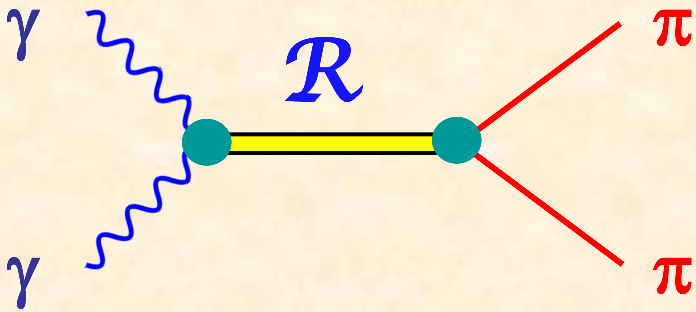
Light by Light

Summer School on Reaction Theory

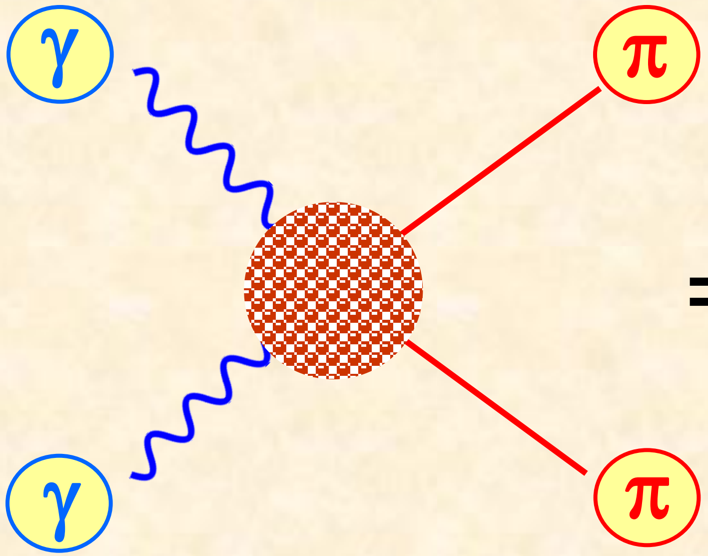
INDIANA UNIVERSITY

Michael Pennington
Jefferson Lab

Amplitude Analysis

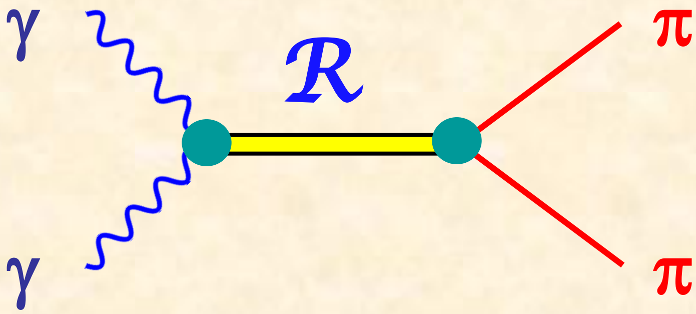


resonances have definite quantum numbers **I, J, P (C)**



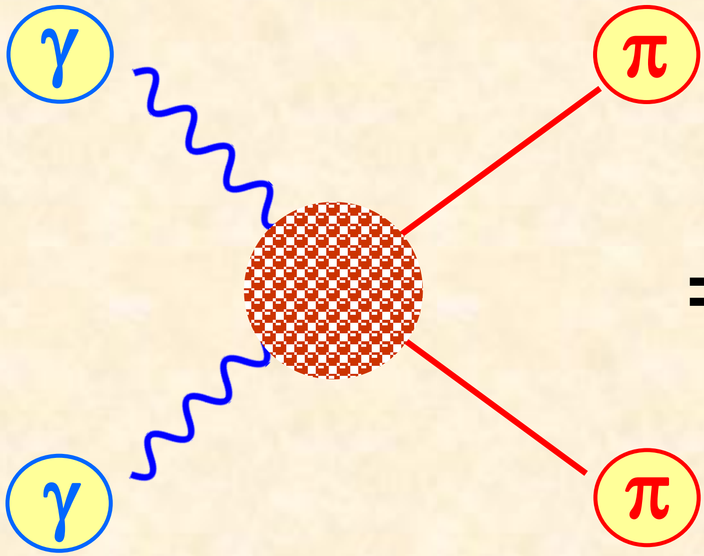
$$= \sum_{J,\lambda} \mathcal{F}_{J\lambda}(s)\ Y_{J\lambda}(\vartheta,\varphi)$$

Amplitude Analysis



resonances have definite quantum numbers I, J, P (C)

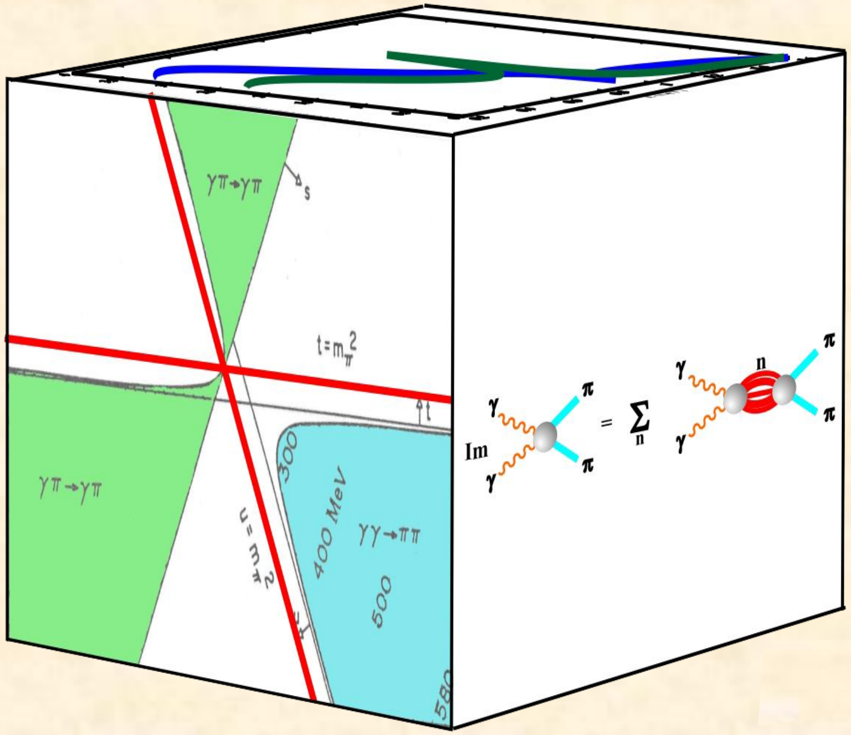
resonances and backgrounds not separable within unitarity



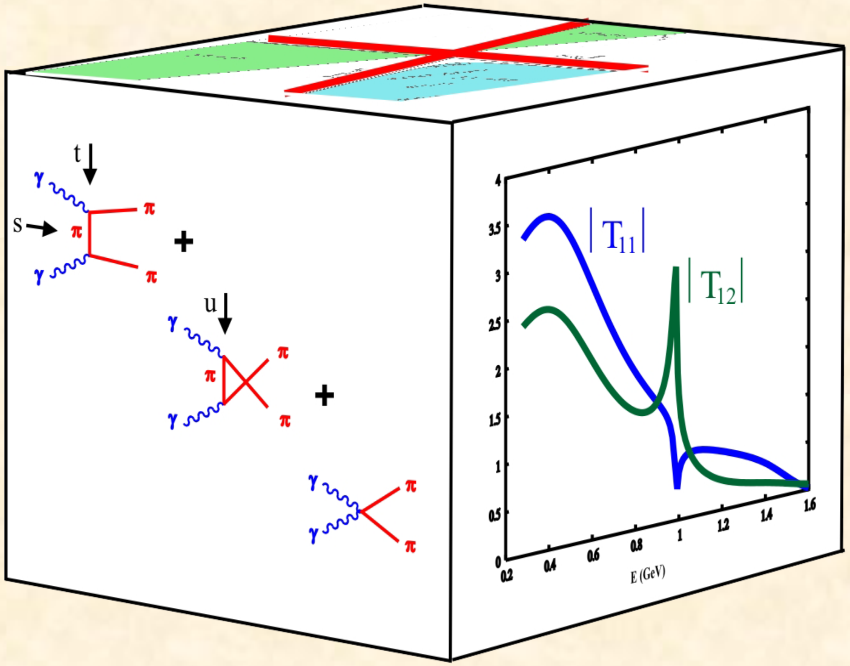
$$= \sum_{J,\lambda} \mathcal{F}_{J\lambda}(s)\, Y_{J\lambda}(\vartheta,\varphi)$$

partial wave amplitudes

To perform a partial wave separation need to know the partial waves at low energy accurately

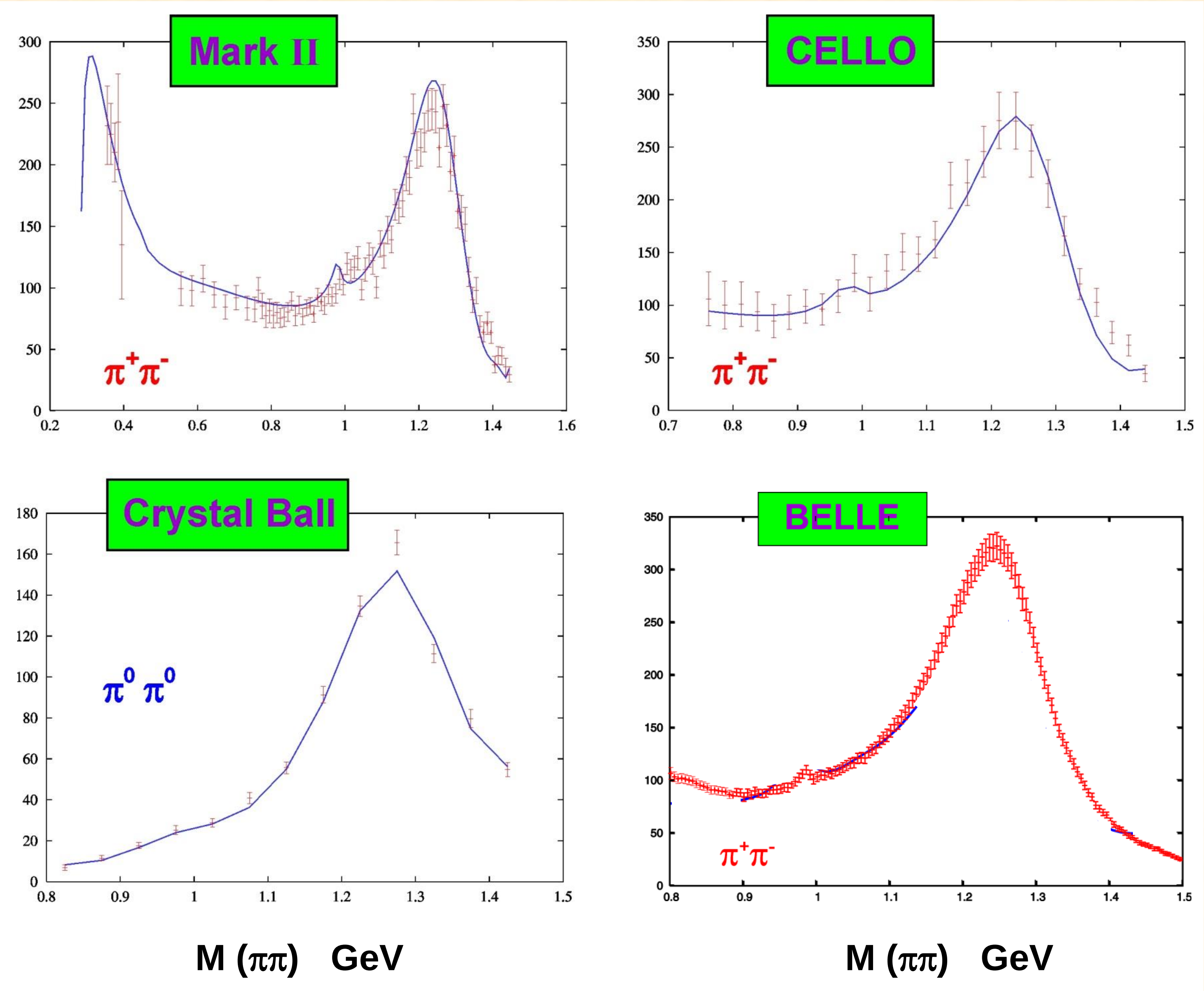


Morgan & P



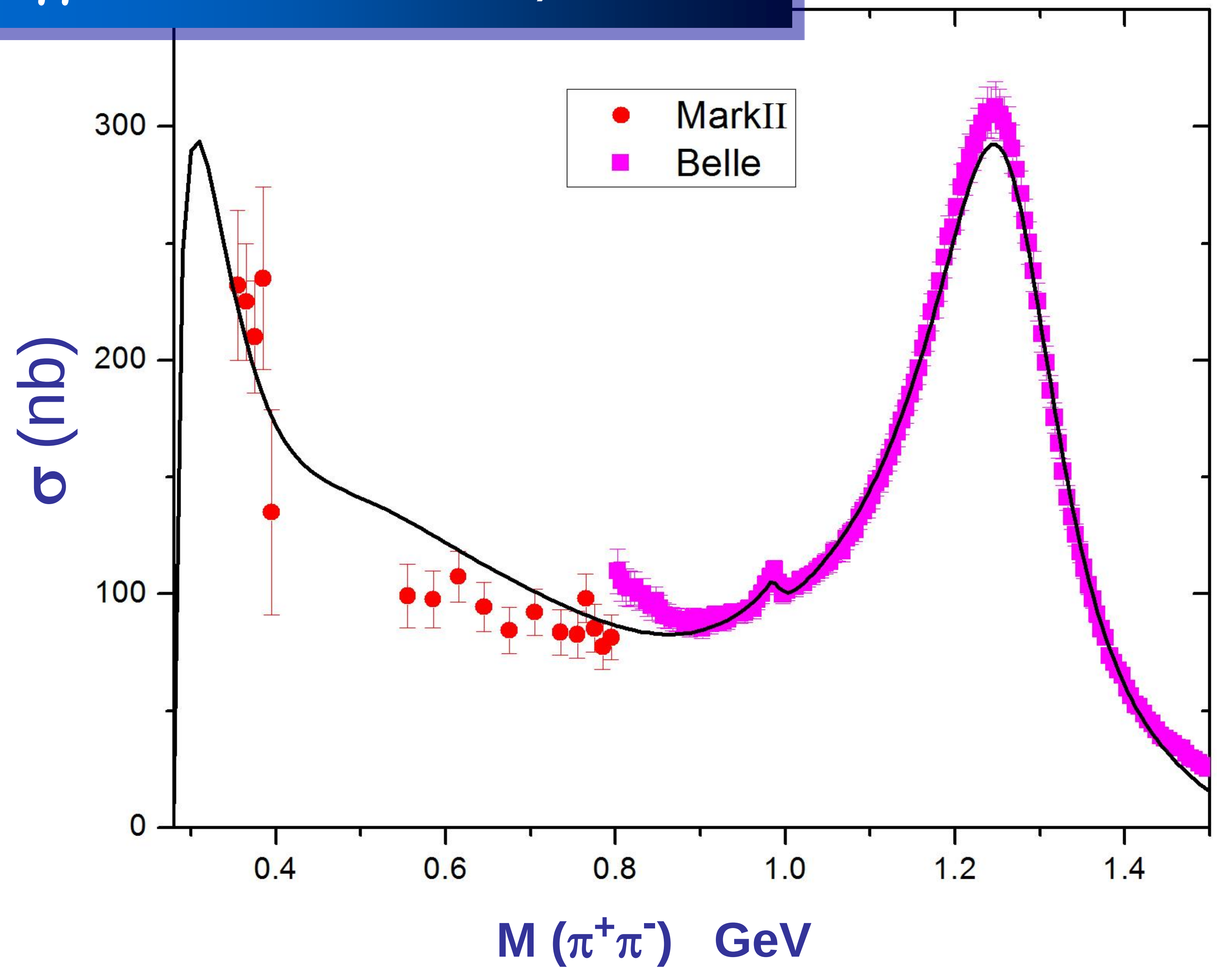
Amplitude Analysis

Lingyun Dai & P 2014

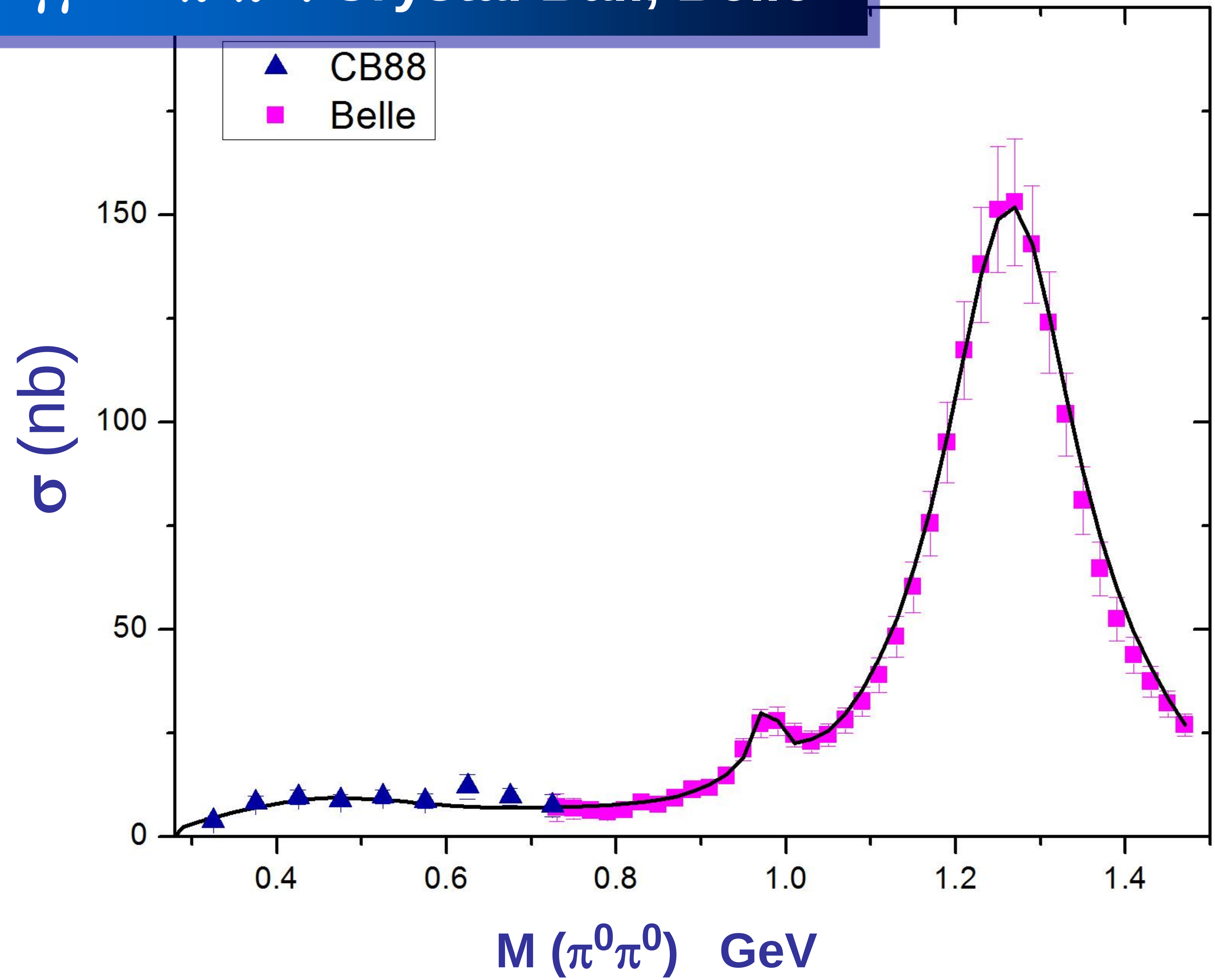


Mark II
CELLO
Crystal Ball
BELLE
π+π-
π0 π0
M (ππ) GeV
M (ππ) GeV

$\gamma\gamma \longrightarrow \pi^+\pi^-$: Mark II, Belle



$\gamma\gamma \longrightarrow \pi^0\pi^0$: Crystal Ball, Belle



Isospin decomposition

$$\mathcal{F}_\pi^{+-}(s) = -\sqrt{\frac{2}{3}}\mathcal{F}_\pi^{I=0}(s) - \sqrt{\frac{1}{3}}\mathcal{F}_\pi^{I=2}(s)$$

$$\mathcal{F}_\pi^{00}(s) = -\sqrt{\frac{1}{3}}\mathcal{F}_\pi^{I=0}(s) + \sqrt{\frac{2}{3}}\mathcal{F}_\pi^{I=2}(s)$$

$\gamma\gamma \longrightarrow \pi\pi$

One Pion Exchange

$\mathcal{B}^{I=0}(s) = -\sqrt{2/3}\,\mathcal{B}(s)$

$\mathcal{B}^{I=2}(s) = -\sqrt{1/3}\,\mathcal{B}(s)$

$\Rightarrow$

$B^{+-}(s) = B(s)$

$B^{00}(s) = 0$

Isospin decomposition

$$\mathcal{F}_{\pi}^{+-}(s) = -\sqrt{\frac{2}{3}}\mathcal{F}_{\pi}^{I=0}(s) - \sqrt{\frac{1}{3}}\mathcal{F}_{\pi}^{I=2}(s)$$

$$\mathcal{F}_{\pi}^{00}(s) = -\sqrt{\frac{1}{3}}\mathcal{F}_{\pi}^{I=0}(s) + \sqrt{\frac{2}{3}}\mathcal{F}_{\pi}^{I=2}(s)$$

$$\mathcal{F}_{K}^{+-}(s) = -\sqrt{\frac{1}{2}}\mathcal{F}_{K}^{I=0}(s) - \sqrt{\frac{1}{2}}\mathcal{F}_{K}^{I=1}(s)$$

$$\mathcal{F}_{K}^{00}(s) = -\sqrt{\frac{1}{2}}\mathcal{F}_{K}^{I=0}(s) + \sqrt{\frac{1}{2}}\mathcal{F}_{K}^{I=1}(s)$$

$\gamma\gamma \longrightarrow \pi\pi$

$\gamma\gamma \longrightarrow \bar{K}K$

$$\mathcal{B}^{I=0}(s) = -\sqrt{2/3}\,\mathcal{B}(s)$$

$$\mathcal{B}^{I=2}(s) = -\sqrt{1/3}\,\mathcal{B}(s)$$

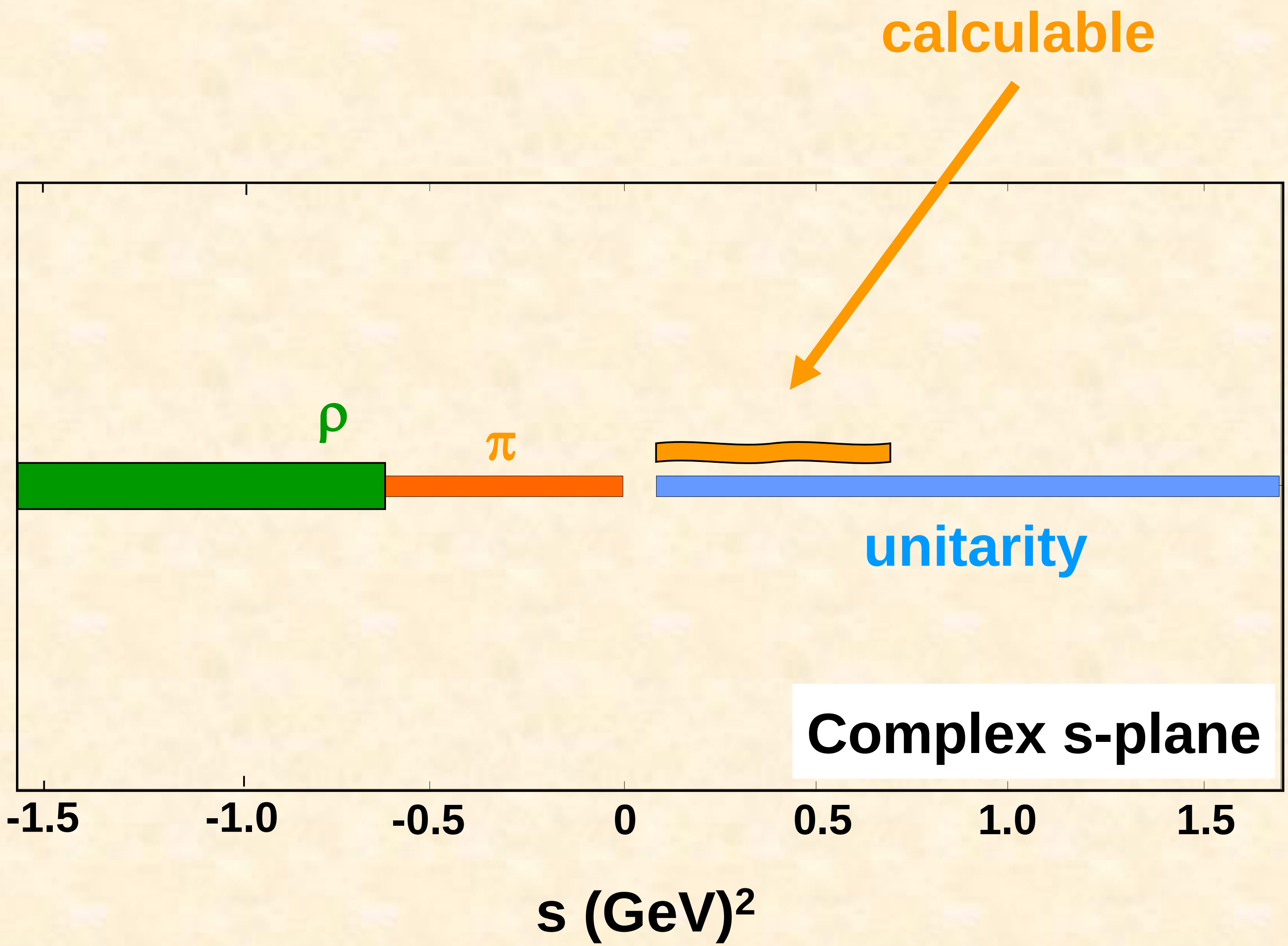
$\Rightarrow$

$$B^{+-}(s) = B(s)$$

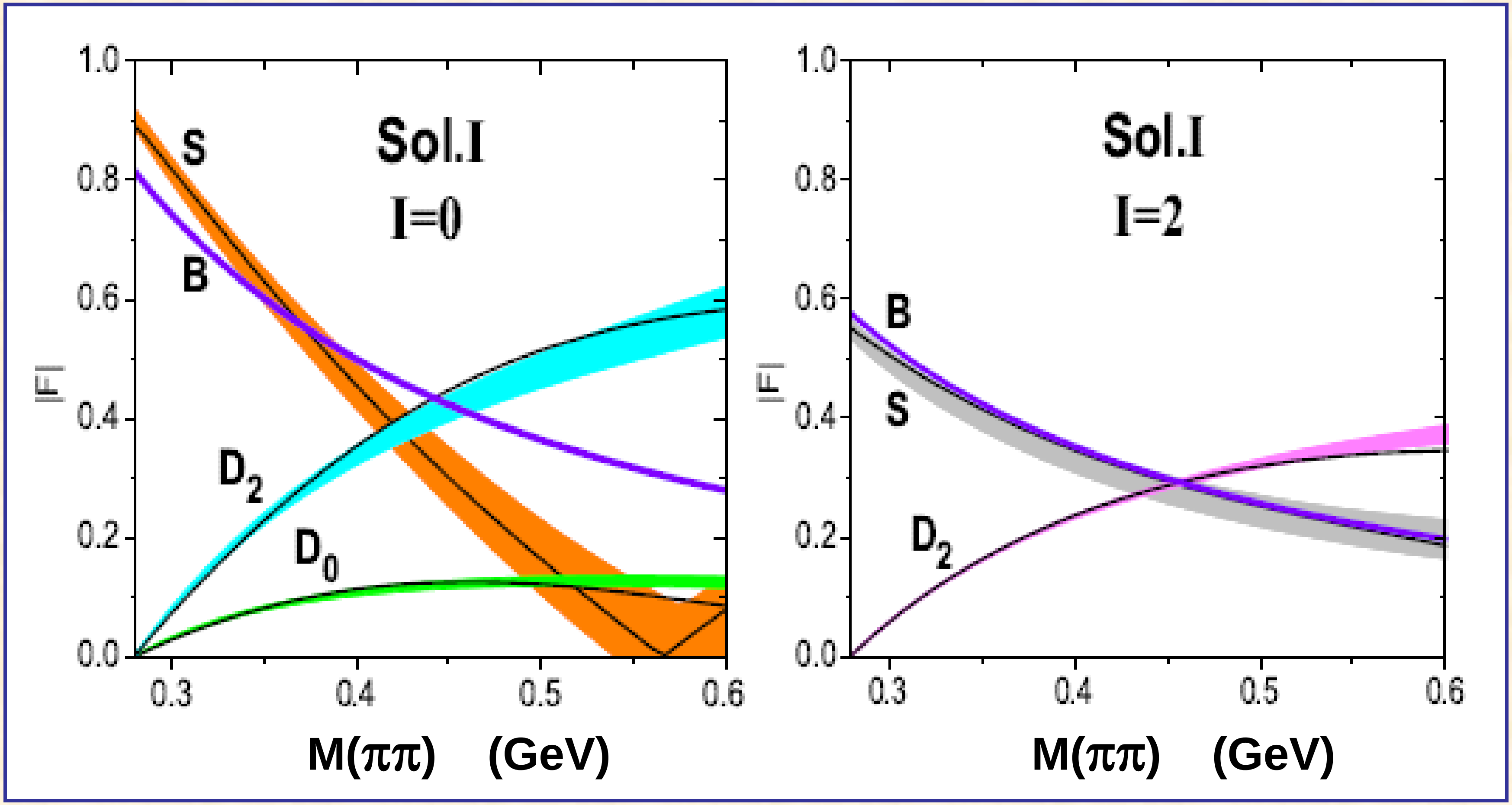
$$B^{00}(s) = 0$$

$\gamma\gamma \longrightarrow \pi\pi$

$\mathcal{F}(s)$ for each **I, J, λ**



Dispersive calculation of low energy partial waves



Unusual feature: large D-waves near threshold, I=2 as large as I=0

$$\mathcal{F}(s) \equiv \mathcal{H}(\mathbf{s}) = \mathcal{B}(\mathbf{s}) + \mathcal{L}(\mathbf{s})$$

along left hand cut

For $J = \lambda = 0$, consider $(\mathcal{F}(s) - \mathcal{B}(s))\,\Omega^{-1}(s)$ with $I = 0,2$

$$\mathcal{F}^I_{00}(s) = \mathcal{B}^I_{00}(s) + b^I s\, \Omega^I_{00}(s) + \frac{s^2\, \Omega^I_{00}(s)}{\pi} \int_L ds' \frac{\mathrm{Im}\left[\mathcal{L}^I_{00}(s')\right]\, \Omega^I_{00}(s')^{-1}}{s'^2(s'-s)}$$

$$- \frac{s^2\, \Omega^I_{00}(s)}{\pi} \int_R ds' \frac{\mathcal{B}^I_{00}(s')\, \mathrm{Im}\left[\Omega^I_{00}(s')^{-1}\right]}{s'^2(s'-s)}$$

with subtraction constants b^I

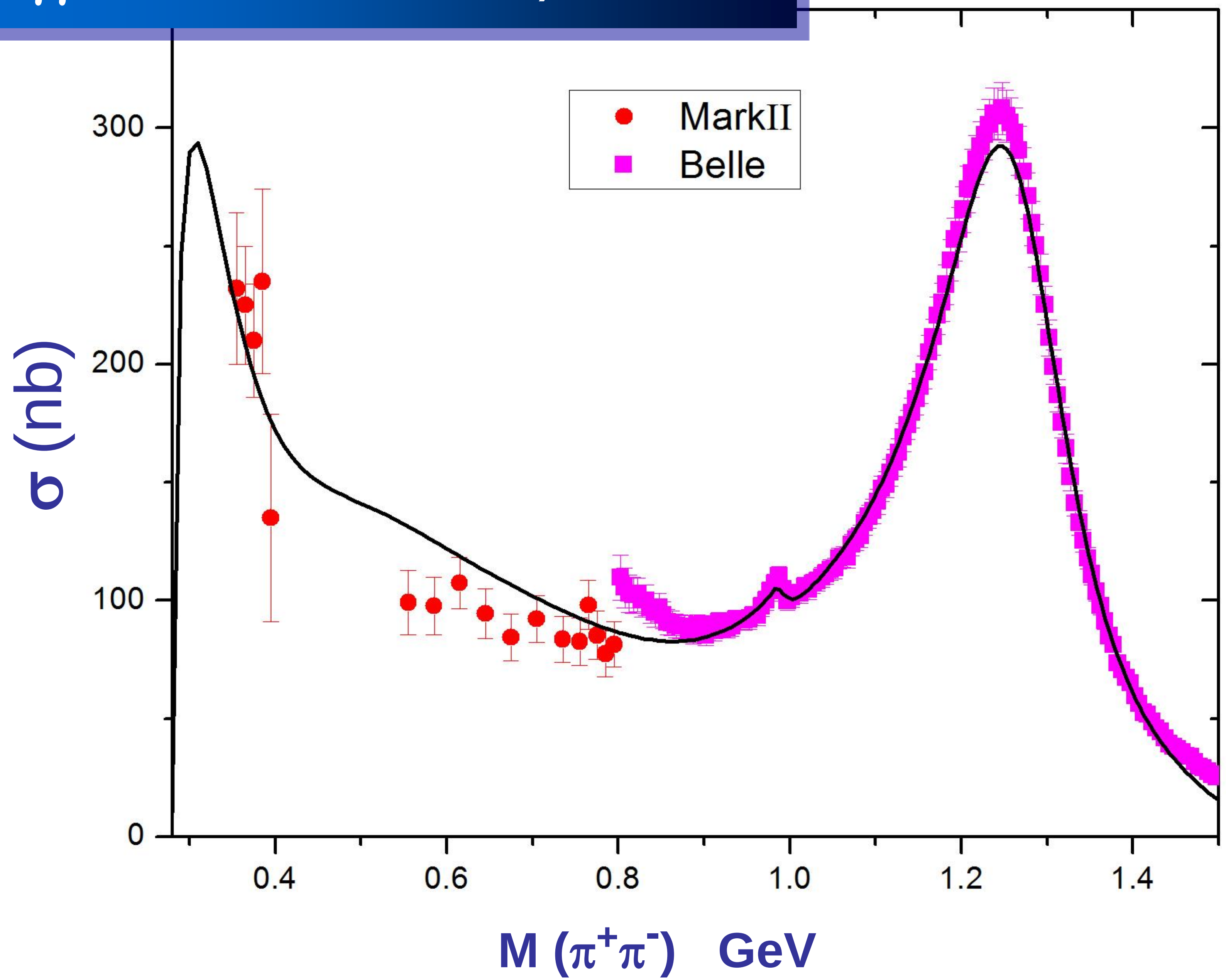
Consider $\left(\mathcal{F}(s) - \mathcal{B}(s) \right) \Omega^{-1}(s) \,/\, s^n (s - 4m_\pi^2)^{J/2}$

with $n = 2 - \lambda/2$, and $J > 0$, $\lambda = 0,2$, $I = 0,2$

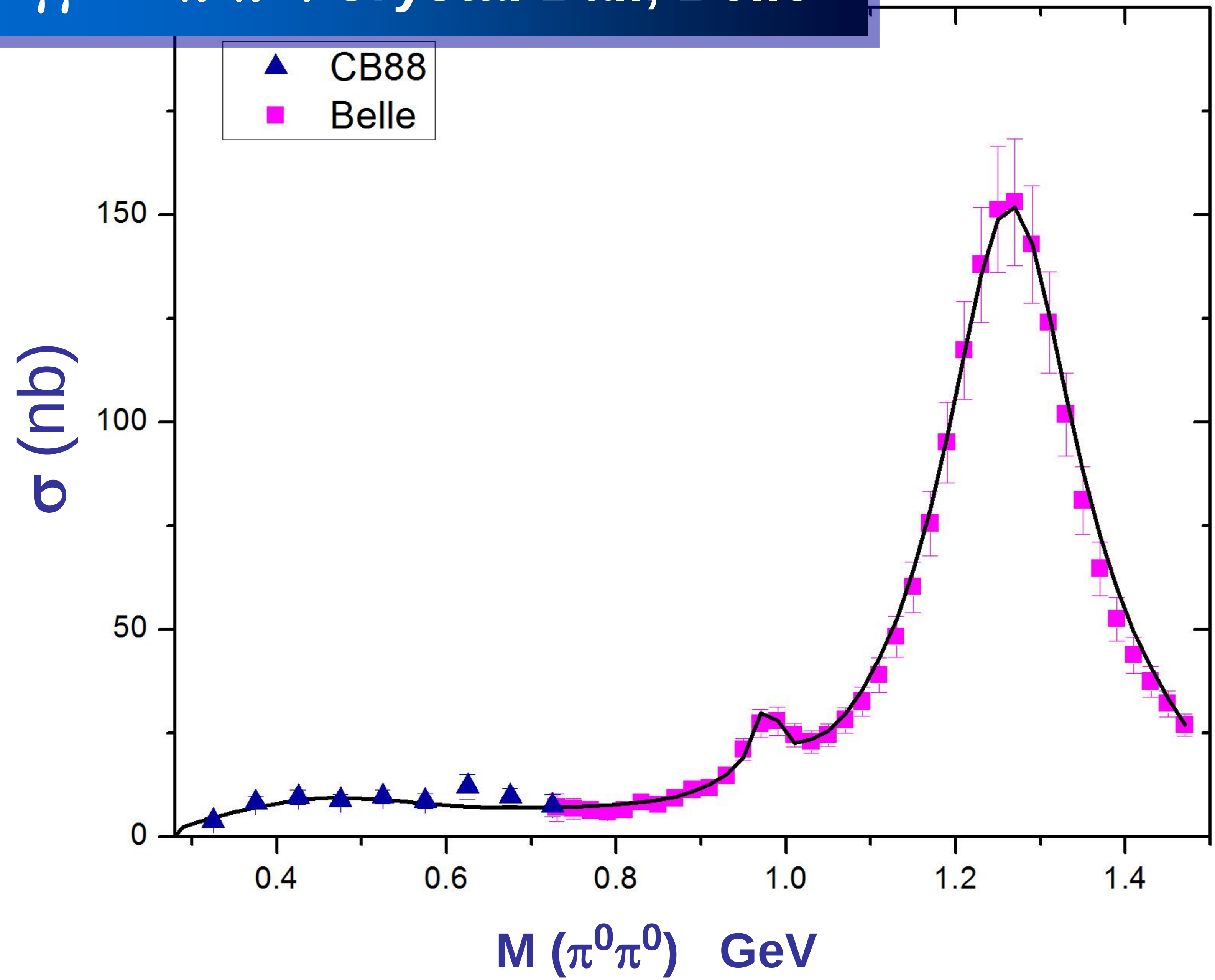
$$\mathcal{F}^I_{J\lambda}(s) \; = \mathcal{B}^I_{J\lambda}(s) + \frac{s^n(s-4m_\pi^2)^{J/2}}{\pi}\,\Omega^I_{J\lambda}(s) \int_L ds' \frac{\mathrm{Im}\left[\mathcal{L}^I_{J\lambda}(s')\right]\, \Omega^I_{J\lambda}(s')^{-1}}{s'^n(s'-4m_\pi^2)^{J/2}(s'-s)}$$

$$- \frac{s^n(s-4m_\pi^2)^{J/2}}{\pi}\,\Omega^I_{J\lambda}(s) \int_R ds' \frac{B^I_{J\lambda}(s')\,\mathrm{Im}\left[\Omega^I_{J\lambda}(s')^{-1}\right]}{s'^m(s'-4m_\pi^2)^{J/2}(s'-s)}$$

