Dispersion Relations

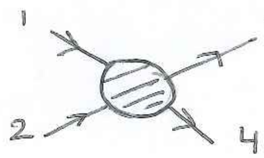
Causality and crossing symmetry imply analyticity.

Gribov Sec 2.1 p3

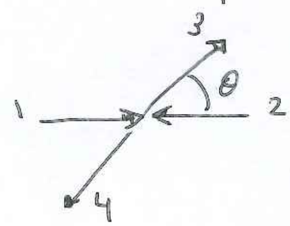
Collins Sec 1.4 p11

Simplified Proof:

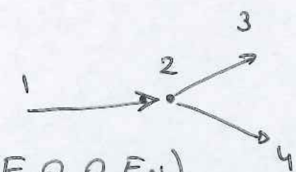
Consider $1+2 \to 3+4$



c.o.m. frame



Lab frame



$1 \equiv (E, 0, 0, Ev)$

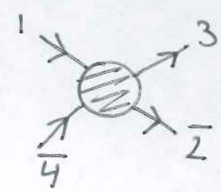
$s = m_1^2 + m_2^2 + 2 m_2 E$

Represent scattering amplitude as

$$M(E) = \int_0^\infty dz \int_z^\infty dt \; e^{iE(t - vz)} f(z,t)$$

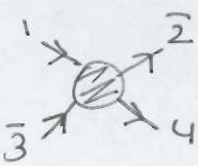
causality $\Rightarrow$ $t - vz > 0$ $\Rightarrow$ $\boxed{M(E) \text{ analytic for } \operatorname{Im} E > 0}$

crossing symmetry: Same function for



u-channel

and



t-channel

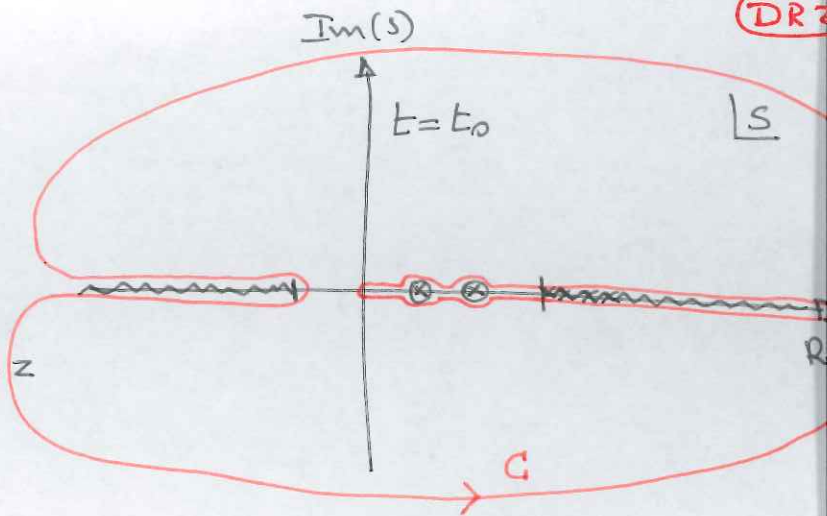
$M(s)$ analytic for $\operatorname{Im}(s) > 0$ but also $M(u)$ analytic for $\operatorname{Im}(u) > 0$

or $u = -s - t + \sum_i m_i^2$ $\Rightarrow$ $\boxed{M(s) \text{ analytic for } \operatorname{Im}(s) \neq 0}$

Only possible singularities are on real axis.

DR2

Cauchy:



$$A(s,t_0) = \oint_C \frac{dz}{2\pi i} \frac{A(z,t_0)}{z-s}$$

$$= \frac{1}{\pi}\int_0^{\infty} \frac{\mathrm{Im}\,A(z,t_0)}{z-s}\,dz - \frac{1}{\pi}\int_{-\infty}^{0} \frac{\mathrm{Im}\,A(z,t_0)}{z-s}\,dz$$

Use $\frac{1}{2i}\left[A(s_+,t) - A(s_-,t)\right] = \mathrm{Im}\,A(s_+,t) \qquad s_\pm = s \pm i\varepsilon$

Converge for $\mathrm{Im}\,A(s,t) \lesssim |s|^{-\epsilon} \quad \epsilon > 0$.

