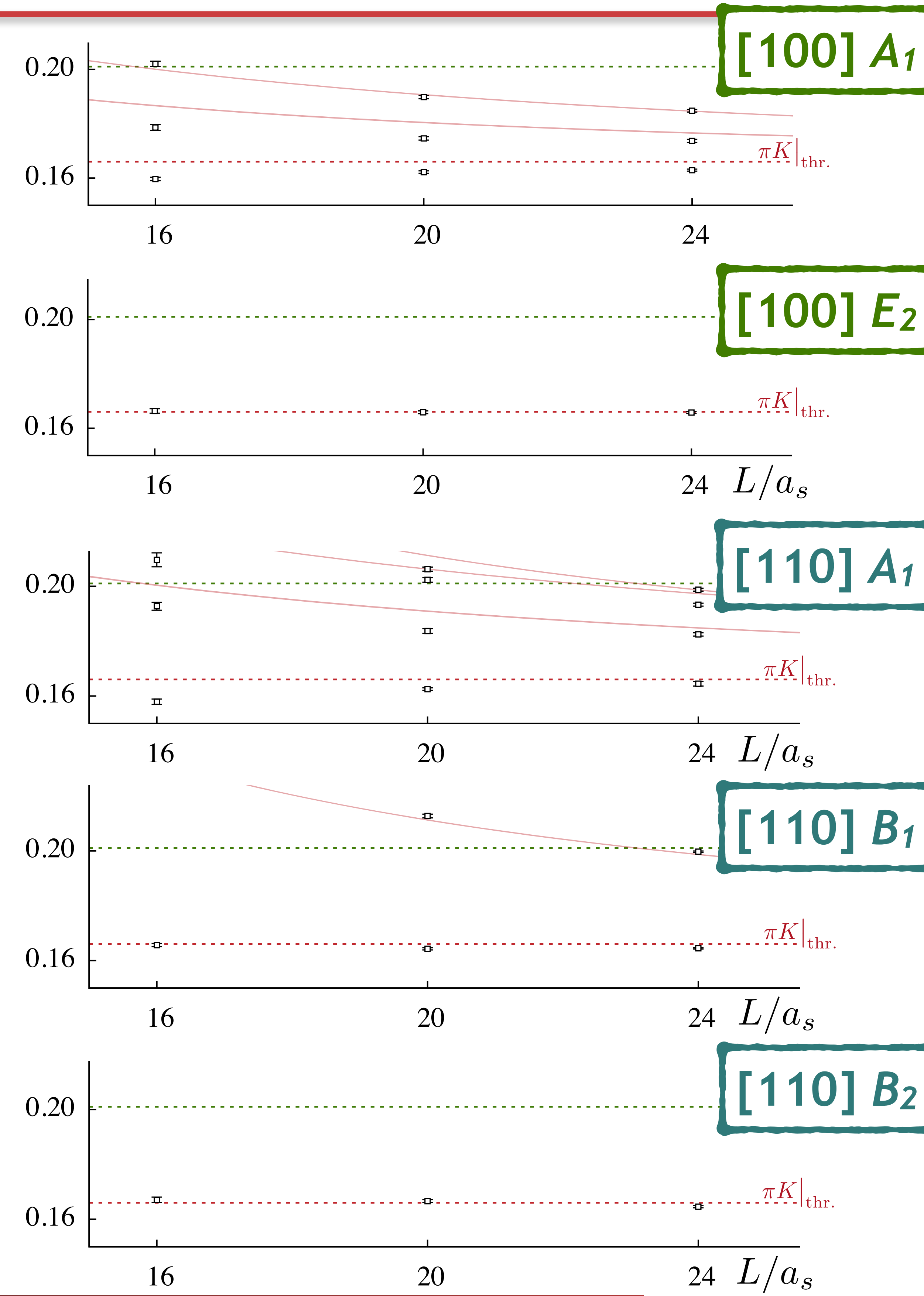


LASS, NPB296 493

P-wave scattering

every irrep containing the *P*-wave has a level very near the threshold