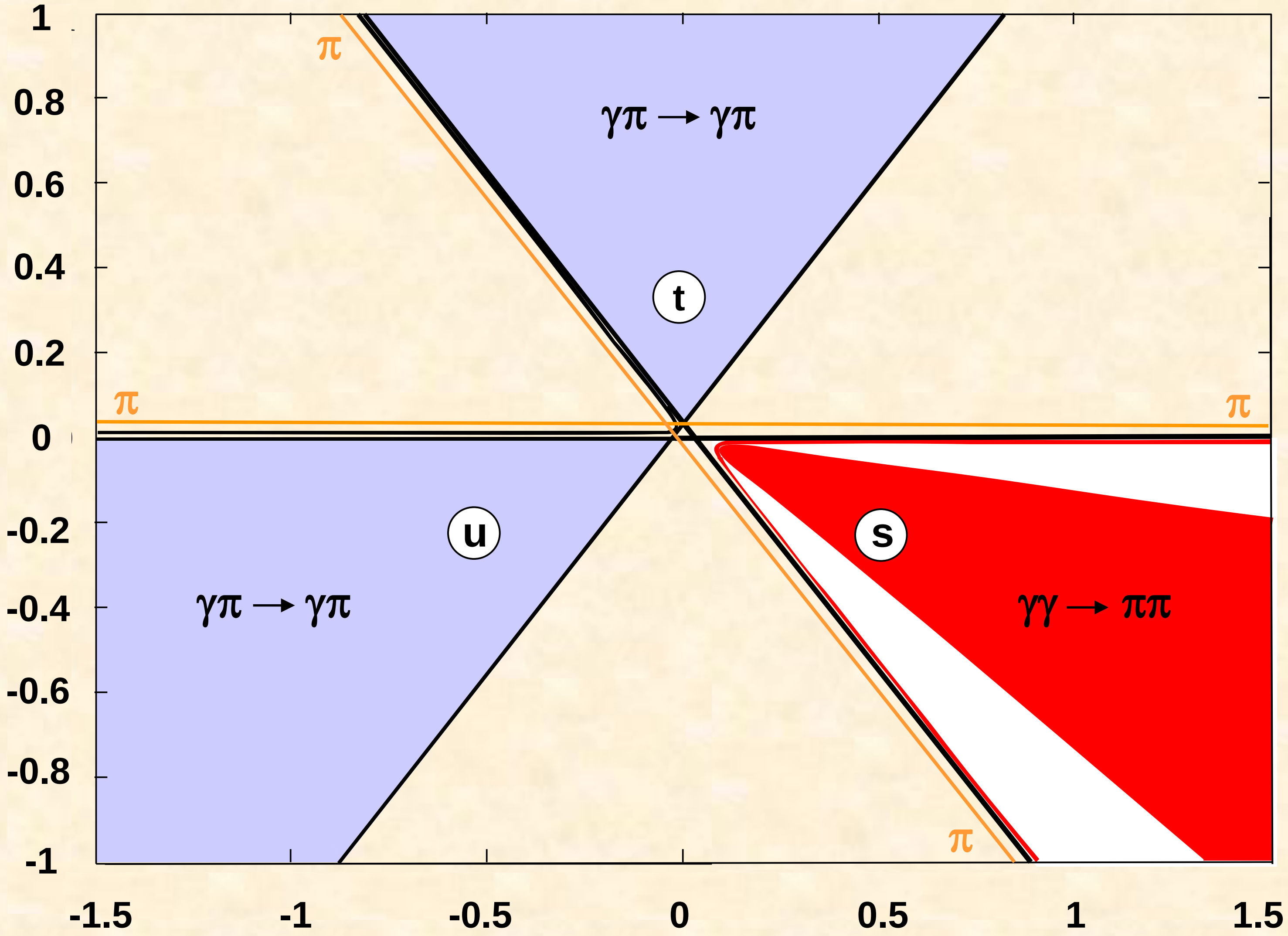
γγ → π⁺π⁻ : Mark II, Belle



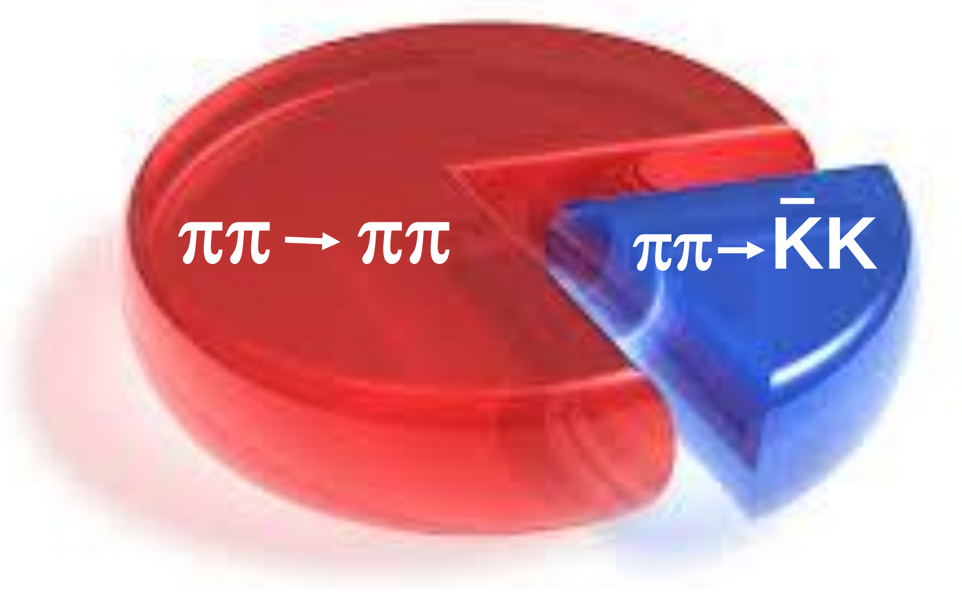
$\gamma\gamma \rightarrow \pi^0\pi^0$: Crystal Ball, Belle



Mandelstam Plane

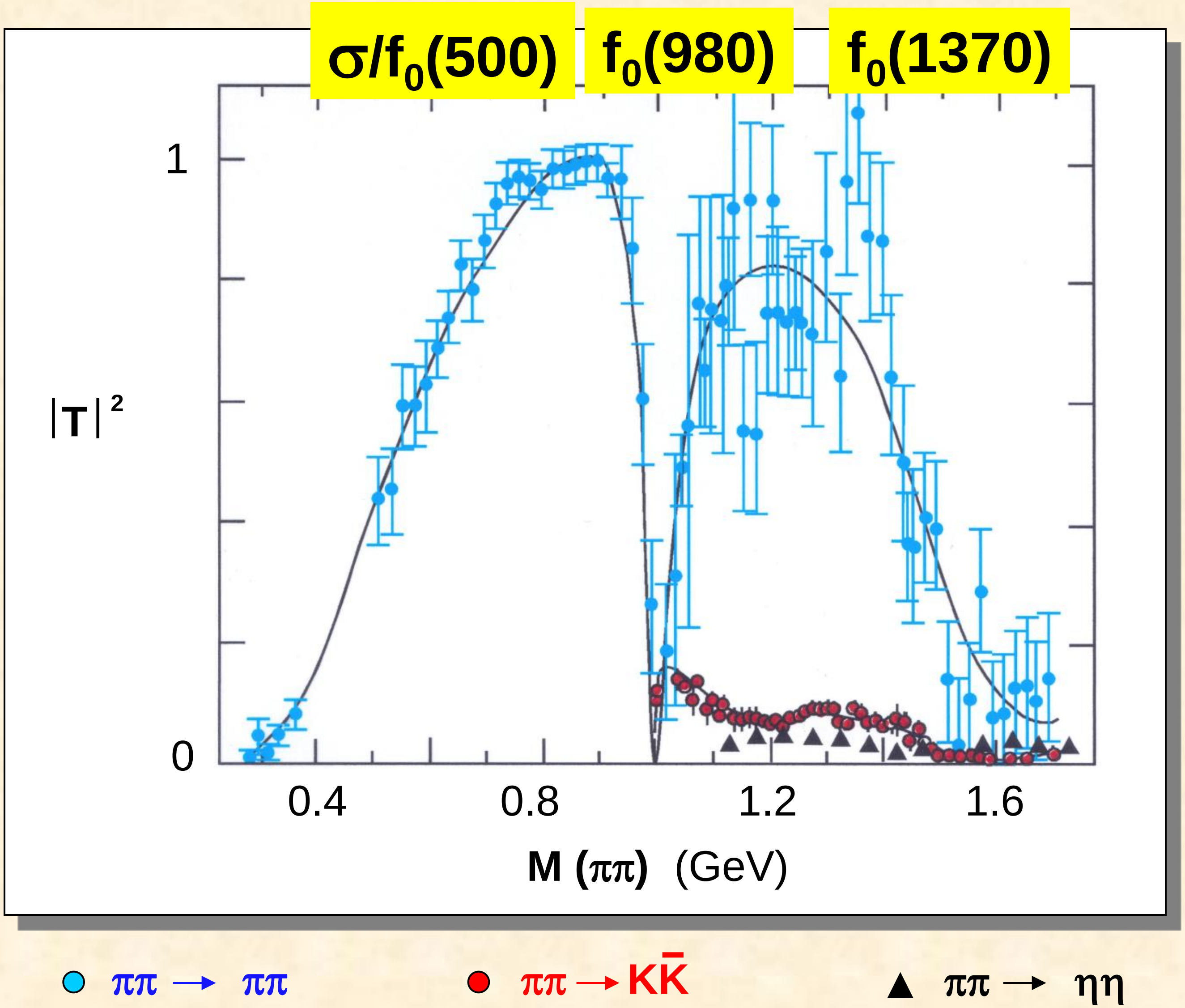


Conservation of probability



Sum of probabilities $= \sum_i P_i = 1$

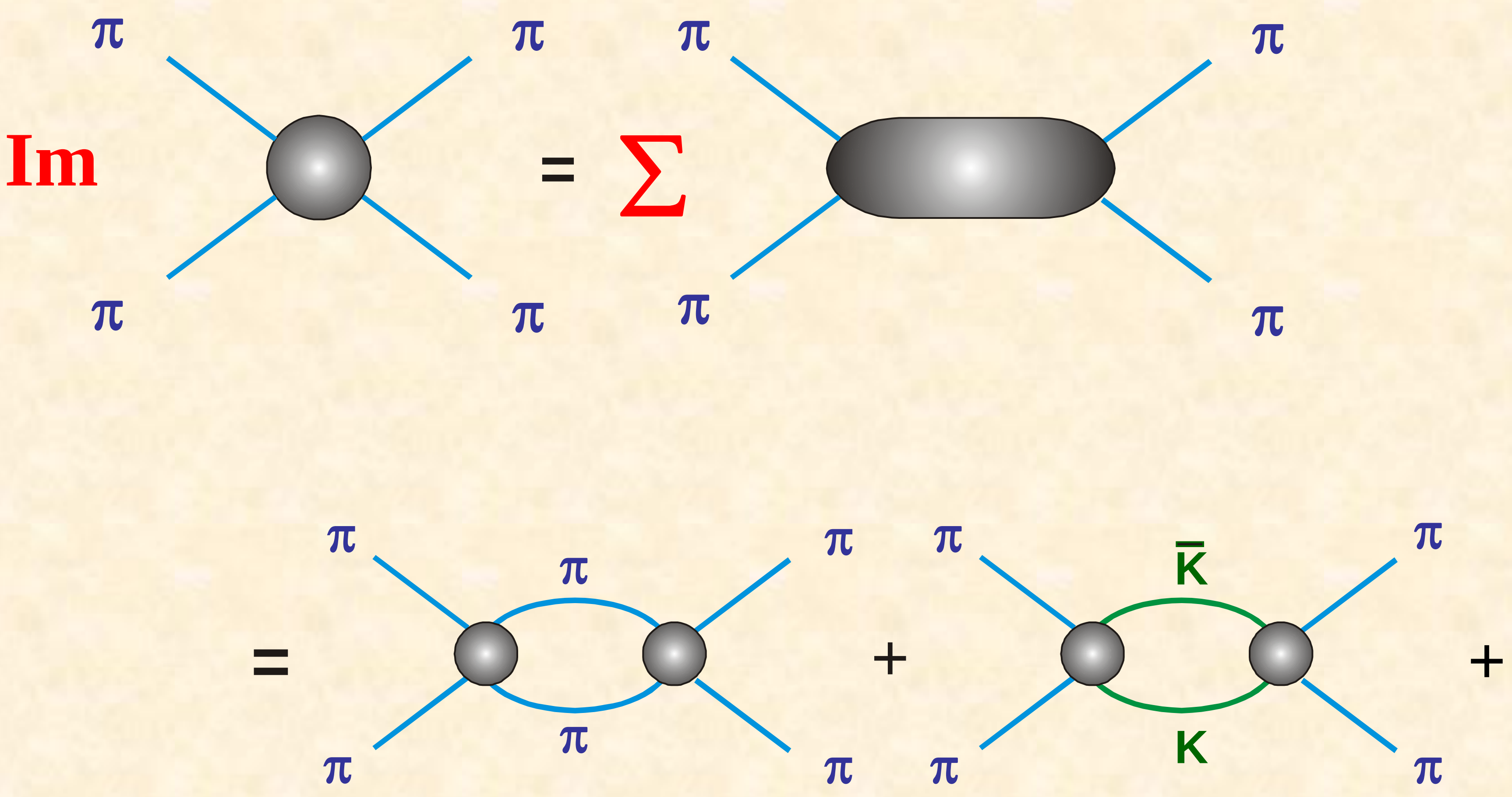
I = J = 0



Unitarity

definite J^{PC}

$\rho_i = k_i / E$



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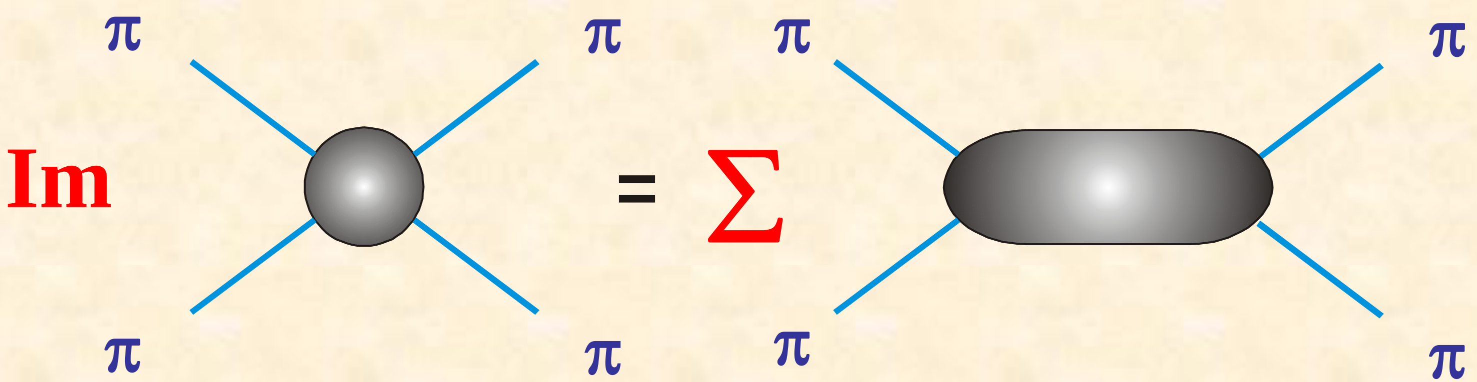
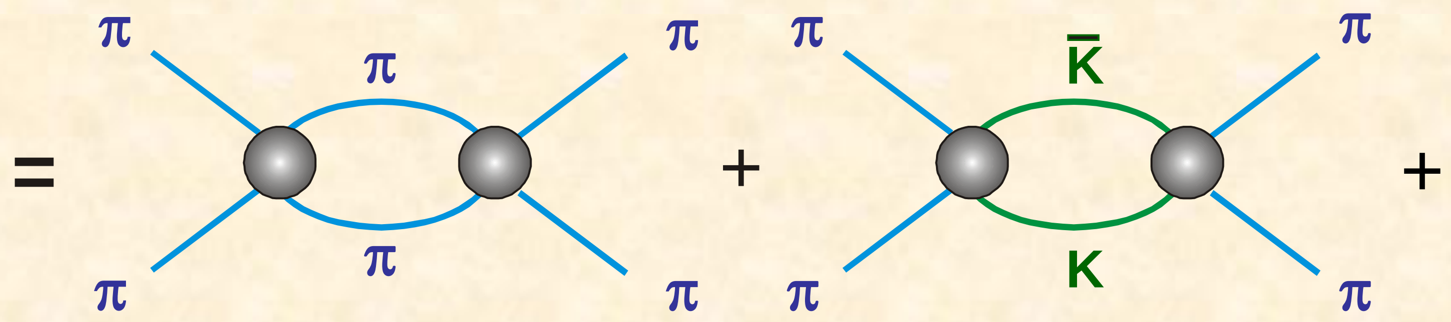


$$\mathrm{Im}\, \mathbf{T_{ij}} \;=\; \rho_1\, \mathbf{T^*_{i1}}\, \mathbf{T_{1j}} + \rho_2\, \mathbf{T^*_{i2}}\, \mathbf{T_{2j}} + \cdots$$



Elastic Unitarity

$$\text{Im } T_{11}(s) = \rho_1(s) \; T_{11}^*(s) \, T_{11}(s)$$

$$\frac{1}{T_{11}} = \frac{T_{11}^*}{|T_{11}|^2} \qquad \text{Im}\frac{1}{T_{11}} = \text{Im}\frac{T_{11}^*}{|T_{11}|^2} = -\rho_1$$

Elastic Unitarity

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$$\mathrm{Re}\frac{1}{T_{11}} = \frac{1}{K_{11}}$$

where **K** is real for real $s > 4m^2$

Elastic Unitarity

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$$\frac{1}{T_{11}} = \frac{T_{11}^*}{|\,T_{11}\,|^2}$$

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where K is real for real $s > 4m^2$

$$\frac{1}{T_{11}} = \frac{1}{K_{11}} - i\,\rho_1$$

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$$\frac{1}{T_{11}} = \frac{1}{K_{11}} - i\,\rho_1$$

$$T_{11} = \frac{K_{11}}{1 - i\,\rho_1 K_{11}}$$

Elastic Unitarity

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$$T_{11} = \frac{K_{11}}{1 - i\,\rho_1 K_{11}}$$

$$K_{11} = \frac{1}{\rho_1}\tan\delta$$

Elastic Unitarity

$$T_{11} = \frac{K_{11}}{1 - i\,\rho_1 K_{11}}$$

$$K_{11} = \frac{1}{\rho_1}\tan\delta$$

Physical particles are poles of the **S**-Matrix
on nearby unphysical sheet (s).

These are given by the zeros of

$$1 - i\,\rho_1 K_{11}$$

Elastic Unitarity

$$T_{11} = \frac{K_{11}}{1 - i\,\rho_1 K_{11}}$$

$$K_{11} = \frac{1}{\rho_1} \tan\delta$$

Physical particles are poles of the S-Matrix on nearby unphysical sheet (s).

These are given by the zeros of $1 - i\,\rho_1 K_{11}$

The K-matrix has no physical significance (except in models).
It is just a convenient way to implement unitarity.
In particular poles in K are not poles of the S-matrix

Elastic Unitarity

$$T_{11} = \frac{K_{11}}{1 - i\,\rho_1 K_{11}}$$

$$K_{11} = \frac{1}{\rho_1} \tan \delta$$

Example :

$$K_{11} = \frac{M\,\Gamma}{M^2 - s}$$

$$\Rightarrow$$

$$T_{11} = \frac{M\,\Gamma}{M^2 - s - i\,\rho\, M\,\Gamma}$$

Breit-Wigner form

Elastic Unitarity

$$T_{11} = \frac{K_{11}}{1 - i\,\rho_1 K_{11}}$$

$$K_{11} = \frac{1}{\rho_1} \tan\delta$$

Example :

$$K_{11} = \frac{M\,\Gamma}{M^2 - s} \quad \Rightarrow \quad T_{11} = \frac{M\,\Gamma}{M^2 - s - i\,\rho\,M\,\Gamma}$$

Breit-Wigner form

The K-matrix has no physical significance (except in models).
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Elastic Unitarity & Analyticity

$$\mathrm{Im}\,\frac{1}{T_{11}} = \mathrm{Im}\,\frac{T_{11}^{*}}{|\,T_{11}\,|^{2}} = -\,\rho_1$$

$$\mathrm{Re}\,\frac{1}{T_{11}} = \frac{1}{K_{11}}$$

where K is real
for real $s > 4m^2$

$$\frac{1}{T_{11}} = \frac{1}{K_{11}} - i\,\rho_1$$

Elastic Unitarity & Analyticity

$$\mathrm{Im}\,\frac{1}{T_{11}} = \mathrm{Im}\,\frac{T_{11}^{*}}{|T_{11}|^2} = -\rho_1$$

$$\mathrm{Re}\,\frac{1}{T_{11}} = \frac{1}{K_{11}}$$

where K is real for real $s > 4m^2$

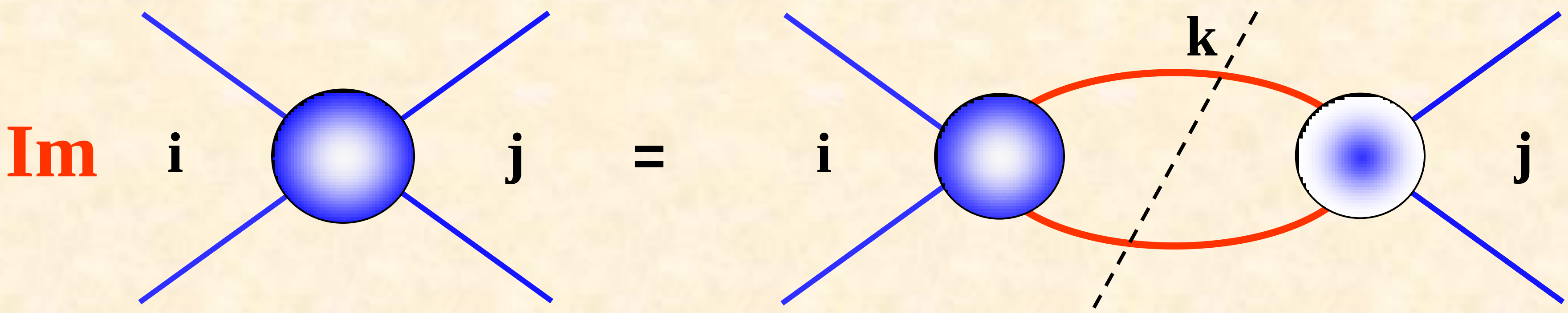
$$\frac{1}{T_{11}} = \frac{1}{K_{11}} - i\rho_1$$

$$\frac{1}{T_{11}} = \frac{1}{K_{11}} + \frac{\rho_1}{\pi}\, \ln\left(\frac{\rho_1+1}{\rho_1-1}\right)$$

corrects the analyticity on right hand cut ONLY

Multi-channel Unitarity

Amplitude with definite J^{PC}

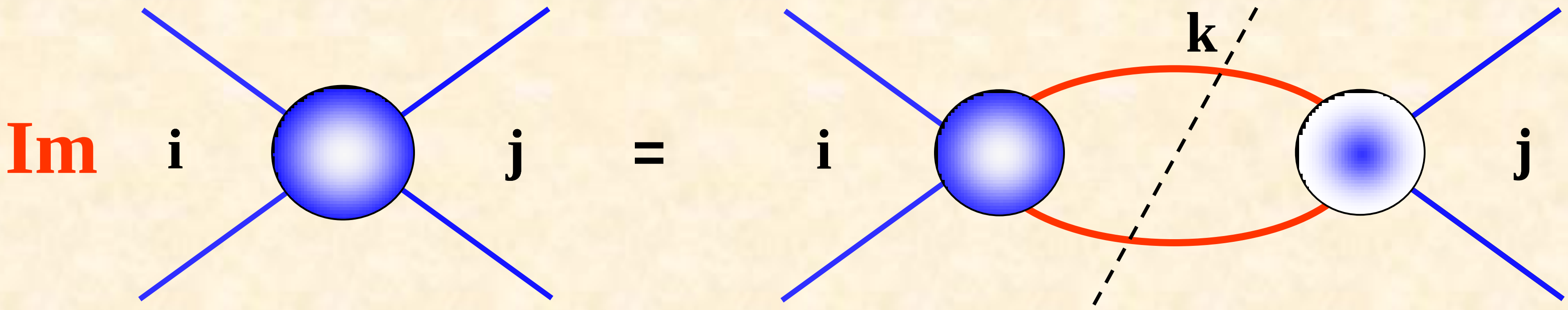


$1 = \pi\pi$
$2 = K\bar{K}$

$$\mathrm{Im}\ T_{ij}(s) = \sum_k \rho_k(s)\ T_{ik}^*(s)\ T_{kj}(s)$$

Multi-channel Unitarity

Amplitude with definite J^{PC}



1 = $\pi\pi$

2 = $K\bar{K}$

$$\text{Im } T_{ij}(s) = \sum_k \rho_k(s) \, T_{ik}^*(s) \, T_{kj}(s)$$

$$\text{Im } T_{11} = \rho_1 T_{11}^* T_{11} + \rho_2 T_{12}^* T_{21}$$

$$\text{Im } T_{12} = \rho_1 T_{11}^* T_{12} + \rho_2 T_{12}^* T_{22}$$

$$\text{Im } T_{22} = \rho_1 T_{21}^* T_{12} + \rho_2 T_{22}^* T_{22}$$

Multi-channel Unitarity

Amplitude with definite J^{PC}

$1 = \pi\pi$

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$$\mathrm{Im}\, T_{11} = \rho_1 T_{11}^{*} T_{11} + \rho_2 T_{12}^{*} T_{21}$$

$$\mathrm{Im}\, T_{12} = \rho_1 T_{11}^{*} T_{12} + \rho_2 T_{12}^{*} T_{22}$$

$$\mathrm{Im}\, T_{22} = \rho_1 T_{21}^{*} T_{12} + \rho_2 T_{22}^{*} T_{22}$$

1-channel : $\mathrm{Im} \dfrac{1}{T_{11}} = -\rho_1$

n-channel : $\mathrm{Im}\; T^{-1} = -\rho$

$$\rho = \begin{vmatrix} \rho_1 & & & \\ & \rho_2 & & \\ & & \rho_3 & \\ & & & \rho_4 \end{vmatrix}$$

Multi-channel Unitarity

Amplitude with definite J^{PC}

$1 = \pi\pi$

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$$\mathrm{Im}\, T_{11} = \rho_1 T_{11}^* T_{11} + \rho_2 T_{12}^* T_{21}$$

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1-channel : $\mathrm{Im}\, \frac{1}{T_{11}} = -\rho_1$

n-channel : $\mathrm{Im}\, T^{-1} = -\rho$

$$\mathrm{Re}\, T^{-1} = K^{-1}$$

$$T^{-1} = K^{-1} - i\rho$$

Multi-channel Unitarity

Amplitude with definite J^{PC}

$1 = \pi\pi$
$2 = K\bar{K}$

$$\mathrm{Im}\, T_{11} = \rho_1 T_{11}^* T_{11} + \rho_2 T_{12}^* T_{21}$$

$$\mathrm{Im}\, T_{12} = \rho_1 T_{11}^* T_{12} + \rho_2 T_{12}^* T_{22}$$

$$\mathrm{Im}\, T_{22} = \rho_1 T_{21}^* T_{12} + \rho_2 T_{22}^* T_{22}$$

2-channel :

$$T_{11} = \frac{K_{11} - i\,\rho_2 \det K}{\Delta}$$

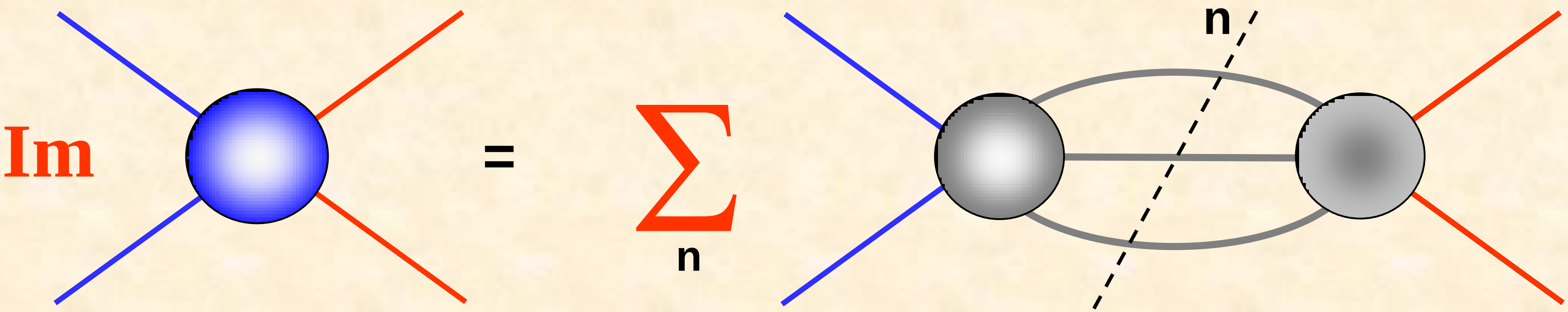
$$T_{12} = \frac{K_{12}}{\Delta}$$

$$T_{22} = \frac{K_{22} - i\,\rho_1 \det K}{\Delta}$$

where $\Delta = 1 - i\,\rho_1 K_{11} - i\,\rho_2 K_{22} - \rho_1 \rho_2 \det K$

Unitarity

Amplitude with definite J^{PC}

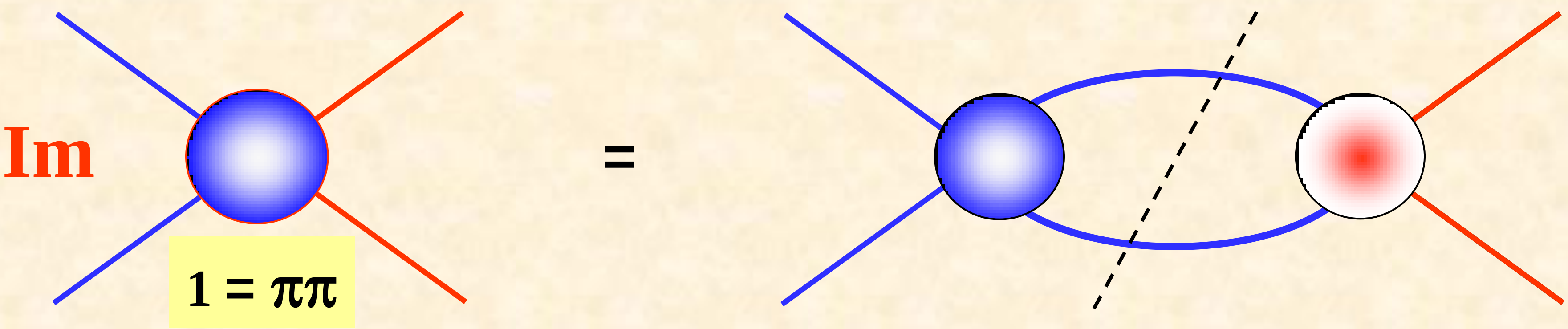


What happens when the final state particles are hadrons,
but the incoming particles are not hadrons, ***eg*** e^+e^- **,** $\gamma\gamma$

Sum **n** **only need include hadrons**
as initial state particles are suppressed by powers of α

