Lectures 2+3: Bound states in field theory

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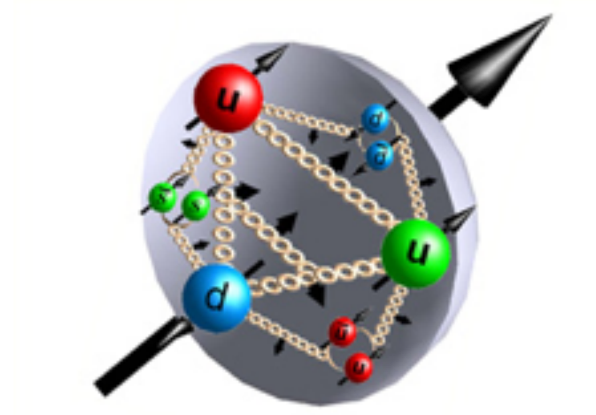


Positronium — QED

Proton — QCD

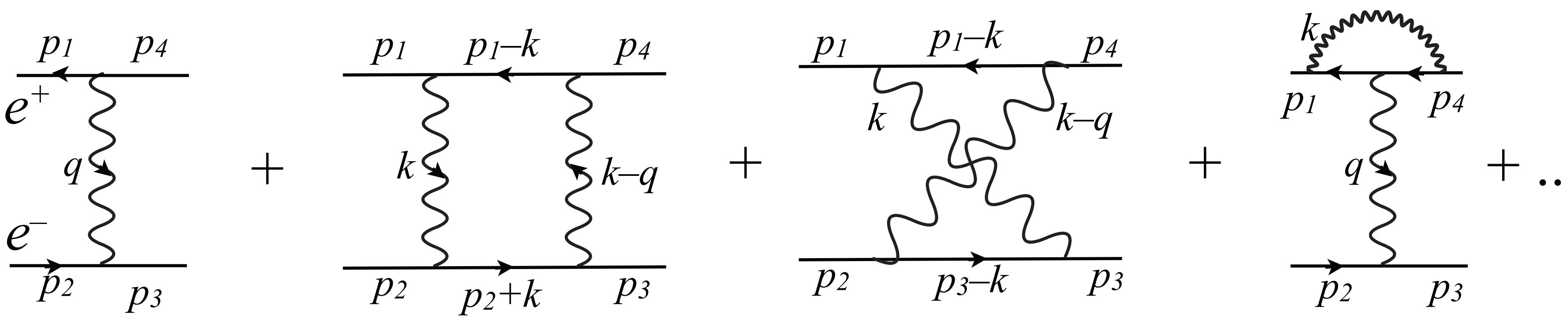
Bound States at lowest order in $\hbar$

Bound state studies

Bound states are both familiar and poorly known to students of QFT

- First courses in Quantum Mechanics explain the *H*-atom.
- Field Theory courses (generally) do not discuss bound states.

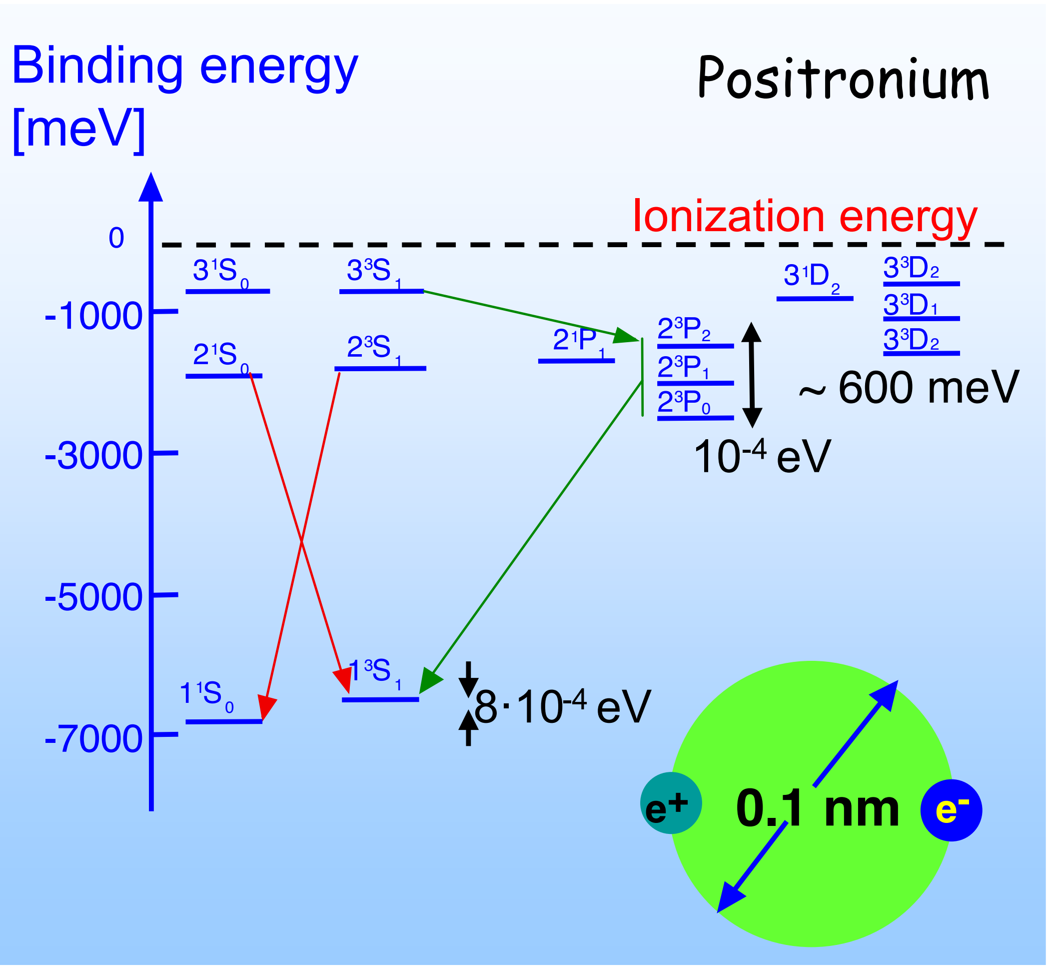
Perturbative calculations of atoms require summing an infinite series in α



- Why are higher order diagrams not suppressed by α?
- How can the expansion nevertheless be useful?
- Need to study principles of atoms before considering hadrons.

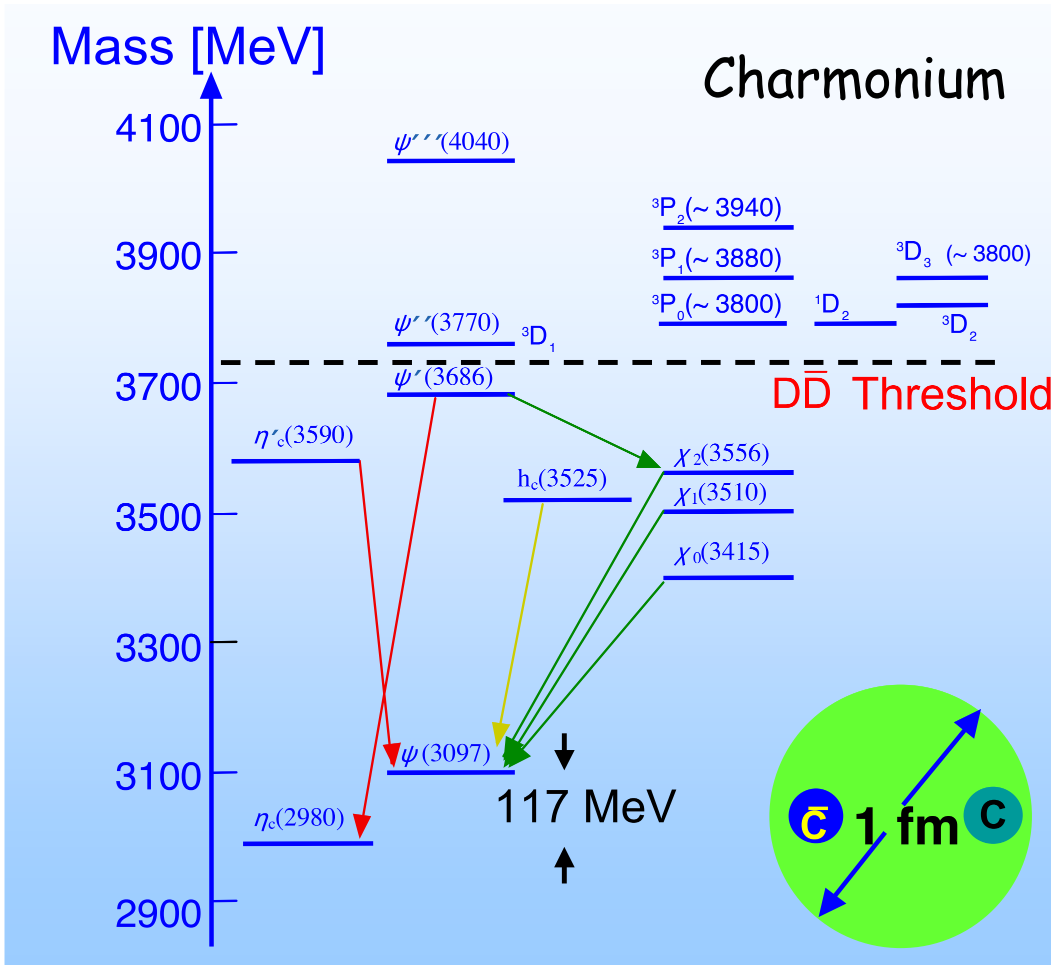
"The J/ψ is the Hydrogen atom of QCD"

QED



$$V(r) = -\frac{\alpha}{r}$$

QCD



$$V(r) = c\,r - \frac{4}{3}\frac{\alpha_s}{r}$$

Positronium provides precision tests of QED

Orthopositronium: $J^{PC} = 1^{--}$ Parapositronium: $J^{PC} = 0^{-+}$

$\Delta E = E(\text{ortho}) - E(\text{para})$ $\Delta\nu = \Delta E/2\pi\hbar$

$$\begin{aligned}\Delta\nu_{QED} &= m_e\alpha^4\left\{\frac{7}{12} - \frac{\alpha}{\pi}\left(\frac{8}{9} + \frac{\ln 2}{2}\right)\right.\\ &+ \frac{\alpha^2}{\pi^2}\left[-\frac{5}{24}\pi^2\ln\alpha + \frac{1367}{648} - \frac{5197}{3456}\pi^2 + \left(\frac{221}{144}\pi^2 + \frac{1}{2}\right)\ln 2 - \frac{53}{32}\zeta(3)\right]\\ &\left. - \frac{7\alpha^3}{8\pi}\ln^2\alpha + \frac{\alpha^3}{\pi}\ln\alpha\left(\frac{17}{3}\ln 2 - \frac{217}{90}\right) + \mathcal{O}\left(\alpha^3\right)\right\} = 203.39169(41)\ \text{GHz}\end{aligned}$$

Note log α

A. Penin, PoS LL2014 (2014) 074

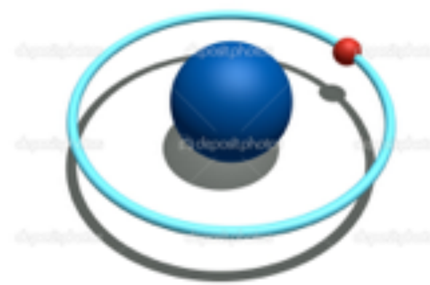
$\Delta\nu_{EXP}$ = 203.3941± .003 GHz (2013) A. Ishida et al, 1310.6923

Bound state masses can be expanded in powers of α and log α.