If not need substraction(s).

$$A(s,t_0) = A(s_0,t_0) + \frac{s-s_0}{\pi}\int_0^{\infty} \frac{\mathrm{Im}\,A(z,t_0)}{(z-s)(s_0-s)}\,dz + \cdots$$

Poles can be included as $\mathrm{Im}\,A(s,t) \subset g_p\,\delta(s-s_p)$

Introduce crossing variable $\nu = \frac{s-u}{4m_2} = \frac{2s+t-\Sigma}{4m_2} = E_{Lab} + \frac{t+(m_1^2-m_3^2)+(m_2^2-\cdots}{4m_2}$

$$A(\nu,t_0) = \frac{1}{\pi}\int_0^{\infty}\left(\frac{\mathrm{Im}\,A(z,t_0)}{z-\nu} + \frac{\mathrm{Im}\,A(-z,t_0)}{z+\nu}\right)dz$$

Decompose $A(\nu,t) = A^+(\nu,t) + A^-(\nu,t)$

$$A^\pm(\nu,t) = \pm A^\pm(-\nu,t)$$

$$A^+(\nu,t_0) = \frac{2}{\pi}\int_0^{\infty} \frac{\mathrm{Im}\,A^+(z,t_0)}{z^2-\nu^2}\,z\,dz$$

$$A^-(\nu,t_0) = \frac{2}{\pi}\int_0^{\infty} \frac{\mathrm{Im}\,A^-(z,t_0)}{z^2-\nu^2}\,\nu\,dz$$

Sommerfeld-Watson Representation

Gribov sec 7.1 p153 DR 3

Collins sec 2.9 p73

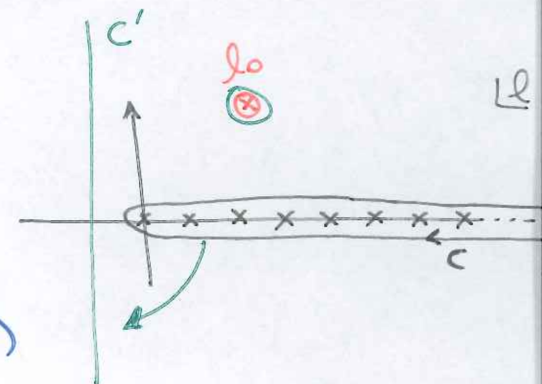
PW expansion in t-channel:

$$A(s,t) = \sum_{\ell \geq 0} (2\ell+1)\, A_\ell(t)\, P_\ell(z_t) \quad ; \quad z_t = 1 + \frac{2s}{t-4m^2} = \frac{4m\nu}{t-4m^2}$$

for identical spinless particles.

Analytic continuation of $A_\ell(t) \to A(\ell,t)$

$$A(s,t) = \frac{1}{2i} \oint_C (2\ell+1)\, \frac{A(\ell,t)}{\sin \pi\ell}\, P_\ell(-z_t)\, d\ell$$



C include all positive integers

Assume $A(\ell,t) = \dfrac{g(t)}{\ell-\alpha(t)}$ around $\ell = \alpha(t)$

For $\ell \in \mathbb{N}$ $\quad A_\ell(t) = \dfrac{\tilde{g}(t)}{m_\ell^2 - t}$

For fixed $t = t_0$, $A(\ell,t)$ has a pole at $\alpha(t_0) = \ell_0$

Deform the contour $C \to C'$ and get

$$A(s,t) = \pi(2\alpha+1)\, \frac{g(t)}{\sin \pi\alpha}\, P_\alpha(-z_t) + \underbrace{\int_{C'} (\dots)}_{O(1/\sqrt{s})}$$

Introducing signature Collins sec 2.5 p122

$$A^{\pm}(s,t) = \pi\, g(t)\, \frac{2\alpha+1}{\sin \pi\alpha} \left[P_\alpha(-z_t) \pm P_\alpha(z_t) \right] + O(1/\sqrt{s})$$

$$\boxed{R^{\pm}(\nu,t) = -\beta(t)\, \frac{e^{-i\pi\alpha(t)} \pm 1}{\sin \pi\alpha(t)}\, \nu^{\alpha(t)}}$$

for $\nu \gg$

$P_\alpha(\nu) \sim \nu^\alpha$

convention such that $\text{Im}\, R^{\pm}(\nu,t) = \beta(t)\, \nu^{\alpha(t)}$ DR 4