Unitarity

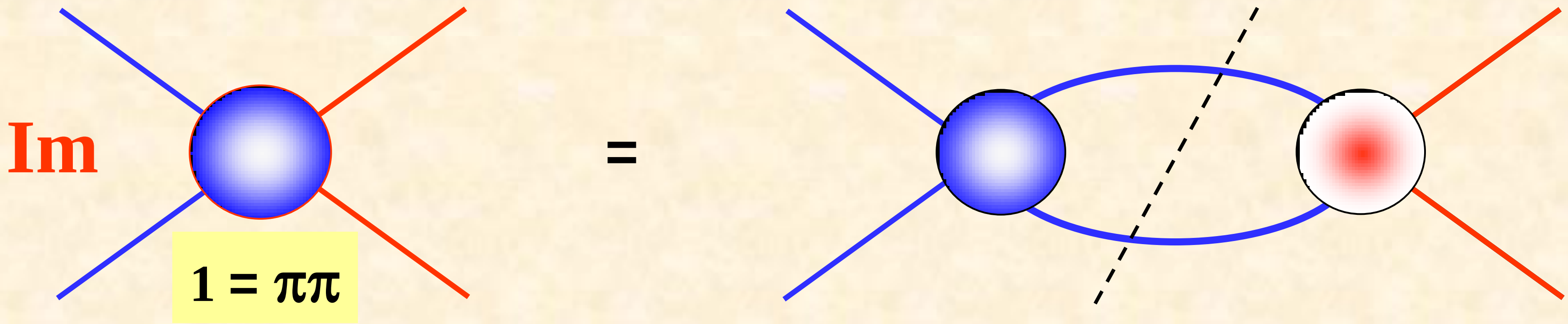
Amplitude with definite J^{PC}



$$\mathrm{Im}\,\mathcal{F}_1(s) = \rho_1(s)\,\mathcal{F}_1^*(s)\,T_{11}(s) = \rho_1(s)\,\mathcal{F}_1(s)\,T_{11}^*(s)$$

Unitarity

Amplitude with definite J^{PC}



$$\text{Im}\ \mathcal{F}_1(s) = \rho_1(s)\ \mathcal{F}_1^*(s)\ T_{11}(s) = \rho_1(s)\ \mathcal{F}_1(s)\ T_{11}^*(s)$$

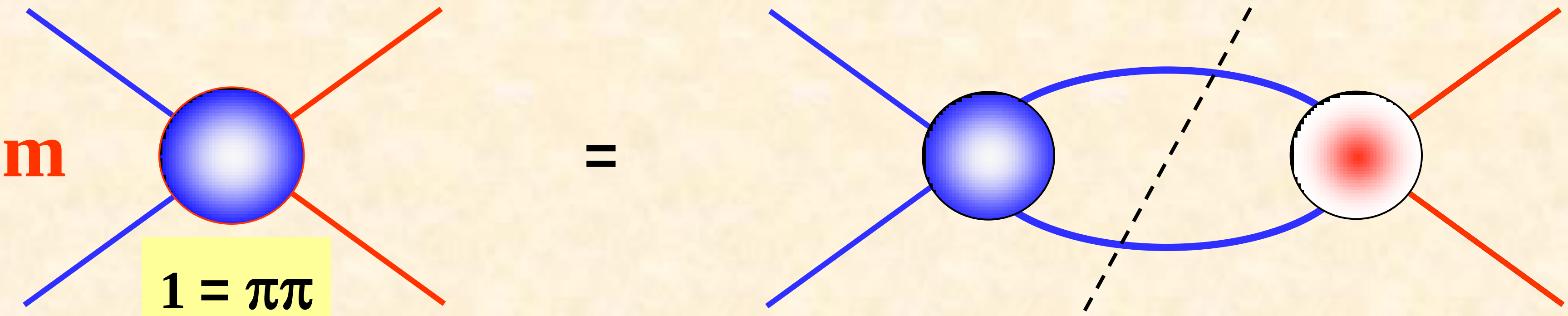
let $\mathcal{F}_1 = |\mathcal{F}_1|\ e^{i\varphi}$

recall $T_{11} = \dfrac{1}{\rho_1} \sin\delta\ e^{i\delta}$

Unitarity

Amplitude with definite J^{PC}

Im



$$\mathrm{Im}\, \mathcal{F}_1(s) = \rho_1(s)\, \mathcal{F}_1^*(s)\, T_{11}(s) = \rho_1(s)\, \mathcal{F}_1(s)\, T_{11}^*(s)$$

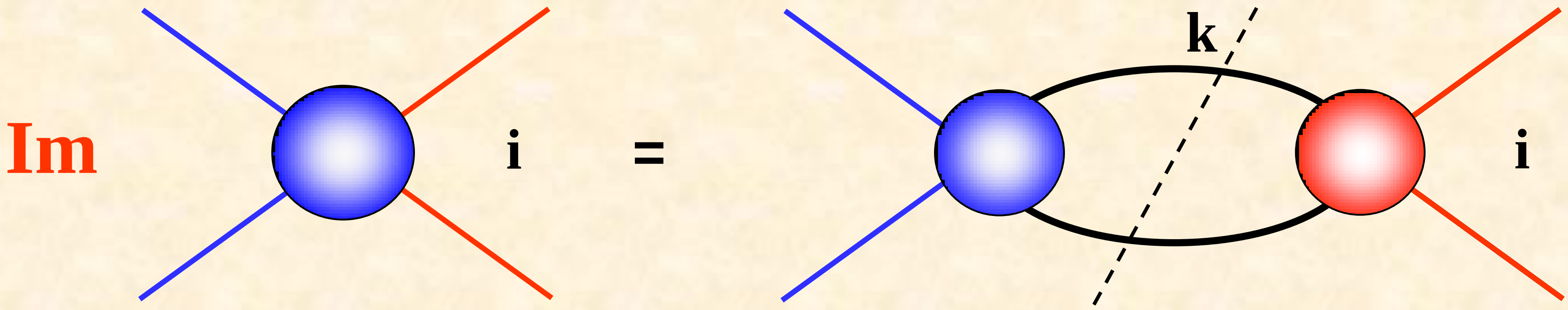
let $\mathcal{F}_1 = |\mathcal{F}_1|\, e^{i\varphi}$ recall $T_{11} = \frac{1}{\rho_1} \sin\delta\, e^{i\delta}$

⟹ $\sin\varphi = \sin\delta$

Watson's final state interaction theorem: $\varphi = \delta \;\; (+n\pi)$

Multi-channel Unitarity

Amplitude with definite J^{PC}



1 = ππ
2 = $K\bar{K}$

$$\text{Im}\ \mathcal{F}_i(s) = \sum_k \rho_k(s)\ \mathcal{F}_k^*(s)\, T_{ki}(s)$$

$$\text{Im}\ \mathcal{F}_1 = \rho_1 \mathcal{F}_1^* T_{11} + \rho_2 \mathcal{F}_2^* T_{21}$$

$$\text{Im}\ \mathcal{F}_2 = \rho_1 \mathcal{F}_1^* T_{12} + \rho_2 \mathcal{F}_2^* T_{22}$$

e.g. $\mathcal{F}(\gamma\gamma \longrightarrow$ hadron channel i)

Single-channel Unitarity

Amplitude with definite J^{PC}

$$\mathrm{Im}\, T_{11} = \rho_1\, T_{11}^{*}\, T_{11}$$

$$\mathrm{Im}\, \mathcal{F}_1 = \rho_1\, \mathcal{F}_1^{*}\, T_{11}$$

$$T_{11} = \frac{K_{11}}{1 - i\,\rho_1\, K_{11}}$$

$$\mathcal{F}_1 = \frac{P_1}{1 - i\,\rho_1\, K_{11}}$$

Like K_{11}, P_1 must be real for real $s > 4m^2$

Resonances are the poles in T_{11} and $\mathcal{F}_1$,
i.e. the common zeros of $1 - i\,\rho_1\, K_{11}$

Single-channel Unitarity

Amplitude with definite J^{PC}

$$\text{Im}\, T_{11} = \rho_1 T_{11}^{*} T_{11}$$

$$\text{Im}\, \mathcal{F}_1 = \rho_1 \mathcal{F}_1^{*} T_{11}$$

$$T_{11} = \frac{K_{11}}{1 - i\rho_1 K_{11}}$$

$$\mathcal{F}_1 = \frac{P_1}{1 - i\rho_1 K_{11}}$$

$$\mathcal{F}_1(s) = \alpha_1(s)\, T_{11}(s)$$

with $\alpha_1(s)$ real for real $s > 4m^2$

Multi-channel Unitarity

Amplitude with definite J^{PC}

$$\mathrm{Im}\,\mathcal{F}_1 = \rho_1 \mathcal{F}_1^* T_{11} + \rho_2 \mathcal{F}_2^* T_{21}$$

$$\mathrm{Im}\,\mathcal{F}_2 = \rho_1 \mathcal{F}_1^* T_{12} + \rho_2 \mathcal{F}_2^* T_{22}$$

1-channel :

$$T_{11} = \frac{K_{11}}{1 - i\,\rho_1 K_{11}} \qquad \mathcal{F}_1 = \frac{P_1}{1 - i\,\rho_1 K_{11}}$$

2-channel :

$$T_{11} = \frac{K_{11} - i\,\rho_2 \det K}{\Delta} \qquad \mathcal{F}_1 = \frac{P_1 - i\,\rho_2 Q_1 \det K}{\Delta}$$

etc.

with P_i and Q_i real for real $s > 4m^2$

Multi-channel Unitarity

Amplitude with definite J^{PC}

$$\text{Im}\,\mathcal{F}_1 = \rho_1 \mathcal{F}_1^* T_{11} + \rho_2 \mathcal{F}_2^* T_{21}$$

$$\text{Im}\,\mathcal{F}_2 = \rho_1 \mathcal{F}_1^* T_{12} + \rho_2 \mathcal{F}_2^* T_{22}$$

1-channel :

$$T_{11} = \frac{K_{11}}{1 - i\,\rho_1 K_{11}} \qquad \mathcal{F}_1 = \frac{P_1}{1 - i\,\rho_1 K_{11}}$$

2-channel :

$$T_{11} = \frac{K_{11} - i\,\rho_2 \det K}{\Delta} \qquad \mathcal{F}_1 = \frac{P_1 - i\,\rho_2 Q_1 \det K}{\Delta}$$

etc.

Solution : $P_1 = K_{11} Q_1 + K_{12} Q_2\,, \quad P_2 = K_{12} Q_1 + K_{22} Q_2$

Multi-channel Unitarity

Amplitude with definite J^{PC}

$$\mathrm{Im}\,\mathcal{F}_1 = \rho_1 \mathcal{F}_1^* T_{11} + \rho_2 \mathcal{F}_2^* T_{21}$$

$$\mathrm{Im}\,\mathcal{F}_2 = \rho_1 \mathcal{F}_1^* T_{12} + \rho_2 \mathcal{F}_2^* T_{22}$$

2-channel :

$$\mathcal{F}_1 = \frac{Q_1 \left[K_{11} - i\,\rho_2 \det K \right] + Q_2 K_{12}}{\Delta}$$

$$\mathcal{F}_2 = \frac{Q_1 K_{12} + Q_2 \left[K_{22} - i\,\rho_1 \det K \right]}{\Delta}$$

where $\Delta = 1 - i\,\rho_1 K_{11} - i\,\rho_2 K_{22} - \rho_1 \rho_2 \det K$

Multi-channel Unitarity

Amplitude with definite J^{PC}

2-channel :

$$\mathcal{F}_1 = \frac{Q_1 \left[K_{11} - i\,\rho_2 \det K \right] + Q_2 K_{12}}{\Delta}$$

$$\mathcal{F}_2 = \frac{Q_1 K_{12} + Q_2 \left[K_{22} - i\,\rho_1 \det K \right]}{\Delta}$$

2-channel :

$$T_{11} = \frac{K_{11} - i\,\rho_2 \det K}{\Delta}$$

$$T_{12} = \frac{K_{12}}{\Delta}$$

$$T_{22} = \frac{K_{22} - i\,\rho_1 \det K}{\Delta}$$

Multi-channel Unitarity

Amplitude with definite J^{PC}

2-channel :

$$\mathcal{F}_1 = Q_1 T_{11} + Q_2 T_{21}$$

$$\mathcal{F}_2 = Q_1 T_{12} + Q_2 T_{22}$$

$Q_n \rightarrow \alpha_n$

2-channel :

$$T_{11} = \frac{K_{11} - i\,\rho_2 \det K}{\Delta}$$

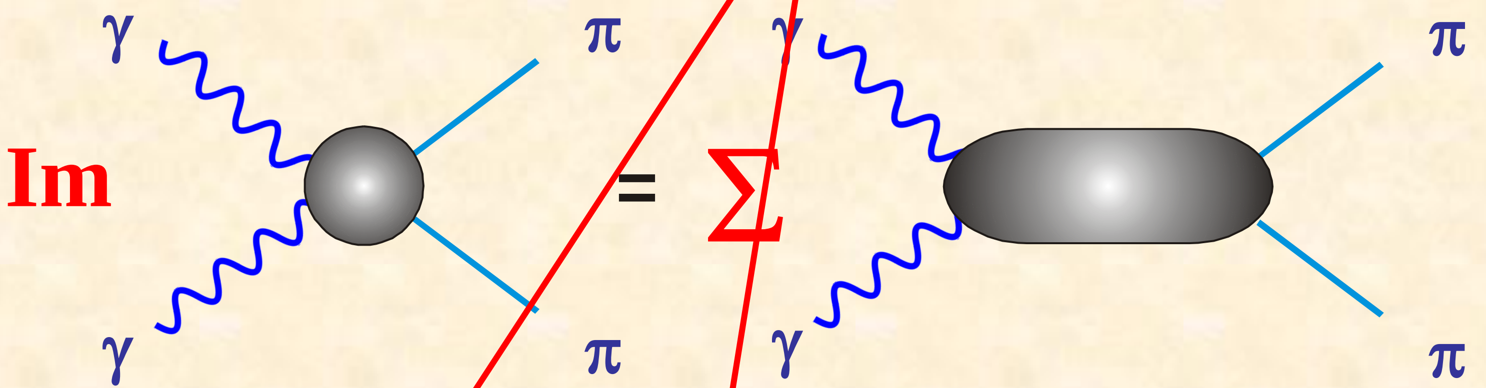
$$T_{12} = \frac{K_{12}}{\Delta}$$

$$T_{22} = \frac{K_{22} - i\,\rho_1 \det K}{\Delta}$$

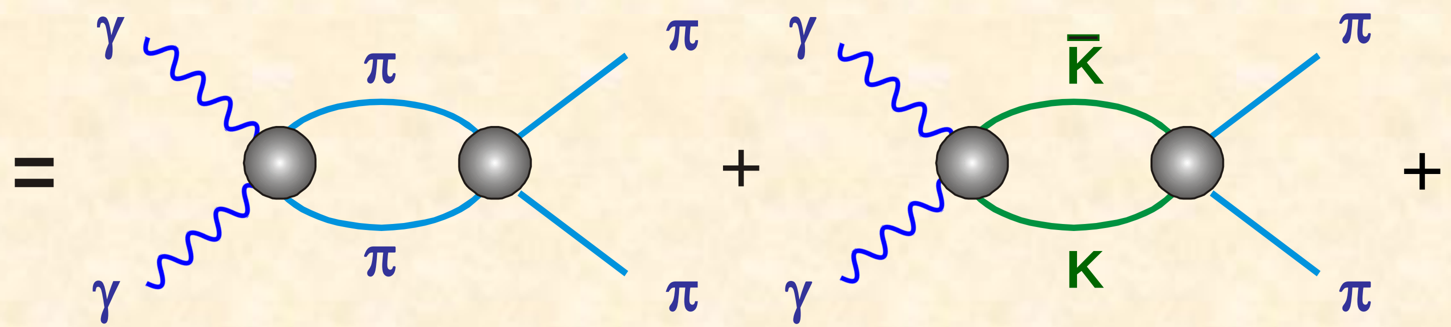
Multi-channel Unitarity

definite J^{PC}

real *coupling* functions



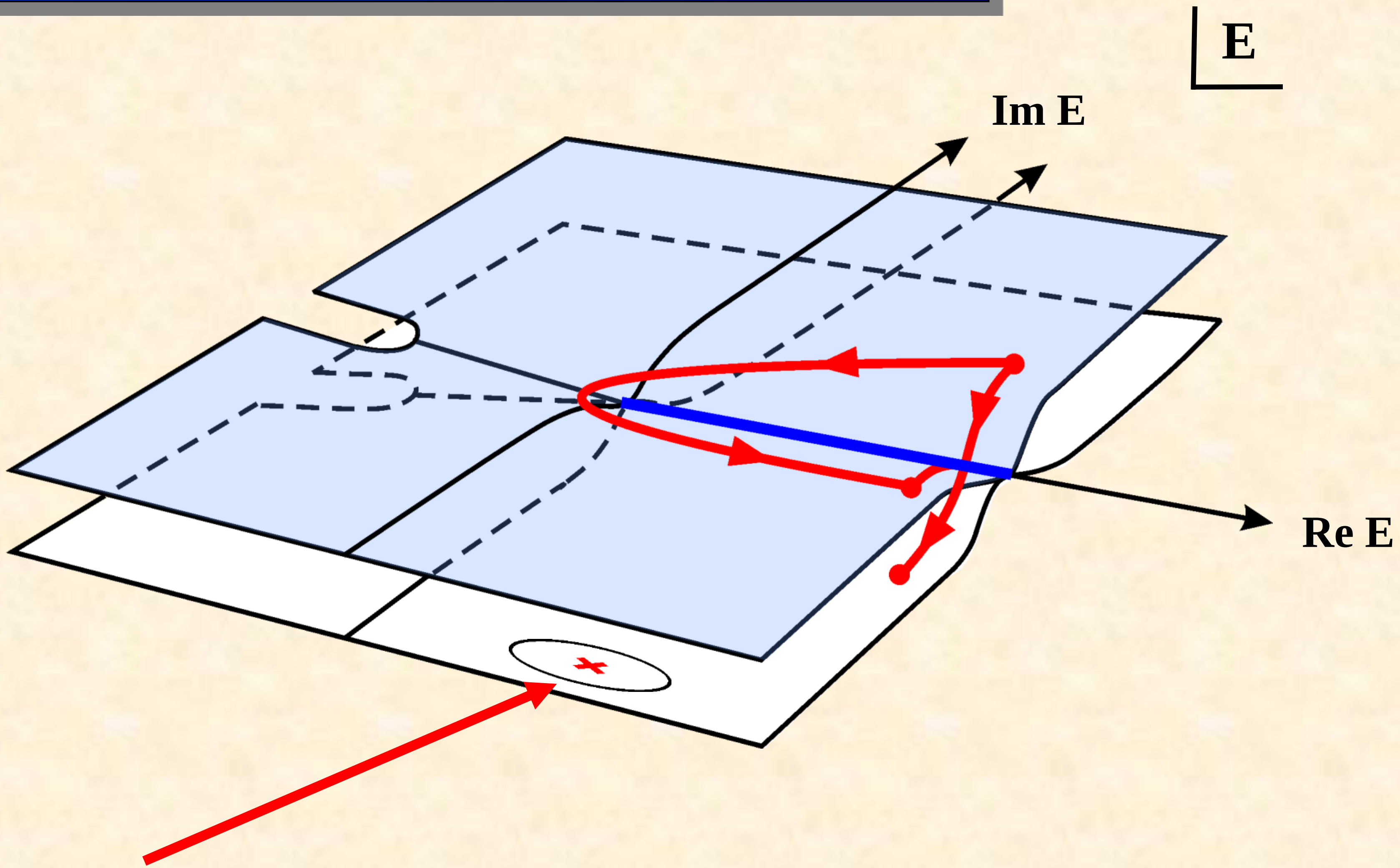
$$\mathcal{F}_j = \alpha_1 \mathbf{T}_{1j} + \alpha_2 \mathbf{T}_{2j} + \cdots$$ on right hand cut



$1 = \pi\pi$

$2 = K\bar{K}$

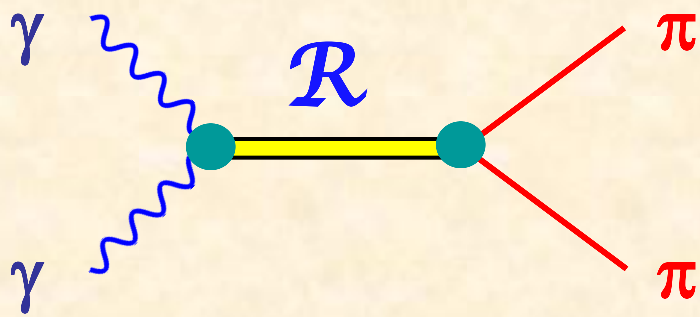
analyticity & complex energy plane



resonance pole

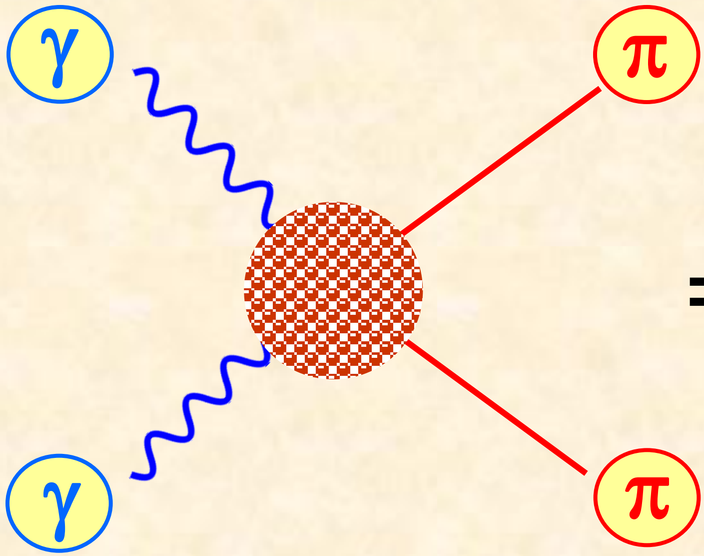
Universal: process independent

Amplitude Analysis



resonances have definite quantum numbers I, J, P (C)

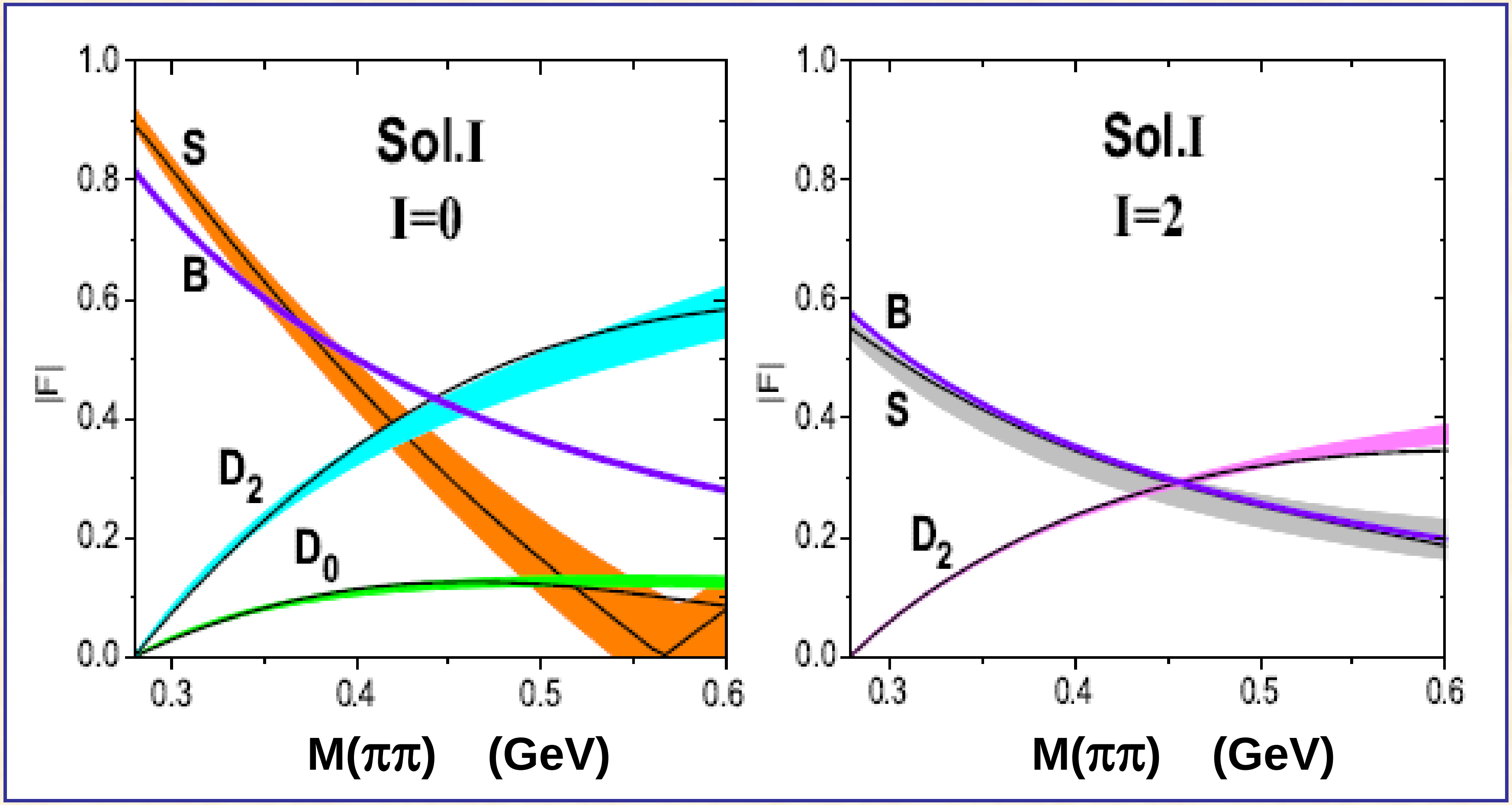
resonances and backgrounds not separable within unitarity



$$= \sum_{J,\lambda} \boxed{\mathcal{F}_{J\lambda}(s)}\, Y_{J\lambda}(\vartheta,\varphi)$$

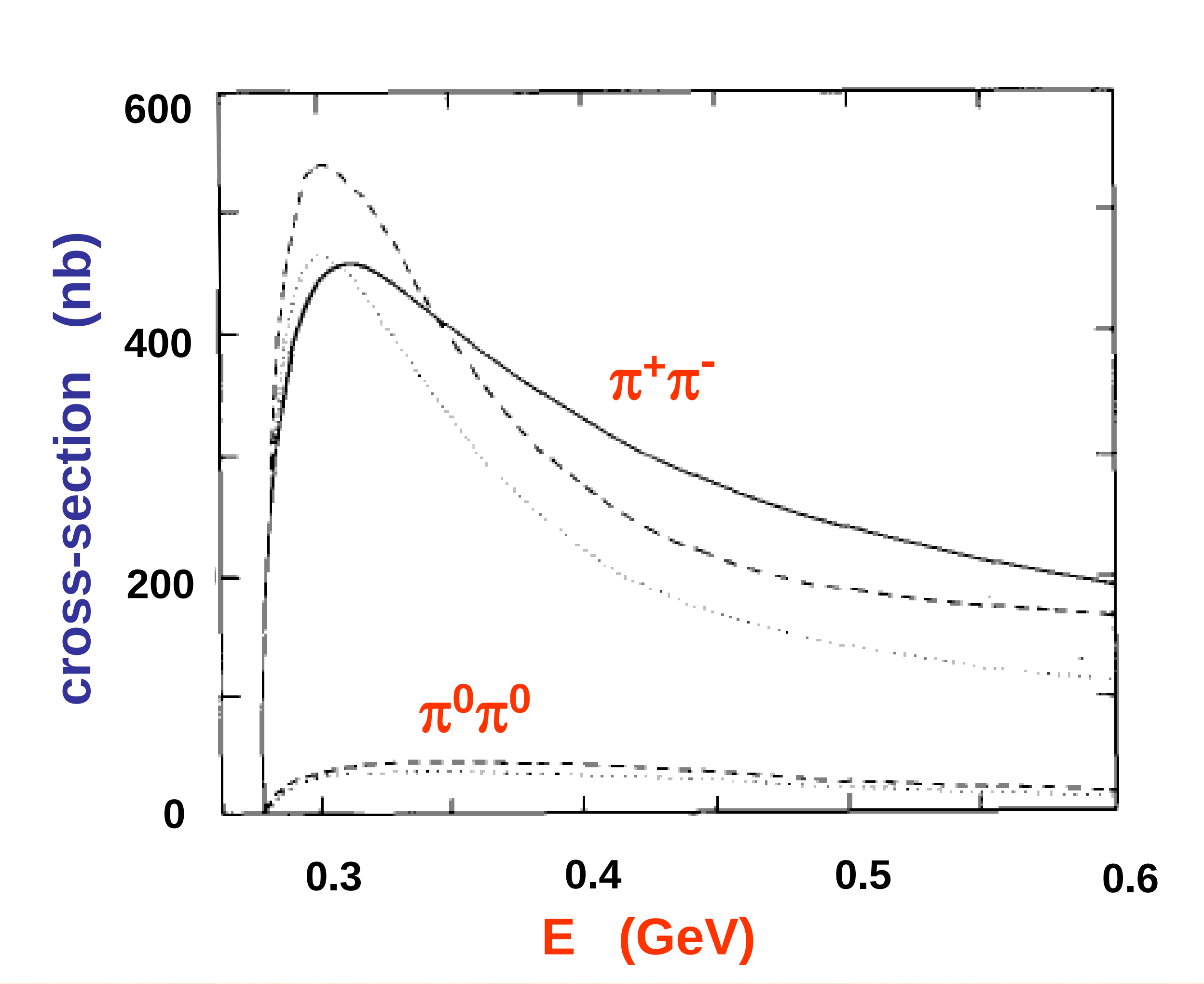
partial wave amplitudes

Dispersive calculation of low energy partial waves



Unusual feature: large D-waves near threshold, I=2 as large as I=0

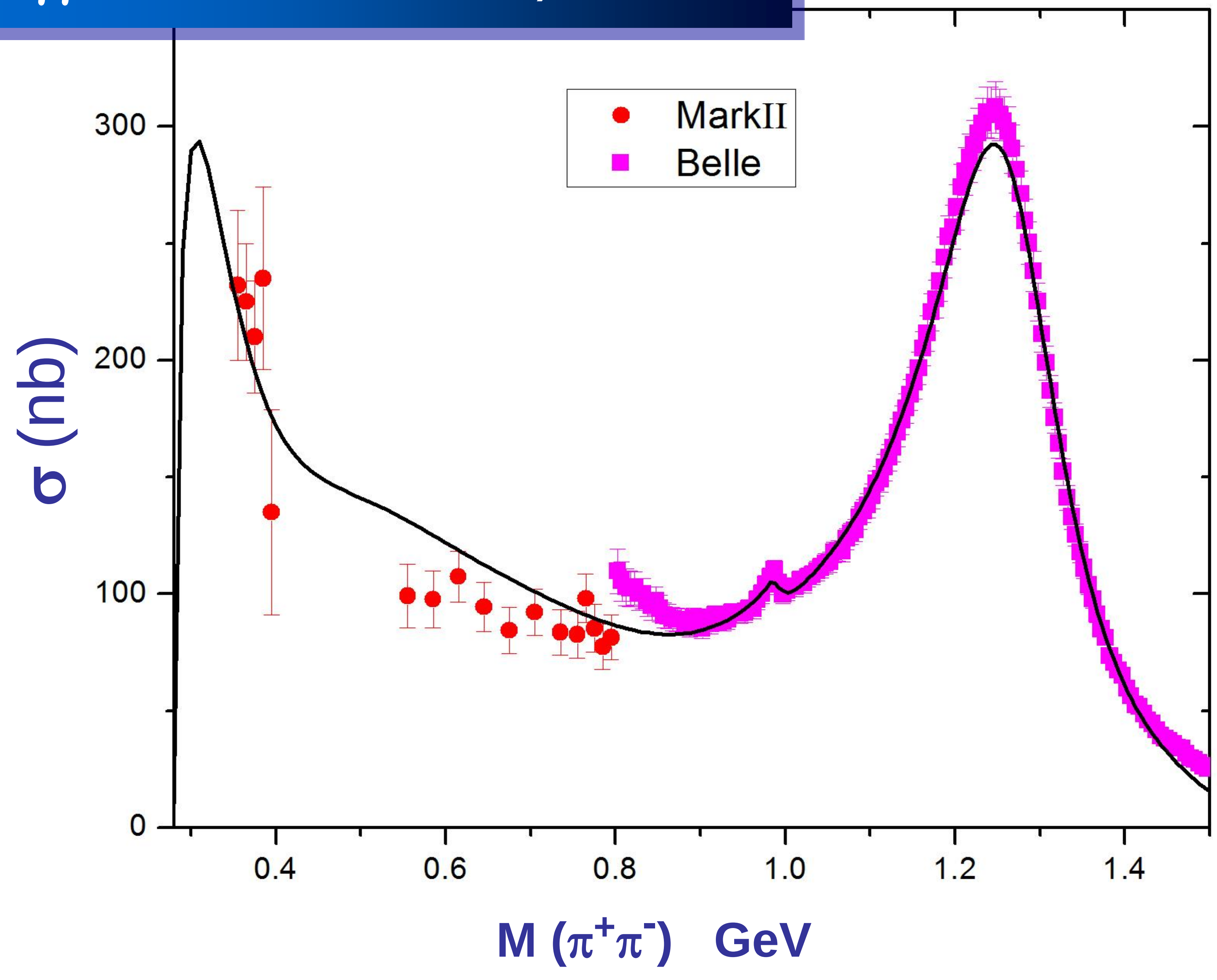
Born amplitude modified by final state interactions



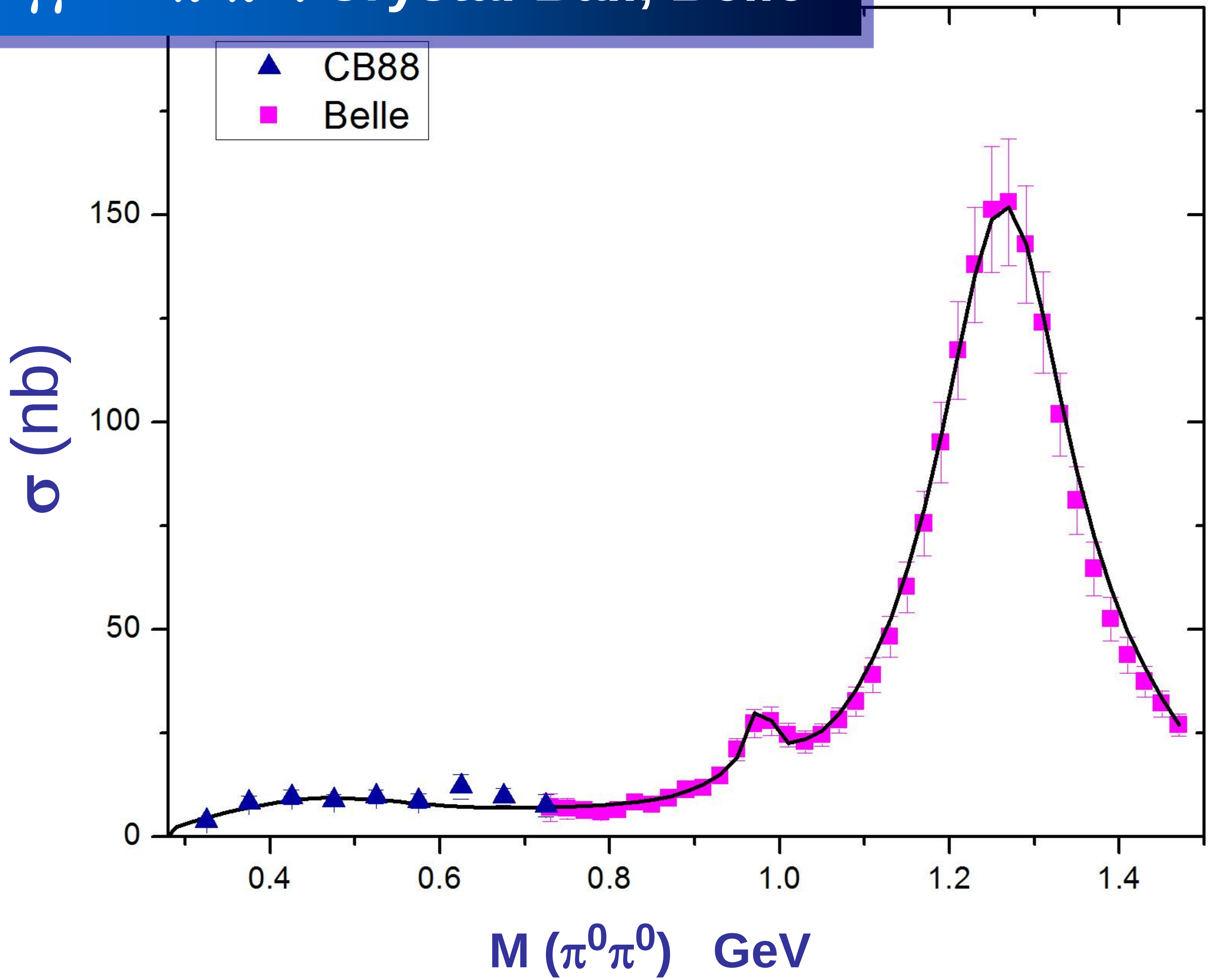
Datasets

Experiment	Process	Int. X-sect.	$\lvert\cos\theta\rvert_{max}$	Ang. distrib.	$\lvert\cos\theta\rvert_{max}$
Mark II	$\gamma\gamma \to \pi^+\pi^-$	81	0.6	63	0.6
Crystal Ball	$\gamma\gamma \to \pi^0\pi^0$	36	0.8 (CB88) 0.7 (CB92)	90	0.8
CELLO	$\gamma\gamma \to \pi^+\pi^-$	28	0.6	104 (Harjes) 201 (Behrend)	0.55 - 0.8
Belle	$\gamma\gamma \to \pi^+\pi^-$	128	0.6	1536	0.6
	$\gamma\gamma \to \pi^0\pi^0$	36	0.8	684	0.6 - 0.8