Perturbation theory is our most precise, analytic tool for gauge theories.

QM I: The Hydrogen atom at lowest order in α

Schrödinger equation (postulated):



$$\Big[-\frac{\boldsymbol{\nabla}^2}{2m_e}-\frac{\alpha}{|\boldsymbol{x}|}\Big]\Phi(\boldsymbol{x}) = E_b\,\Phi(\boldsymbol{x})$$

Ground state binding energy: $E_b = -\frac{1}{2}m_e\,\alpha^2$ $\qquad$ $\mathcal{O}(\alpha^2)$

Wave function: $\Phi(\boldsymbol{x}) = N\exp(-\alpha\, m_e\,|\boldsymbol{x}|)$ $\qquad$ all orders of α

How do we describe relativistic bound states in field theory?

How does the Schrödinger equation emerge from Perturbative QED?

Definition of bound states

A (stable) bound state is stationary in time, i.e., it is an eigenstate of the time translation generator, the Hamiltonian H,

$$H(t)\,|E,t\rangle = E\,|E,t\rangle \qquad \Rightarrow \qquad |E,t\rangle = \exp(-iEt)\,|E,t=0\rangle$$

The Hamiltonian is determined by the time translation invariance of the action. In QED the interaction term $\bar{\psi}\not{A}\psi$ implies that H can create/destroy photons and e^+e^- pairs. Therefore the eigenstate $|E,t\rangle$ must have all Fock components:

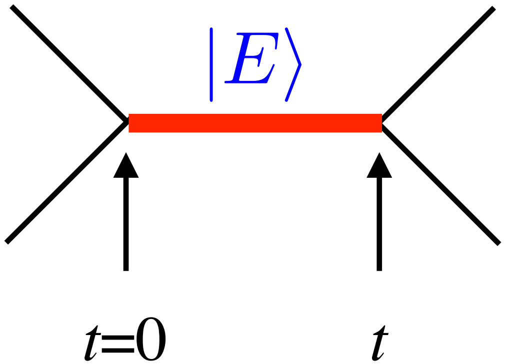
$$|E,t\rangle = \varphi_{e^+e^-}\,\left|e^+e^-\right\rangle + \varphi_{e^+e^-\gamma}\,\left|e^+e^-\gamma\right\rangle + \ldots + \varphi_{\ldots}\,\left|e^+e^-e^+e^-\right\rangle + \ldots$$

The wave functions describe the distributions of the constituents of each Fock state, for example:

$$\varphi_{e^+e^-}\,\left|e^+e^-\right\rangle = \int d\boldsymbol{p}\,d\boldsymbol{p}'\sum_{\lambda,\lambda'}\varphi_{\lambda,\lambda'}(\boldsymbol{p},\boldsymbol{p}')b^\dagger(\boldsymbol{p},\lambda)d^\dagger(\boldsymbol{p}',\lambda')\,|0\rangle$$

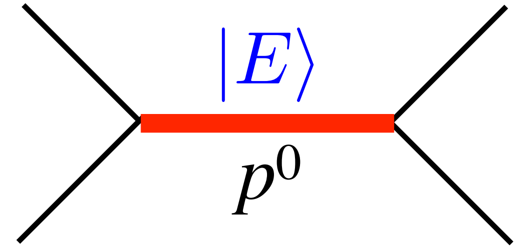
where $b^\dagger(\boldsymbol{p},\lambda)$ creates an electron of momentum $\boldsymbol{p}$ and helicity λ, similarly for positrons and photons. Only the superposition of Fock states is an eigenstate of H.

Bound states as poles in scattering amplitudes



Fourier transform

$t \longrightarrow p^0$

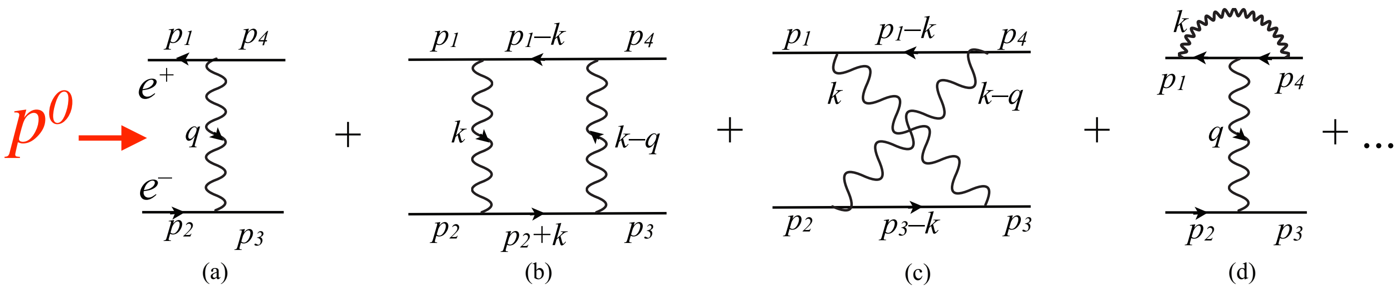


$$\int_0^\infty dt\, e^{i(p^0+i\varepsilon)t}\exp(-iEt) = \frac{-i}{p^0-E+i\varepsilon}$$

Bound states appear as poles in scattering amplitudes (4-momentum space).

By evaluating amplitudes perturbatively (via Feynman diagrams) we can determine the bound states.

$e^+e^- \to e^+e^-$: Positronium poles



$$= \frac{|\varphi_{e^+e^-}|^2}{p^0 - E + i\varepsilon} + \ldots$$

Rest frame: $E = 2m_e - \frac{1}{4}m_e\alpha^2 + \mathcal{O}\left(\alpha^4\right)$

LHS: $\sum_{n=2}^{\infty} c_n \alpha^n$

RHS: Not polynomial in α

$\Rightarrow$

Bound state pole can arise only through a divergence of the perturbative series ($n \to \infty$)

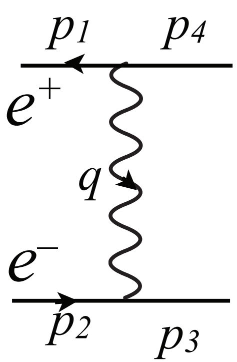
Why does the QED perturbative series diverge (for any α)?

Which diagrams cause the divergence?

Ladder diagrams (rest frame)

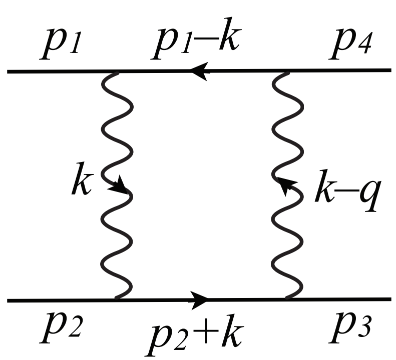
The Bohr momentum scale is $|\boldsymbol{p}| \sim \alpha m$, kinetic energy $|\boldsymbol{p}|^2/2m \sim \alpha^2 m \sim E_B$

With momenta ∝ α, the propagators bring inverse powers of α :



$$\sim \frac{e^2}{q^2} \sim \frac{\alpha}{\boldsymbol{q}^2} \sim \frac{1}{\alpha}$$

Note: $q^0 \sim \alpha^2 \ll |\boldsymbol{q}| \sim \alpha$



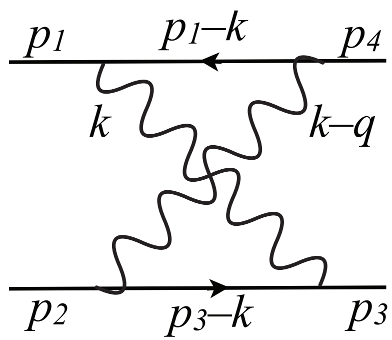
$$\sim \int dk^0\, d^3\boldsymbol{k}\, \frac{e^4}{(\boldsymbol{k}^2)^2\,(\Delta E_e)^2} \sim \alpha^2\,\alpha^3\,\frac{\alpha^2}{(\alpha^2)^2\,(\alpha^2)^2} \sim \frac{1}{\alpha}$$

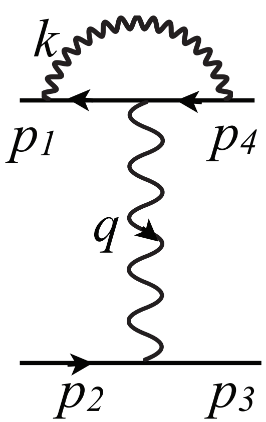
All "ladder diagrams" are of order 1/α ⟹ Sum can diverge!

Exercise 2.1

Using the above counting, show that ***all*** ladder diagrams are of $\mathcal{O}(\alpha^{-1})$

Non-ladders are suppressed by α





These diagrams have the same number of propagators and vertices as the 2-photon ladder. A similar counting would again give ~ 1/α .