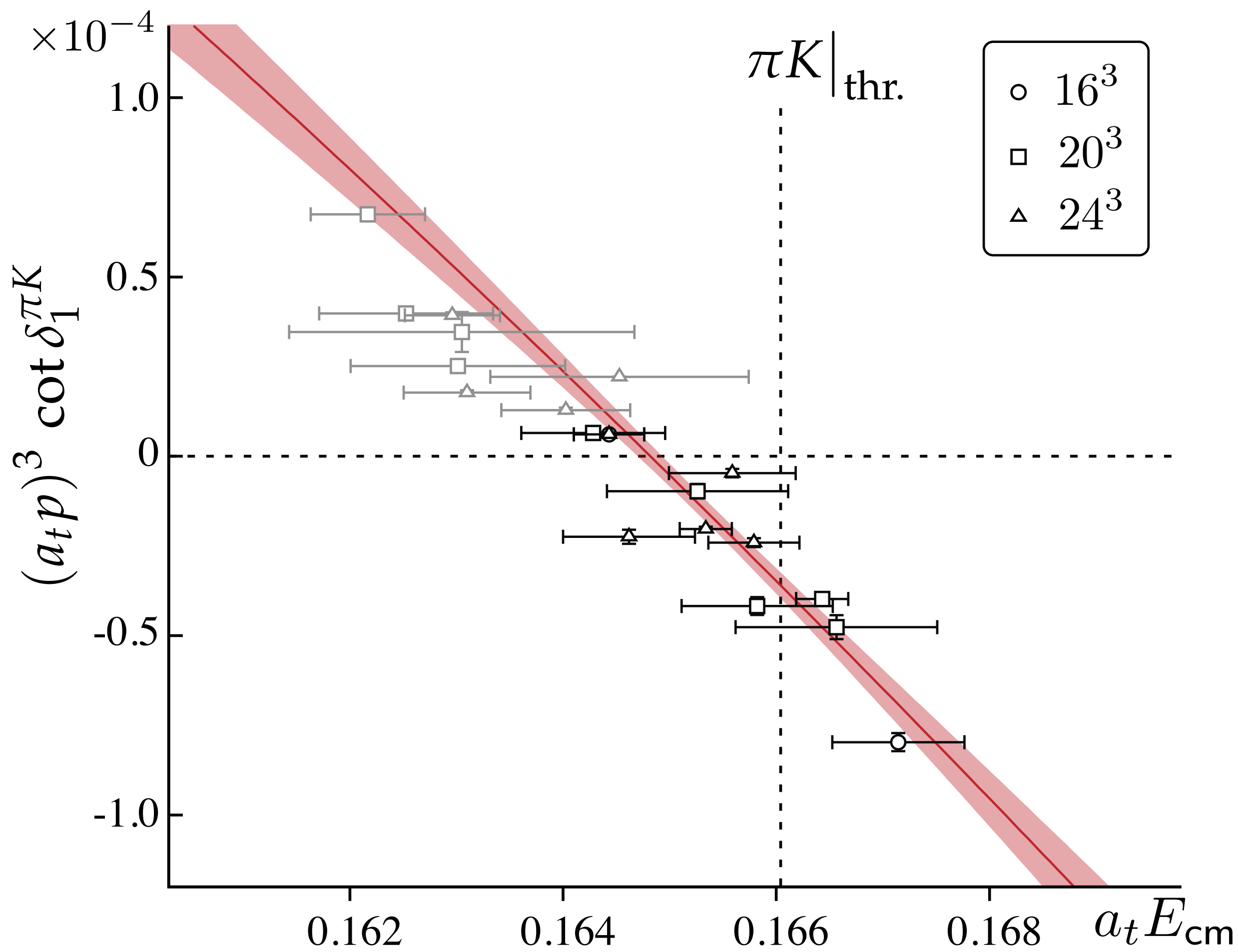


even when there isn't a non-interacting level nearby

suggests a **bound state** near threshold

$\pi K/\eta K$ scattering amplitudes