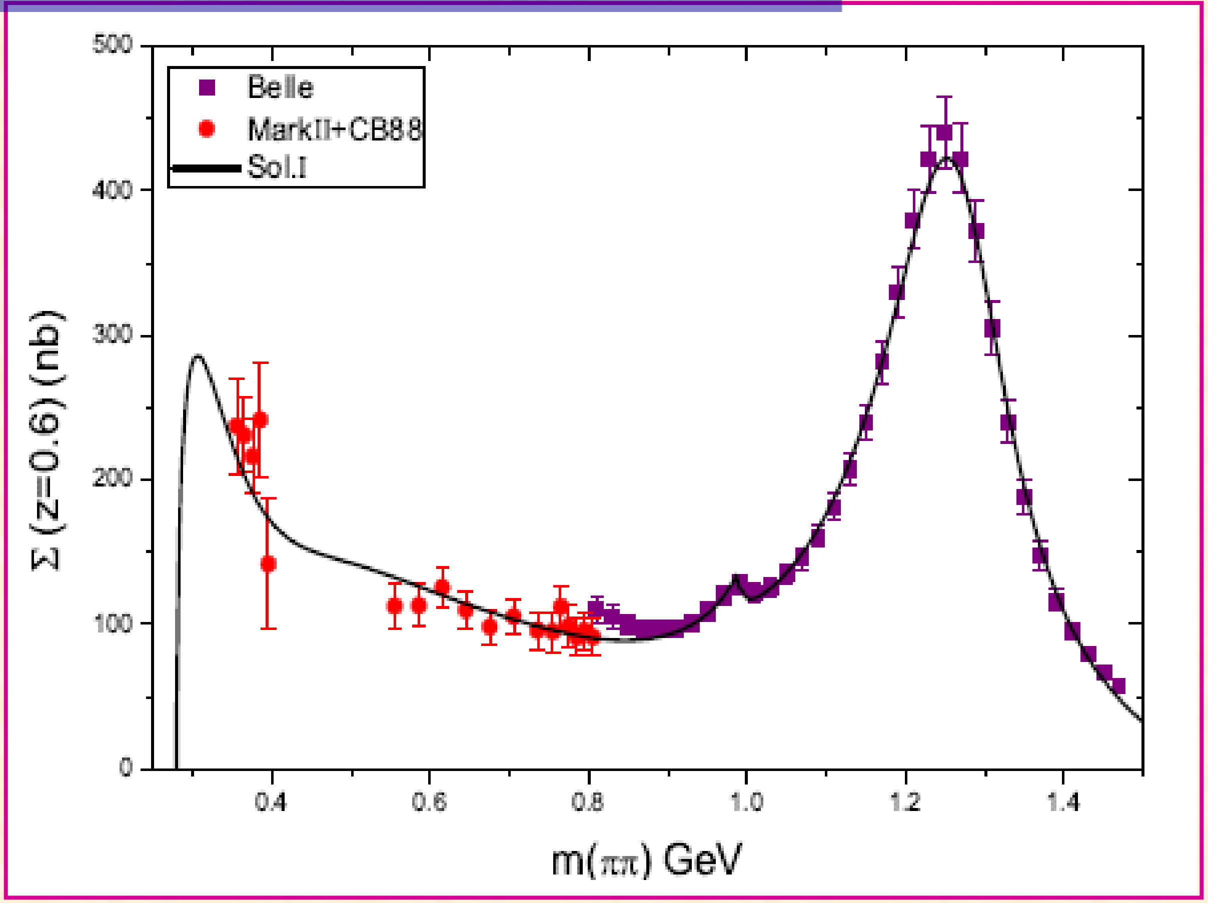
$\gamma\gamma \longrightarrow \pi^+\pi^-$: Mark II, Belle



$\gamma\gamma \longrightarrow \pi^0\pi^0$: Crystal Ball, Belle



$\gamma\gamma \longrightarrow \pi^+\pi^-$, $\pi^0\pi^0$ added



$$\Sigma(Z) \equiv \int_0^Z dz \left[\frac{d\sigma}{dz}(\gamma\gamma \to \pi^+\pi^-) + \frac{d\sigma}{dz}(\gamma\gamma \to \pi^0\pi^0)\right]$$

$$= \frac{2\pi\alpha^2}{s}\rho(s) \sum_{J,J',\lambda} \left[\mathcal{I}^\lambda_{JJ'}(Z) \left(\mathcal{F}^{*0}_{J\lambda}\mathcal{F}^0_{J'\lambda} + \mathcal{F}^{*2}_{J\lambda}\mathcal{F}^2_{J'\lambda}\right)\right]$$

$$\mathcal{I}^\lambda_{JJ'}(Z) = \int_0^Z dz\, P^\lambda_J(z)\, P^{\lambda'}_{J'}(z)$$

Multi-channel Unitarity

Amplitude with definite J^{PC}

2-channel :

$$\mathcal{F}_1 = \alpha_1 T_{11} + \alpha_2 T_{21}$$ ⟸ $\gamma\gamma \to \pi\pi$

$$\mathcal{F}_2 = \alpha_1 T_{12} + \alpha_2 T_{22}$$ ⟸ $\gamma\gamma \to \bar{K}K$

2-channel :

$$T_{11} = \frac{K_{11} - i\,\rho_2 \det K}{\Delta}$$

$$T_{12} = \frac{K_{12}}{\Delta}$$

$$T_{22} = \frac{K_{22} - i\,\rho_1 \det K}{\Delta}$$

Multi-channel Unitarity

Amplitude with definite J^{PC}

2-channel :

$$\mathcal{F}_1 = \alpha_1 T_{11} + \alpha_2 T_{21}$$

$$\mathcal{F}_2 = \alpha_1 T_{12} + \alpha_2 T_{22}$$

α_1, α_2 have left hand cuts only

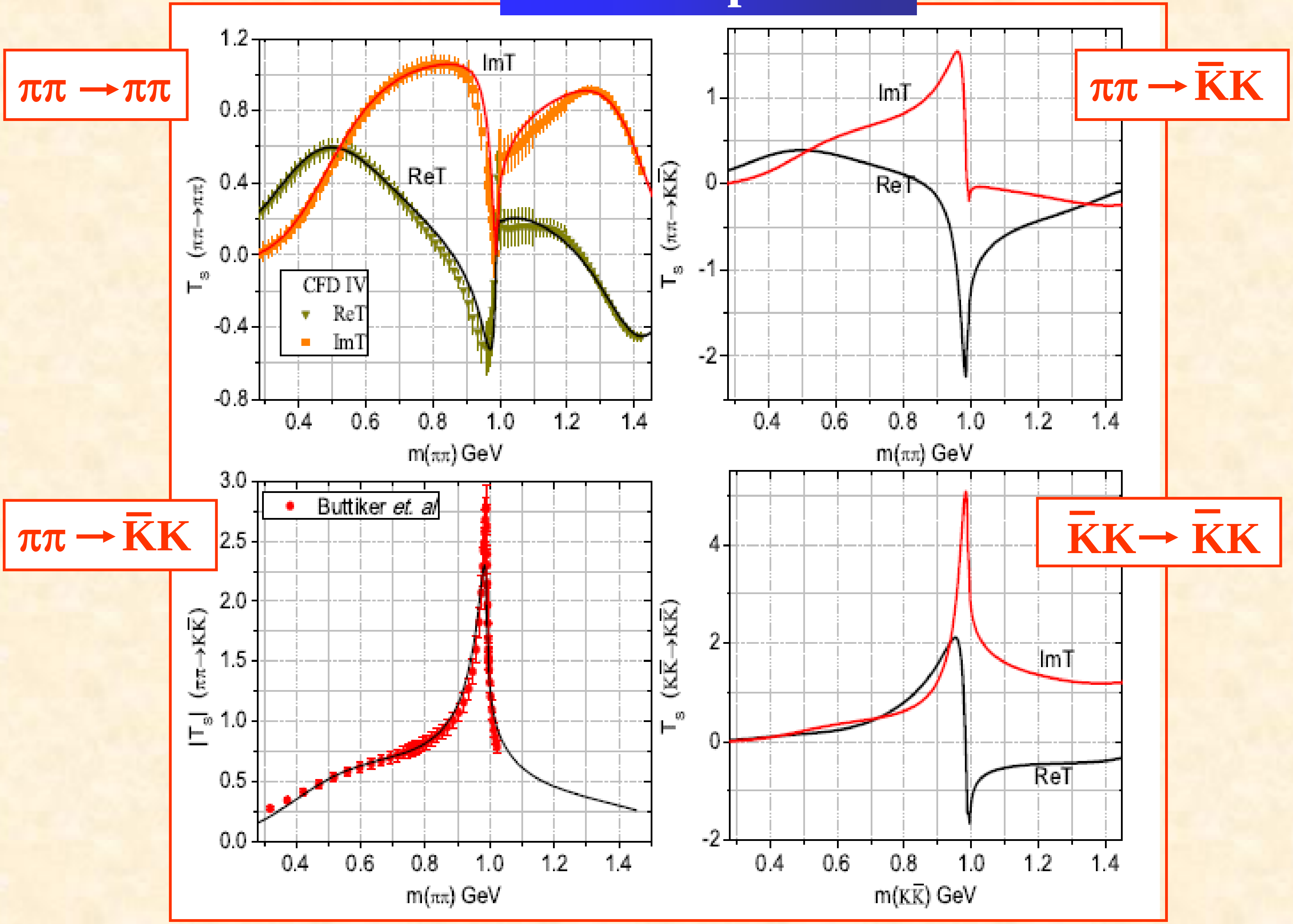
2-channel :

$$T_{11} = \frac{K_{11} - i\,\rho_2 \det K}{\Delta}$$

$$T_{12} = \frac{K_{12}}{\Delta}$$

$$T_{22} = \frac{K_{22} - i\,\rho_1 \det K}{\Delta}$$

I=J=0 amplitudes



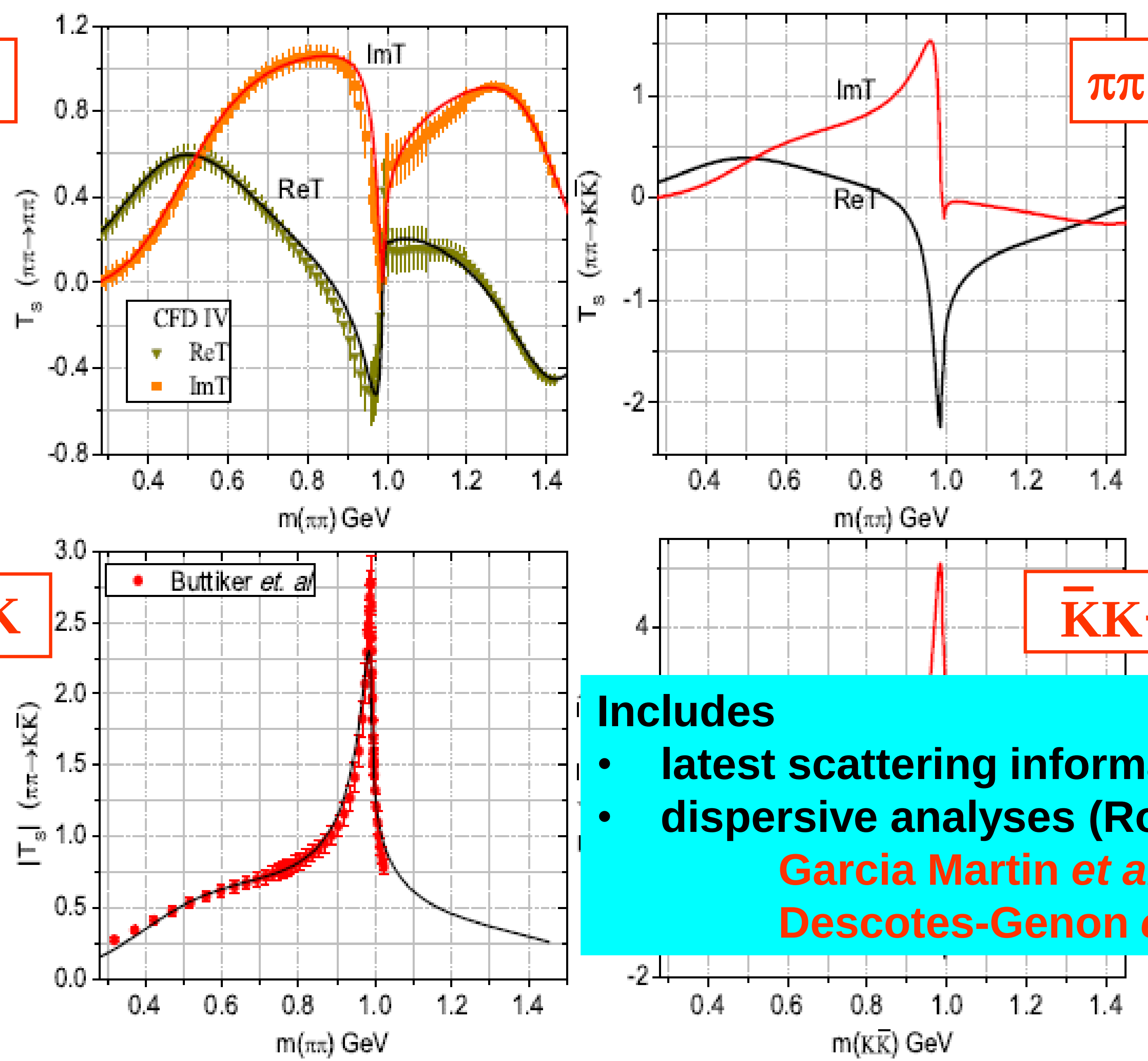
I=J=0 amplitudes

$\pi\pi \rightarrow \pi\pi$

$\pi\pi \rightarrow \bar{K}K$

$\pi\pi \rightarrow \bar{K}K$

$\bar{K}K \rightarrow \bar{K}K$



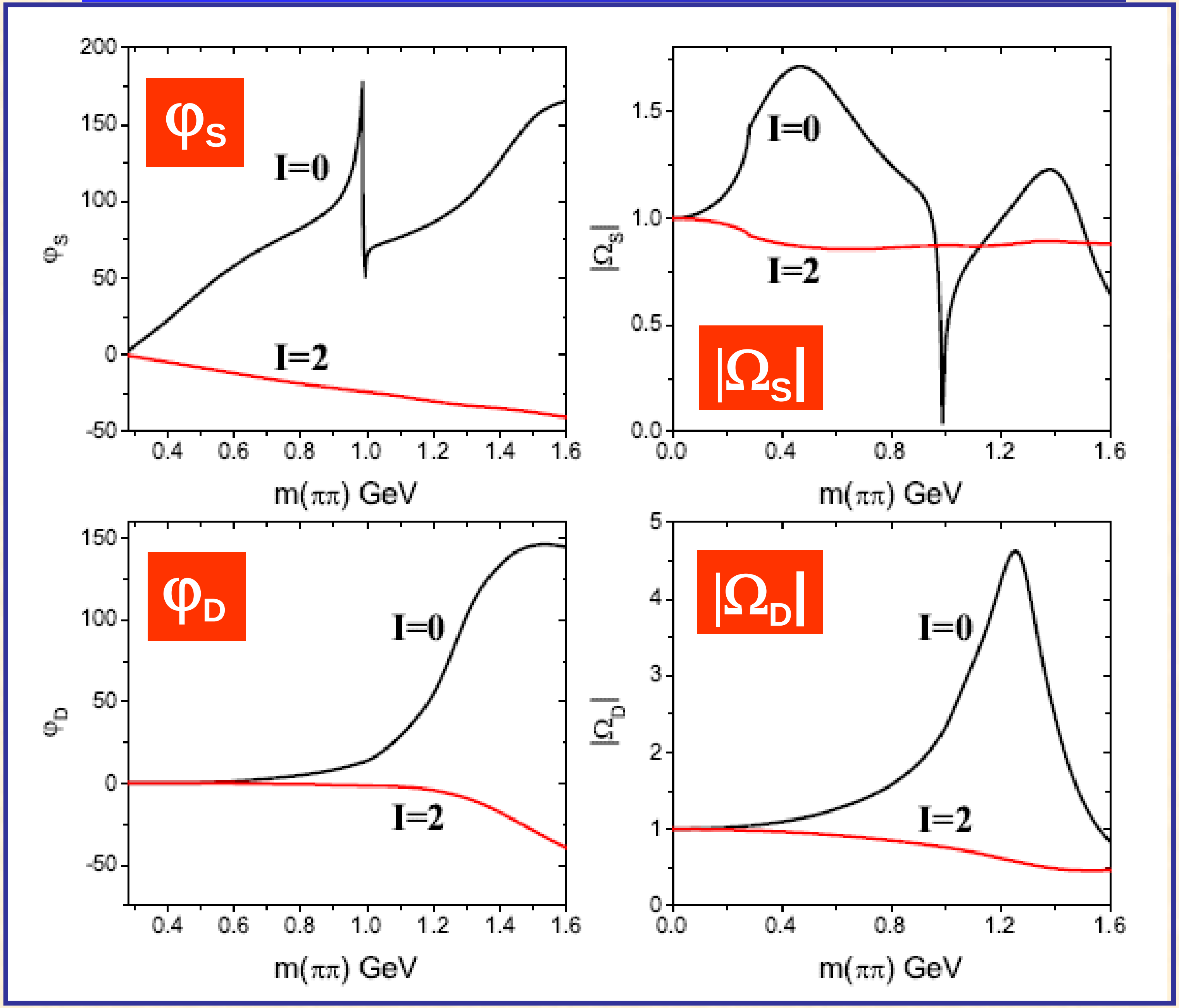
Includes

- **latest scattering information**
- **dispersive analyses (Roy-type)**

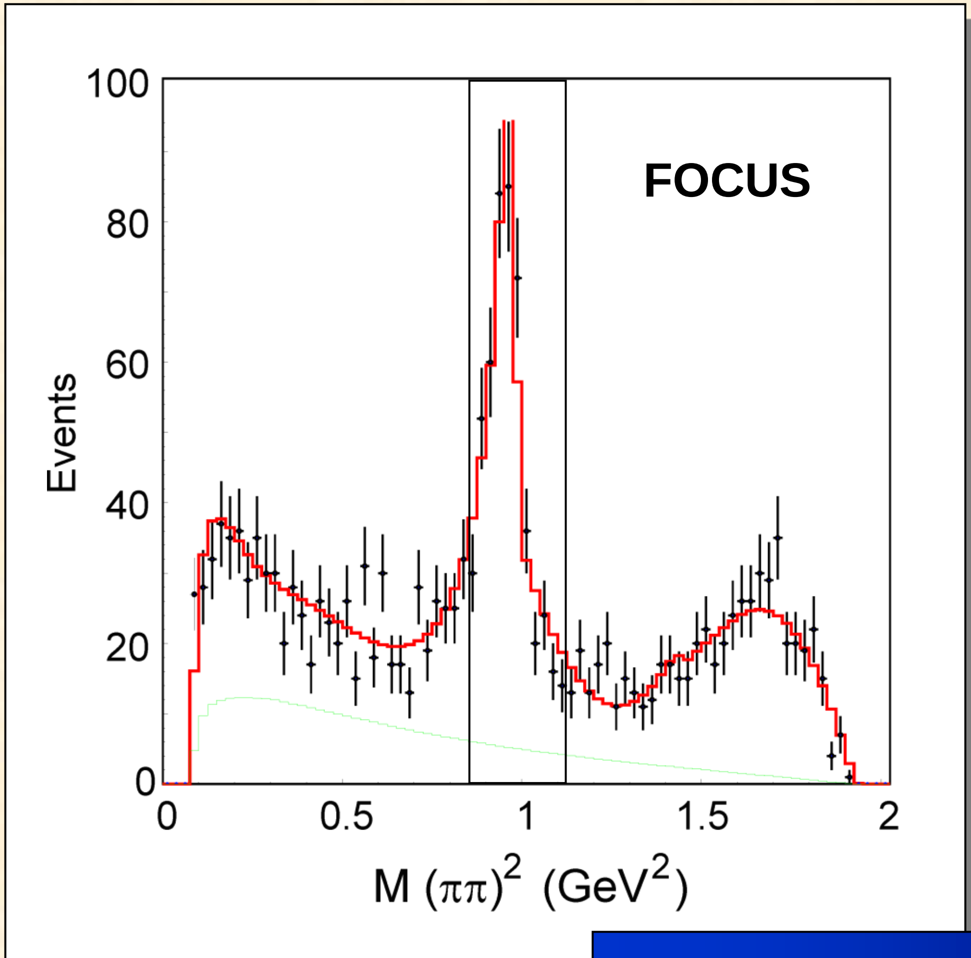
Garcia Martin *et al.*,

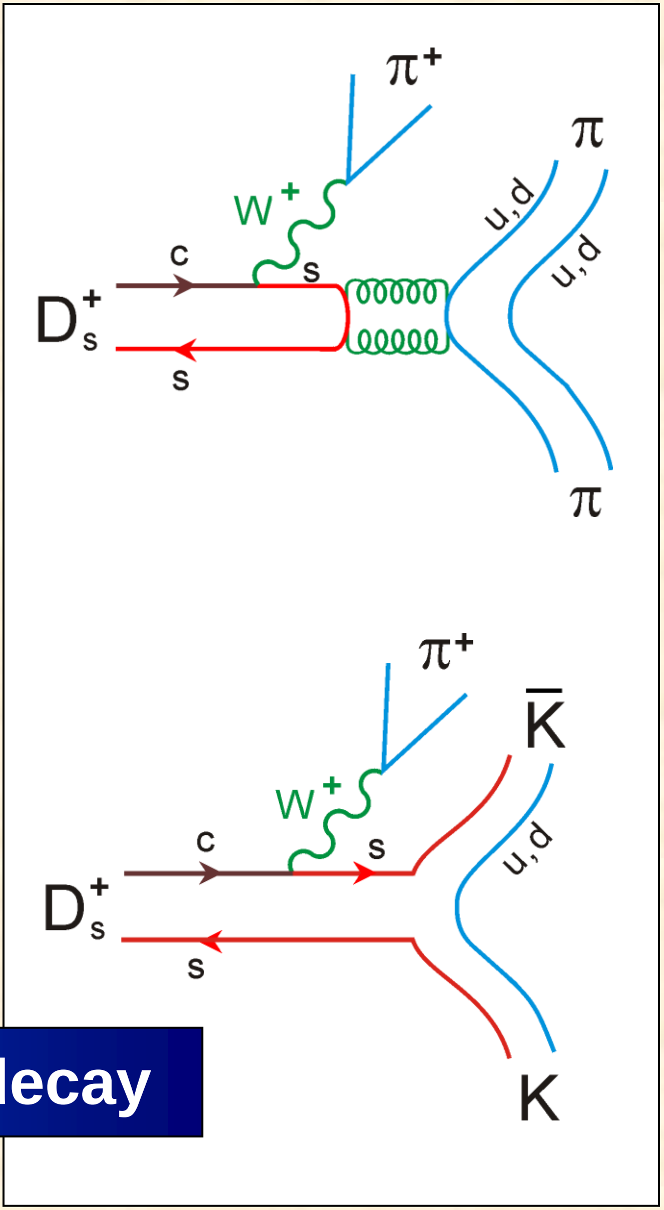
Descotes-Genon *et al.*, …

I=0,2, J=0,2 phases & Omnes functions



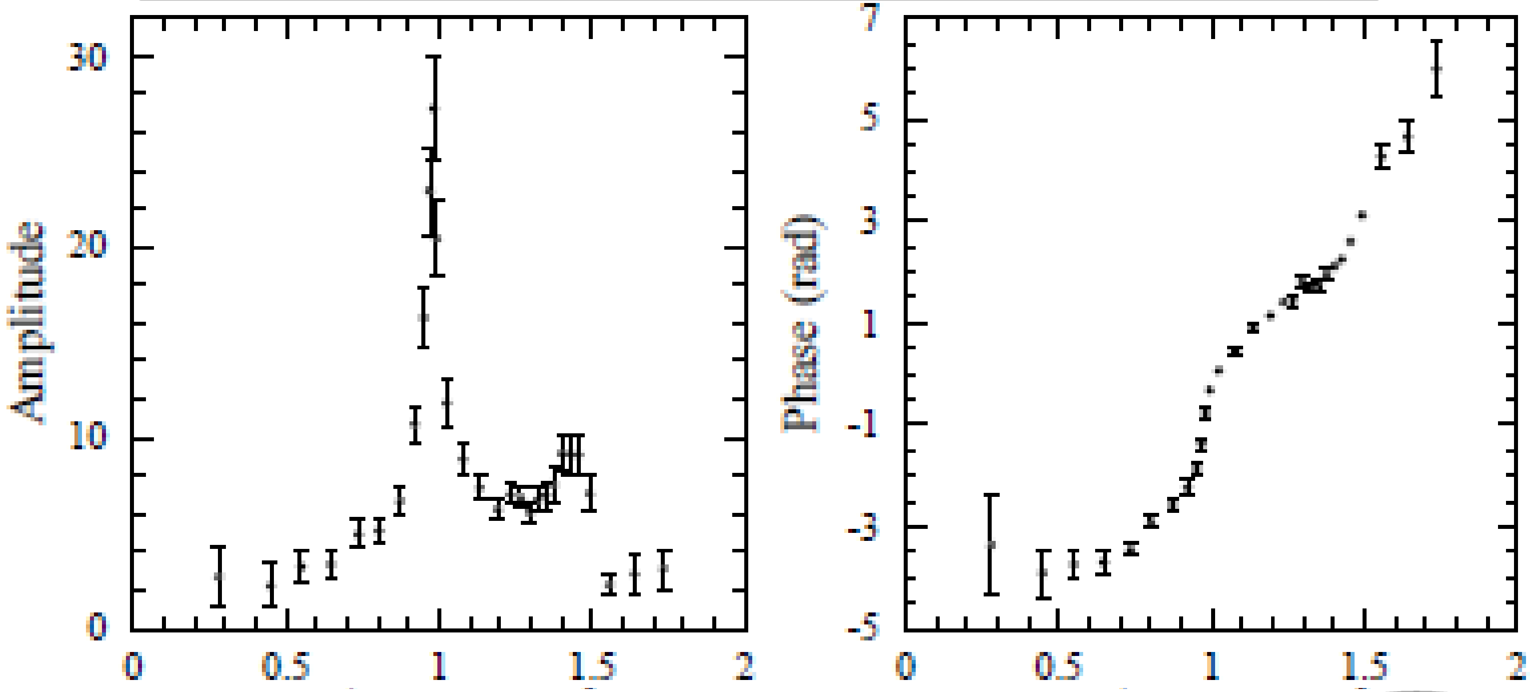
Heavy flavour decays





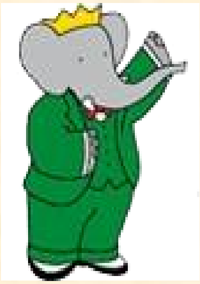
$D_s \rightarrow (MM)\ \pi$ decay

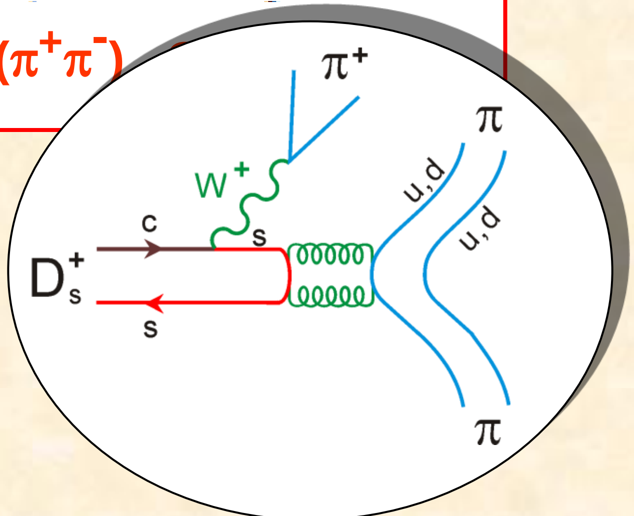
$\pi\pi$ S-wave in $D_s \to \pi(\pi\pi)$ decay



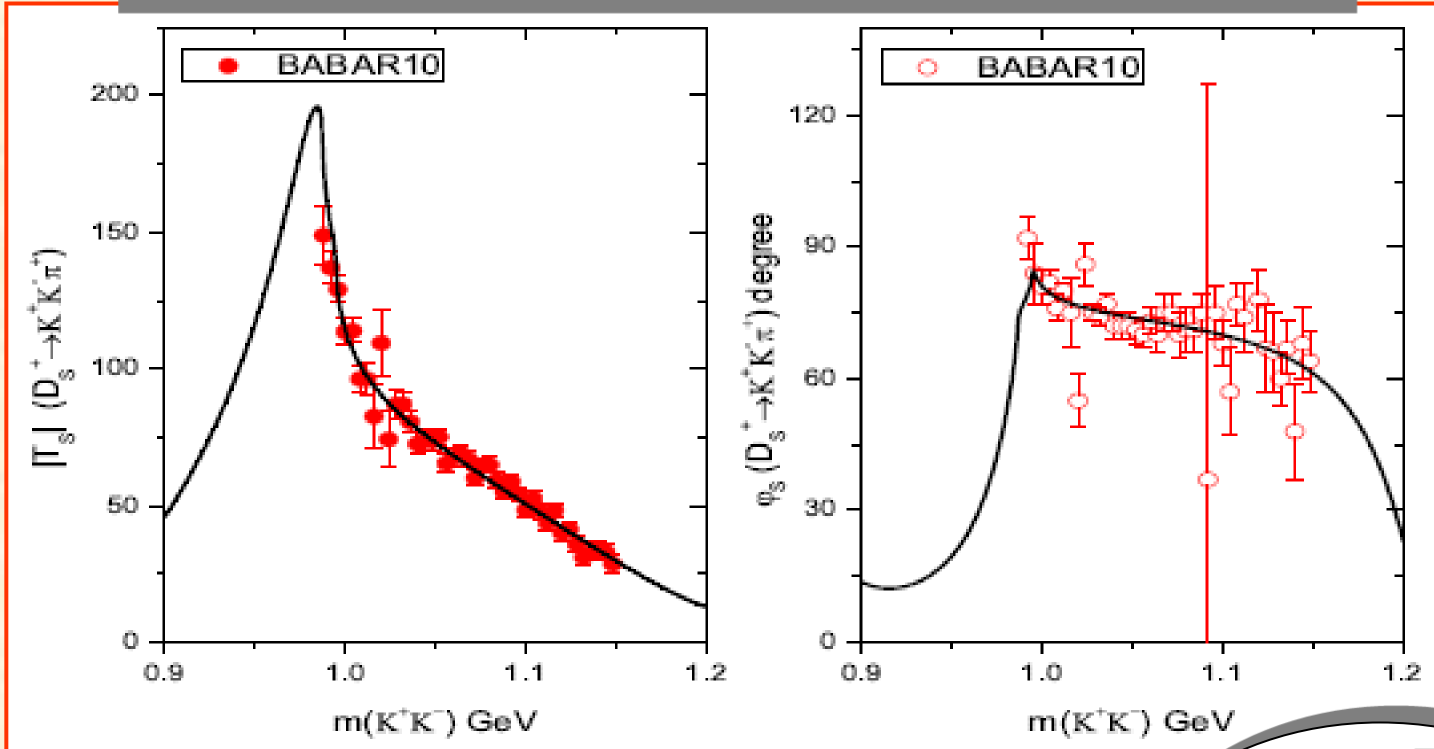
$M(\pi^+\pi^-)$ GeV

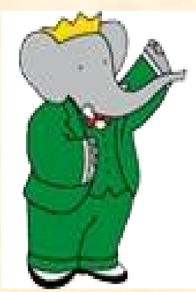
$M(\pi^+\pi^-)$

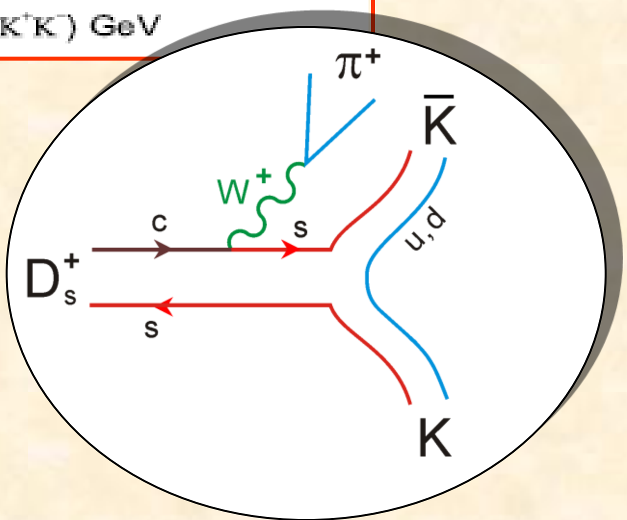




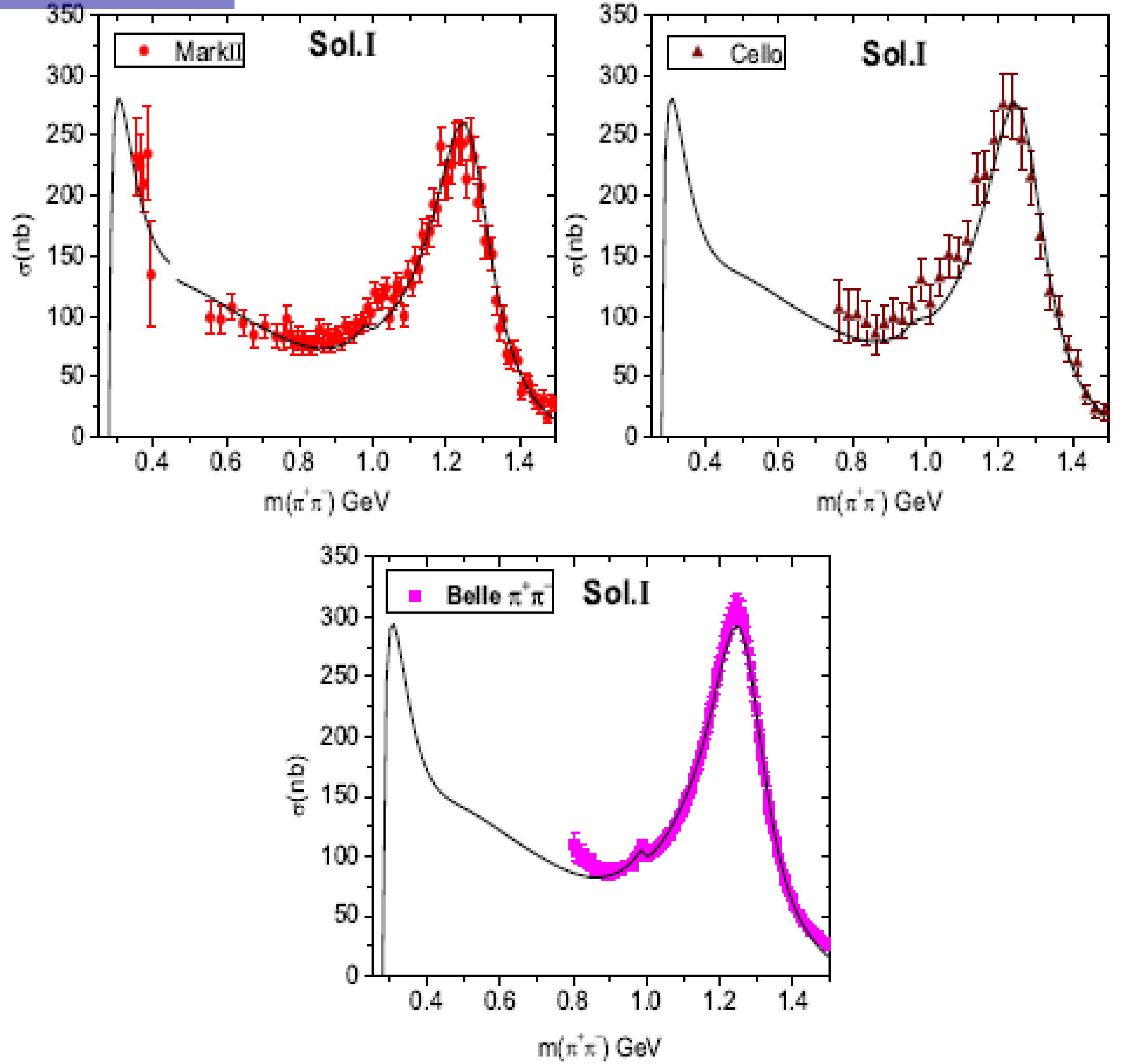
$\bar{K}K$ S-wave in $D_s \to \pi(\bar{K}K)$ decay