However, the O(1/α) term vanishes:

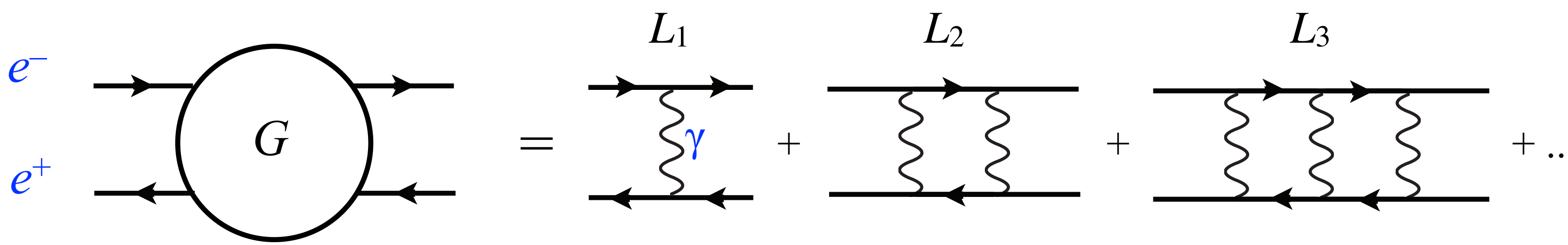
$$\propto \int \frac{dk^0}{2\pi} \frac{1}{(k^0 - a + i\varepsilon)(k^0 - b + i\varepsilon)} = 0$$

In the straight ladders the integration contour is pinched:

$$\propto \int \frac{dk^0}{2\pi} \frac{1}{(k^0 - a + i\varepsilon)(k^0 - b - i\varepsilon)} \neq 0$$

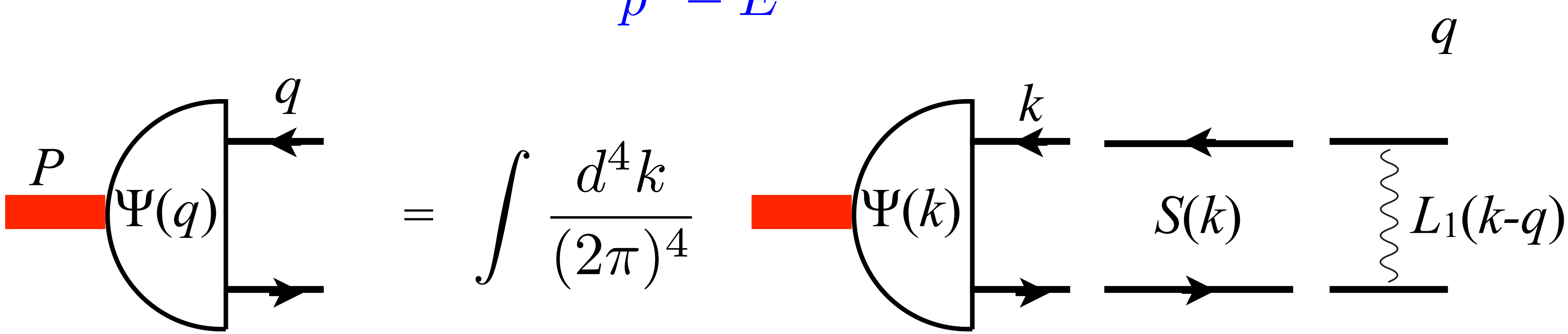
⇒ Only straight ladders are of the leading order, 1/α .

The Bethe-Salpeter equation



$$G = L_1 + G\,S\,L_1 = L_1 + L_1\,S\,L_1 + G\,S\,L_1 S\,L_1 = \ldots$$

At a bound state pole: $G(p^0) \sim \dfrac{\Psi^\dagger \Psi}{p^0 - E}$ $\Rightarrow$ $\Psi = \Psi\, S\, L_1$



This is the Bethe-Salpeter equation for a single photon kernel L_1 .

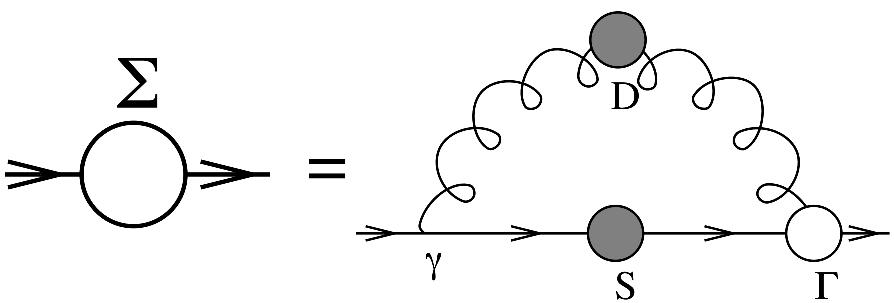
In the rest frame it reduces to the Schrödinger equation as $\alpha \to 0$.

It is valid in any frame (since Feynman diagrams are Lorentz covariant).

Dyson-Schwinger equations

We can establish identities between Green functions, valid to all orders in α.

These are called Dyson-Schwinger equations. Eg., for the quark propagator,



The circles contain quark and gluon loops to all orders.

The same diagrams occur on both sides.

There are analogous identities for the gluon propagator and the vertices.

They imply formally exact Bethe-Salpeter equations for bound states.

Unfortunately, the D-S equations do not close. Truncations are needed to fix the order of summing the Feynman diagrams.

The only model independent truncation is based on powers of α (PQCD).

The NRQED approach

Atomic electrons are non-relativistic: v ≃ $p/m \sim \alpha << 1$.

Expanding the QED Lagrangian in loops, giving inverse powers of p/m, find the effective L:

T. Kinoshita and G. P. Lepage, in *Quantum Electrodynamics (1990)*

$$
\begin{aligned}
\mathcal{L}_{\rm NNRQED} \quad = \quad & -1/4(F^{\mu\nu})^2 + \psi^\dagger \Big\{ i\partial_t - eA_0 + \mathbf{D}^2/2m + \mathbf{D}^4/8m^3 \\
& + c_1\, e/2m\, \boldsymbol{\sigma}\cdot\mathbf{B} + c_2 e/8m^2\, \nabla\cdot\mathbf{E} \\
& + c_3\, ie/8m^2\, \boldsymbol{\sigma}\cdot(\mathbf{D}\times\mathbf{E} - \mathbf{E}\times\mathbf{D}) + \ldots \Big\}\psi \\
& + d_1/m^2\, (\psi^\dagger\psi)^2 + d_2/m^2\, (\psi^\dagger\boldsymbol{\sigma}\psi)^2 + \ldots \\
& + \text{positron and positron-electron terms.}
\end{aligned}
$$

NRQED is the the most efficient method for higher order calculations in α.

It gives up explicit Lorentz covariance, making use of physics: $p/m << 1$

Why does perturbation theory diverge for atoms?

No bound state pole is generated at finite order in α.

⇒ Finite order diagrams are "infinitely wrong" close to the bound states.

Recall the perturbative expansion of the *S*-matrix:

$$S_{fi} = {}_{\textbf{out}}\langle f| \left\{ \mathrm{T} \exp\left[-i \int_{-\infty}^{\infty} dt\, H_I(t) \right] \right\} |i\rangle_{\textbf{in}}$$

The *in* and *out* states are $\mathcal{O}(\alpha^0)$, non-interacting states at $t = \pm\infty$.

They get dressed by H_I as they propagate from the asymptotic times.

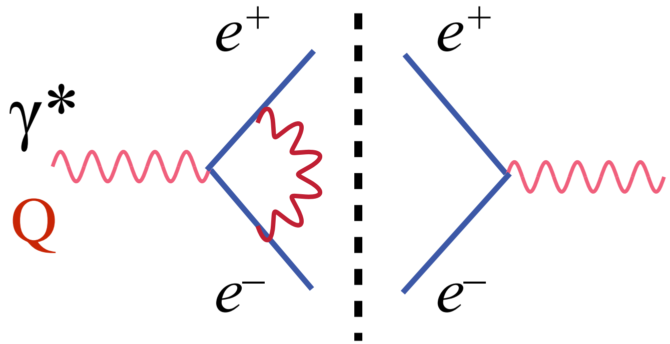
The lack of an EM field around the *in* and *out* electrons implies that we expand around unphysical states. Their matrix elements do not satisfy the field equations:

Gauss' law $-\boldsymbol{\nabla}^2 A^0(t,\boldsymbol{x}) = e\psi^\dagger(t,\boldsymbol{x})\psi(t,\boldsymbol{x})$

requires a charge to be accompanied by an EM field.

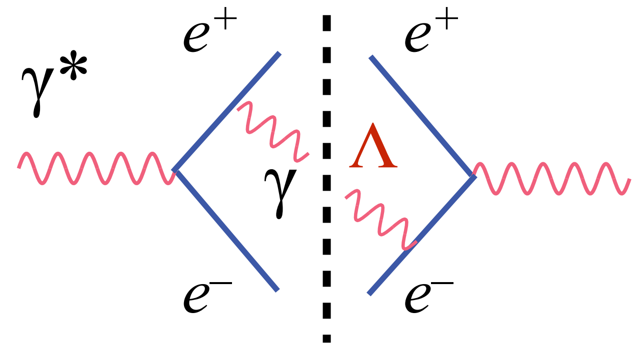
The IR singularities

The absence of an EM field in the *in* and *out* states causes infrared divergencies, These are cured (order-by-order) by adding (the missing) soft photons. E.g.:



$\gamma^* \to e^+e^-$

+



$\gamma^* \to e^+e^-\gamma$

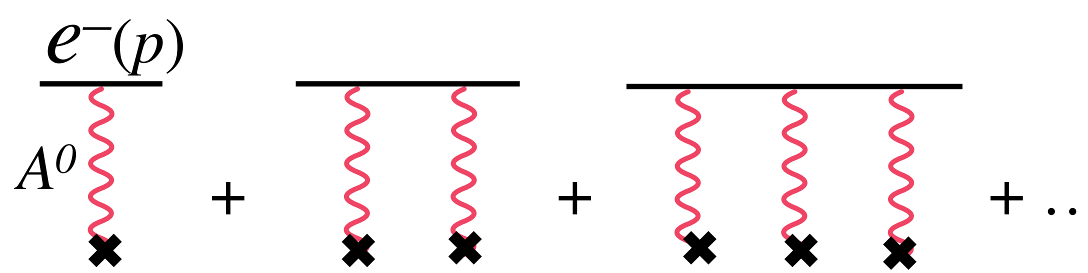
For bound states the soft photons are essential: They provide the binding!

The ladder sum regenerates the omitted classical Coulomb field: $V(r) = -\alpha/r$

Born level = Classical gauge field

The Born term is the lowest order contribution in $\hbar$ to any perturbative amplitude. It is given by tree diagrams (no loops).

The Schrödinger atom is described by tree diagrams, e^- scattering from the classical photon field

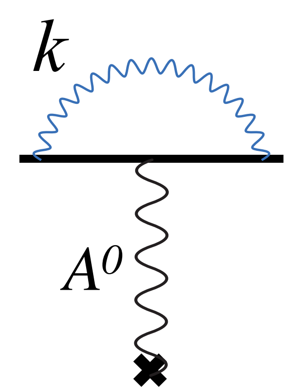


$\Rightarrow$

At Born level, states are bound by a classical gauge field.

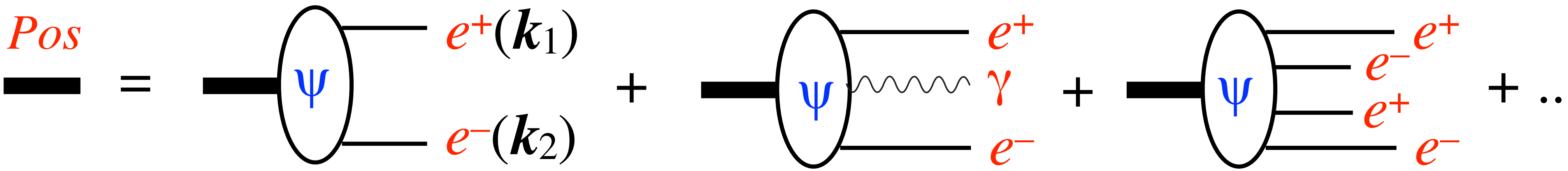
Higher orders in $\hbar$ involve loop diagrams.