Satisfy dispersion relation:

$$R^{\pm}(\nu,t) = \frac{1}{\pi}\int_0^{\infty} \underbrace{\text{Im}\, R^{\pm}(\nu',t)}_{\beta \nu'^{\alpha}} \left(\frac{1}{\nu'-\nu} \pm \frac{1}{\nu'+\nu}\right) d\nu'$$

<u>Finite Energy</u> <u>Sum Rules</u>:

Assume $A^{\pm} = R^{\pm}$ for $\nu \geq \Lambda$:

$$A^{\pm}(\nu,t) = \frac{1}{\pi}\int_0^{\Lambda} \text{Im}\, A^{\pm}(z,t)\left[\frac{1}{z-\nu} \pm \frac{1}{z+\nu}\right] dz + \frac{1}{\pi}\int_{\Lambda}^{\infty} \beta z^{\alpha}\left(\frac{1}{z-\nu} \pm \frac{1}{z+\nu}\right) d\nu$$

$$= R^{\pm}(\nu,t) \quad \text{for } \nu \geq \Lambda$$

$$\Rightarrow \int_0^{\Lambda}\left[\text{Im}\, A^{\pm}(z,t) - \beta z^{\alpha}\right]\left[\frac{1}{z-\nu} \pm \frac{1}{z+\nu}\right] dz = 0 \quad \text{for } \nu \geq \Lambda \geq z$$

Expand $(z^2 - \nu^2)^{-1} = -\nu^{-2}\left[1 + \left(\frac{z}{\nu}\right)^2 + \left(\frac{z}{\nu}\right)^4 + \cdots\right]$

$$\int_0^{\Lambda}\left[\text{Im}\, A^{\pm}(z,t) - \beta z^{\alpha}\right]\left(-\frac{1}{\nu}\right)\sum_{k=0}^{\infty}\left[1 \mp (-1)^k\right]\left(\frac{z}{\nu}\right)^k dz = 0$$

$$\boxed{\frac{1}{\Lambda^k}\int_0^{\Lambda} \text{Im}\, A^{\pm}(z,t) . z^k\, dz = \frac{\beta \Lambda^{\alpha+1}}{\alpha+k+1}}$$

k odd for A^+ ; k even for A^-

Classification of Regge Poles in QCD

In QCD, isospin I, G-parity G and parity P are good quantum numb

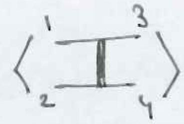
DR introduced Signature $\tau = (-1)^J$, parity implies naturality $\eta = P(-$

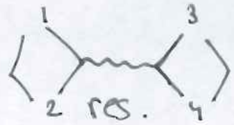
⇒ For mesonic trajectories we have $\underset{\tau}{2}\times\underset{\eta}{2}\times\underset{G}{2}\times\underset{I=0,1}{2} \;+\; \underset{\tau}{2}\times\underset{\eta}{2}\times\underset{I=1/2}{1}$