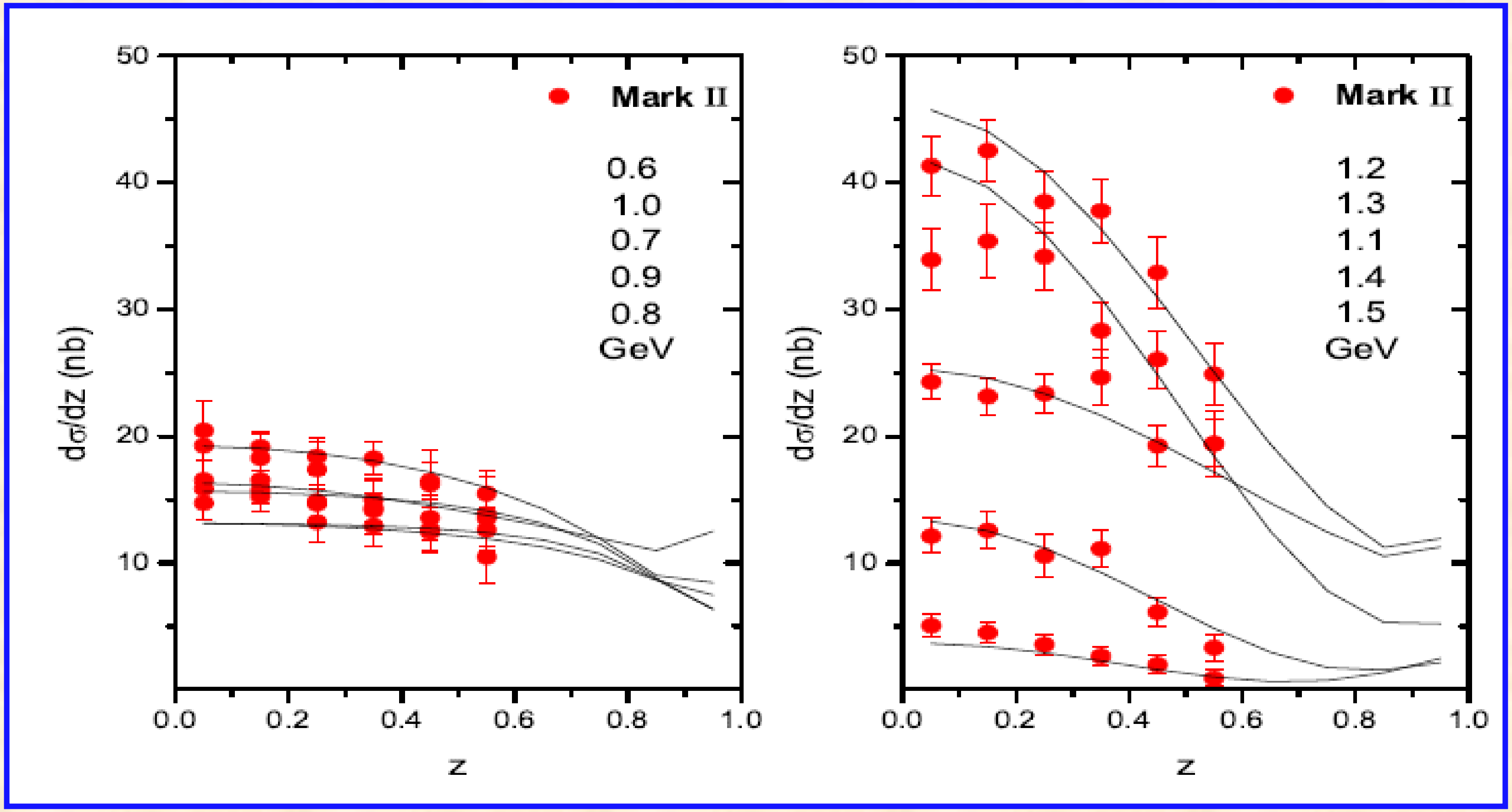




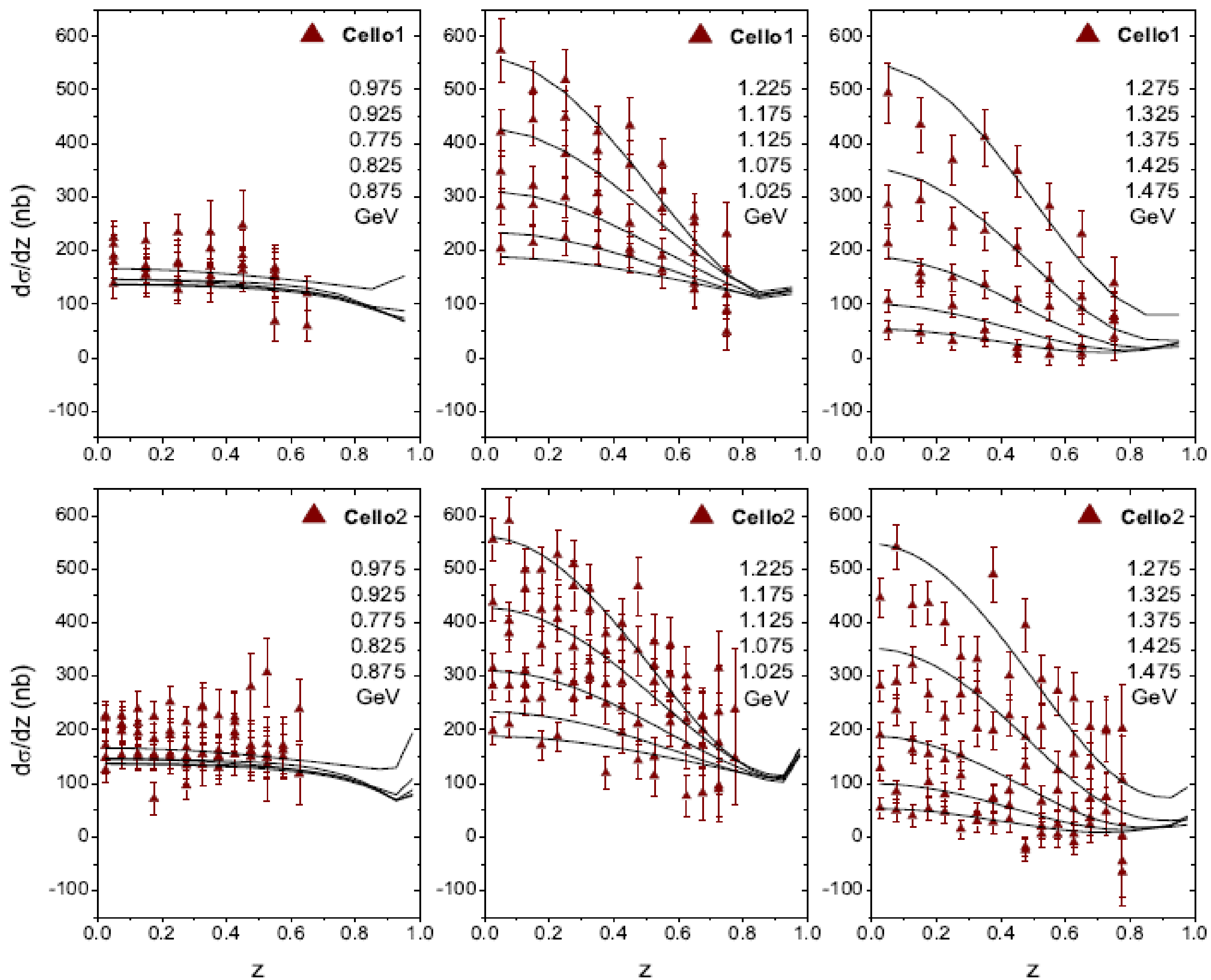
$\gamma\gamma \longrightarrow \pi^+\pi^-$



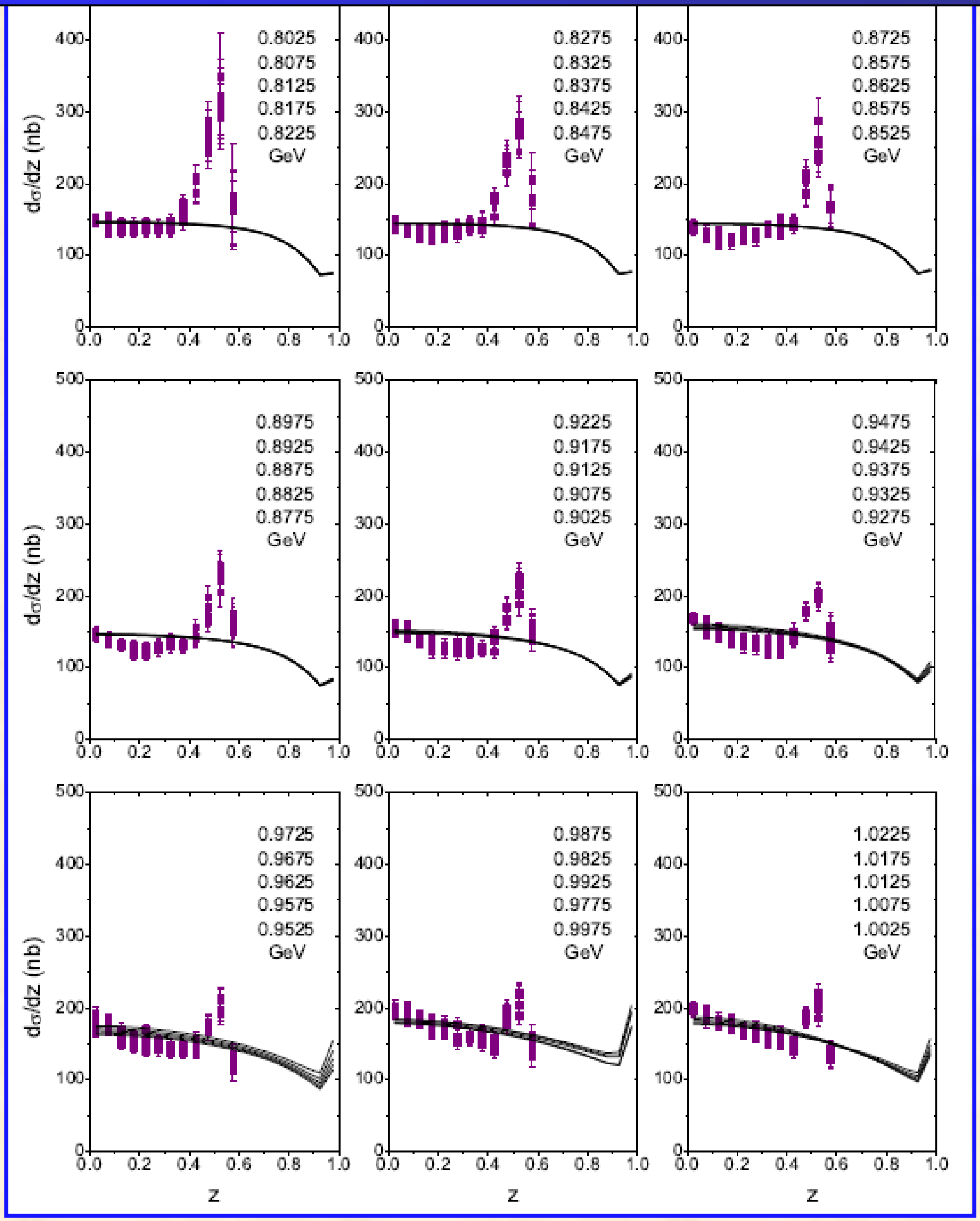
$\gamma\gamma \longrightarrow \pi^+\pi^-$: angular distributions

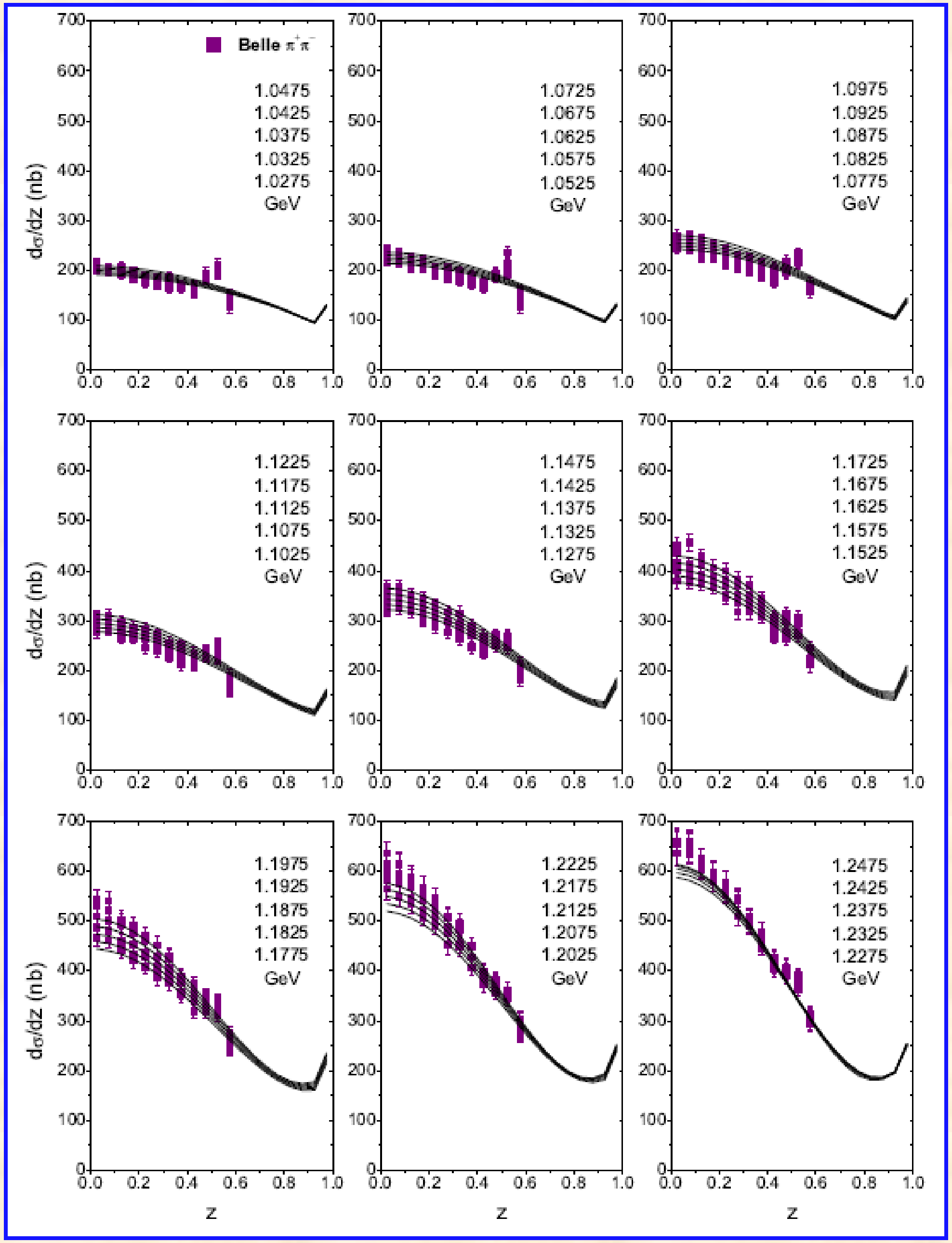


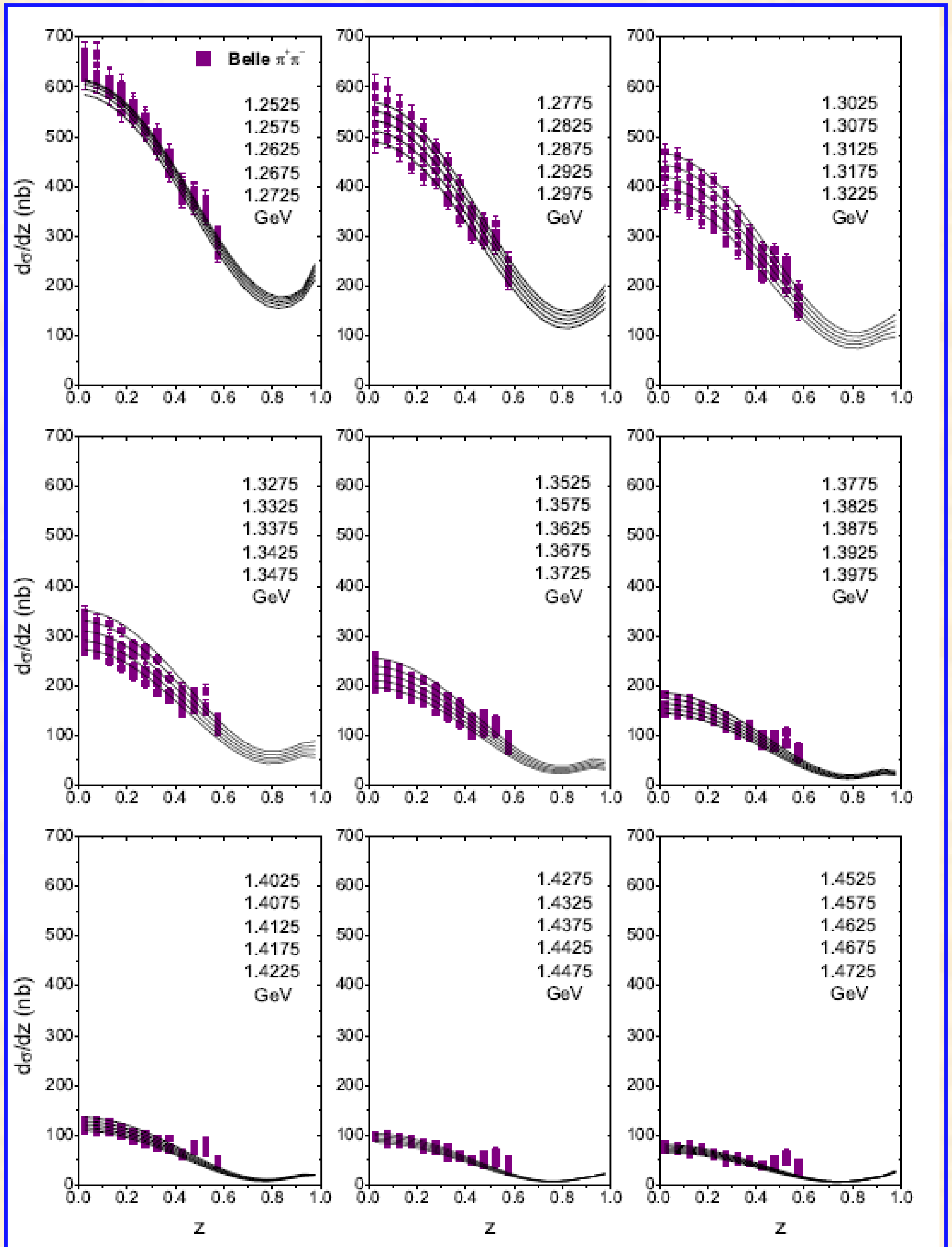
$\gamma\gamma \longrightarrow \pi^+\pi^-$: angular distributions



$\gamma\gamma \longrightarrow \pi^+\pi^-$: angular distributions




Belle $\pi^+\pi^-$
1.0475
1.0425
1.0375
1.0325
1.0275
GeV
1.0725
1.0675
1.0625
1.0575
1.0525
GeV
1.0975
1.0925
1.0875
1.0825
1.0775
GeV
1.1225
1.1175
1.1125
1.1075
1.1025
GeV
1.1475
1.1425
1.1375
1.1325
1.1275
GeV
1.1725
1.1675
1.1625
1.1575
1.1525
GeV
1.1975
1.1925
1.1875
1.1825
1.1775
GeV
1.2225
1.2175
1.2125
1.2075
1.2025
GeV
1.2475
1.2425
1.2375
1.2325
1.2275
GeV
dσ/dz (nb)
z


Belle $\pi^+\pi^-$
1.2525
1.2575
1.2625
1.2675
1.2725
GeV
1.2775
1.2825
1.2875
1.2925
1.2975
GeV
1.3025
1.3075
1.3125
1.3175
1.3225
GeV
1.3275
1.3325
1.3375
1.3425
1.3475
GeV
1.3525
1.3575
1.3625
1.3675
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GeV
1.3775
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1.3875
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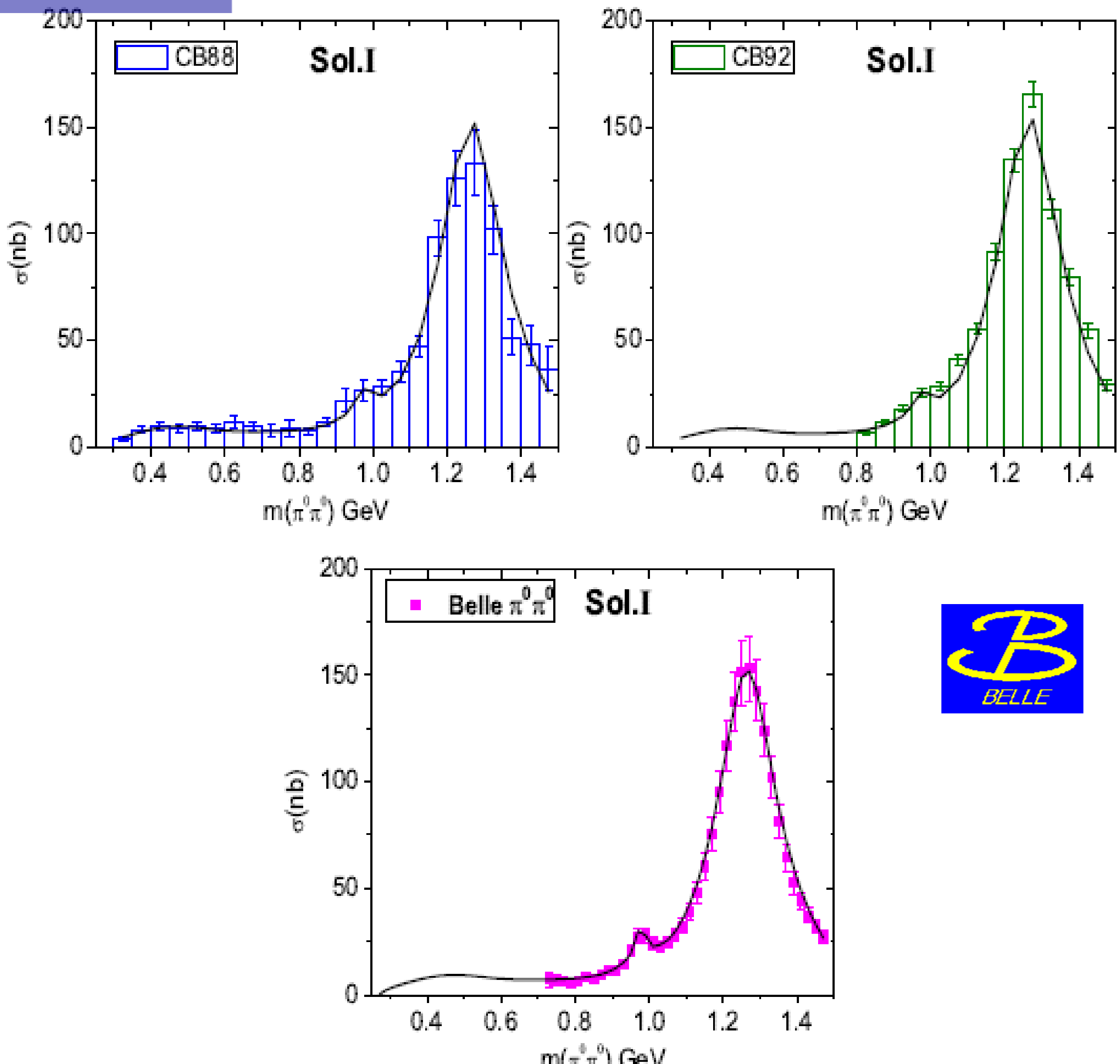
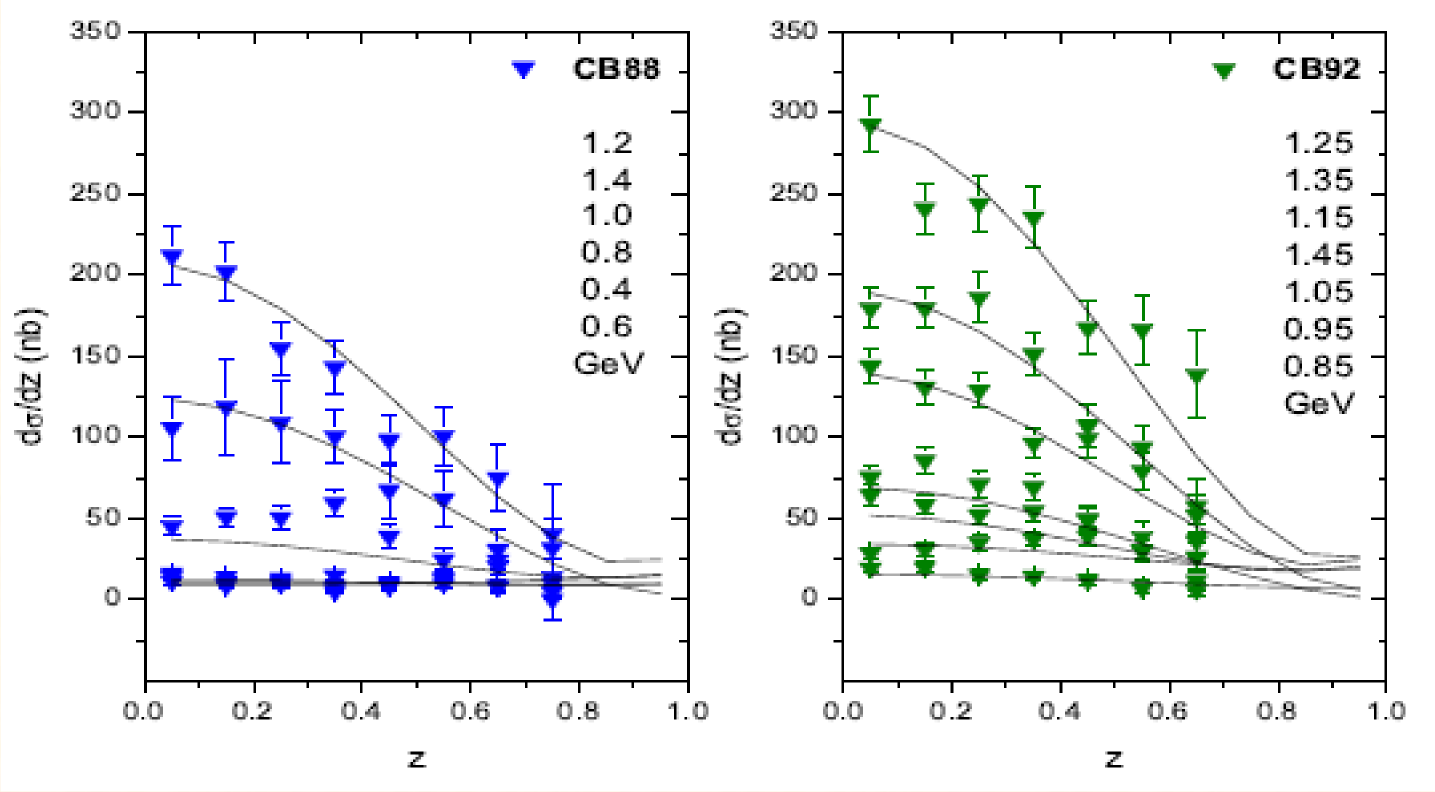
1.3975
GeV
1.4025
1.4075
1.4125
1.4175
1.4225
GeV
1.4275
1.4325
1.4375
1.4425
1.4475
GeV
1.4525
1.4575
1.4625
1.4675
1.4725
GeV
dσ/dz (nb)
z

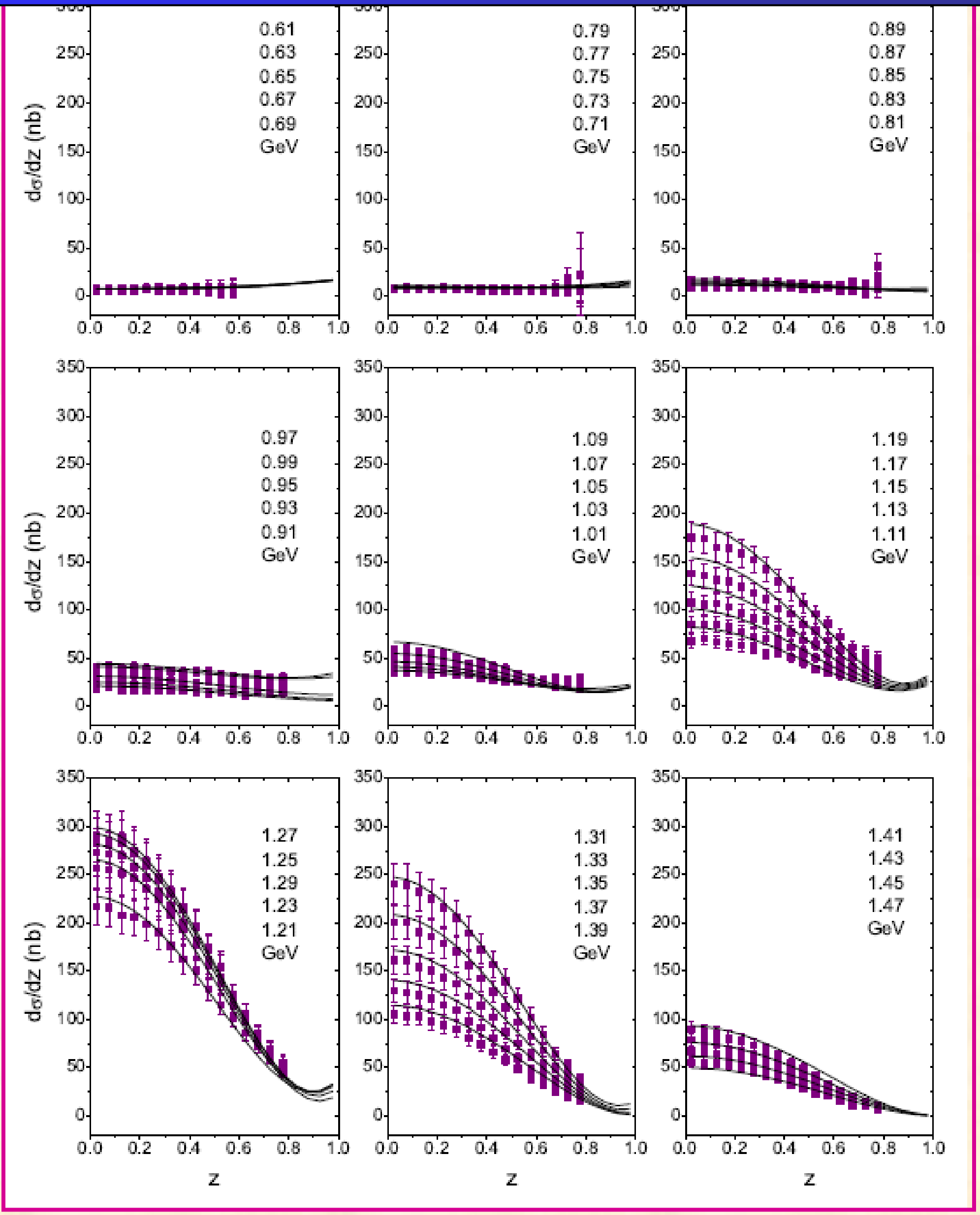
$\gamma\gamma \longrightarrow \pi^0\pi^0$



$\gamma\gamma \longrightarrow \pi^0\pi^0$: angular distributions



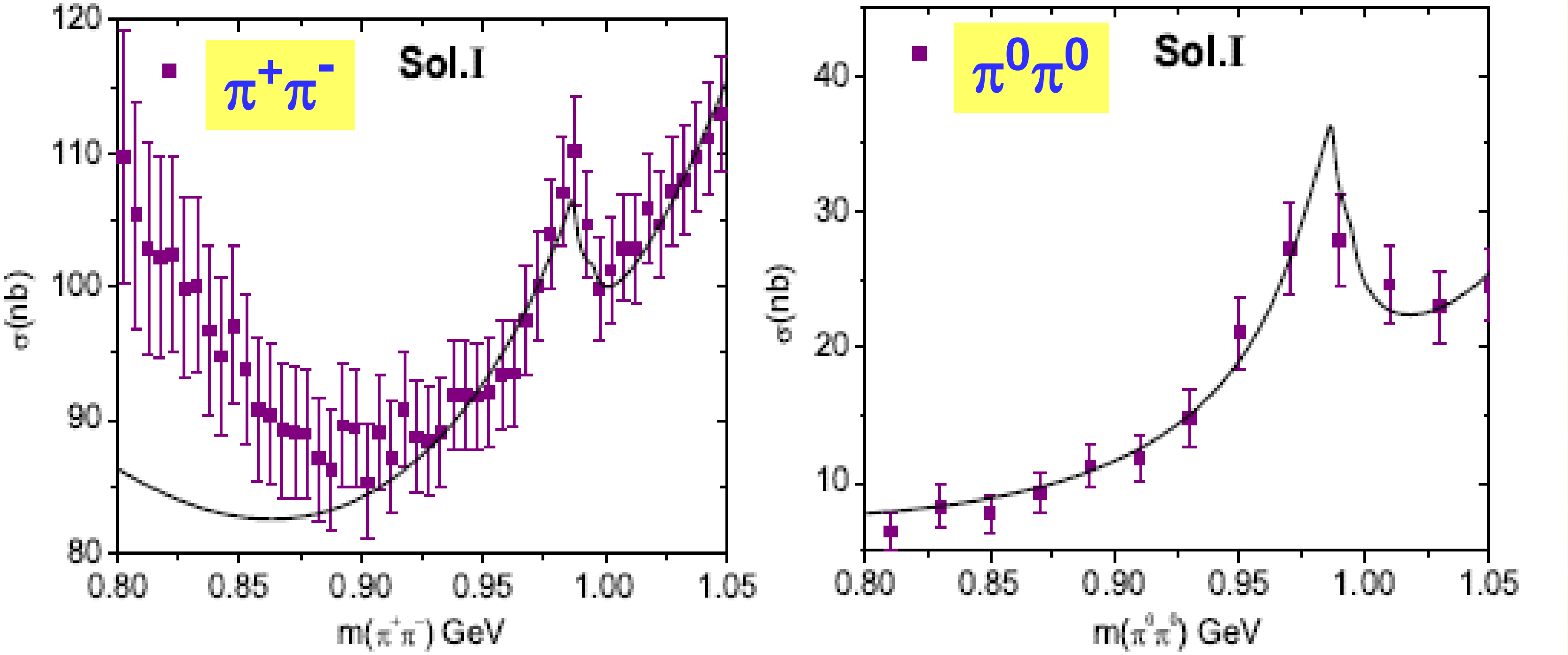
$\gamma\gamma \longrightarrow \pi^0\pi^0$: angular distributions