The photon k is a quantum fluctuation, not a classical field.



Hamiltonian approach at $O(\hbar^0)$

Any state can be expanded on its Fock components, e.g., for Positronium:



For non-relativistic Positronium at rest the e^+e^- Fock state dominates:

$$|M, \boldsymbol{P}=0\rangle = \int d^3\boldsymbol{x}_1 d^3\boldsymbol{x}_2\, \bar\psi(t,\boldsymbol{x}_1)\Phi(\boldsymbol{x}_1-\boldsymbol{x}_2)\psi(t,\boldsymbol{x}_2)\,|0\rangle$$

where $\psi(t,\boldsymbol{x})$ is the electron field.

The state has an electron at $\boldsymbol{x}_1$ and a positron at $\boldsymbol{x}_2$, distributed according to the (4×4, c-numbered) wave function Φ, to be determined from the bound state condition:

$$H\,|M,\boldsymbol{P}=0\rangle = M\,|M,\boldsymbol{P}=0\rangle$$

QED Hamiltonian with the classical potential

The QED action $\mathcal{S} = \int d^4x\,\big[\bar\psi(i\partial\!\!\!/ - eA\!\!\!/ - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\big]$

implies the field equations of motion $\dfrac{\delta\mathcal{S}}{\delta A^\nu(x)} = -e\bar\psi\gamma_\nu\psi + \partial^\mu F_{\mu\nu} = 0$

which allow to express the field energy as

$$\int d^4x\big(-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}\big) = \tfrac{1}{2}\int d^4x\,A_\nu\partial_\mu F^{\mu\nu} = \tfrac{1}{2}\int d^4x\,\bar\psi eA\!\!\!/\psi$$

Using this the action becomes: $\mathcal{S} = \int d^4x\,\bar\psi(i\partial\!\!\!/ - \frac{1}{2}eA\!\!\!/ - m)\psi$

This gives the Hamiltonian for a state with a classical gauge field A^μ:

$$H = \int d^3\boldsymbol{x}\,\bar\psi(t,\boldsymbol{x})(-i\boldsymbol{\nabla}\cdot\boldsymbol{\gamma} + m + \tfrac{1}{2}eA\!\!\!/)\psi(t,\boldsymbol{x})$$

The factor 1/2 takes into account the energy of the EM field.

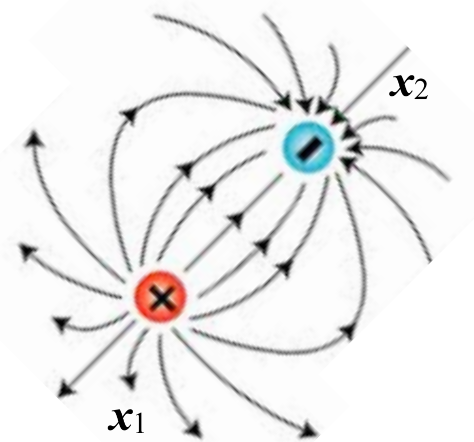
Bound State Equation for Positronium

Impose $H\,|M, \boldsymbol{P}=0\rangle = M\,|M, \boldsymbol{P}=0\rangle$ on the bound state

$$|M, \boldsymbol{P}=0\rangle = \int d^3\boldsymbol{x}_1 d^3\boldsymbol{x}_2\, \bar{\psi}(t,\boldsymbol{x}_1)\Phi(\boldsymbol{x}_1-\boldsymbol{x}_2)\psi(t,\boldsymbol{x}_2)\,|0\rangle$$

The classical EM field in H for each Fock component is

$$|e^-(\boldsymbol{x}_1)\, e^+(\boldsymbol{x}_2)\rangle : \quad eA^0(\boldsymbol{x};\boldsymbol{x}_1,\boldsymbol{x}_2) = \frac{\alpha}{|\boldsymbol{x}-\boldsymbol{x}_1|} - \frac{\alpha}{|\boldsymbol{x}-\boldsymbol{x}_2|}$$



Using $\{\psi(t,\boldsymbol{x}), \psi^\dagger(t,\boldsymbol{y})\} = \delta^3(\boldsymbol{x}-\boldsymbol{y})$ and $H\,|0\rangle = 0$

we get the bound state equation for the 4×4 wave function $\Phi(\boldsymbol{x}_1 - \boldsymbol{x}_2)$:

$$i\boldsymbol{\nabla}\cdot\{\gamma^0\boldsymbol{\gamma}, \Phi(\boldsymbol{x})\} + m\,[\gamma^0, \Phi(\boldsymbol{x})] = [M - V(\boldsymbol{x})]\,\Phi(\boldsymbol{x})$$

where $V(\boldsymbol{x}) = -\dfrac{\alpha}{|\boldsymbol{x}|}$

In the NR limit ($\alpha \to 0$) this reduces to the Schrödinger eq., with $M = 2m + E_b$.

Exercise 2.2

a) Derive the bound state equation for $\Phi(\boldsymbol{x})$,

$$i\boldsymbol{\nabla}\cdot\left\{\gamma^0\boldsymbol{\gamma},\Phi(\boldsymbol{x})\right\}+m\left[\gamma^0,\Phi(\boldsymbol{x})\right]=\left[M-V(\boldsymbol{x})\right]\Phi(\boldsymbol{x})$$

from the condition $H\,|M,\boldsymbol{P}=0\rangle = M\,|M,\boldsymbol{P}=0\rangle$

b) Show that it reduces to the non-relativistic Schrödinger equation as $\alpha \to 0$.

Hint: Proceed as in the case of the Dirac equation. Write $M = 2m+E_b$, and take $\nabla = \mathcal{O}(\alpha m)$; $E_b, V = \mathcal{O}(\alpha^2 m)$. Express the 4×4 wave function $\Phi(\boldsymbol{x})$ in block 2×2 form:

$$\Phi=\begin{bmatrix}\Phi_{11} & \Phi_{12}\\ \Phi_{21} & \Phi_{22}\end{bmatrix}$$

$\Phi_{12}(\boldsymbol{x})$ is the leading component (if the γ's are in Dirac representation.

Summary: Schrödinger eq. in QED

- A H formulation requires an equal-time definition of the bound state.
- The QED Hamiltonian with a classical photon field (Born level).

$$\Longrightarrow \qquad i\boldsymbol{\nabla}\cdot\{\gamma^0\boldsymbol{\gamma},\Phi(\boldsymbol{x})\}+m\left[\gamma^0,\Phi(\boldsymbol{x})\right]=\left[M-V(\boldsymbol{x})\right]\Phi(\boldsymbol{x})$$

Method can be used in any frame: $H\,|E,\boldsymbol{P}\rangle = E\,|E,\boldsymbol{P}\rangle$ with $\boldsymbol{P} \neq 0$

The frame dependence of bound states is non-trivial:
Wave function Lorentz contracts and $E=\sqrt{\boldsymbol{P}^2+(2m+E_b)^2}$

All this can be done for QCD as well. What about the linear term in the quarkonium potential?

$$V(r) = c\,r - \frac{4}{3}\frac{\alpha_s}{r}$$

A Homogeneous solution of Gauss' law

For a state with e^- at x_1 and e^+ at x_2 $\quad \bar\psi(t,x_1)\psi(t,x_2)|0\rangle$

Gauss' law for the classical A^0 field is (in QED)

$$-\boldsymbol{\nabla}^2 A^0(t,\boldsymbol{x}) = e\big[\delta^3(\boldsymbol{x}-\boldsymbol{x}_1) - \delta^3(\boldsymbol{x}-\boldsymbol{x}_2)\big]$$

There is also a homogeneous solution, with $\varkappa$ independent of $\boldsymbol{x}$:

$$A^0(t,\boldsymbol{x}) = \kappa\,\boldsymbol{x}\cdot(\boldsymbol{x}_1-\boldsymbol{x}_2)$$

In QED this is excluded by a boundary condition:

$$\lim_{|\boldsymbol{x}|\to\infty} A^0(\boldsymbol{x}) = 0$$

The field energy density is independent of $\boldsymbol{x}$

$$\big[\boldsymbol{\nabla} A^0\big]^2 = \kappa^2\,(\boldsymbol{x}_1-\boldsymbol{x}_2)^2$$

The infinite field energy is irrelevant provided it is a universal constant.

The linear potential

Requiring:

- Translation invariance: $V(\boldsymbol{x}+\boldsymbol{a}) = V(\boldsymbol{x})$
- A universal field density as $|\boldsymbol{x}| \to \infty$ $\big[\boldsymbol{\nabla} A^0(\boldsymbol{x})\big]^2 = \Lambda^4$

suffices to specify the potential. In U(1) gauge theory:

$$V(\boldsymbol{x}_1,\boldsymbol{x}_2) \equiv \tfrac{1}{2} g\big[A^0(t,\boldsymbol{x}_1) - A^0(t,\boldsymbol{x}_2)\big] = \tfrac{1}{2} g\Lambda^2|\boldsymbol{x}_1-\boldsymbol{x}_2|$$

Only neutral states are translation invariant: $g_1 = -g_2 \equiv g$

The bound state equation has the same form as above for Positronium.

Usual perturbation theory has charged $\mathcal{O}(\alpha^0)$ states: electrons, quarks, gluons

Then the linear potential would break translation symmetry.

The above solution is unique, up to the single parameter Λ

At the Born (no loop) level, a dimensionful parameter can only be introduced through a boundary condition.

Exercise 2.3

a) Determine ϰ in the expression $A^0(t, \boldsymbol{x}) = \kappa\, \boldsymbol{x} \cdot (\boldsymbol{x}_1 - \boldsymbol{x}_2)$

from the requirement that the field energy density is universal.

b) Show that the potential arising from $H\,|M, \boldsymbol{P} = 0\rangle = M\,|M, \boldsymbol{P} = 0\rangle$

where $|M, \boldsymbol{P} = 0\rangle = \int d^3\boldsymbol{x}_1 d^3\boldsymbol{x}_2\, \bar{\psi}(t, \boldsymbol{x}_1)\Phi(\boldsymbol{x}_1 - \boldsymbol{x}_2)\psi(t, \boldsymbol{x}_2)\,|0\rangle$

and $H = \int d^3\boldsymbol{x}\, \bar{\psi}(t, \boldsymbol{x})(-i\boldsymbol{\nabla} \cdot \boldsymbol{\gamma} + m + \tfrac{1}{2}e\not{A})\psi(t, \boldsymbol{x})$ $(A^j = 0)$

is linear: $V(\boldsymbol{x}_1, \boldsymbol{x}_2) = \tfrac{1}{2}e\Lambda^2\,|\boldsymbol{x}_1 - \boldsymbol{x}_2|$

Hint: The canonical commutation relations are $\left\{\psi_\alpha(t, \boldsymbol{x}), \psi_\beta^\dagger(t, \boldsymbol{y})\right\} = \delta_{\alpha\beta}\delta^3(\boldsymbol{x} - \boldsymbol{y})$

The linear potential in QCD

For SU(3) there is a solution only for color singlet mesons:

$$V_{\mathcal{M}}(\boldsymbol{x}_1 - \boldsymbol{x}_2) = \tfrac{1}{2}\sqrt{C_F}\, g\Lambda^2 |\boldsymbol{x}_1 - \boldsymbol{x}_2|$$

and for color singlet baryons:

$$V_{\mathcal{B}}(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3) = \tfrac{1}{2\sqrt{2}}\sqrt{C_F}\, g\Lambda^2 \sqrt{(\boldsymbol{x}_1 - \boldsymbol{x}_2)^2 + (\boldsymbol{x}_2 - \boldsymbol{x}_3)^2 + (\boldsymbol{x}_3 - \boldsymbol{x}_1)^2}$$

Note: $V_{\mathcal{B}}(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_2) = V_{\mathcal{M}}(\boldsymbol{x}_1 - \boldsymbol{x}_2)$

The quark-diquark potential V_B agrees with quark-antiquark V_M.