P-WAVE πK SCATTERING

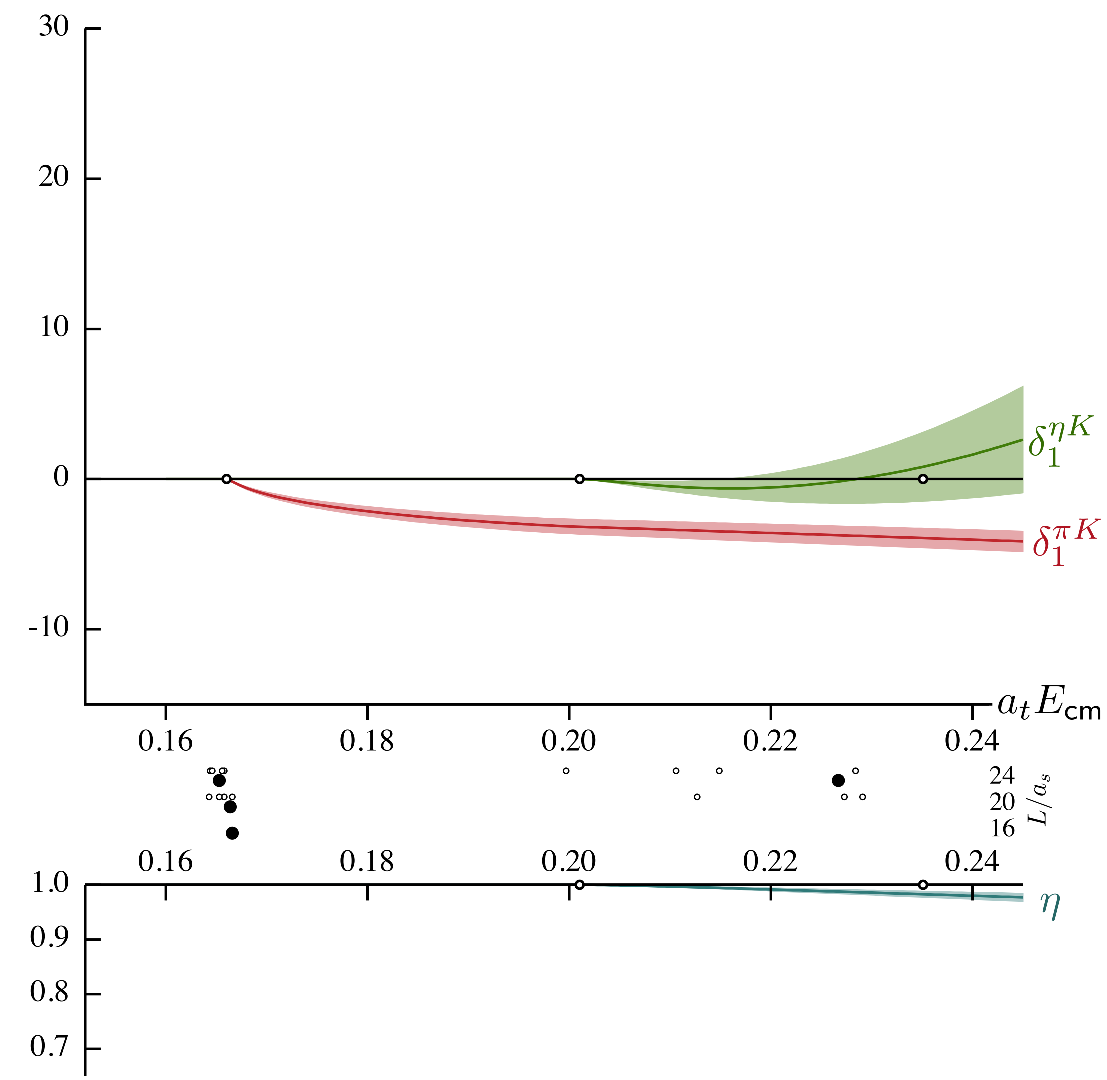


use a Breit-Wigner with
a subthreshold mass

$$a_t m(K^\star) = 0.16482(15)$$
$$g = 5.93(30)$$