$\gamma\gamma \longrightarrow \pi^+\pi^-, \pi^0\pi^0$

BELLE

800 to 1050 MeV



Multi-channel Unitarity

Amplitude with definite J^{PC}

2-channel :

$$\mathcal{F}_1 = \alpha_1 T_{11} + \alpha_2 T_{21} \quad \Leftarrow \quad \gamma\gamma \to \pi\pi$$

$$\mathcal{F}_2 = \alpha_1 T_{12} + \alpha_2 T_{22} \quad \Leftarrow \quad \gamma\gamma \to \bar{K}K$$

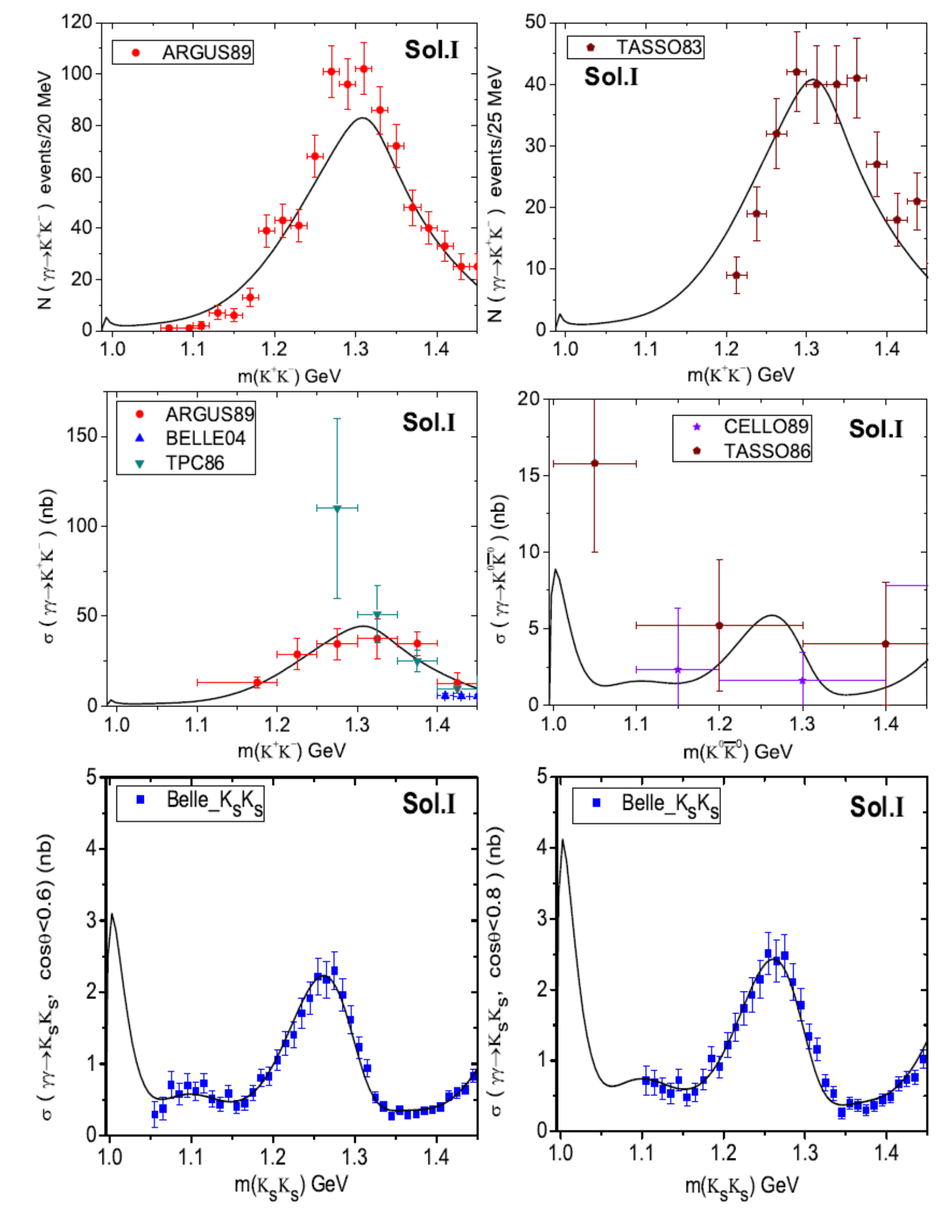
2-channel :

$$T_{11} = \frac{K_{11} - i\,\rho_2 \det K}{\Delta}$$

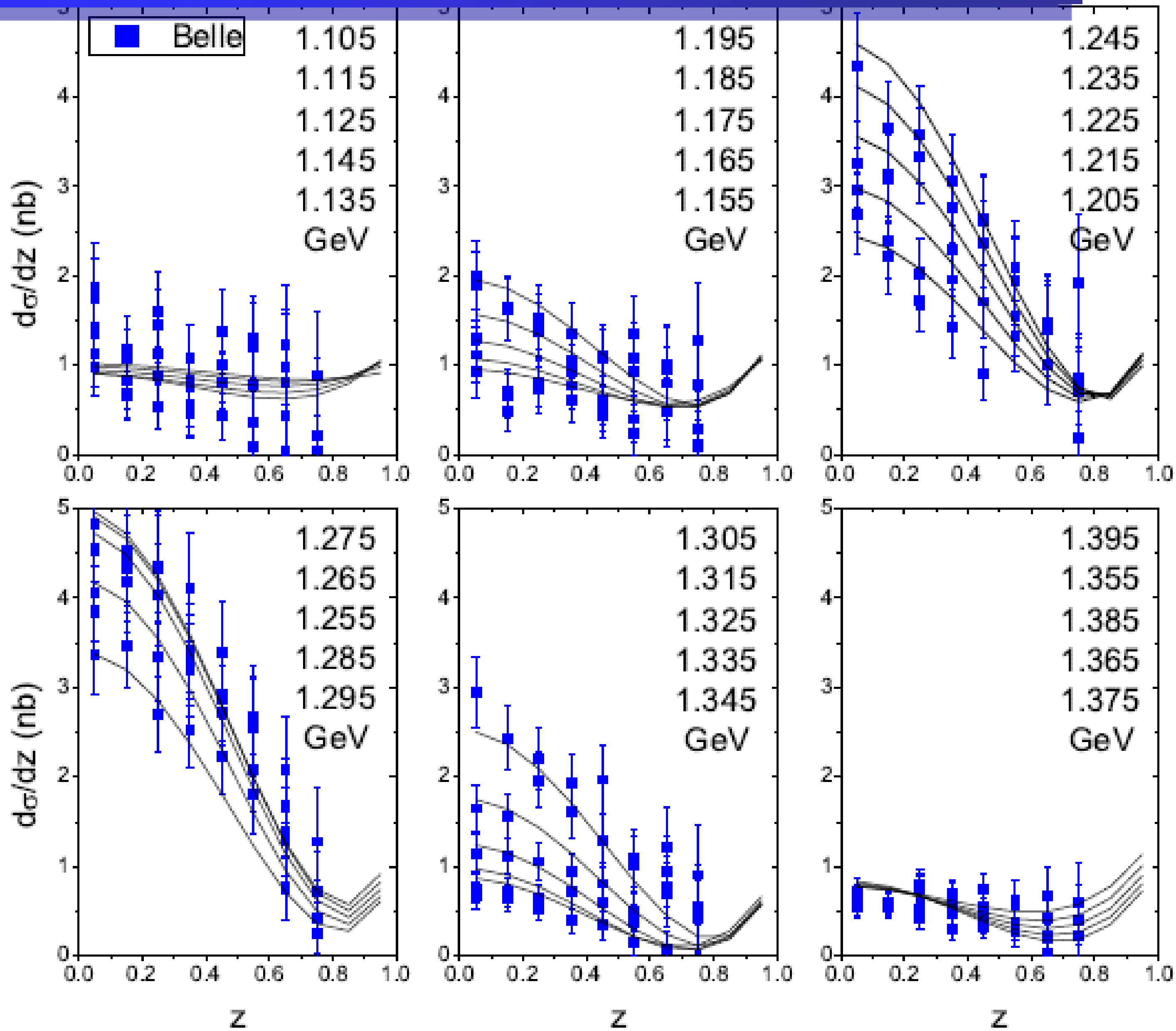
$$T_{12} = \frac{K_{12}}{\Delta}$$

$$T_{22} = \frac{K_{22} - i\,\rho_1 \det K}{\Delta}$$

$\gamma\gamma \longrightarrow K^+K^-$, K_sK_s : cross-sections

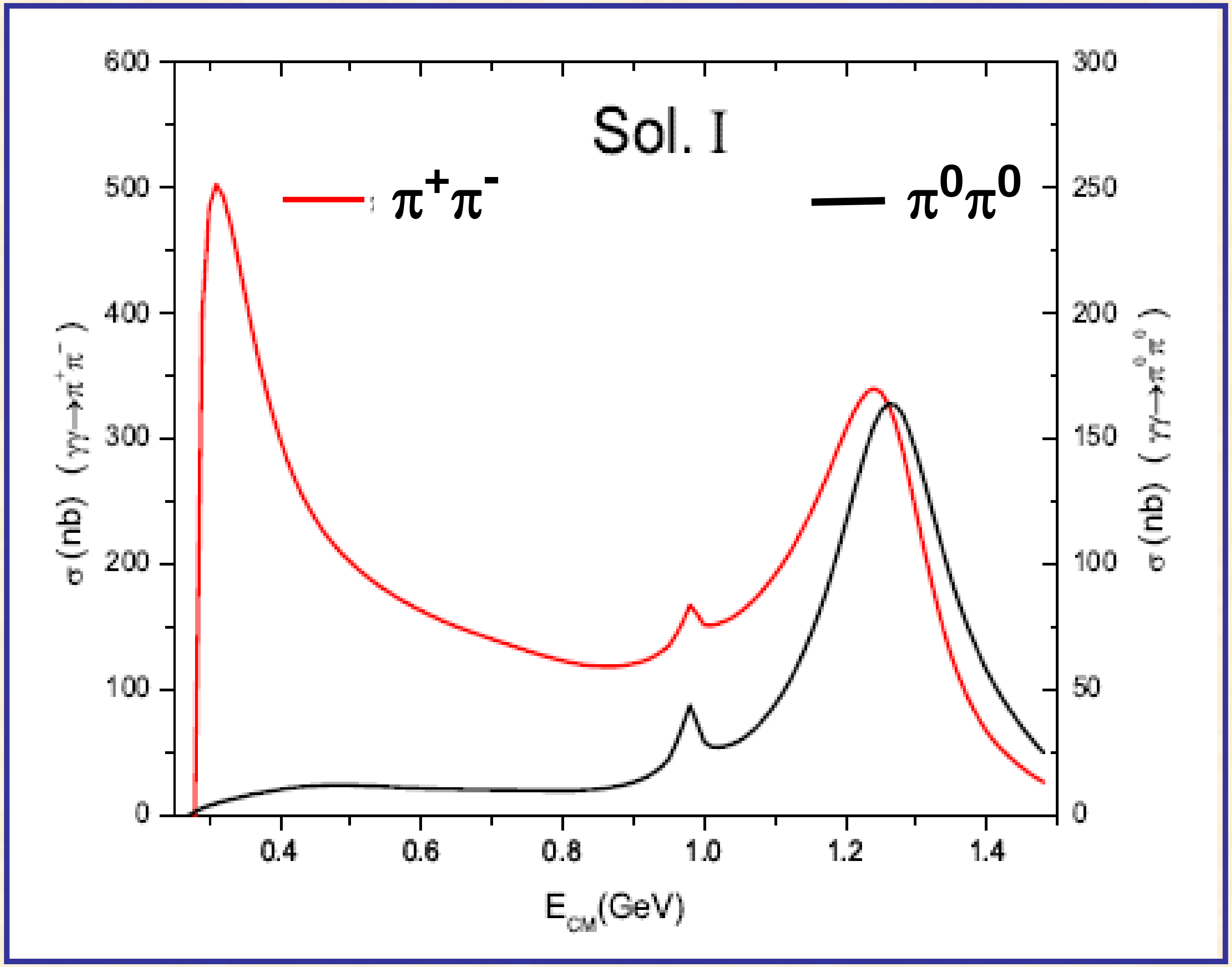


$\gamma\gamma \longrightarrow K_s K_s$: angular distributions

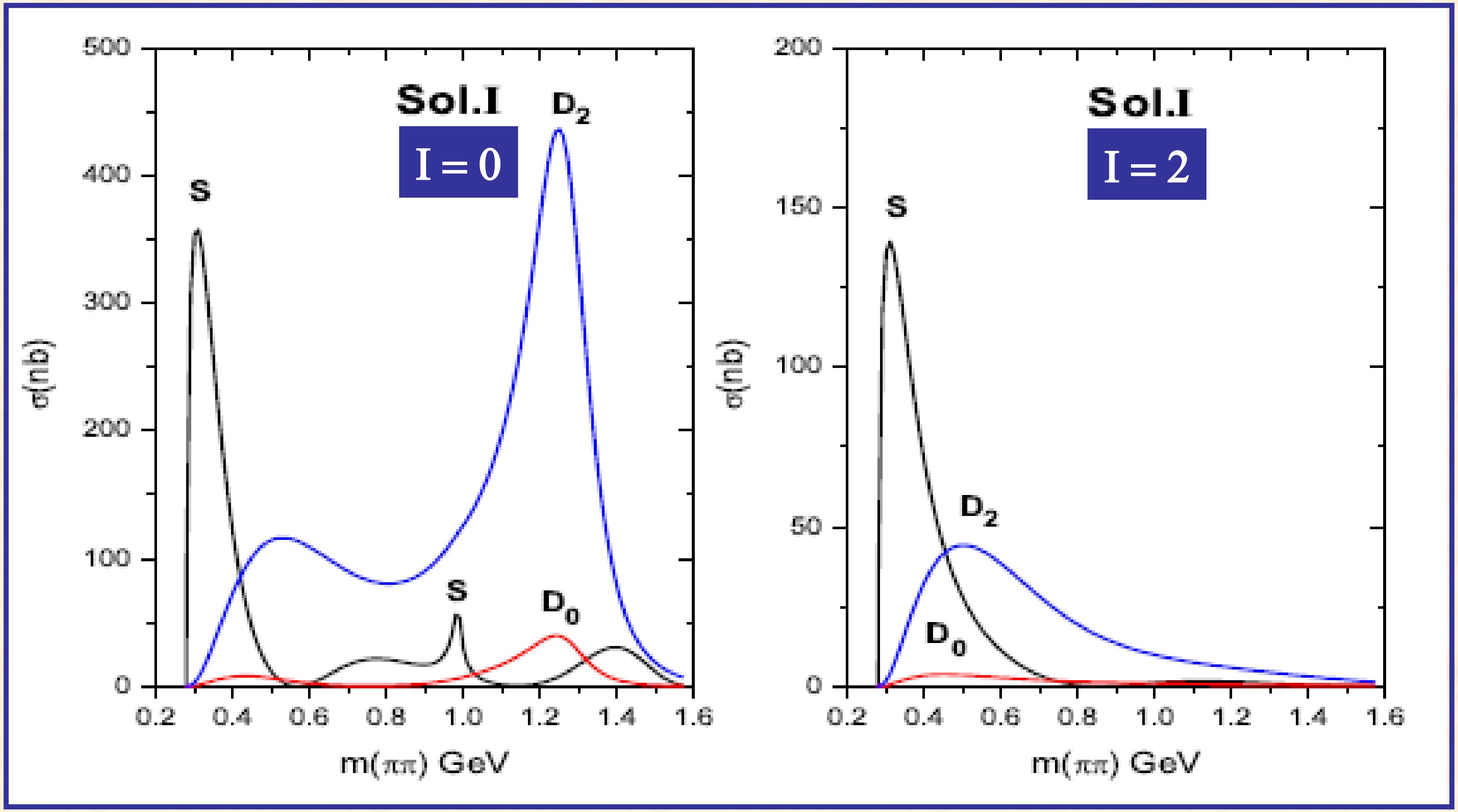


$\gamma\gamma \longrightarrow \pi^+\pi^-$, $\pi^0\pi^0$

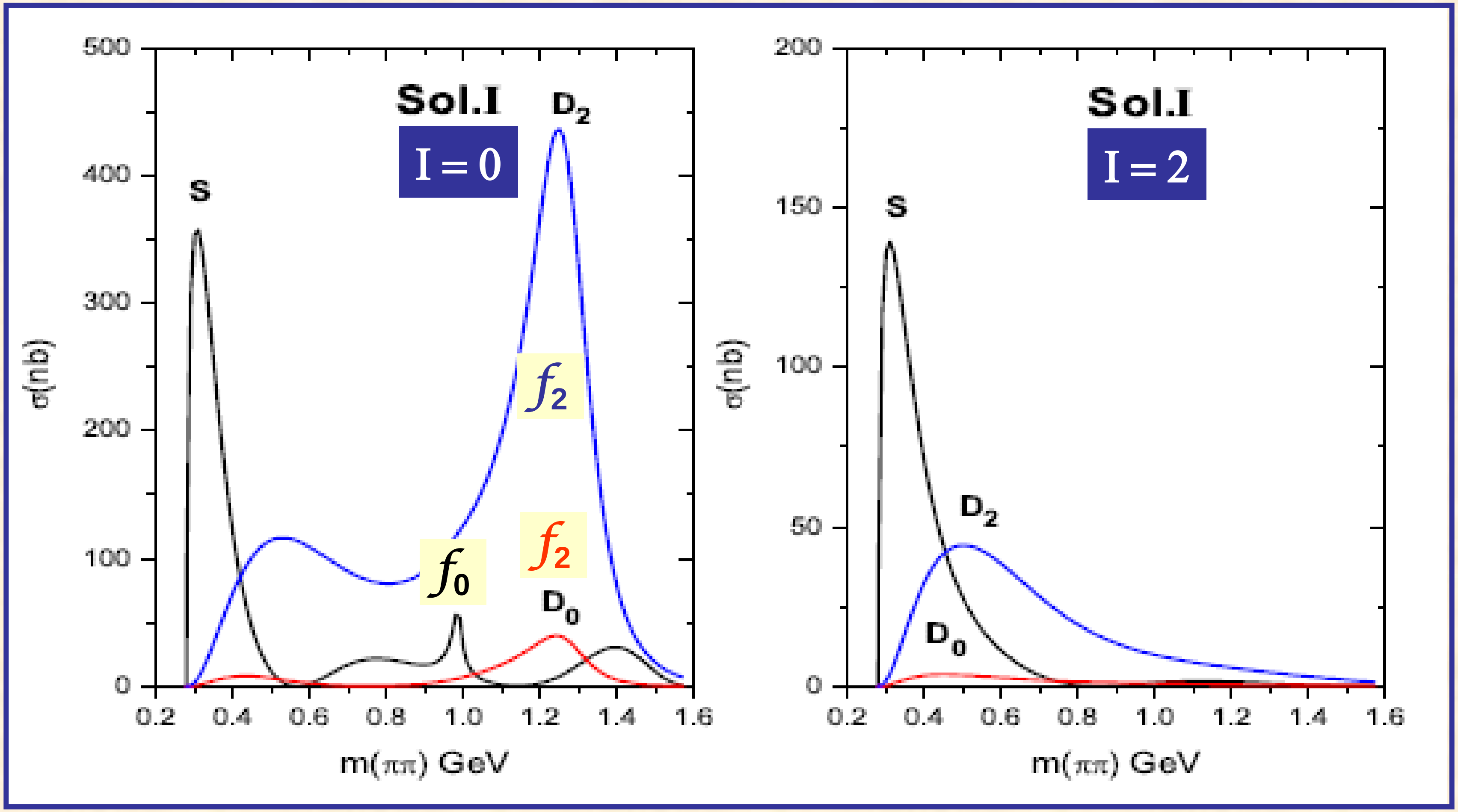
Integrated cross-sections



$\gamma\gamma \longrightarrow \pi\pi$ partial wave cross-sections

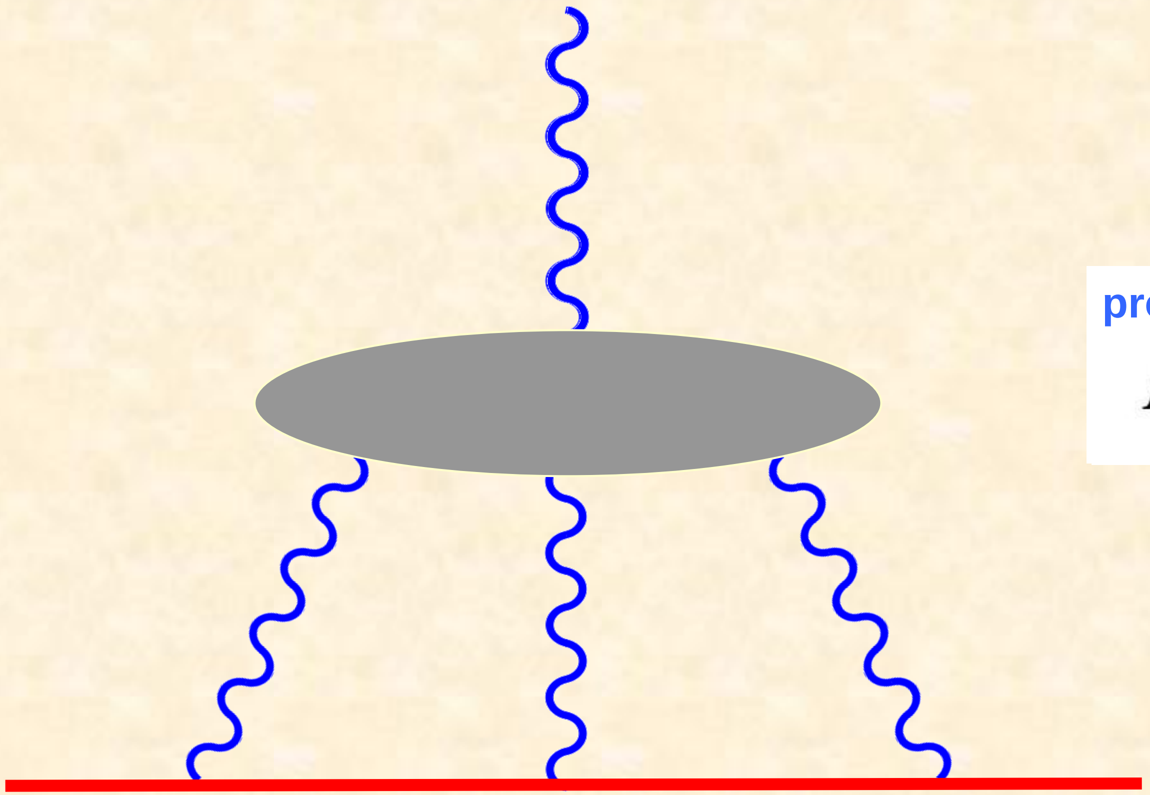


$\gamma\gamma \longrightarrow \pi\pi$ partial wave cross-sections



input into dispersion relation for LbL

Light by Light

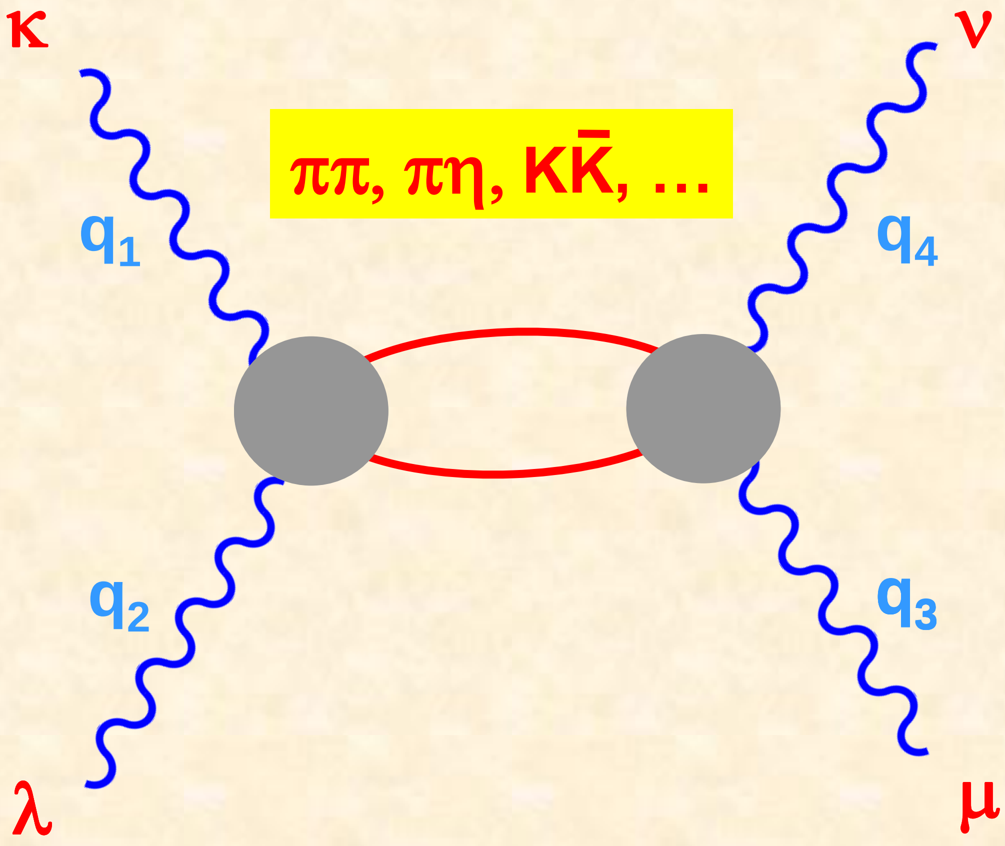


projector

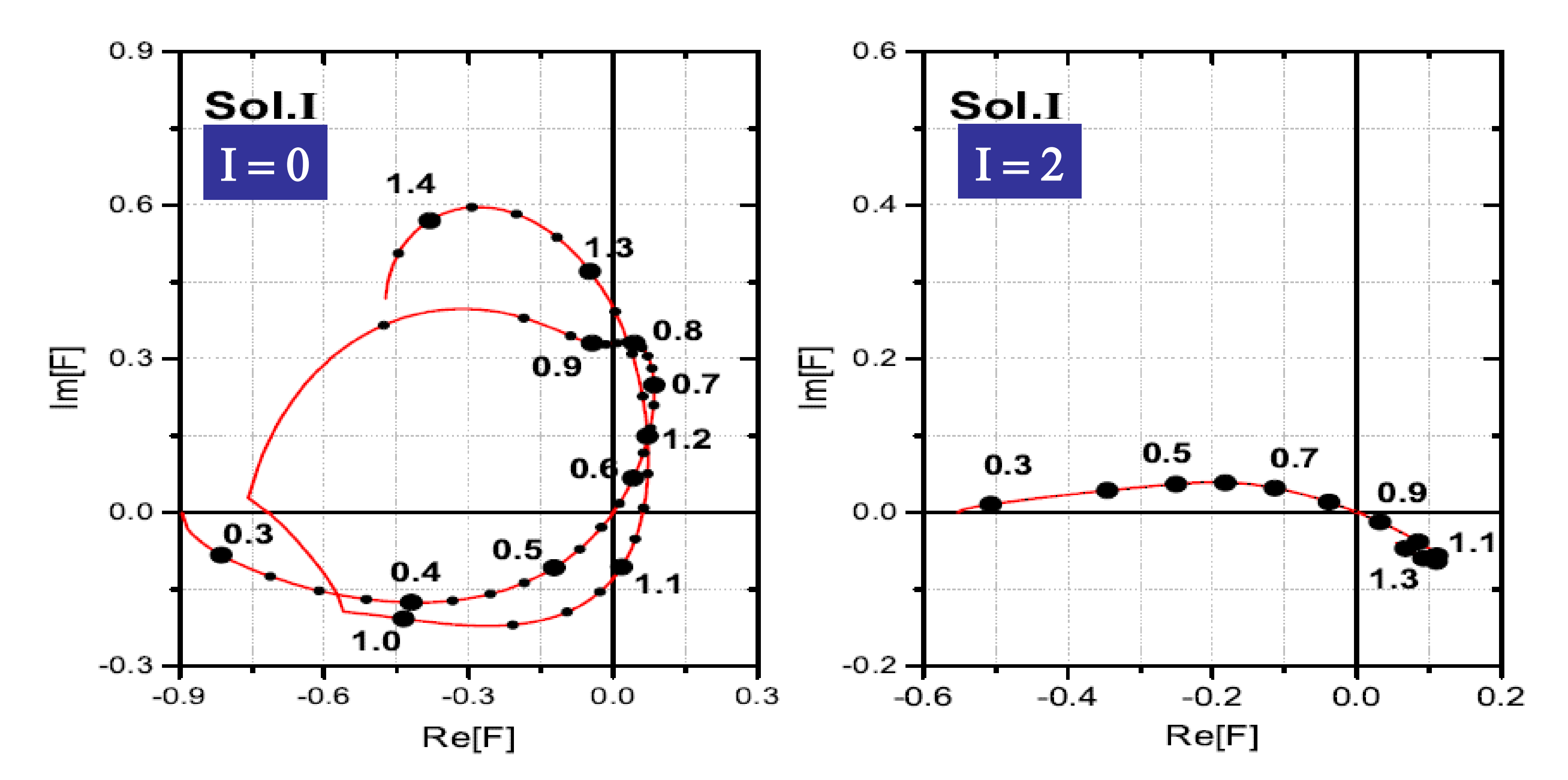
$$\Lambda^\nu(p',p)$$

$$a_\mu = \lim_{k\to 0} \mathrm{Tr}\left\{(\not p + m)\,\Lambda^\nu(p',p)\,(\not p' + m)\,\Gamma_\nu(p',p)\right\}$$

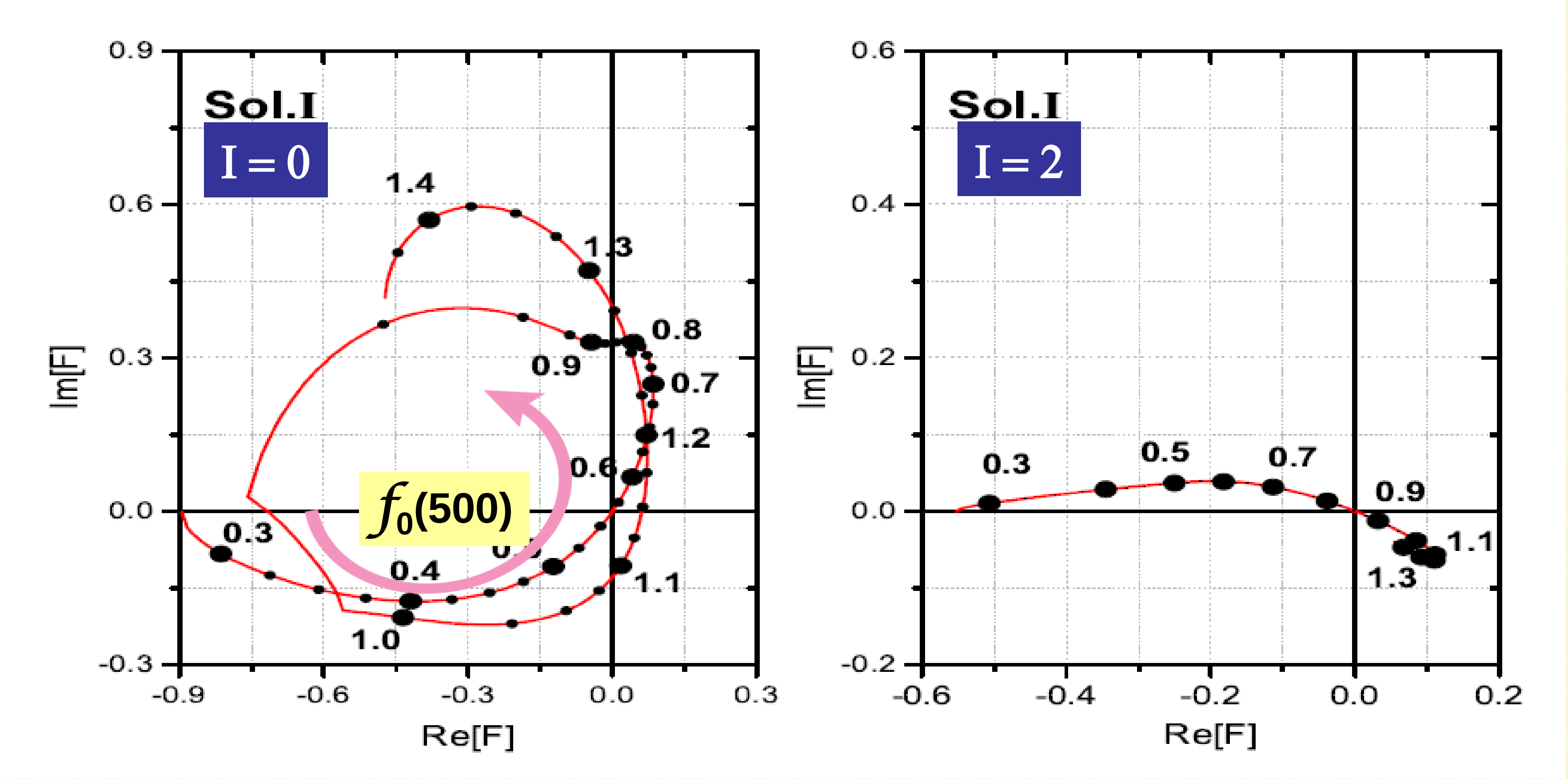
Light by Light



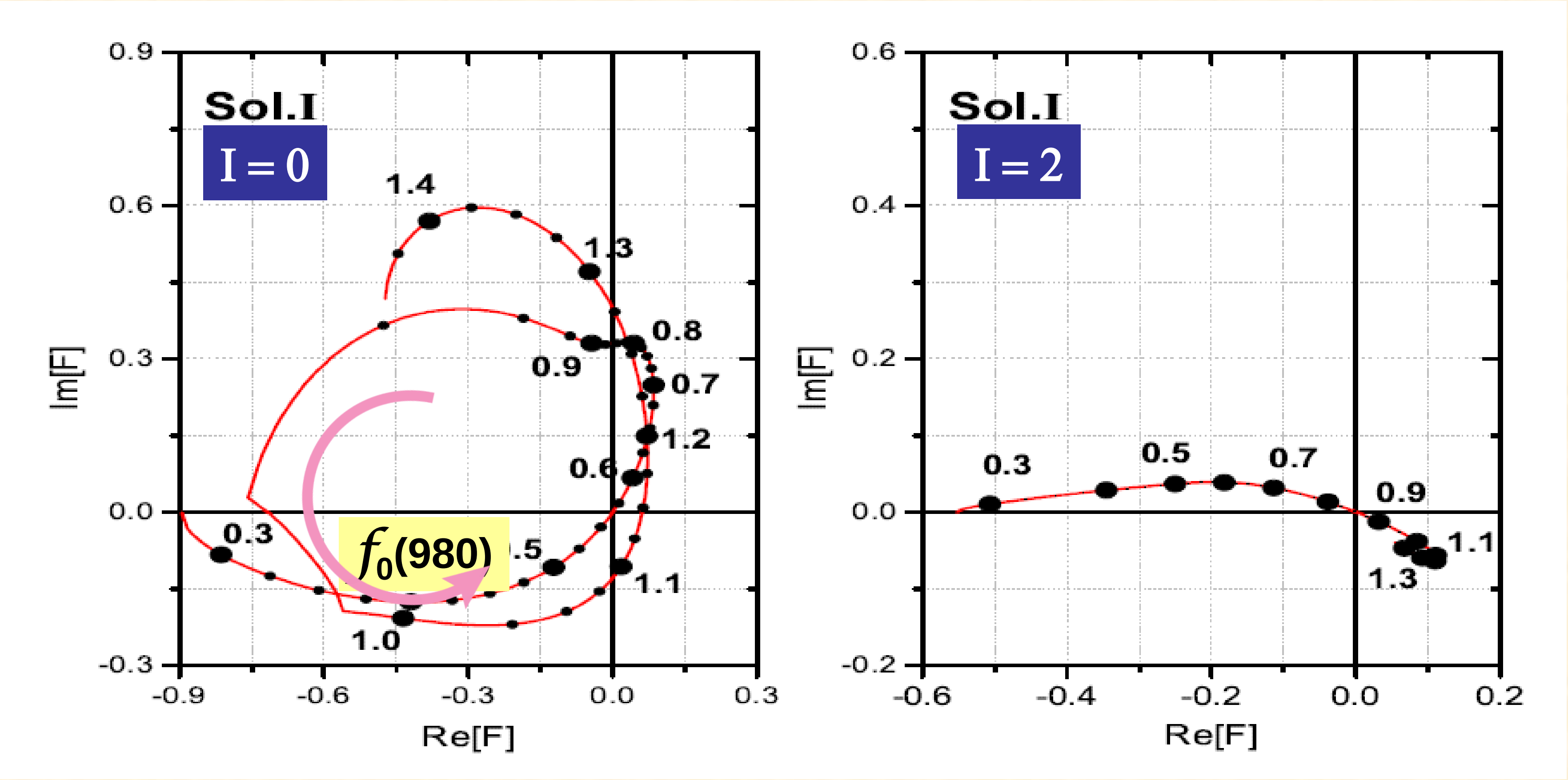
$\gamma\gamma \longrightarrow \pi\pi$ S-waves