Relativistically Bound States

Hadrons are ultrarelativistic states: $\dfrac{M_p}{2m_u + m_d} \simeq 50$

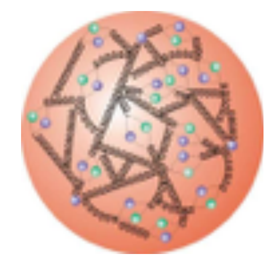
$\Longrightarrow$ They have Fock states with many sea quarks and gluons

$$|\text{proton}\rangle = \phi_{uud}\,|uud\rangle + \phi_{uudg}\,|uud\,g\rangle + \phi_{uudq\bar{q}}\,|uud\,q\bar{q}\rangle + \ldots$$

Nevertheless, hadron quantum numbers reflect valence quarks only

?

$\Longleftrightarrow$



An example of this "paradox" is provided by the Dirac equation: A relativistic electron bound in an external field.

- The Dirac wave function has the degrees of freedom of a single electron
- Its $E < 0$ components show the presence of e^+e^- pairs (*cf*. Klein paradox)

What **state** does the Dirac wave function actually describe?

J. P. Blaizot & PH

The states described by Dirac wave functions

For any given A^μ, the Dirac equation has solutions with positive and negative energy eigenvalues,

$$\big(-i\boldsymbol{\nabla}\cdot\boldsymbol{\gamma}+m+e\not{A}\big)\phi_n(\boldsymbol{x}) = E_n\gamma^0\phi_n(\boldsymbol{x}) \qquad E_n > 0$$

$$\big(-i\boldsymbol{\nabla}\cdot\boldsymbol{\gamma}+m+e\not{A}\big)\bar\phi_n(\boldsymbol{x}) = -\bar{E}_n\gamma^0\bar\phi_n(\boldsymbol{x}) \qquad \bar{E}_n > 0$$

The corresponding eigenstates of the Dirac Hamiltonian

$$H = \int d^3\boldsymbol{x}\,\psi^\dagger(t,\boldsymbol{x})\big[-i\boldsymbol{\nabla}\cdot\gamma^0\boldsymbol{\gamma}+m\gamma^0+e\not{A}(\boldsymbol{x})\big]\psi(t,\boldsymbol{x})$$ are

$$H(t)\,|n,t\rangle = E_n\,|n,t\rangle\ ; \qquad |n,t\rangle = \int d\boldsymbol{x}\,\psi^\dagger_\alpha(t,\boldsymbol{x})\phi_{n\,\alpha}(\boldsymbol{x})\,|\Omega\rangle \equiv c_n^\dagger\,|\Omega\rangle$$

$$H(t)\,|\bar{n},t\rangle = \bar{E}_n\,|\bar{n},t\rangle\ ; \qquad |\bar{n},t\rangle = \int d\boldsymbol{x}\,\bar\phi^\dagger_{n\,\alpha}(\boldsymbol{x})\psi_\alpha(t,\boldsymbol{x})\,|\Omega\rangle \equiv \bar{c}_n^\dagger\,|\Omega\rangle$$

Note that all *states* have positive energy eigenvalues.

The creation operators of Dirac states

The creation operators $c_n^\dagger,\ \bar{c}_n^\dagger$ of the eigenstates can be expressed as

$$c_n = \sum_{\boldsymbol{p},\lambda} \phi_n^\dagger(\boldsymbol{p})\big[u(\boldsymbol{p},\lambda)b_{\boldsymbol{p},\lambda} + v(-\boldsymbol{p},\lambda)d^\dagger_{-\boldsymbol{p},\lambda}\big]$$

$$\bar{c}_n = \sum_{\boldsymbol{p},\lambda} \big[b^\dagger_{\boldsymbol{p},\lambda}u^\dagger(\boldsymbol{p},\lambda) + d_{-\boldsymbol{p},\lambda}v^\dagger(-\boldsymbol{p},\lambda)\big]\bar{\phi}_n(\boldsymbol{p})$$

$b^\dagger$ creates an electron

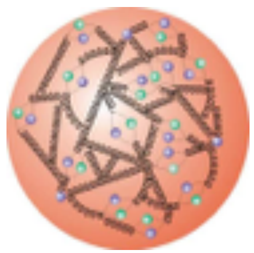
$d^\dagger$ creates a positron

They diagonalize the Dirac Hamiltonian: $H = \sum_n \big[E_n c_n^\dagger c_n + \bar{E}_n \bar{c}_n^\dagger \bar{c}_n\big]$

The ground state (vacuum) $|\Omega\rangle$ is determined by the conditions:

$$c_n\,|\Omega\rangle = \bar{c}_n\,|\Omega\rangle = H\,|\Omega\rangle = 0$$

$|\Omega\rangle$ is given by the Dirac wave functions $\phi_n,\ \bar{\phi}_n$
It is a superposition of any number of e^-e^+ pairs.



Properties of the Dirac wave function in D=1+1

The Dirac matrices can be represented as 2x2 Pauli matrices $\gamma^0 = \sigma_3 \qquad \gamma^0\gamma^1 = \sigma_1$

and the potential is $V(x) = \frac{1}{2}e^2|x|$

The 2-component Dirac spinor then satisfies

$$\left[-i\sigma_1\partial_x + \tfrac{1}{2}e^2|x| + m\sigma_3\right]\begin{bmatrix}\varphi(x)\\ \chi(x)\end{bmatrix} = M\begin{bmatrix}\varphi(x)\\ \chi(x)\end{bmatrix}$$

Eliminating the lower component,

$$\partial_x^2\varphi(x) + \frac{\varepsilon(x)}{2(M-V+m)}\partial_x\varphi(x) + \left[(M-V)^2 - m^2\right]\varphi(x) = 0,$$

The wf oscillates at large *x*: $\varphi(x\to\infty) \sim \exp(\pm i e^2 x^2/4)$

Hence it cannot be normalized, and there is no condition on *M*!

Exercise 2.4

Show that for a linear potential $V(x) = \frac{1}{2}e^2|x|$ the solution of

$$\partial_x^2\varphi(x) + \frac{\varepsilon(x)}{2(M-V+m)}\partial_x\varphi(x) + \left[(M-V)^2 - m^2\right]\varphi(x) = 0,$$

oscillates at large *x*: $\varphi(x \to \infty) \sim \exp(\pm ie^2x^2/4)$

After a non-relativistic reduction to the Schrödinger equation the wave function is instead exponentially suppressed (or enhanced) at large *x*. Explain the reason for this difference.

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The Dirac Electron in Simple Fields*

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(Received June 6, 1932)

The relativity wave equations for the Dirac electron are transformed in a simple manner into a symmetric canonical form. This canonical form makes readily possible the investigation of the characteristics of the solutions of these relativity equations for simple potential fields. If the potential is a polynomial of any degree in x, a continuous energy spectrum characterizes the solutions. If the potential is a polynomial of any degree in $1/x$, the solutions possess a continuous energy spectrum when the energy is numerically greater than the rest-energy of the electron; values of the energy numerically less than the rest-energy are barred. When the potential is a polynomial of any degree in r, all values of the energy are allowed. For potentials which are polynomials in $1/r$ of degree higher than the first, the energy spectrum is again continuous. The quantization arising for the Coulomb potential is an exceptional case.

See also: E. C. Titchmarsh, Proc. London Math. Soc. (3) 11 (1961) 159 and 169; Quart. J. Math. Oxford (2), 12 (1961), 227.

Analytic solution of the Dirac equation

In terms of the variable $\sigma = (M-V)^2 = M^2 - e^2|x|M + \frac{1}{4}e^4x^2$

For $x > 0$, with $\varphi(x)$ real and $\chi(x)$ imaginary, define: $\psi(\sigma) \equiv \varphi(\sigma) + \chi(\sigma)$

$$\psi(\sigma) = \left[(a+ib)\,{}_1F_1\Big(-\frac{im^2}{2},\frac{1}{2},2i\sigma\Big) + (b+ia)2m\,\varepsilon(M-V)\sqrt{\sigma}\,{}_1F_1\Big(\frac{1-im^2}{2},\frac{3}{2},2i\sigma\Big)\right]\exp(-i\sigma)$$

where a and b are real constants and $m \equiv m/e$ is the dimensionless parameter. The solution for $x < 0$ is defined by parity and the continuity condition at $x = 0$ fixes a/b. A solution is found for all M: The spectrum is continuous.