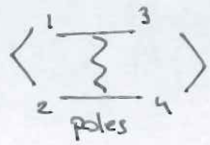
$I^{G\tau\eta}$		J^{PC}	$I^{G\tau\eta}$		J^{PC}
0^{+++}	f	$(0,2,4,\ldots)^{++}$	0^{+--}	$\bar{f}$	$(1,3,5,\ldots)^{++}$
0^{--+}	ω	$(1,3,5,\ldots)^{--}$	0^{-+-}	$\bar{\omega}$	$(0,2,4,\ldots)^{--}$
1^{-++}	a	$(0,2,4,\ldots)^{++}$	1^{---}	$\bar{a}$	$(1,3,5,\ldots)^{++}$
1^{+-+}	ρ	$(1,3,5,\ldots)^{--}$	1^{++-}	$\bar{\rho}$	$(0,2,4,\ldots)^{--}$
0^{++-}	η	$(0,2,4,\ldots)^{-+}$	0^{+-+}	$\bar{\eta}$	$(1,3,5,\ldots)^{-+}$
0^{---}	h	$(1,3,5,\ldots)^{+-}$	0^{-++}	$\bar{h}$	$(0,2,4,\ldots)^{+-}$
1^{-+-}	π	$(0,2,4,\ldots)^{-+}$	1^{--+}	$\bar{\pi}$	$(1,3,5,\ldots)^{-+}$
1^{+--}	b	$(1,3,5,\ldots)^{+-}$	1^{+++}	$\bar{b}$	$(0,2,4,\ldots)^{+-}$
$1/2^{\cdot-+}$	K^*	$(1,3,5,\ldots)^{-}$	$1/2^{\cdot+-}$	K	$(0,2,4,\ldots)^{-}$
$1/2^{\cdot++}$	$\bar{K}^*$	$(0,2,4,\ldots)^{+}$	$1/2^{\cdot--}$	$\bar{K}$	$(1,3,5,\ldots)^{+}$

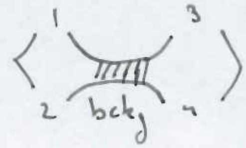
There is also the Pomeron $\mathbb{P}$ 0^{+++}

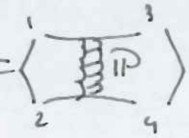
Duality Hypothesis

FESR ⇒  = 

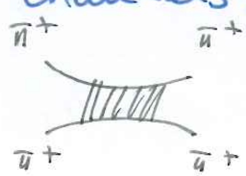
Duality: 

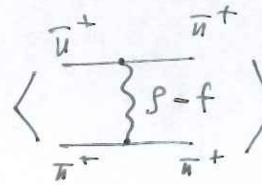
= 

; 

= 

Application to exotic channels:

$\mathrm{Im}(\pi^+\pi^+) = 0$ 

⇒ 

$= 0$

only bckg.

$$\beta^{\rho}_{\pi\pi}(t)\,\nu^{\alpha_\rho(t)} - \beta^{f}_{\pi\pi}(t)\,\nu^{\alpha_f(t)} = 0 \quad \forall t$$

$$\Rightarrow \boxed{\begin{array}{c} \beta^{\rho}_{\pi\pi}(t) = \beta^{f}_{\pi\pi}(t) \\ \alpha^{\rho}(t) = \alpha^{f}(t) \end{array}} \quad \text{EXD}$$

Repeat for $\pi^+\pi^+$, K^+K^+, $K^+\pi^+$, pp, K^+p, ...

$$\boxed{\begin{array}{c} \alpha^{\rho} = \alpha^{f} = \alpha^{\omega} = \alpha^{a} \\ \alpha^{\pi} = \alpha^{b} = \alpha^{h} = \alpha^{\eta} \end{array}}$$

Total cross sections

Optical theorem: $\sigma_{tot}(s) = \dfrac{\mathrm{Im}\, A(s, t=0)}{2 m\, p_{Lab}}$

Exchanges:

$\pi^{\pm} p : \mathbb{P} + f \mp \rho$

$\pi^{\pm} n : \mathbb{P} + f \pm \rho$

$K^{\pm} p : \mathbb{P} + f \mp \omega + a \mp \rho$

$K^{\pm} n : \mathbb{P} + f \mp \omega - a \pm \rho$

$\overset{(-)}{p}\, p : \mathbb{P} + f \mp \omega + a \mp \rho$

$\overset{(-)}{p}\, n : \mathbb{P} + f \mp \omega - a \pm \rho$

hierarchy:

$\mathbb{P} > f, \omega > \rho, a$

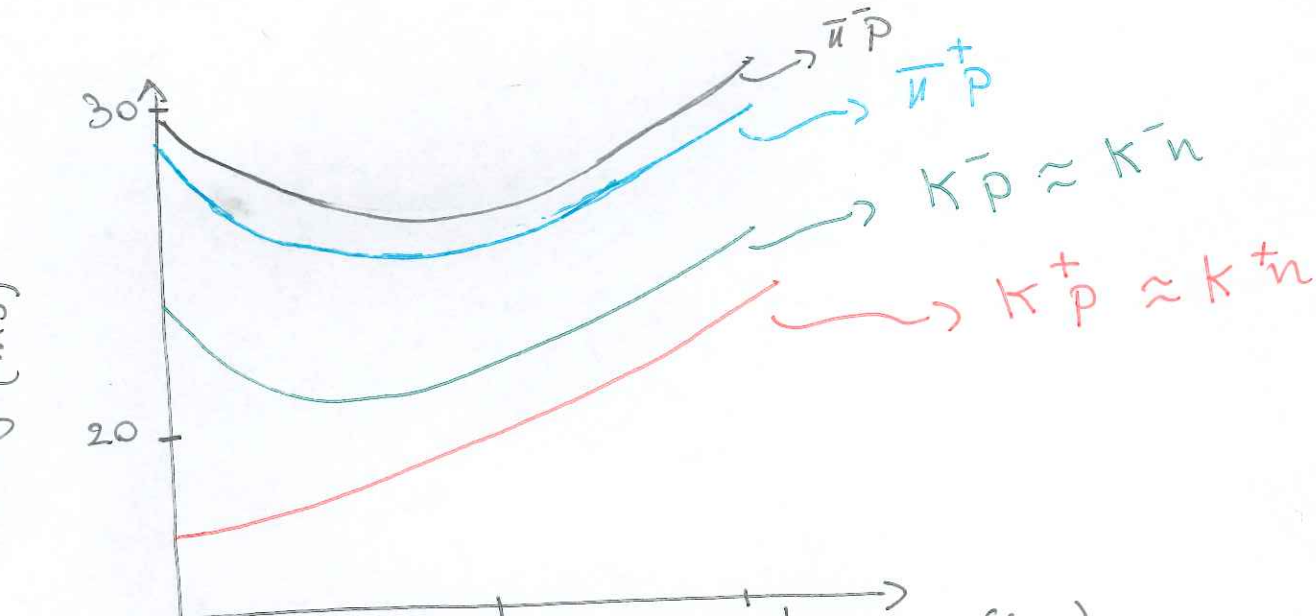
Parametrization of σ_{tot} for large s:

$$\sigma_{tot}(s) = \frac{1}{2 m\, p_{Lab}} \cdot \left[\beta_{\mathbb{P}}\, s^{\alpha_0^{\mathbb{P}}} + \left(\sum_i \beta_i\right) s^{\alpha_0^{R}} \right]$$

Exotic channels:

$K^{+} p : \mathbb{P} + (f - \omega) + (a - \rho) \simeq \mathbb{P}$ idem for $K^{+} n$

$p\, p : \mathbb{P} + (f - \omega) + (a - \rho) \simeq \mathbb{P}$ idem for $p\, n$



Charge Exchange Reactions

$\pi^- p \to \pi^0 n \;:\; \sqrt{2}\,\rho$	$K^+ n \to K^0 p \;:\; \sqrt{2}\,(\rho + a)$
$\pi^- p \to \eta\, n \;:\; \sqrt{2}\, a$	$K^- p \to \bar{K}^0 n \;:\; \sqrt{2}\,(\rho - a)$

$$A_\rho : -\beta_\rho \frac{-1 + e^{-i\pi\alpha_\rho}}{\sin \pi \alpha_\rho} \nu^{\alpha_\rho}$$

$$A_a : -\beta_a \frac{1 + e^{-i\pi\alpha_a}}{\sin \pi \alpha_a} \nu^{\alpha_a}$$

For $\pi^- p \to (\pi^0, \eta)\, n$, $\beta_{\rho,a}$ can be extracted from FESR

$K^+ n \to K^0 p$ and $K^- p \to \bar{K}^0 n$ have same $\frac{d\sigma}{dt} \sim |\rho \pm a|^2$

$$A_\rho + A_a = -\beta \frac{e^{-i\pi\alpha} \nu^\alpha}{\sin \pi \alpha} \times 2$$

$$A_\rho - A_a = -\beta \frac{(-2)\, \nu^\alpha}{\sin \pi \alpha}$$