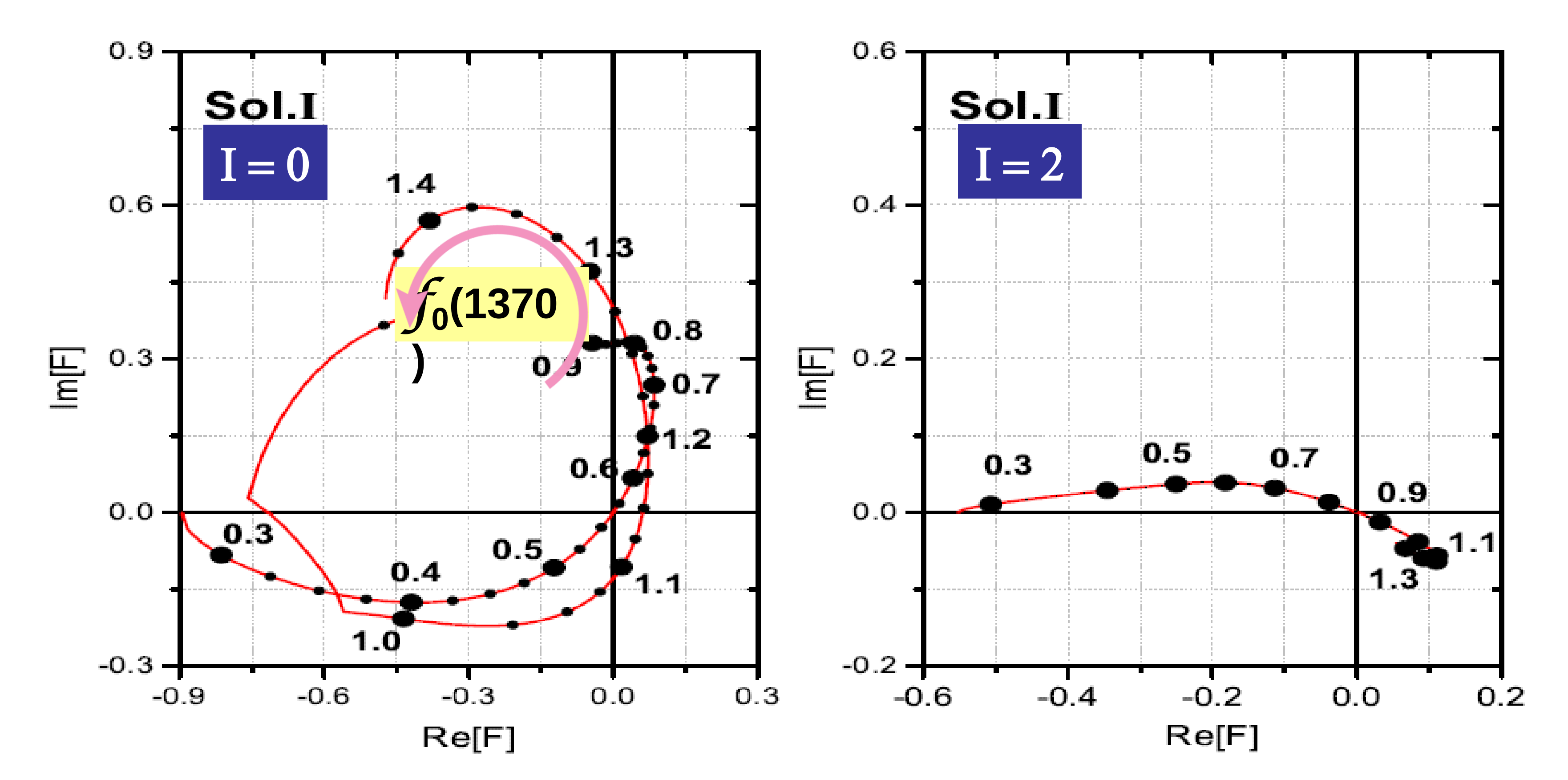
$\gamma\gamma \longrightarrow \pi\pi$ S-waves



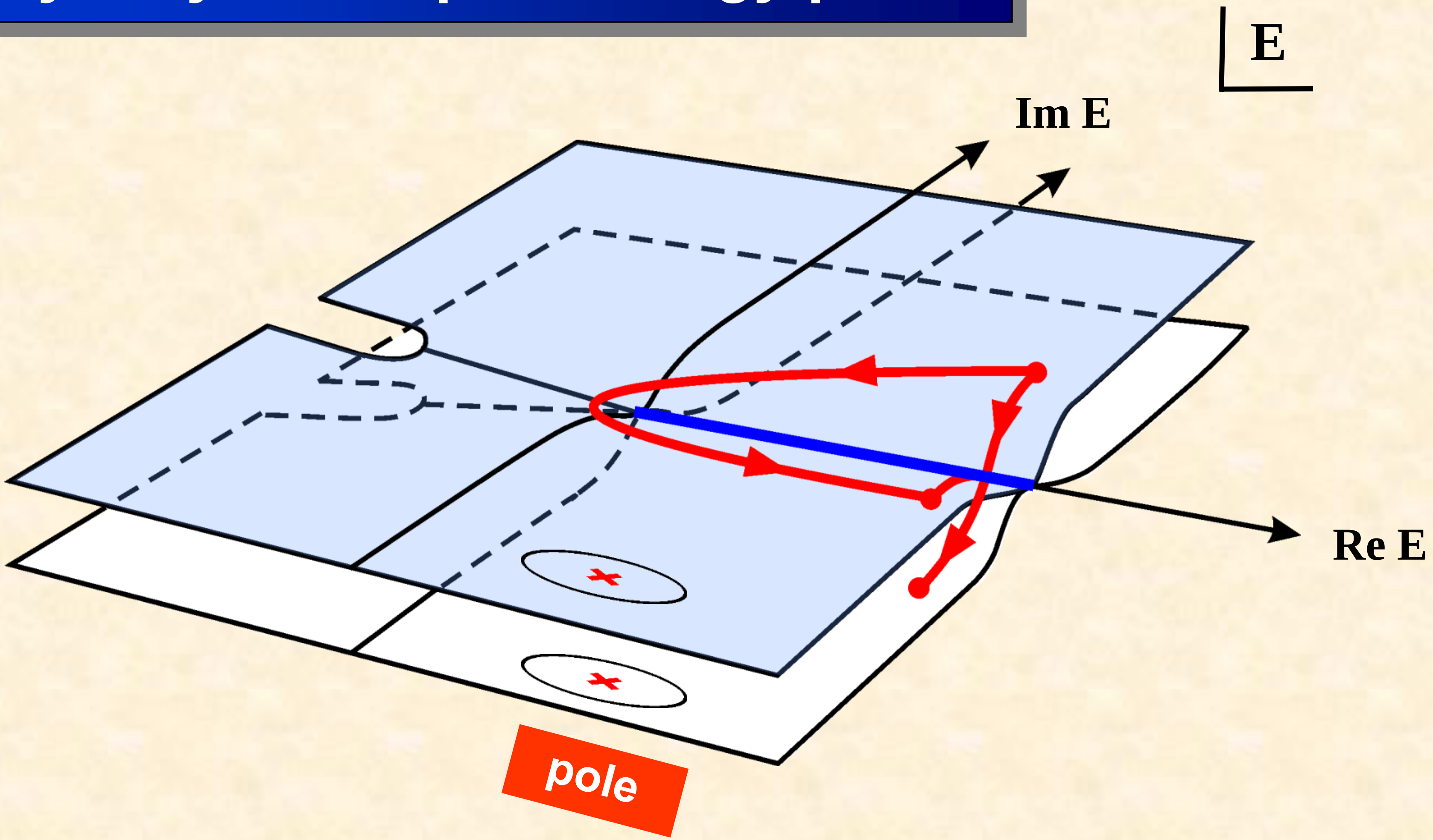
$\gamma\gamma \longrightarrow \pi\pi$ S-waves



$\gamma\gamma \longrightarrow \pi\pi$ S-waves



analyticity & complex energy plane



$$\mathcal{F}^{I}_{J\lambda}(s) = \frac{g_{\pi\pi}\, g_{\gamma\gamma}}{s_R - s} + \ldots$$

model meaning of residue

$$\Gamma(R \to \gamma\gamma) = \frac{\alpha^2}{4\,(2J+1)\,m_R} |g_{\gamma\gamma}|^2$$

γγ couplings

State	Sh	pole locations (GeV)	$g_{\gamma\gamma} = \|g\|e^{i\varphi}$ J_λ	$\|g\|$ (GeV)	φ (°)	$\Gamma(f_J \to \gamma\gamma)$ (keV)	$\lambda = 0$ fraction %	C.L.
$f_2(1270)$	II	$1.270 - i0.081$	D_0	0.37±0.03	172±6	3.49±0.43	8.4±1.4	****
			D_2	1.23±0.08	176±5			
	III	$1.267 - i0.108$	D_0	0.35±0.03	168±6	2.93±0.40	8.7±1.7	****
			D_2	1.13±0.08	173±6			
$a_2(1370)$	IV	$1.313 - i0.053$	D_2	0.72±0.08	174±3	1.04±0.22	$0^\dagger$	**
$f_0(500)$	II	$0.441 - i0.272$	S	0.26±0.01	105±3	2.05±0.21	100	****
$f_0(980)$	II	$0.998 - i0.021$	S	0.16±0.01	-175±5	0.32±0.05	100	****
$f_0(1370)$	II	$1.423 - i0.177$	S	0.96±0.10	8±13	8.6±1.9	100	*
	III	$1.406 - i0.344$	S	0.65±0.15	−146±15	4.0±1.9	100	*

$\gamma\gamma$ couplings

State	Sh	pole locations (GeV)	$g_{\gamma\gamma} = \|g\|e^{i\varphi}$			$\Gamma(f_J \to \gamma\gamma)$ (keV)	$\lambda = 0$ fraction %	C.L.
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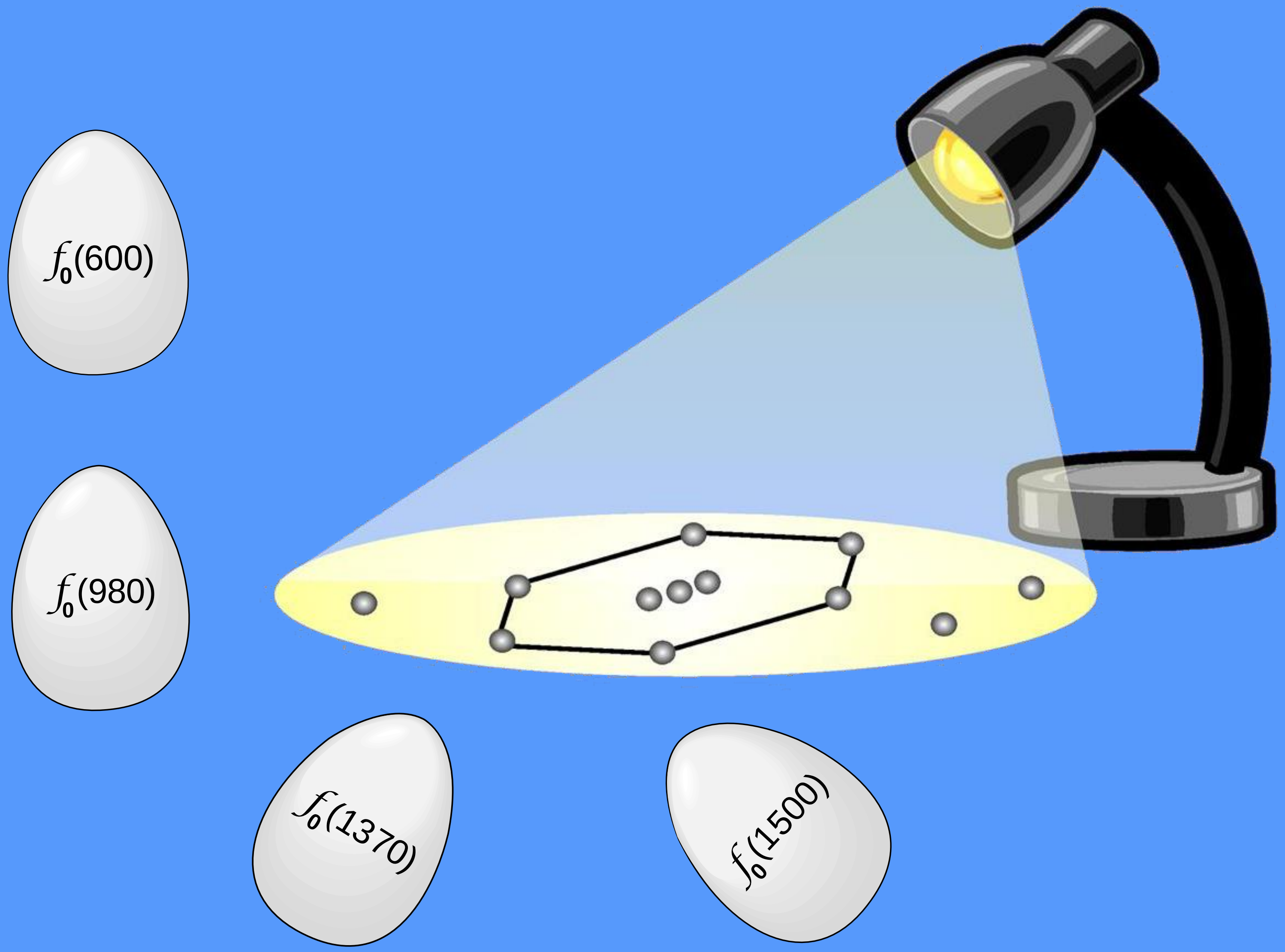
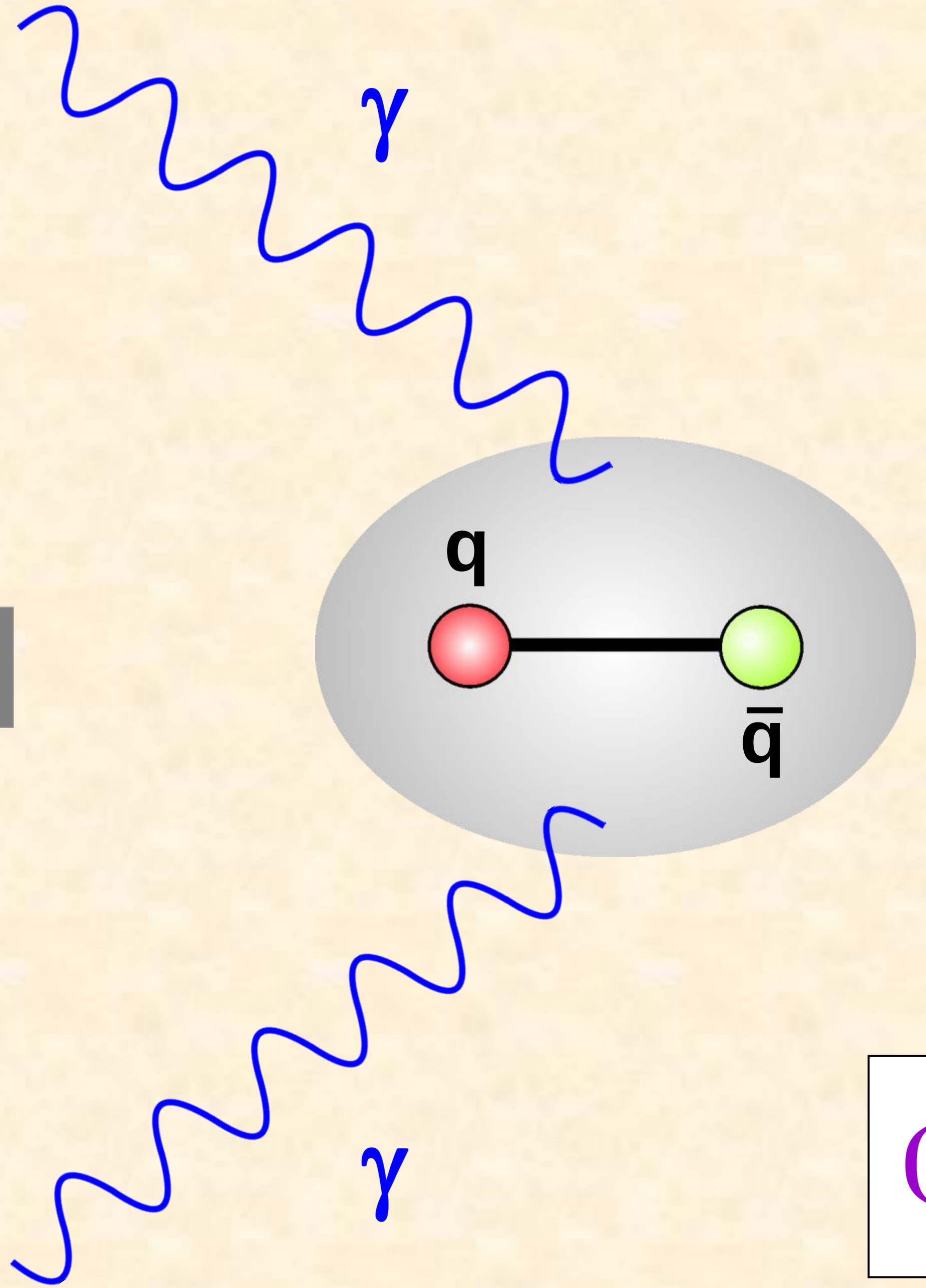
$\Gamma(0^{++} \rightarrow \gamma\gamma)$ model predictions

composition	prediction (keV)	author(s)
$(\bar{u}u + \bar{d}d)/\sqrt{2}$	4.0	Babcock & Rosner [65]
	$< 1^{\dagger}$	Giacosa *et al.* [66]
$\bar{s}s$	0.2	Barnes [67]
	0.062	Giacosa *et al.* [68]
$\overline{[ns]}[ns]$	0.27	Achasov *et al.* [69]
$\overline{K}K$	0.6	Barnes [70]
	0.22	Hanhart *et al.* [72]
gg	0.2–0.6	Narison [73]

Shedding light on scalar mesons



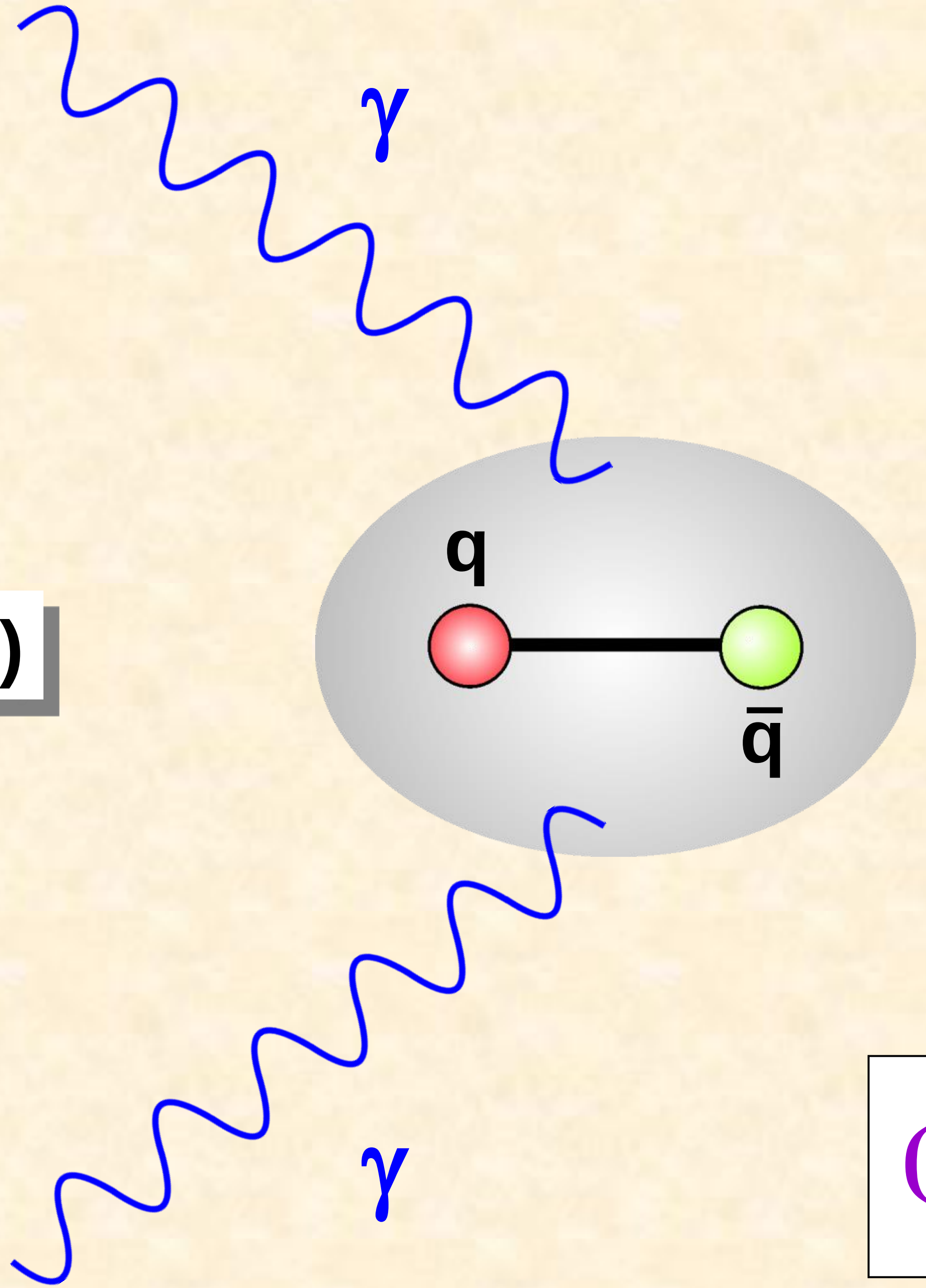
$q\bar{q}$



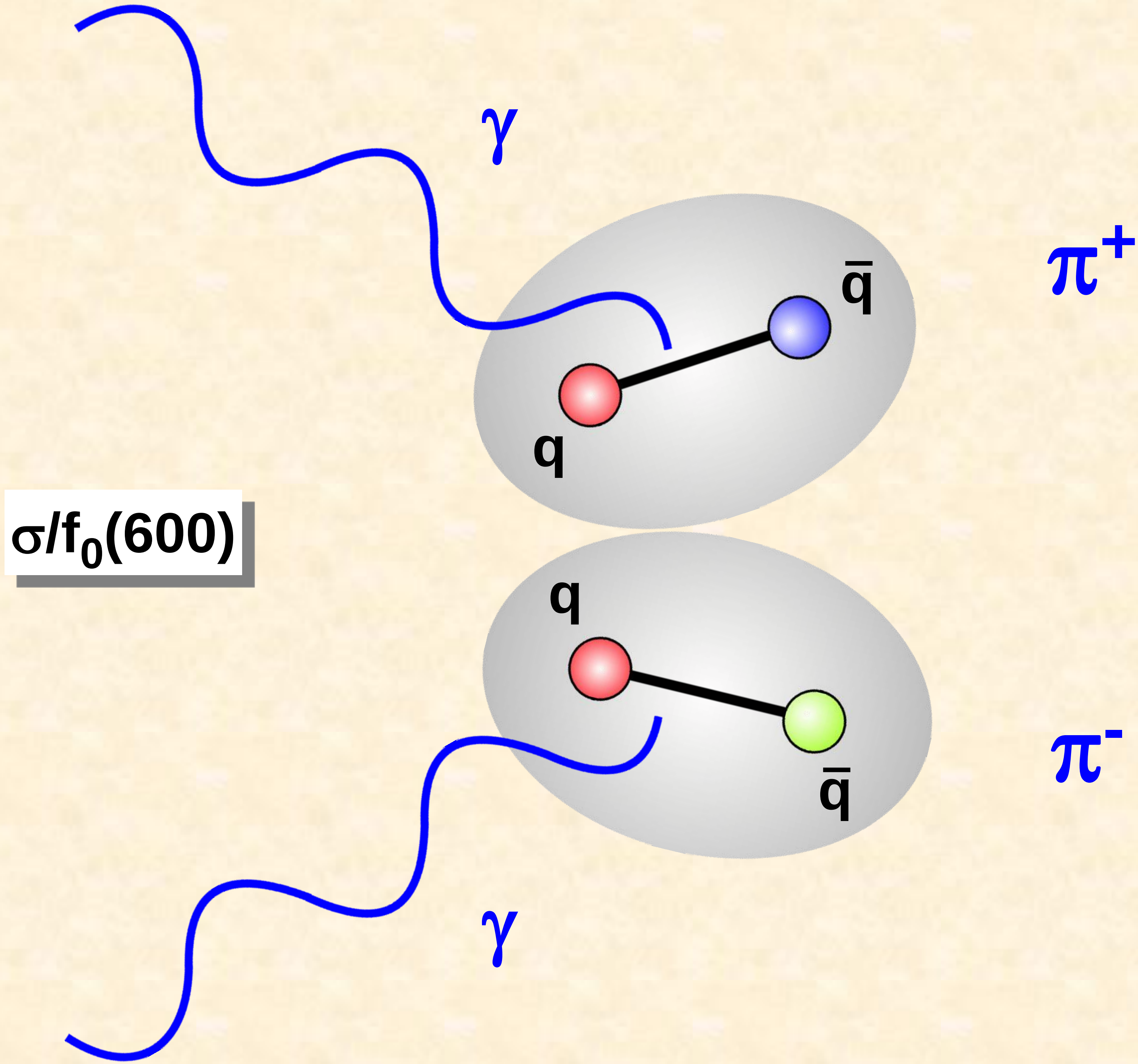
$|\Psi(0)|^2$

$$\left(\sum_q \langle e_q^2 \rangle\right)^2 \Pi_R$$

$f_2(1270)$

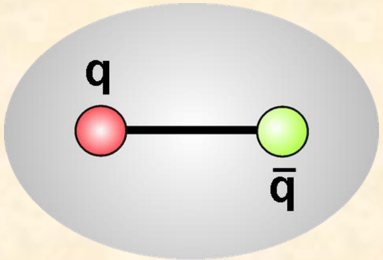


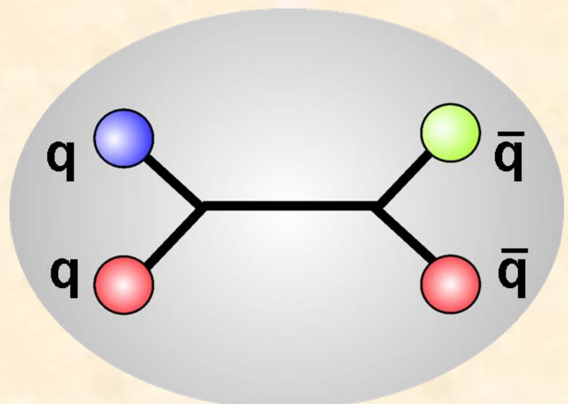
$$\left(\sum_q \langle e_q^{\,2} \rangle \right)^2 \Pi_R$$

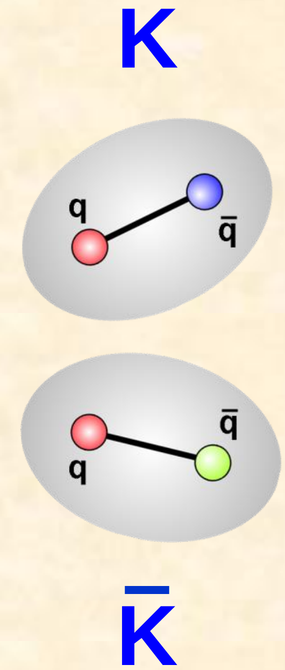

γ
$\bar{q}$
π^+
q
σ/f0(600)
q
π^-
$\bar{q}$
γ

$f_0(980)$

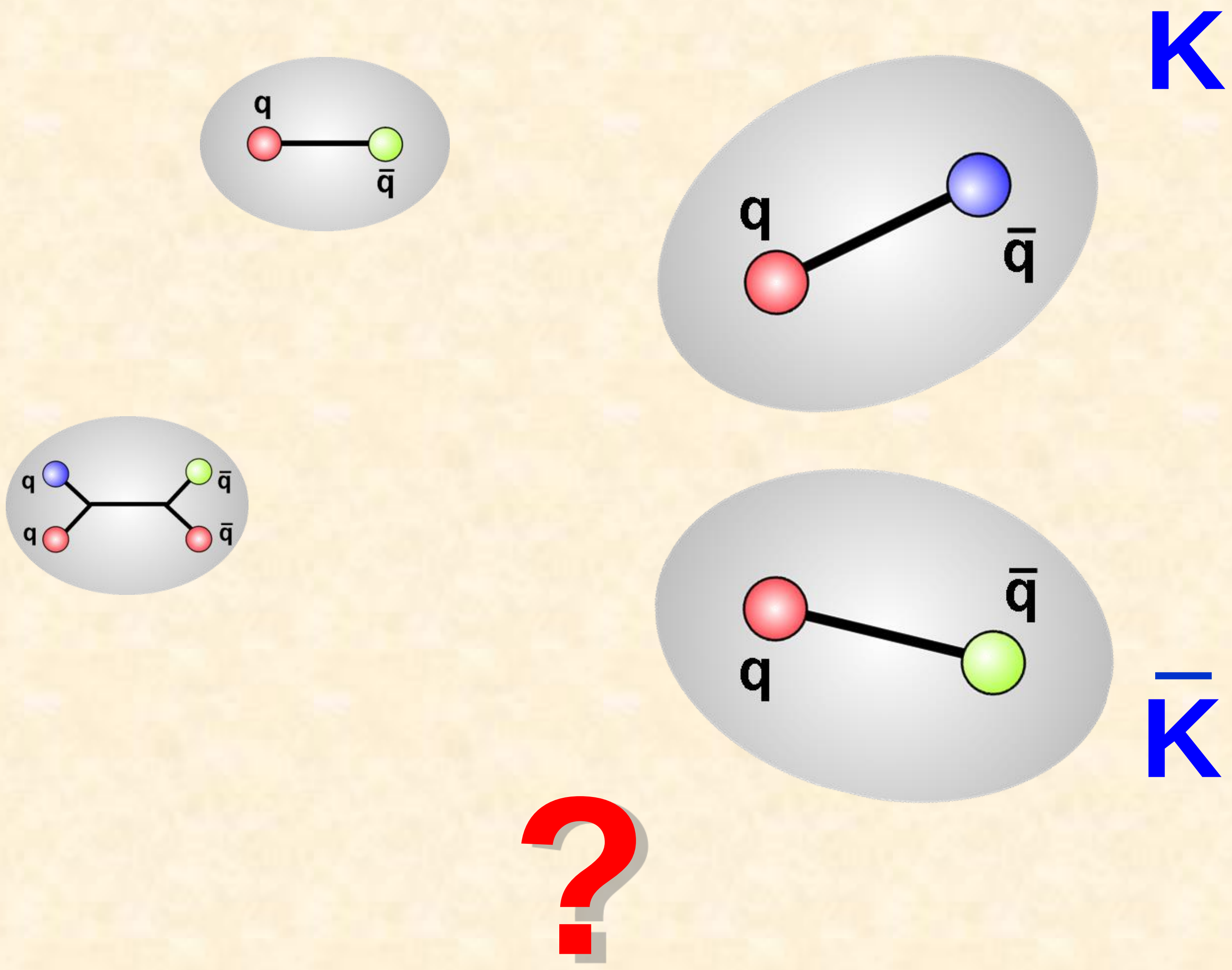
?