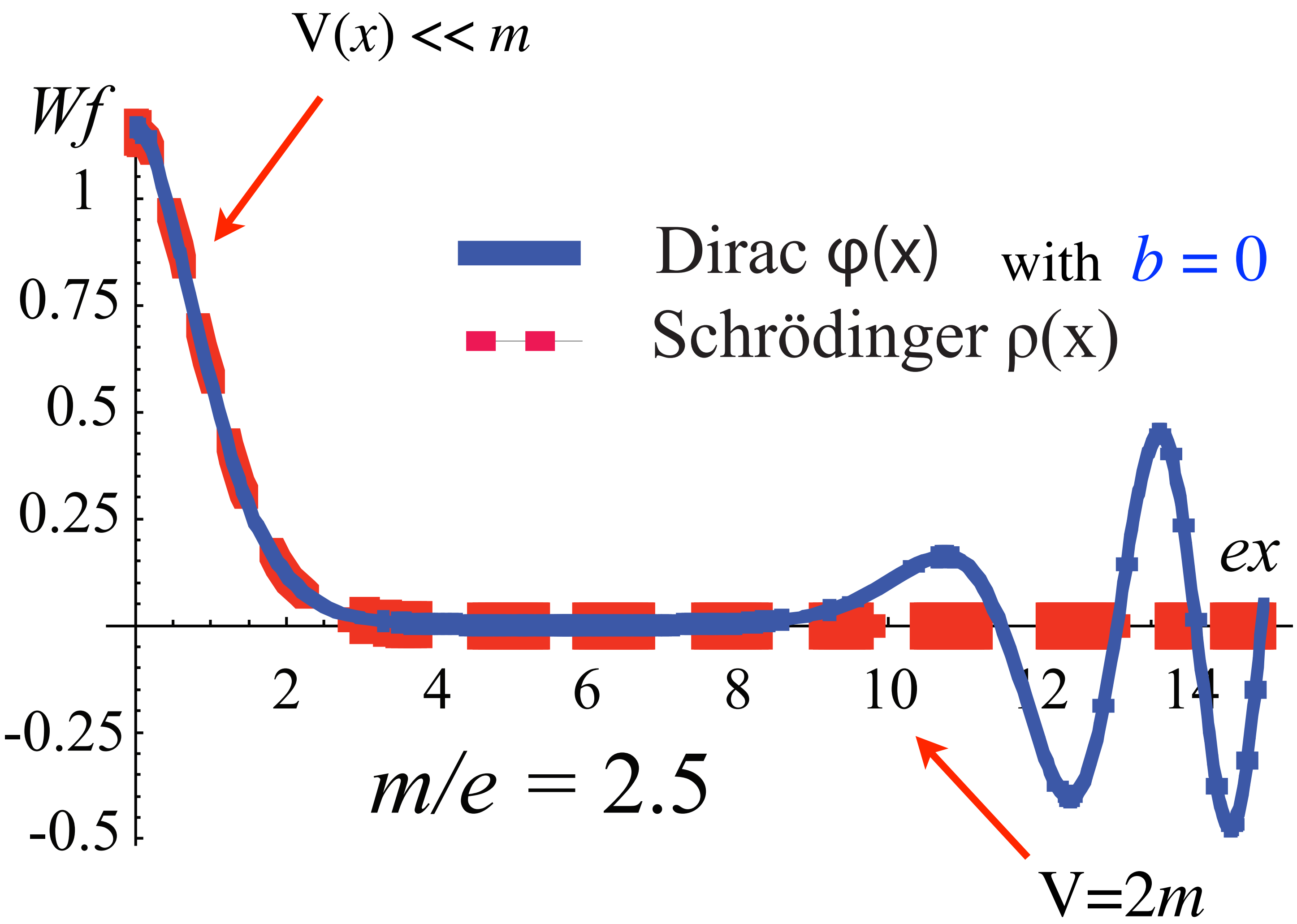
In the NR limit of large m/e, the eigenvalues $M = m + E_b$ become insensitive to a/b, and (for $a+b \neq 0$) the wave function reduces to the Schrödinger solution:

$$\psi(\sigma) = (1+i)(a+b)\sqrt{\pi}m^{1/3}e^{\pi m^2/2 - i\pi/4}\mathrm{Ai}\left[m^{1/3}(|x| - 2E_b)\right]\left[1 + \mathcal{O}\left(m^{-2/3}\right)\right]$$

In the NR limit, the continuous range of M is restricted to $a/b = -1$.

Dirac wave function for m/e = 2.5

Comparison of the Dirac $\varphi(x)$ wave function (for $b = 0$) with the Schrödinger Ai solution $\varrho(x)$:



The oscillations at V > 2m describe positrons, which are repelled by the potential. The amplitude of the oscillations depends on a/b, but is always non-zero.

$f\,\bar{f}$ bound states in D=1+1

A state with two fermions of energy E and momentum $P^1 = P$:

2x2 c-numbered wf.

$$|E,P\rangle = \int dx_1 dx_2\, \bar\psi(t,x_1) \exp\left[\tfrac{1}{2}iP(x_1+x_2)\right]\Phi(x_1-x_2)\psi(t,x_2)|0\rangle$$

field operators

Unlike the Dirac eq., this is an eigenstate also of the space translation generator:

$$\hat{P}^1|E,P\rangle = P|E,P\rangle$$ Bound state has momentum P

$$\hat{P}^0|E,P\rangle = E|E,P\rangle$$ Bound state equation for $\Phi(x)$ from QED action:

$$i\partial_x\left\{\sigma_1,\Phi(x)\right\} + \left[-\tfrac{1}{2}P\sigma_1 + m\sigma_3, \Phi(x)\right] = \left[E - V(x)\right]\Phi(x)$$

where $V(x) = \frac{1}{2}e^2|x|$ and $\gamma^0 = \sigma_3, \quad \gamma^1 = i\sigma_2, \quad \gamma^0\gamma^1 = \sigma_1$

Here the CM momentum P is a parameter, thus E and Φ depend on P .

Frame dependence of bound states

Boosts are dynamical transformations: H is not invariant.

In D=1+1 the Poincare Lie algebra is, with K the boost generator, $H = P^0$:

$$\left[P^0, P^1\right] = 0 \qquad \left[P^0, K\right] = iP^1 \qquad \left[P^1, K\right] = iP^0$$

States are defined at equal time in all frames: This is a frame-dependent concept.
The Hamiltonian generates time translations, hence is frame dependent.

Correspondingly, the eigenvalue condition for H has no explicit covariance:

$$i\partial_x \left\{\sigma_1, \Phi(x)\right\} + \left[-\tfrac{1}{2}P\sigma_1 + m\sigma_3, \Phi(x)\right] = \left[E - V(x)\right]\Phi(x)$$

Being derived from a Poincaré invariant action we may expect that it has a dynamical covariance.

Boost covariance

"Miraculously", the state is indeed covariant under boosts:

$$(1 - id\xi K)\,|E, P\rangle = |E + d\xi P, P + d\xi E\rangle$$

This holds only for a linear potential and ensures that $E(P) = \sqrt{P^2 + M^2}$

The P-dependence of the wave function Φ can be explicitly given in terms of an invariant distance :

$$\sigma(x) \equiv (E - V)^2 - P^2$$

$$\Phi^P(\sigma) = e^{\gamma_0\gamma_1\zeta/2}\Phi^{(P=0)}(\sigma)e^{-\gamma_0\gamma_1\zeta/2}$$ Any P

where $dx = -\dfrac{d\sigma}{E - V(x)}$ and $\tanh\zeta = -\dfrac{P}{E - V}$

Relativistic Lorentz contraction

Explicit Lorentz covariance: Bethe-Salpeter approach

The B-S wave function Φ is defined Lorentz covariantly (here D=3+1)

$$\langle\Omega|\, T\left\{\bar{\psi}_\beta(x_2)\psi_\alpha(x_1)\right\}|P\rangle \equiv e^{-iP\cdot(x_1+x_2)/2}\,\Phi^{\boldsymbol{P}}_{\alpha\beta}(x_1-x_2)$$

where $|\,P>$ is any state with total momentum 4-momentum P, and $|\Omega>$ is the vacuum.

The B-S wave function Φ transforms simply under boosts. If $P' = \Lambda P$ then

$$\Phi^{\boldsymbol{P}'}(x_1'-x_2') = S(\Lambda)\Phi^{\boldsymbol{P}}(x_1-x_2)S^{-1}(\Lambda)$$

Since the time difference $x_2^0 - x_1^0$ is frame-dependent, the B-S wf is not simply related to the Fock state wf's of a Hamiltonian approach.

Solutions of the ff bound state equation

To solve the fermion-antifermion bound state equation (here $m_1 = m_2 = m$)

$$i\partial_x\{\sigma_1, \Phi(x)\} + \left[-\tfrac{1}{2}P\sigma_1 + m\sigma_3, \Phi(x)\right] = \left[E - V(x)\right]\Phi(x)$$

we may expand the 2x2 wave function as $\Phi = \Phi_0 + \sigma_1\Phi_1 + \sigma_2\Phi_2 + \sigma_3\Phi_3$.

We get two coupled equations, with no explicit E or P dependence:

$$-2i\partial_\sigma\Phi_1(\sigma) = \Phi_0(\sigma) \qquad -2i\partial_\sigma\Phi_0(\sigma) = \left[1 - \frac{4m^2}{\sigma}\right]\Phi_1(\sigma)$$

The general solution is

$$\Phi_1(\sigma) = \sigma\, e^{-i\sigma/2}\left[a\,{}_1F_1(1 - im^2, 2, i\sigma) + b\,U(1 - im^2, 2, i\sigma)\right]$$

If $b \neq 0$ the wf Φ is singular at $\sigma = 0$. Requiring $b = 0$ the spectrum is discrete.

Note: This constraint only applies for $m \neq 0$.

Non-relativistic limit

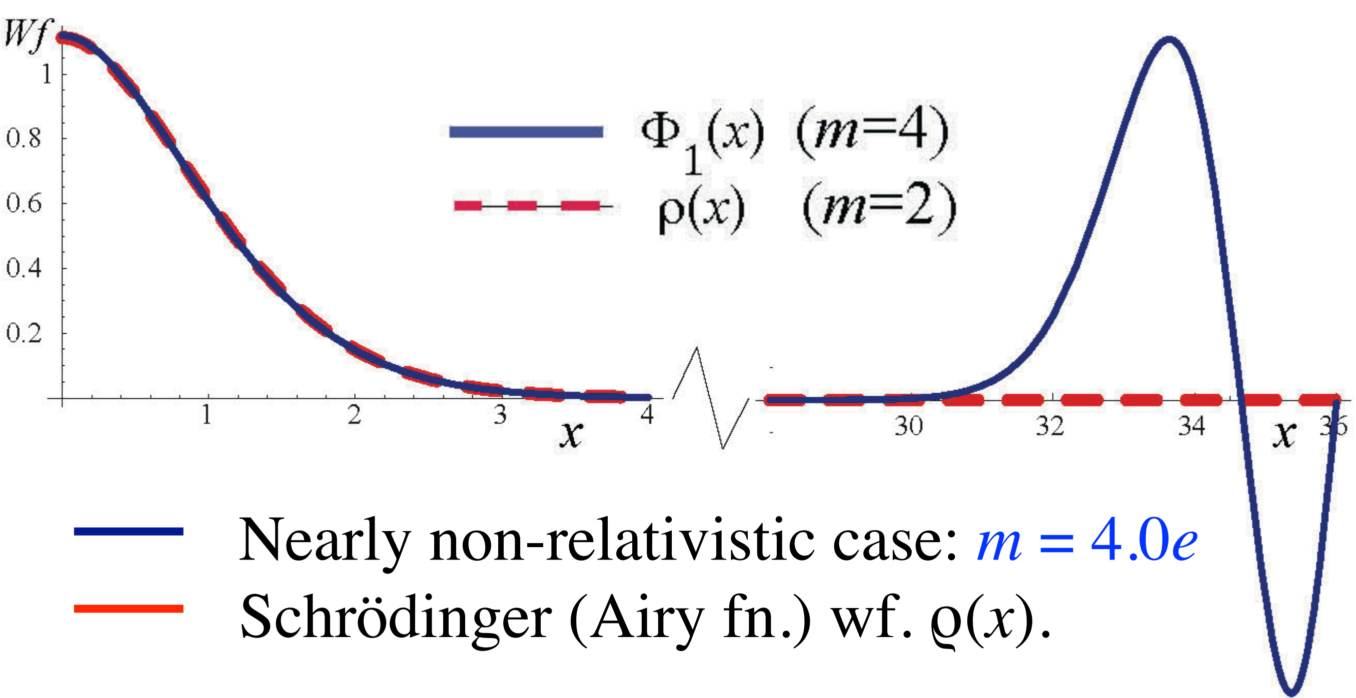
For $m/e \to \infty$ with $E_b = M - 2m$ fixed the Hypergeometric functions become

$$a\,\sigma e^{-i\sigma/2}\,{}_1F_1(1-im^2,2,i\sigma) = \left(\frac{2}{m}\right)^{2/3} e^{\pi m^2}\,\mathrm{Ai}\left[\left(\tfrac{1}{2}m\right)^{1/3}(|x|-2E_b)\right]$$

$$b\,\sigma e^{-i\sigma/2}\,U(1-im^2,2,i\sigma) = -(2m^2)^{2/3}\frac{\pi\,e^{-\pi m^2}}{\Gamma(1-im^2)}\left\{\mathrm{Ai}\left[(\tfrac{1}{2}m)^{1/3}(|x|-2E_b)\right] + i\,\mathrm{Bi}\left[(\tfrac{1}{2}m)^{1/3}(|x|-2E_b)\right]\right\}$$

The solution is normalizable in the NR limit only if $b = 0$.