vector
bound-state

P-WAVE $\pi K/\eta K$ SCATTERING

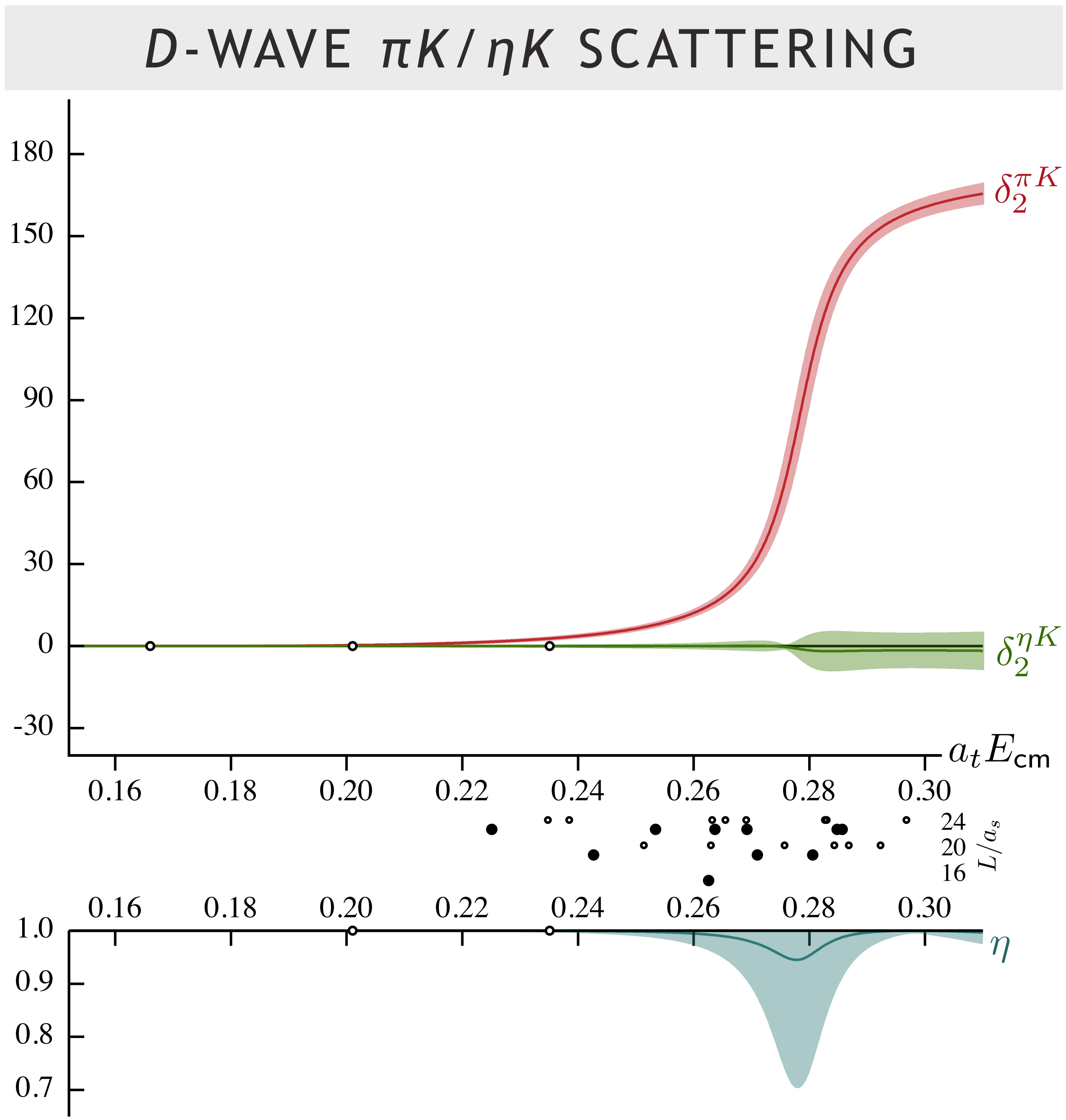


quark mass accident that it lies
so close to threshold ...