Exponentially increasing



Nearly non-relativistic case: $m = 4.0e$

Schrödinger (Airy fn.) wf. $\varrho(x)$.

Oscillations at large ex similar to the Dirac case. Reflect fermions accelerated to high momenta by the linear potential.

Solutions for small fermion mass m

The solution simplifies for $m/e \to 0$

$$\Phi_1(\sigma) = N\sin(\tfrac{1}{2}\sigma)\left[1+\mathcal{O}\left(m^2\right)\right]$$

Linear "Regge trajectories" $M_n^2 = n\pi e^2\left[1+\mathcal{O}\left(m^2\right)\right]$ $(n = 0,1,2,\ldots)$

The parity is $(-1)^{n+1}$: No parity doublets for $m \neq 0$!

Recall: $-2i\partial_\sigma\Phi_0(\sigma) = \left[1-\dfrac{4m^2}{\sigma}\right]\Phi_1(\sigma)$

Wf's that are regular at $\sigma = 0$ have discrete spectrum

Chiral symmetry appears only when $m = 0$ exactly. The wave function is then regular for all M, and parity doublets exist.

String breaking (hadron loops) are probably important at small m.

However, the spectrum breaks chiral symmetry even without string breaking, for any $m \neq 0$.

Infinite Momentum Frame (IMF) ≈ Light Front (LF)

The wf is frame invariant in terms of $\sigma = (E-V)^2 - P^2$. Since $V(x) = \frac{1}{2}|x|$:

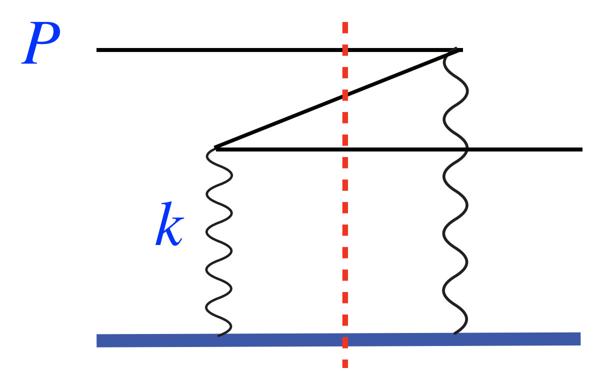
$$x = 2\left(E \pm \sqrt{P^2+\sigma}\,\right)$$

For $P \to \infty$ at fixed σ: $$x \simeq 2(E \pm P) \pm \frac{\sigma}{P} \simeq \begin{cases} 4P + \sigma/P \\ (M^2-\sigma)/P \end{cases}$$

Lower solution: $x \propto 1/P$ Lorentz-contracted "valence" region.

Upper solution: $x \approx 4P \to \infty$ Oscillations (pairs move to infinite x.

Perturbatively: "Z-diagrams" get infinite energy ($k \to \infty$) in the $P \to \infty$ limit.



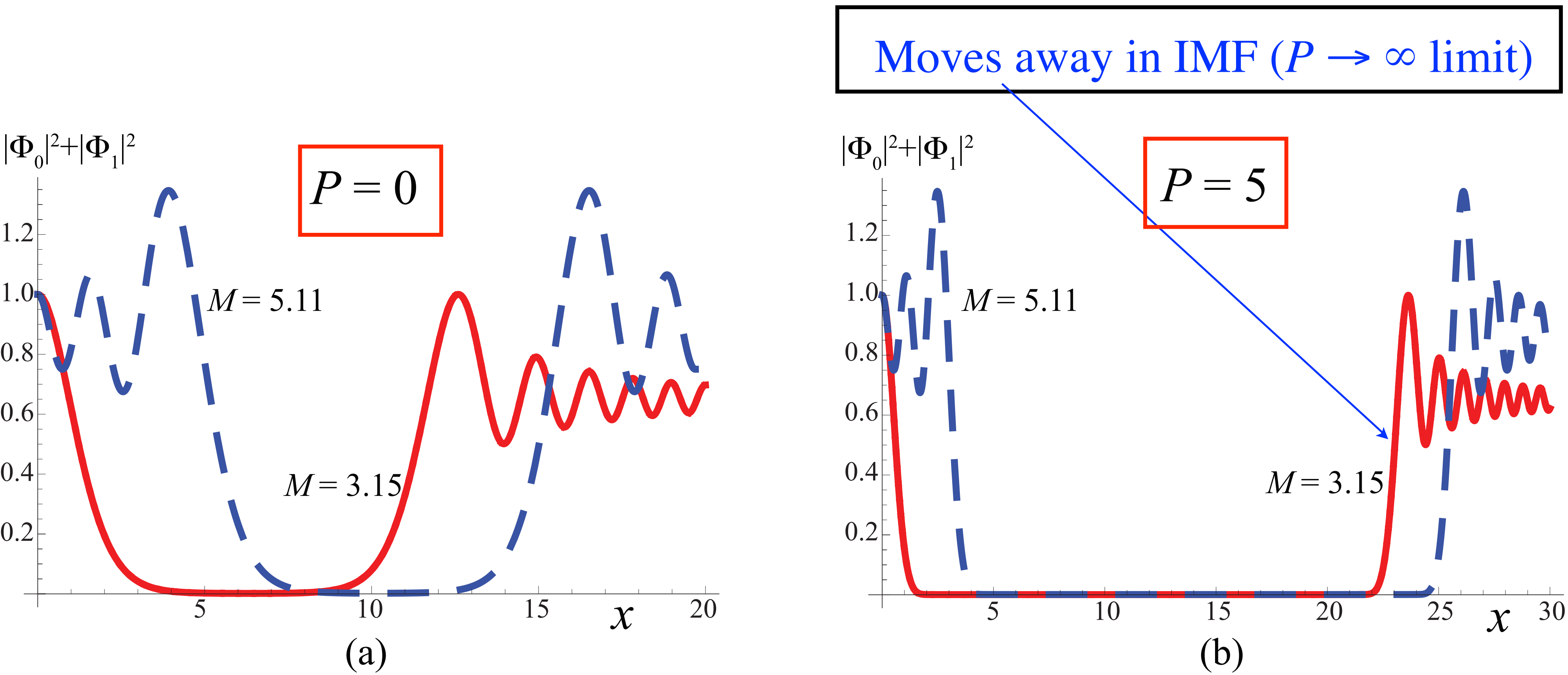
C.f.: $H|0\rangle = 0$ in LF quantization.
$p^+ = 0$ means $p^z \to -\infty$

Explicitly: $$\Phi_{P\to\infty}(\sigma) = 2am\,P\gamma^+ e^{-i\sigma/2}\,{}_1F_1(1-im^2, 2, i\sigma)$$

Frame (P) dependence of the solutions ($m_1 \neq m_2$)

Comparison of ground and excited state wave functions

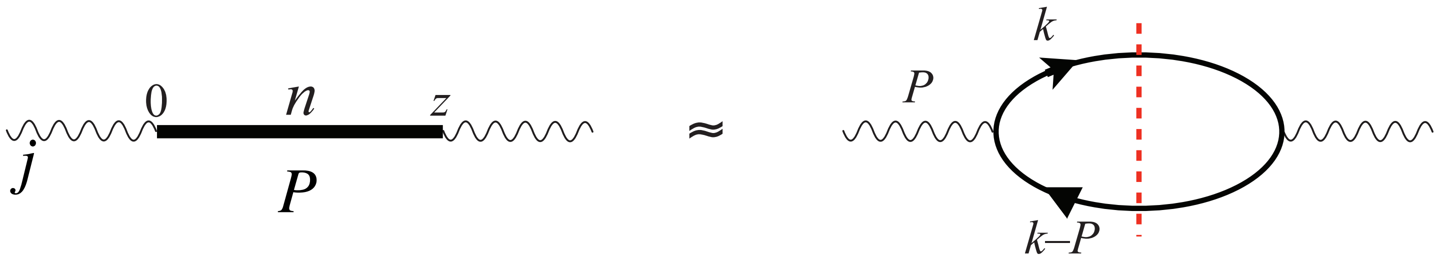
for P=0 (CM frame) and for $P = 5e$. (m_1=1.0e m_2=1.5e)



Note: In the IMF limit, only the normalizable, valence part of the wf remains.