$g_{\text{phys.}} = 5.5(2)$ *PDG*

$\pi K/\eta K$ coupled-channel scattering

clear narrow resonance in *D*-wave scattering



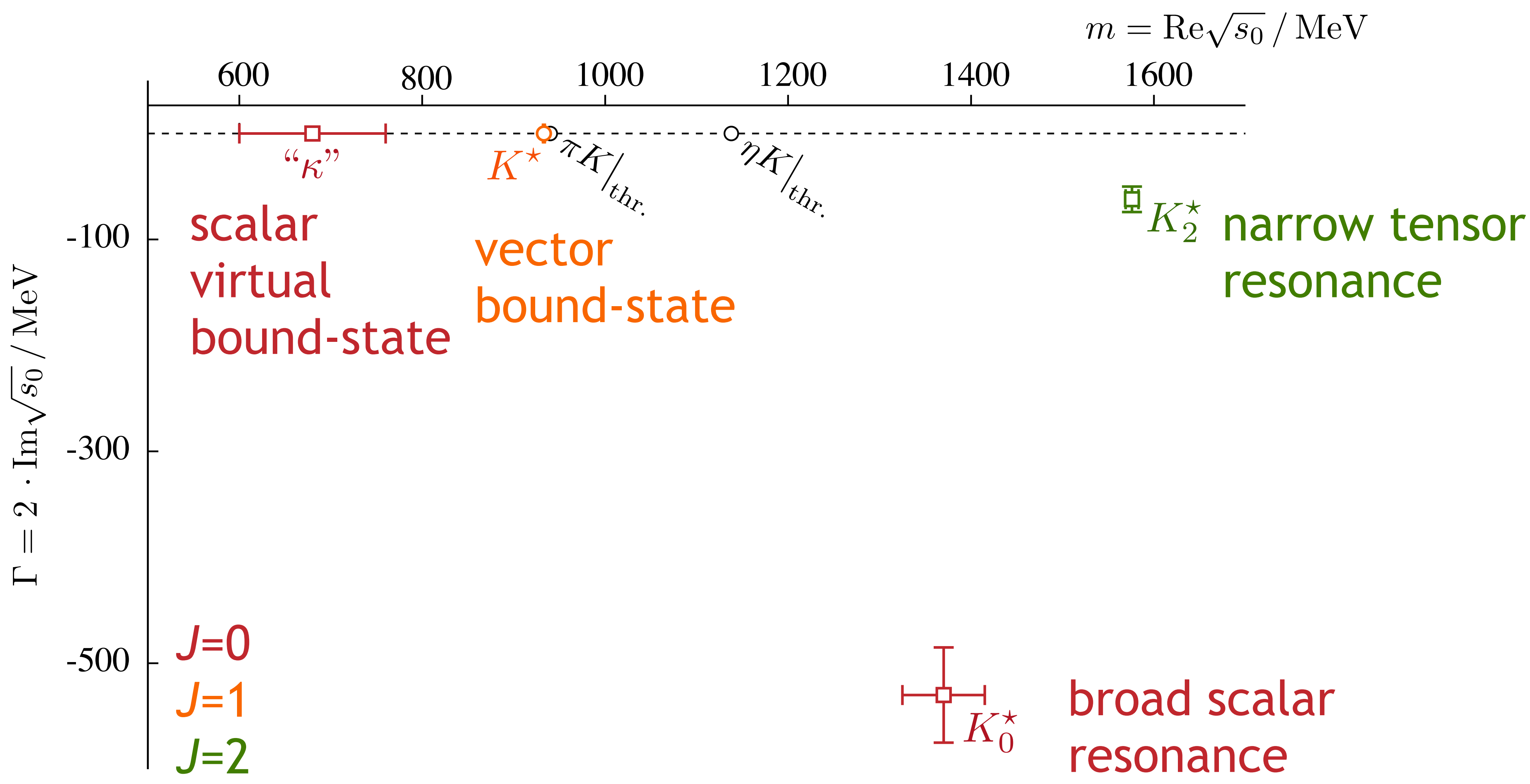
$m_\pi \sim 391\,\mathrm{MeV}$

(you might worry about $\pi\pi K$ in this case)

singularity content

***t*-matrix poles** as least model-dependent characterization of resonances

COMPLEX ENERGY PLANE



$m_\pi \sim 391$ MeV

PRL 113 182001
PRD 91 054008

what else can you do ?