Quark - Hadron duality

The wave functions of highly excited (large mass M) bound states are similar to free $f\bar{f}$ pairs (for $V(x) << M$). This determines their normalization:

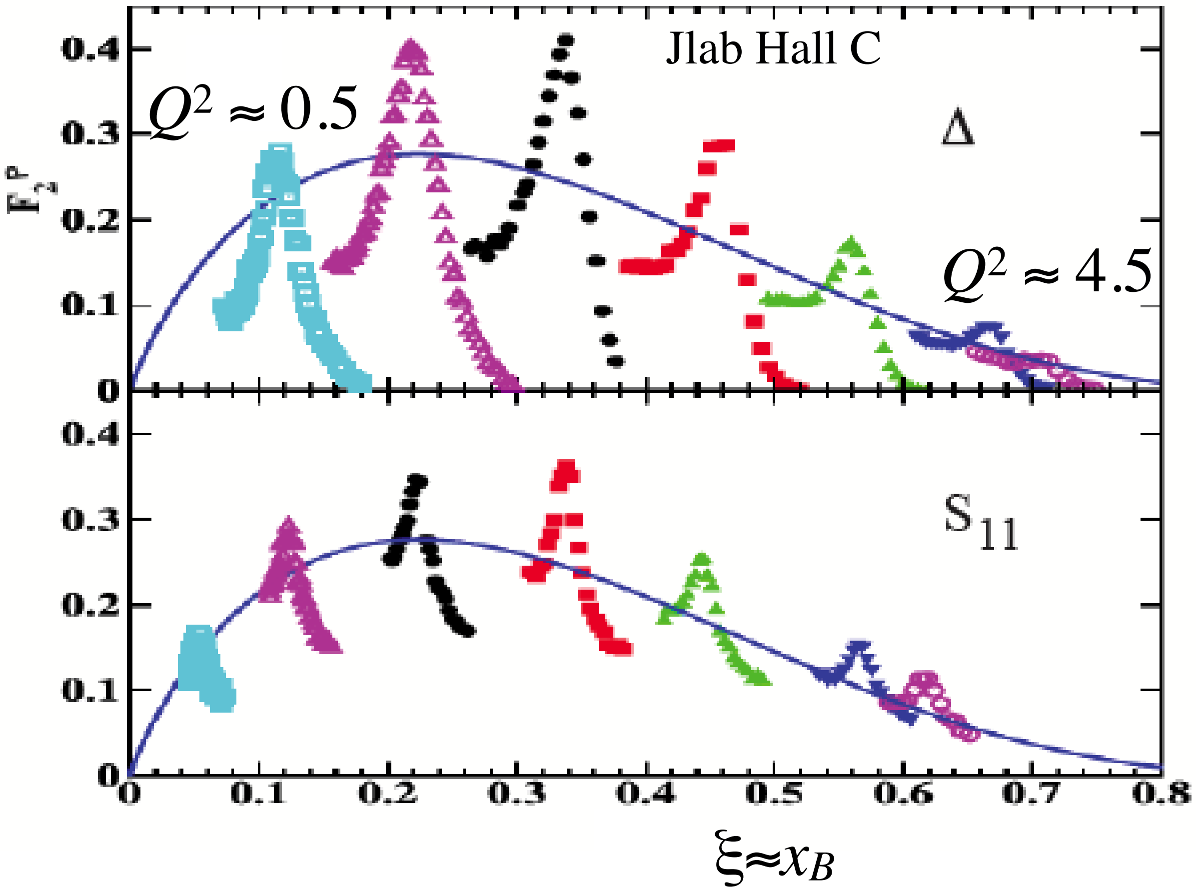


$$\Rightarrow \quad |\Phi_0(x=0)|^2 = |\Phi_1(x=0)|^2 = \pi/2$$

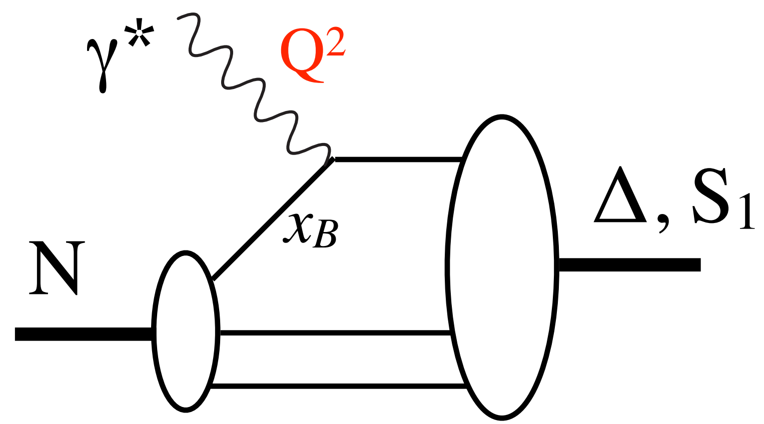
The same result for $j = S, P, V, A$ currents

Bloom-Gilman Duality

W. Melnitchouk et al, Phys. Rep. 406 (2005) 127



Resonance contributions
*ep → eN**
build DIS scaling in
ep → eX



$$m_{N^*}^2 = m_N^2 + Q^2\left(\frac{1}{x_B} - 1\right)$$

Scattering dynamics is built into hadron wave functions.

Requires relativistic bound states in motion.

Partons in bound states

In the parton picture, high energy quarks can be treated as free constituents.
They are momentum eigenstates, described by plane waves.
How does this fit into the bound state wave functions?

Consider a highly excited state (P=0): $M \to \infty$, $V(x) << M$

$$\sigma = (M-V)^2 \approx M^2 - 2MV \to \infty$$

$$\Phi(\sigma \to \infty) \sim \exp(\pm i\sigma/2) = e^{\pm iM^2} \exp(\mp ix\, M/2)$$

Thus oscillations of wf at large σ gives plane wave with $p = \pm M/2$

The operator expression for the state is in this limit:

$$|M, P=0\rangle = \frac{\sqrt{2\pi}}{2M}\big(b_{M/2}^\dagger d_{-M/2}^\dagger + b_{-M/2}^\dagger d_{M/2}^\dagger\big)|\Omega\rangle$$

As in the parton picture, only $E > 0$ particles appear (no b or d operators).

Bound state scattering amplitudes

The perturbative expansion of the S-matrix is defined by

$$S_{fi} = {}_{\mathbf{out}}\langle f|\left\{\mathrm{T}\exp\left[-i\int_{-\infty}^{\infty} dt\, H_I(t)\right]\right\}|i\rangle_{\mathbf{in}}$$

where the *in* and *out* states are $O(\alpha^0)$ asymptotic states at $t = \pm\infty$.

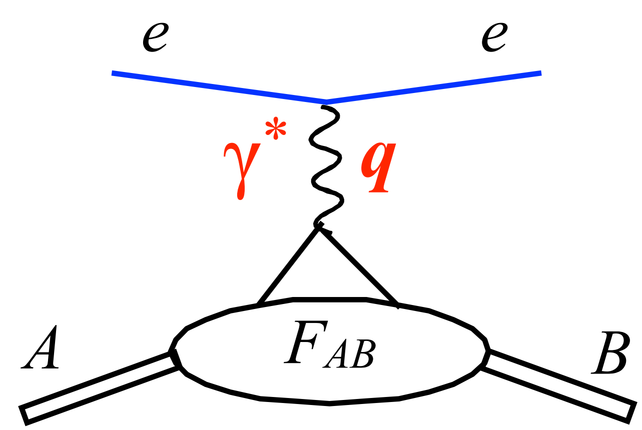
The *ff* states bound by a linear potential are $O(\alpha^0)$ and Poincaré covariant. They can be used as *in* and *out* states, defining the perturbative expansion.

Even the $O(\alpha^0)$ amplitudes have a rich dynamics (string breaking,…).
The feasibility of the perturbative approach to hadrons discussed here requires that the main features of hadron dynamics are described at $O(\alpha^0)$

EM Form Factor (D = 1+1)

$$F^{\mu}_{AB}(z) = \langle B(P_B); t=+\infty | j^{\mu}(z) \, | A(P_A); t=-\infty \rangle$$

A, B: *in* & *out* states



EM current:

$$j^{\mu}(z) = \bar{\psi}(z)\gamma^{\mu}\psi(z) = e^{i\hat{P}\cdot z} j^{\mu}(0) e^{-i\hat{P}\cdot z}$$

Gauge invariance is verified: $\partial_{\mu} F^{\mu}_{AB}(z) = 0$

Poincaré invariance is verified (numerically).

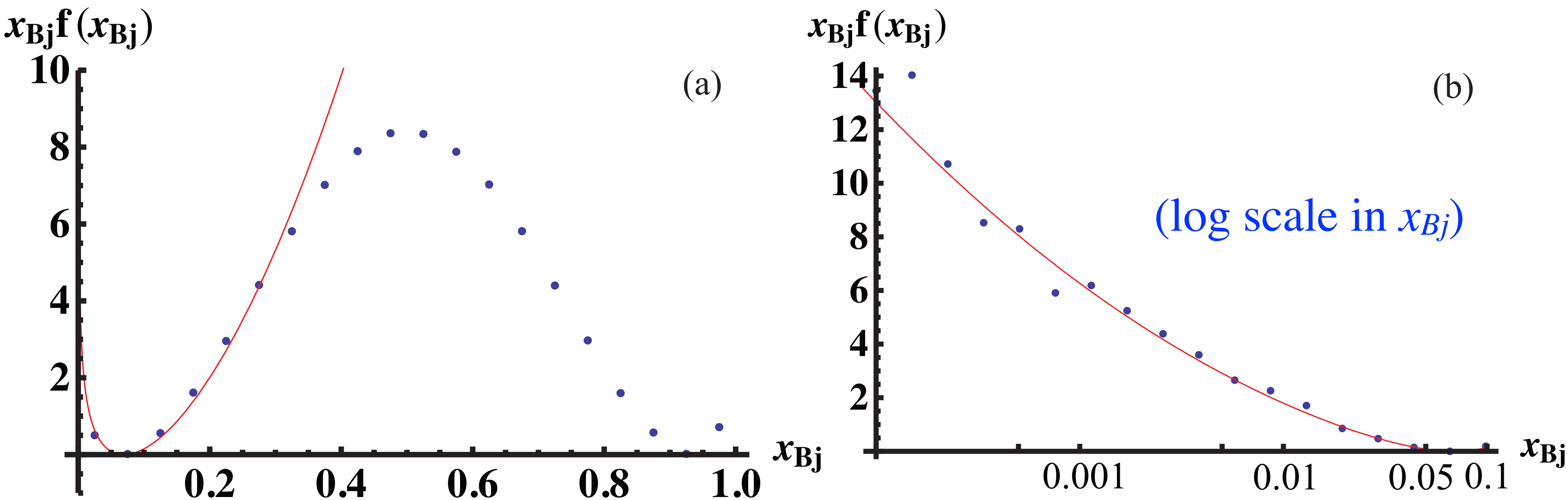
In the Bjorken limit we can calculate the parton distribution.

$$x_{Bj} = \frac{Q^2}{2p_A \cdot q} \qquad M_B^2 = Q^2 \left(\frac{1}{x_{Bj}} - 1 \right) \to \infty$$

Parton distributions have a sea component

The sea component is prominent at low *m/e* :

$$m/e = 0.1$$



The red curve is an analytic approximation, valid in the $x_{Bj} \to 0$ limit.

Note: Enhancement at low x is not due to Φ_A^{IMF} (valence wf.)

Final remarks

- Hadron physics is fortunate: Theory (QCD) is known
  Much data on spectra, couplings, scattering
- Unprecedented features: Confinement, Chiral SB, Ultrarelativistic states
- Suggestive features of data: Hadron spectrum, Couplings, Duality

- Present approach: Assume that regularities are not "accidental"
  They suggest that QCD is perturbative even for Q → 0

- Conclusions: Essential features appear already at $\mathcal{O}\left(\alpha_s{}^0\right)$
  Implies a different expansion point for perturbation theory.
  The approach is strongly constrained by the requirement of a perturbative expansion (single parameter Λ).

PH: arXiv:1409.4703