with the current technology: other two-body coupled-channel problems ...

e.g. $\pi\eta,\ K\overline{K}$ $\pi\eta'$ — a_0 resonance — strongly coupled to both channels
$K\overline{K}$ molecule ?

first study by the end of this year

e.g. $\pi\pi,\ K\overline{K},\ \eta\eta$ — f_0 resonances — $\sigma,\ f_0(980)\ldots$

distant σ pole
$f_0(980)$ as $K\overline{K}$ molecule ?
glueball contributions ?

e.g. $\pi\omega\ldots$ — [ω stable at larger quark masses]

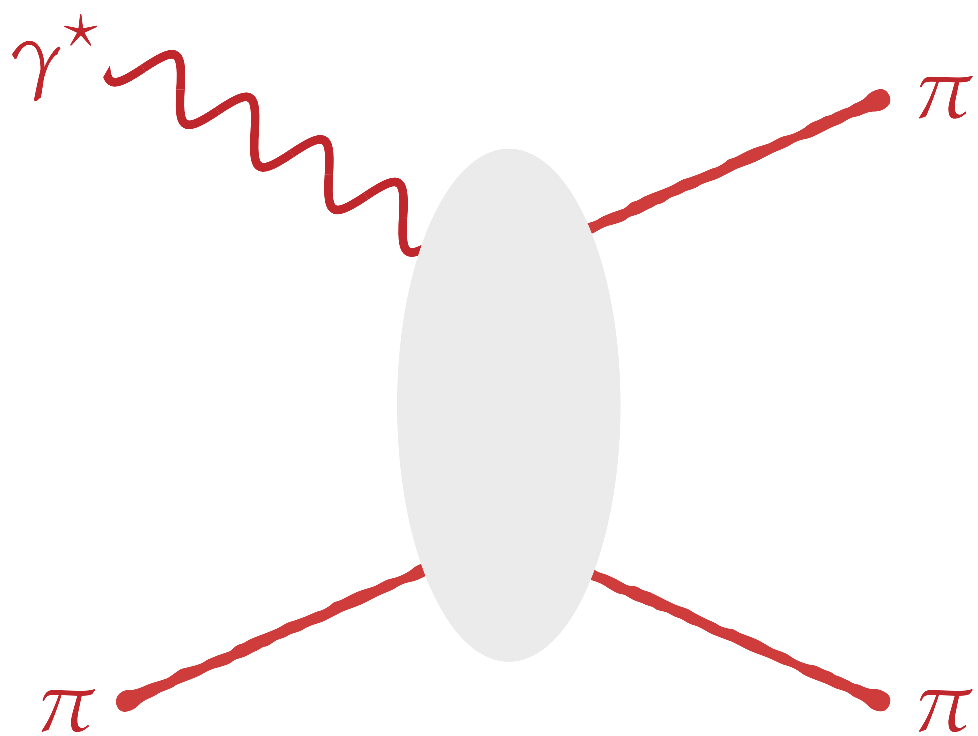
axial resonance physics - coupled S,D-waves

... use quark mass dependence as a tool ...

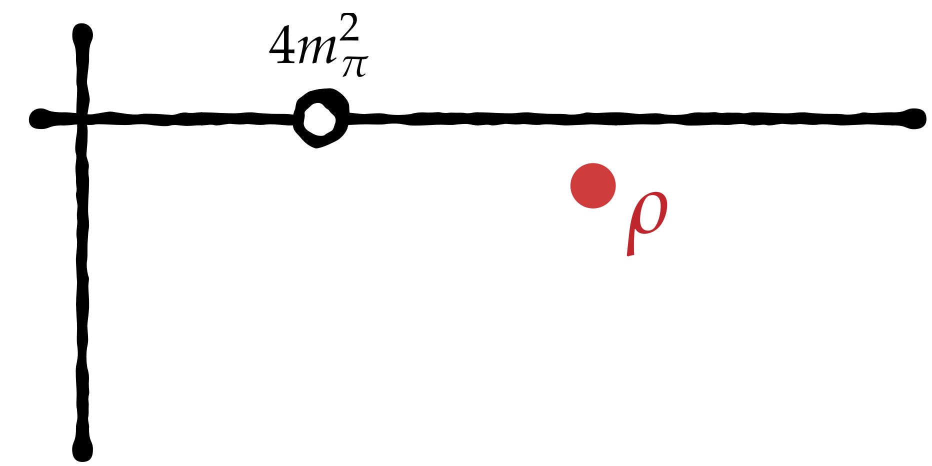
what else can you do ?

coupling to external currents

e.g.



formalism exists to determine the amplitude $A_P^{\gamma^\star\pi\to\pi\pi}(s,Q^2)$



residue at the pole is the (unstable) **ρ→πγ transition form-factor**

$$A_P^{\gamma^\star\pi\to\pi\pi}(s\sim s_\rho, Q^2) \sim \frac{g_{\pi\gamma}^{(\rho)}(Q^2)\, g_{\pi\pi}^{(\rho)}}{}$$

coming soon to the arXiv ...
Raul Briceno et. al.

what are the challenges ?

as you've heard, many resonances appear in three-hadron final states e.g.

$$a_1 \to \pi\pi\pi$$
$$\eta(1295) \to \eta\pi\pi$$
$$N^\star \to N\pi\pi$$

an open problem is:

how are three-body amplitudes related to the spectrum in a box ?

no complete formalism to date … so naturally no explicit calculations

try simple channels first ?

$\pi\pi\pi$ isospin=3 ~ non-resonant

$\pi\pi\pi$ isospin=2
- ~ non-resonant 3-body
- ~ resonant 2-body 'isobars'

to conclude …

lattice QCD is a controlled approximation to QCD

implement numerically on big computers

calculate correlation functions → spectra, matrix-elements

field theories in finite-volume have a discrete spectrum

but that spectrum is related to scattering amplitudes

calculate enough spectra and you can infer the scattering amplitudes

these methods now being applied

$\pi\pi$ elastic scattering

first determination of coupled-channel case: $\pi K, \eta K$

coupling to external currents: first calculation will appear soon $\gamma^{*}\pi \rightarrow \pi\pi$

… in all cases utilize constraints from *S*-matrix theory …

thank you

Jozef Dudek

Old Dominion University

Jefferson Lab

hadron spectrum collaboration

JEFFERSON LAB

Jozef Dudek
Robert Edwards
Balint Joo
David Richards

Dave Wilson
Raul Briceno

TRINITY COLLEGE, DUBLIN

Mike Peardon
Sinead Ryan

TATA, MUMBAI

Nilmani Mathur

CAMBRIDGE UNIVERSITY

Christopher Thomas

U. OF MARYLAND

Steve Wallace

MESON SPECTRUM

PRL103 262001 (2009)	$I=1$
PRD82 034508 (2010)	$I=1,\ K^\star$
PRD83 111502 (2011)	$I=0$
JHEP07 126 (2011)	$c\bar{c}$
PRD88 094505 (2013)	$I=0$
JHEP05 021 (2013)	$D,\ D_s$

BARYON SPECTRUM

PRD84 074508 (2011)	$(N,\Delta)^\star$
PRD85 054016 (2012)	$(N,\Delta)_{\rm hyb}$
PRD87 054506 (2013)	$(N\ldots\Xi)^\star$
PRD90 074504 (2014)	$\Omega^\star_{ccc}$
arXiv:1502.01845	$\Xi^\star_{cc}$

HADRON SCATTERING

PRD83 071504 (2011)	$\pi\pi\ I=2$
PRD86 034031 (2012)	$\pi\pi\ I=2$
PRD87 034505 (2013)	$\pi\pi\ I=1,\ \rho$
PRL113 182001 (2014)	$\pi K, \eta K$
PRD91 054008 (2015)	$\pi K, \eta K$

"TECHNOLOGY"

PRD79 034502 (2009)	lattices
PRD80 054506 (2009)	distillation
PRD85 014507 (2012)	$\vec{p}>0$

MATRIX ELEMENTS

arXiv:1501.07457	$M'\to\gamma M$
PRD90 014511 (2014)	$f_{\pi^\star}$