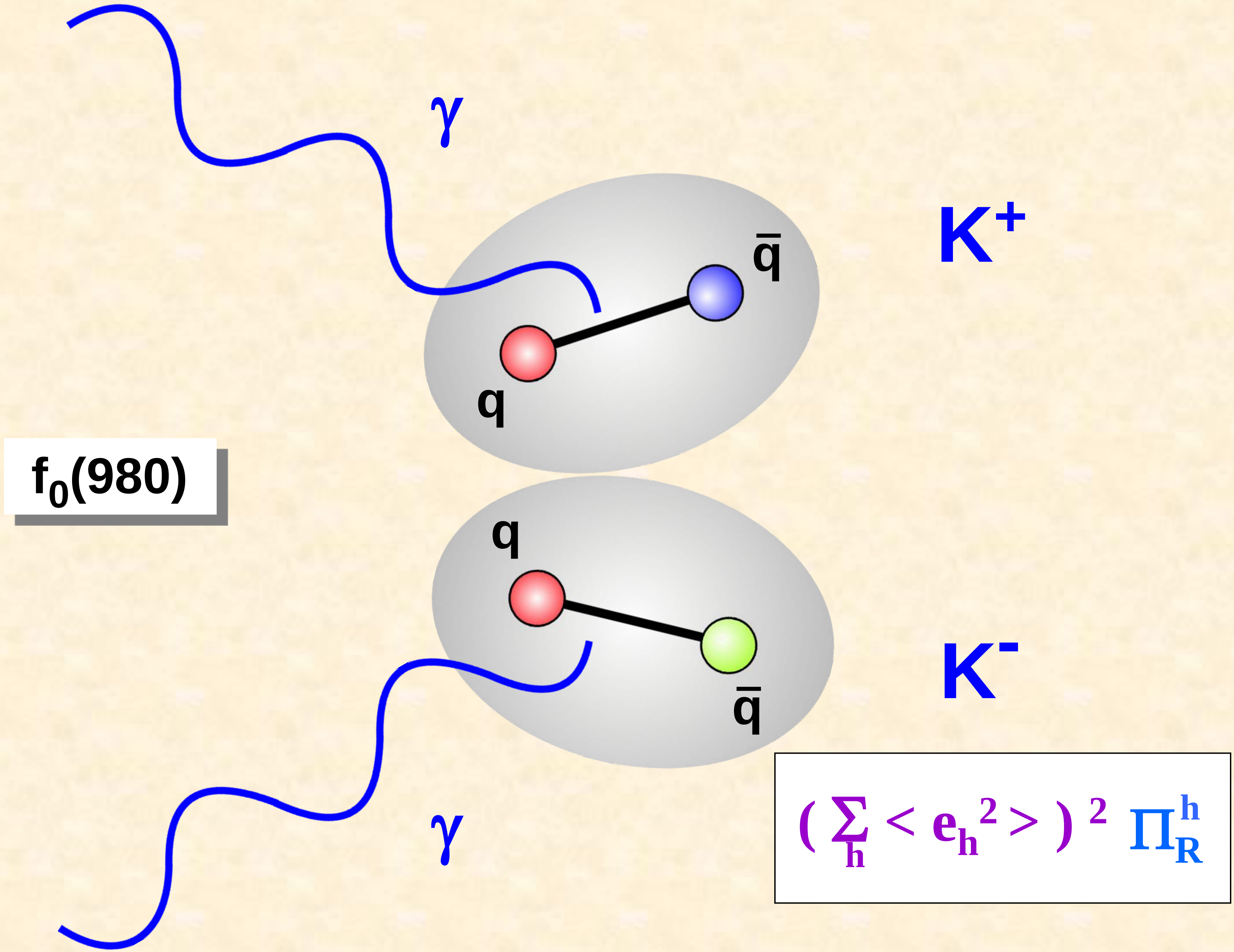






$f_0(980)$




γ
K+
q̄
q
f0(980)
q
K-
q̄
γ
(Σh < eh2 >) 2 ΠhR

$\Gamma(0^{++} \to \gamma\gamma)$ model predictions

composition	prediction (keV)	author(s)
$(\bar{u}u + \bar{d}d)/\sqrt{2}$	4.0	Babcock & Rosner [65]
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$\bar{K}K$	0.6	Barnes [70]
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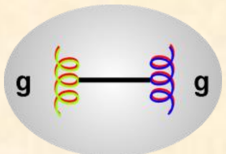
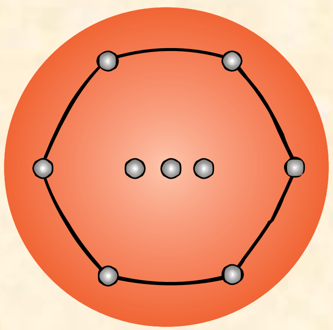
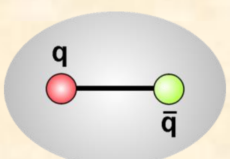
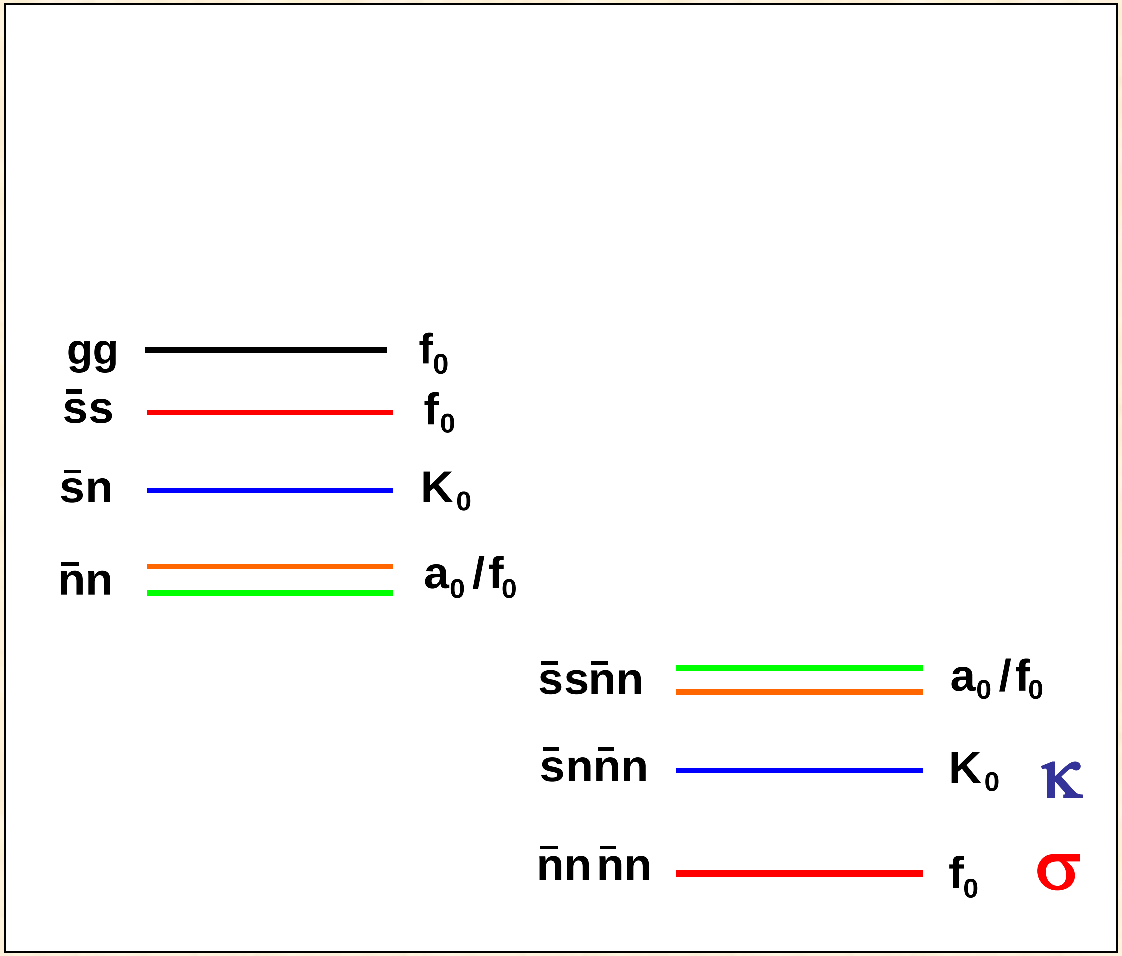
gg	0.2–0.6	Narison [73]

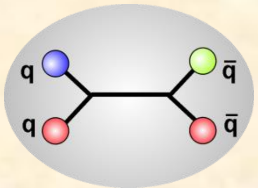
Scalar mesons

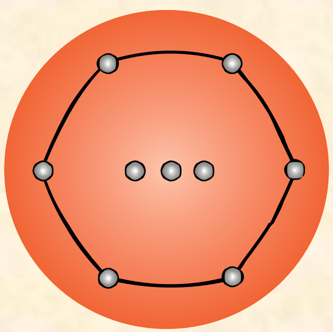






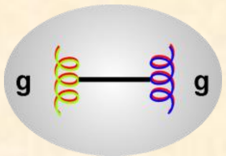


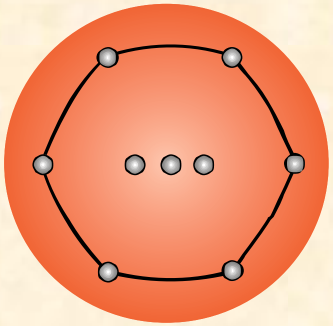


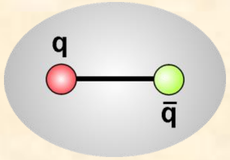


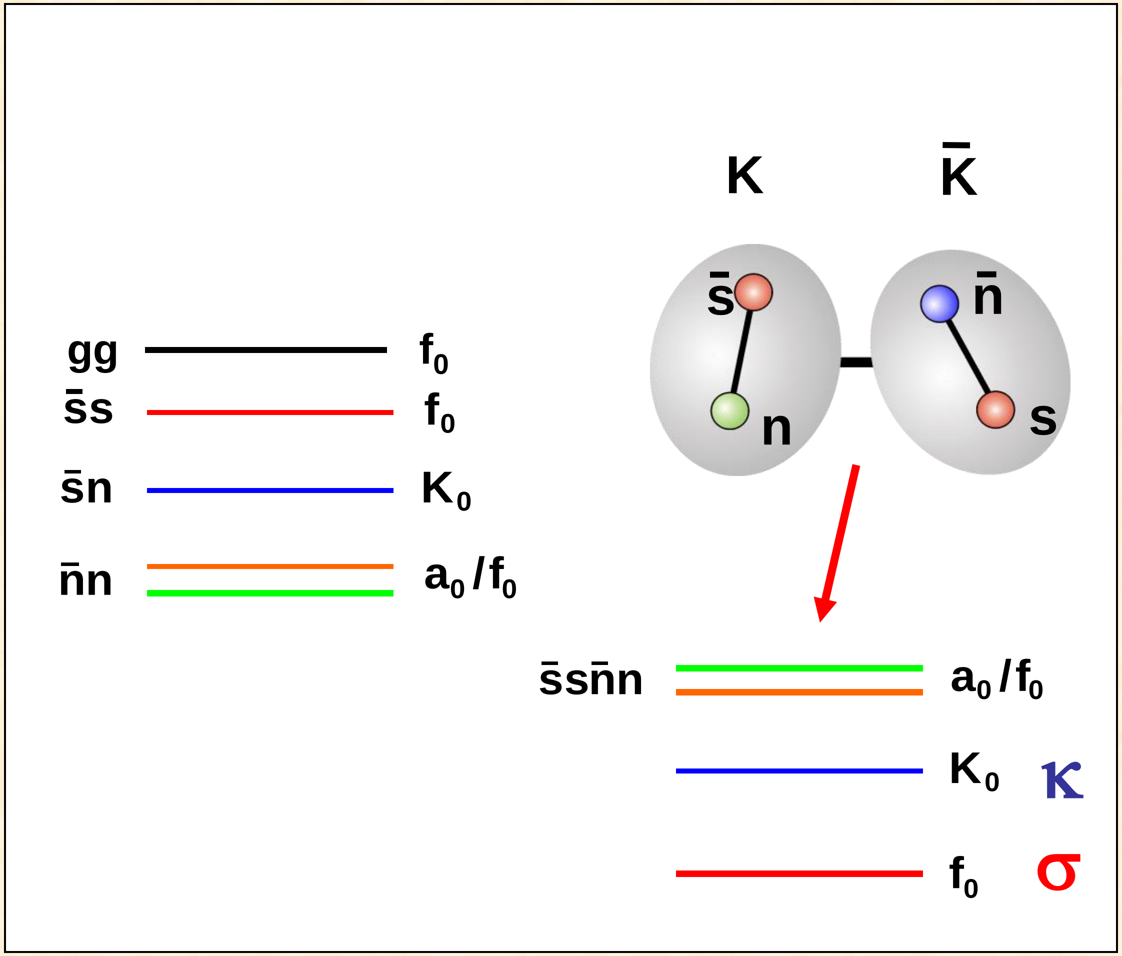
n = u,d

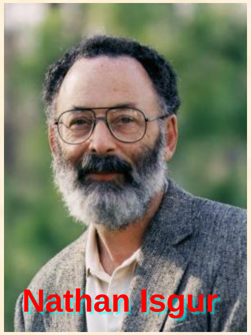
Scalar mesons





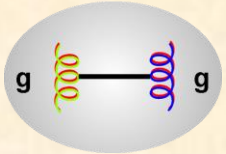


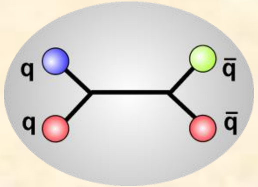


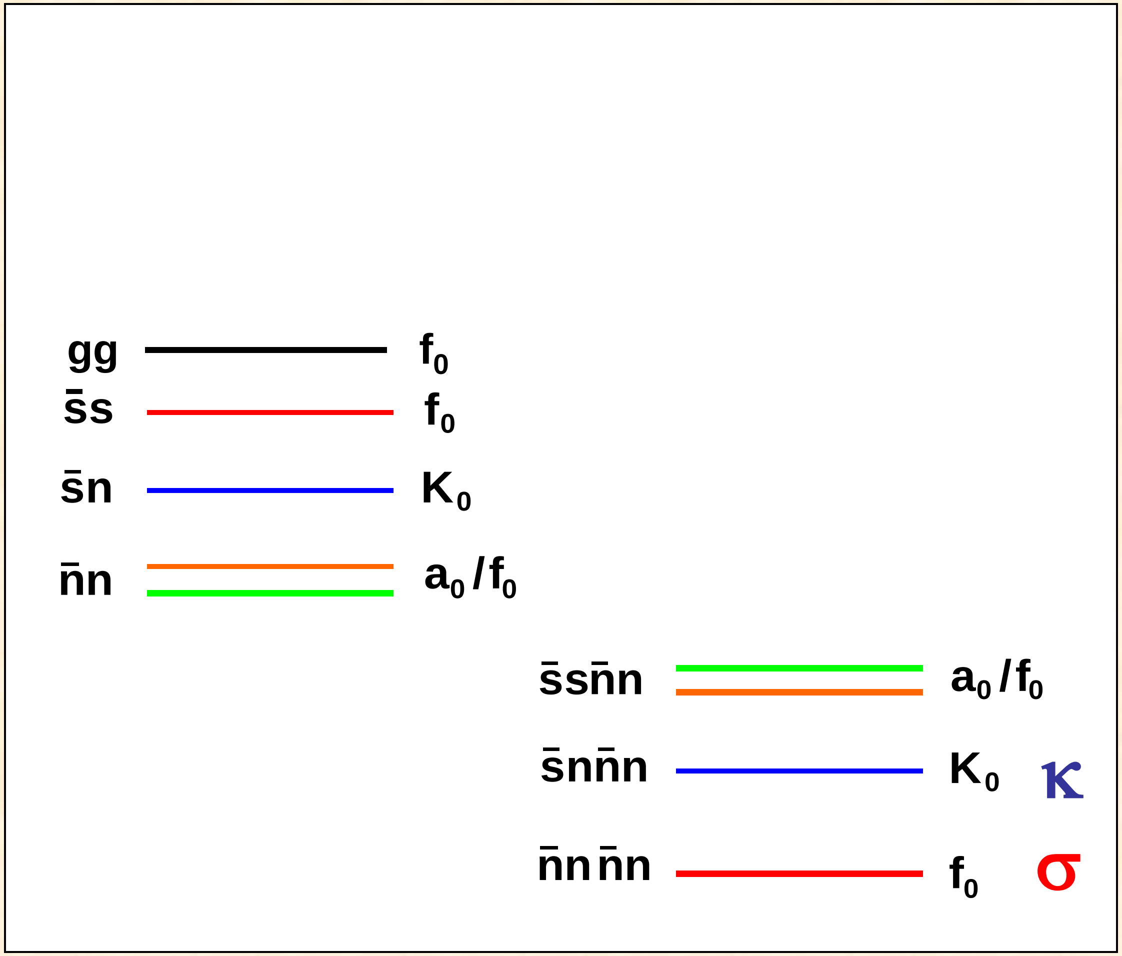


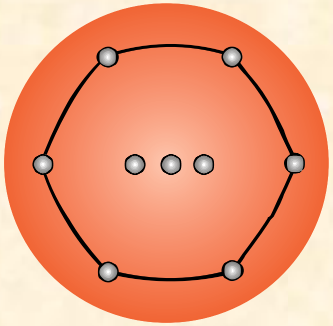
n = u,d

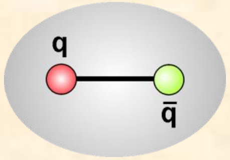
Scalar mesons

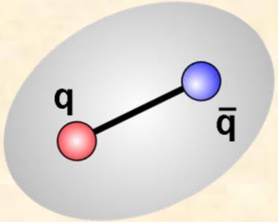


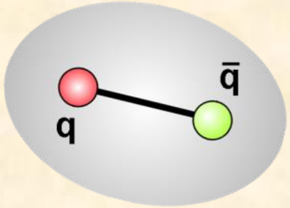


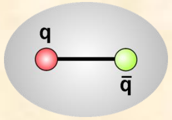


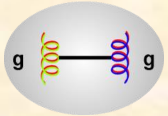




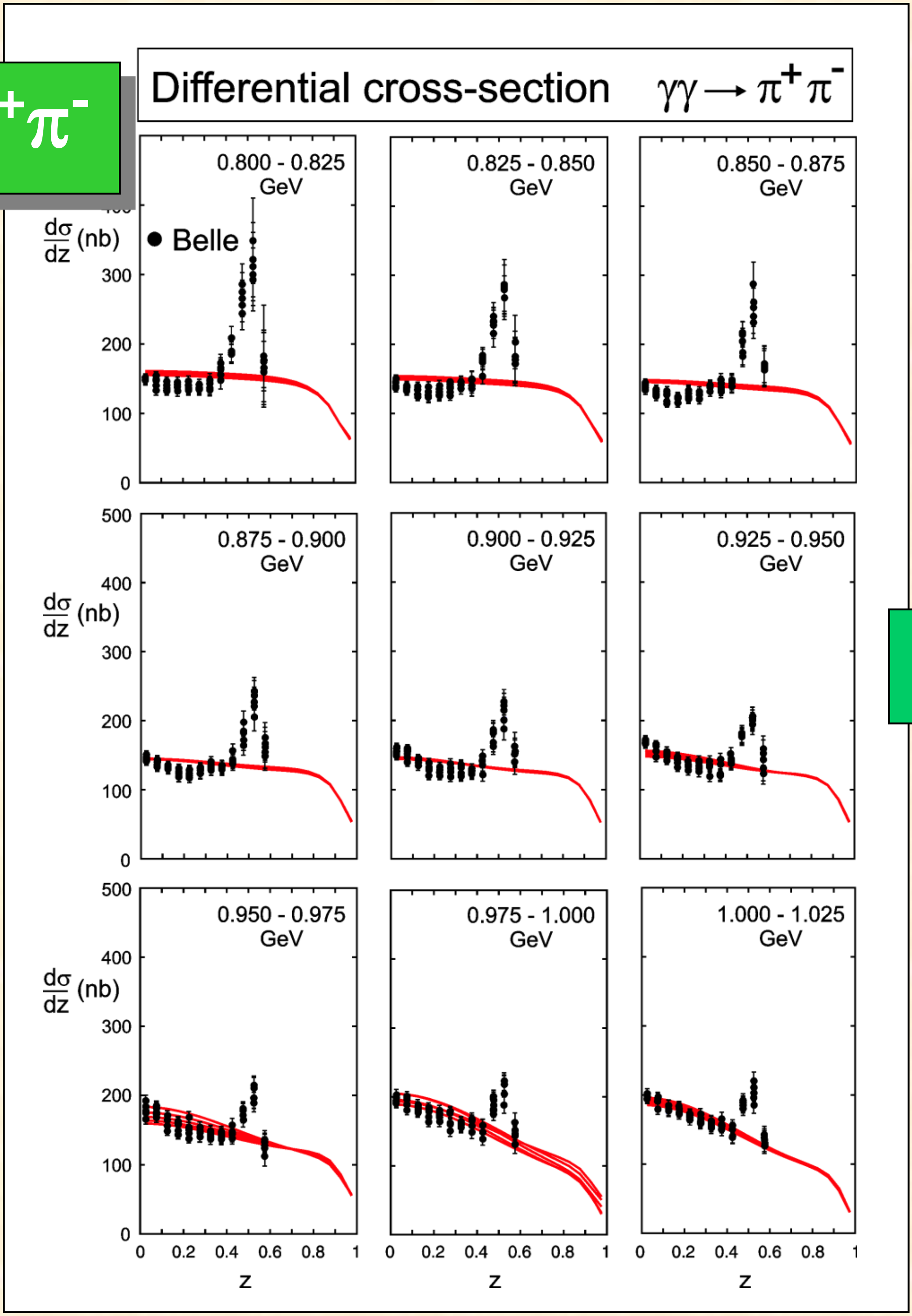








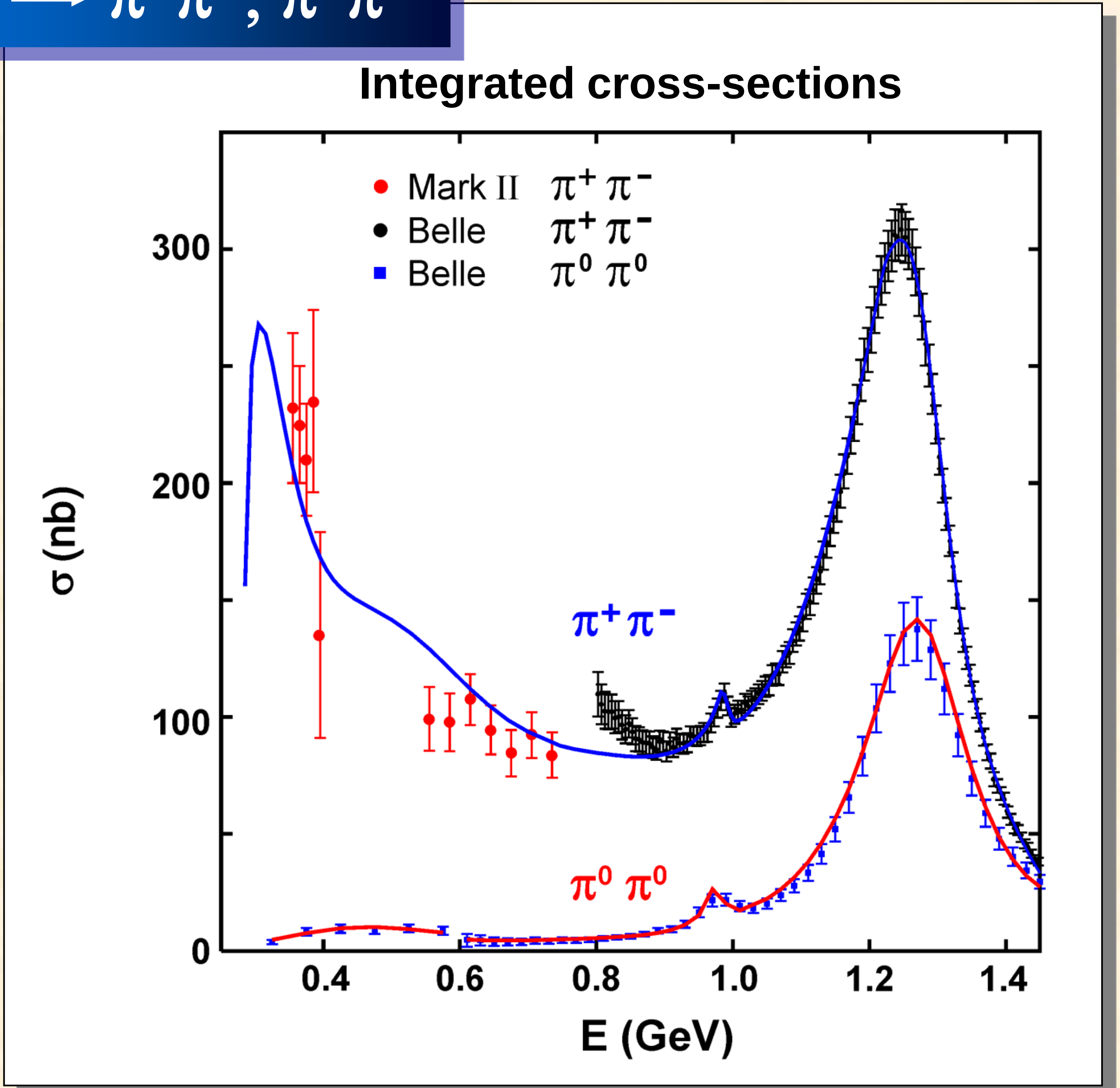
$\gamma\gamma \longrightarrow \pi^+\pi^-$



$\mu^+\mu^-$ contamination

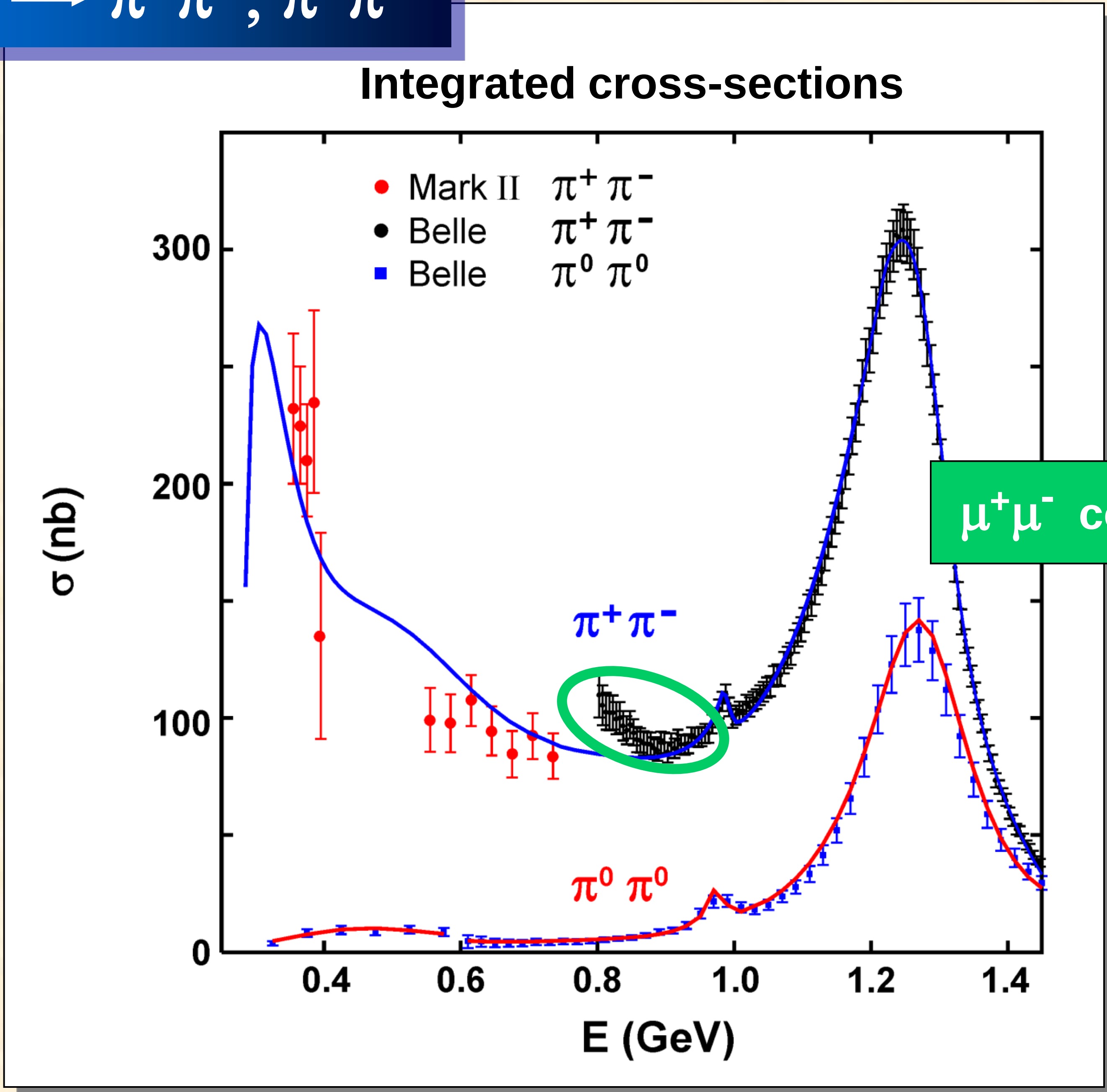
800-1025 MeV

$\gamma\gamma \longrightarrow \pi^+\pi^-, \pi^0\pi^0$



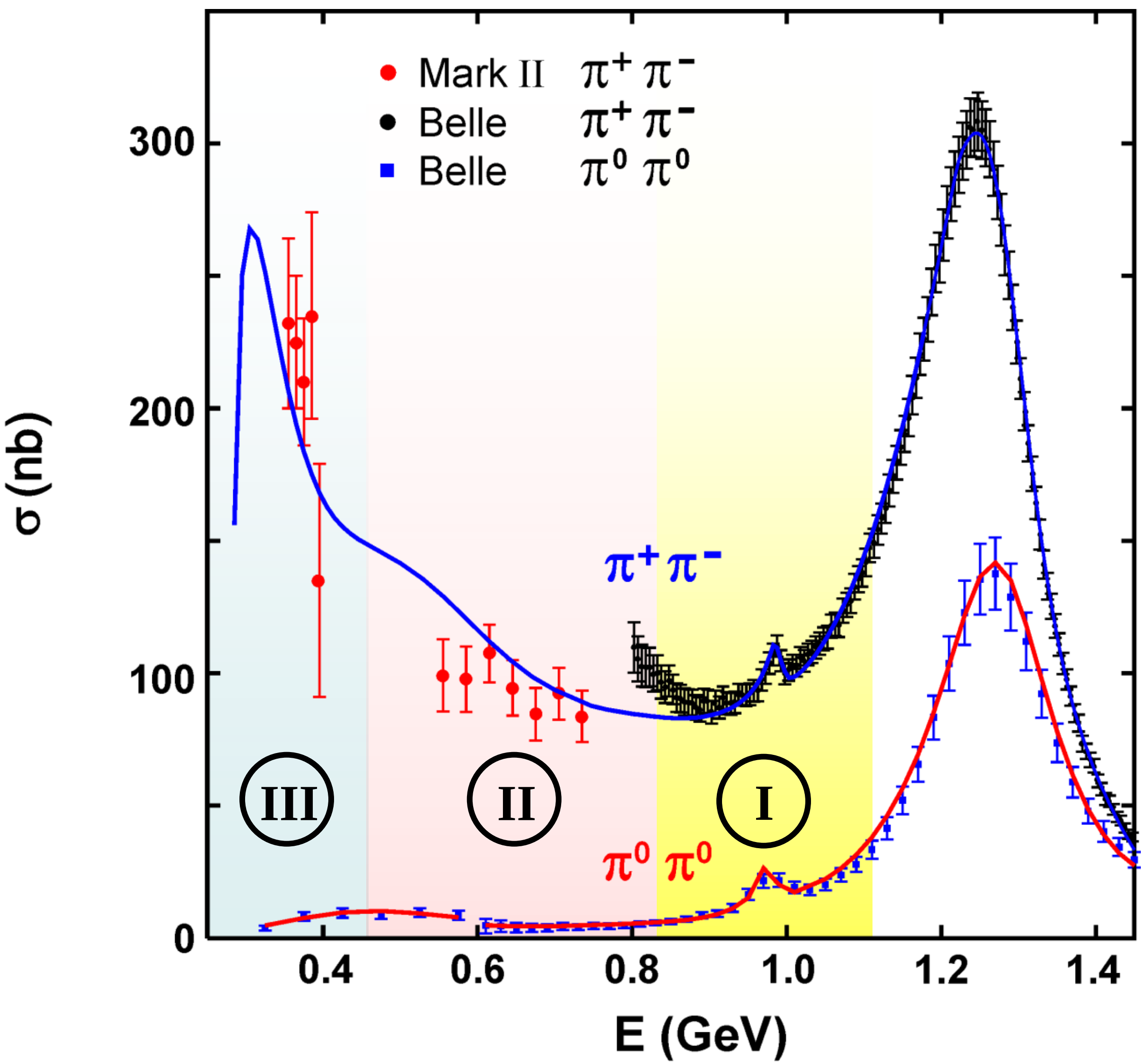
Dai & P

$\gamma\gamma \longrightarrow \pi^+\pi^-, \pi^0\pi^0$

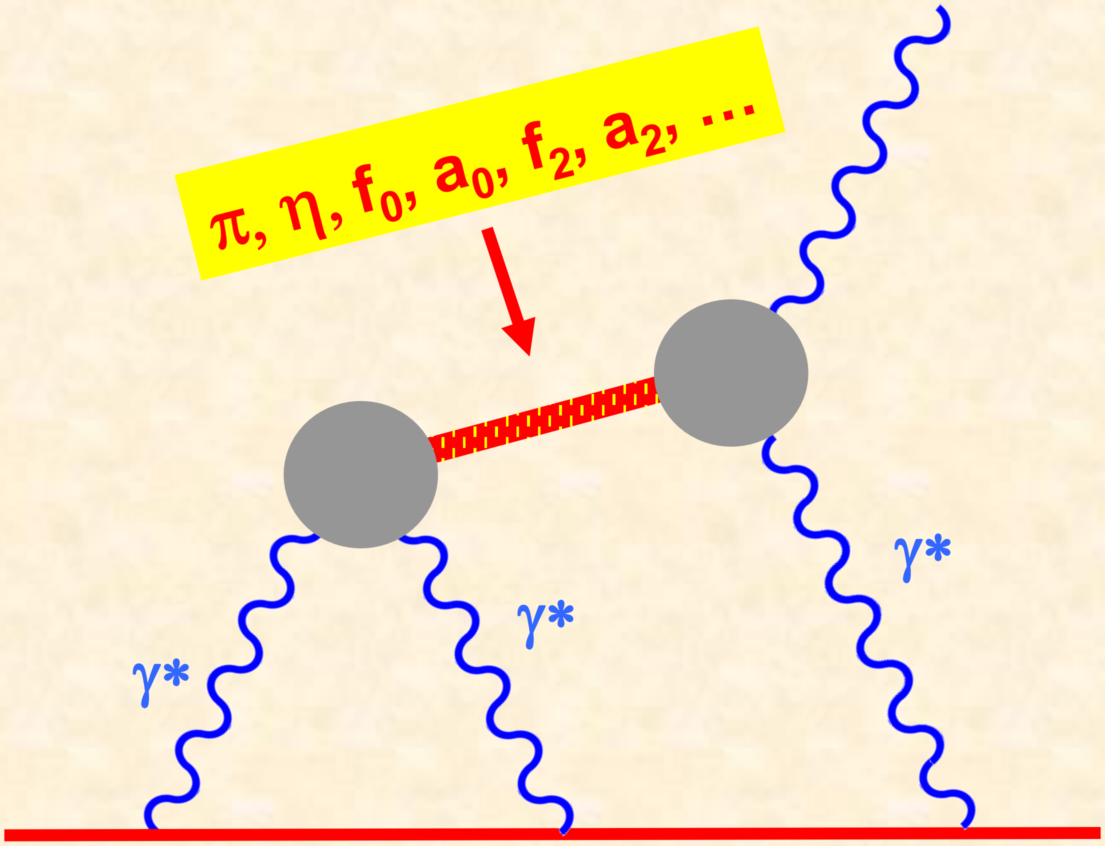


$\mu^+\mu^-$ contamination

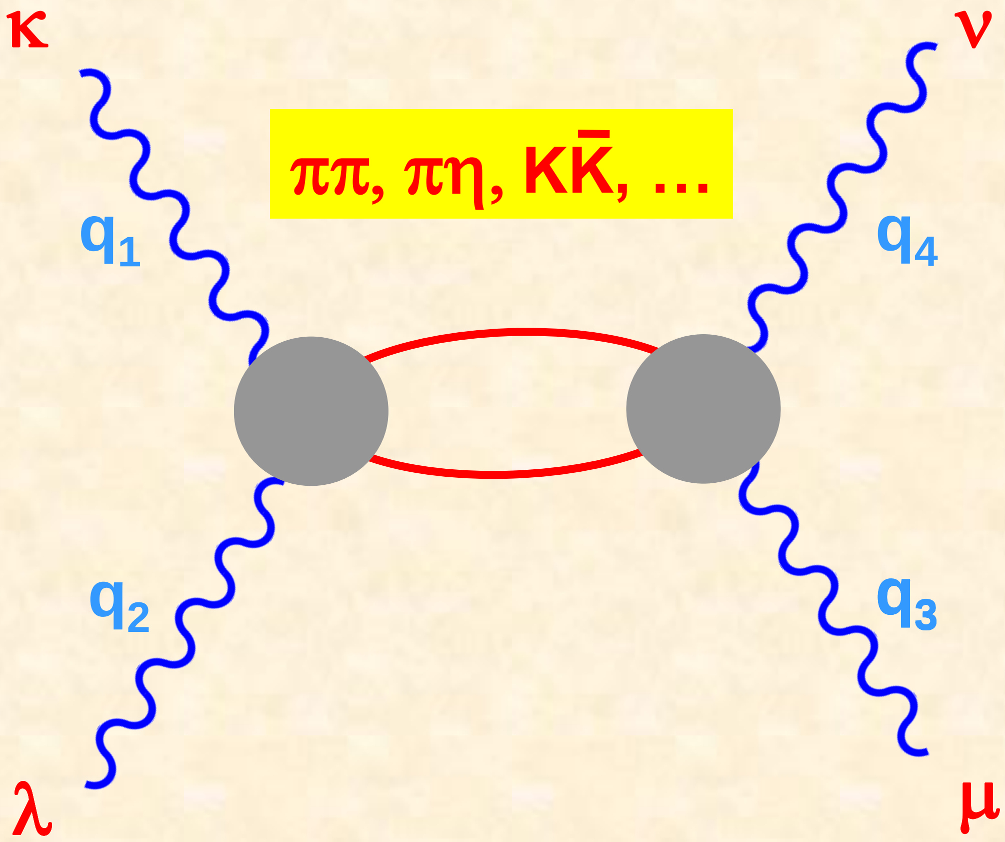
Integrated cross-section



Light by Light

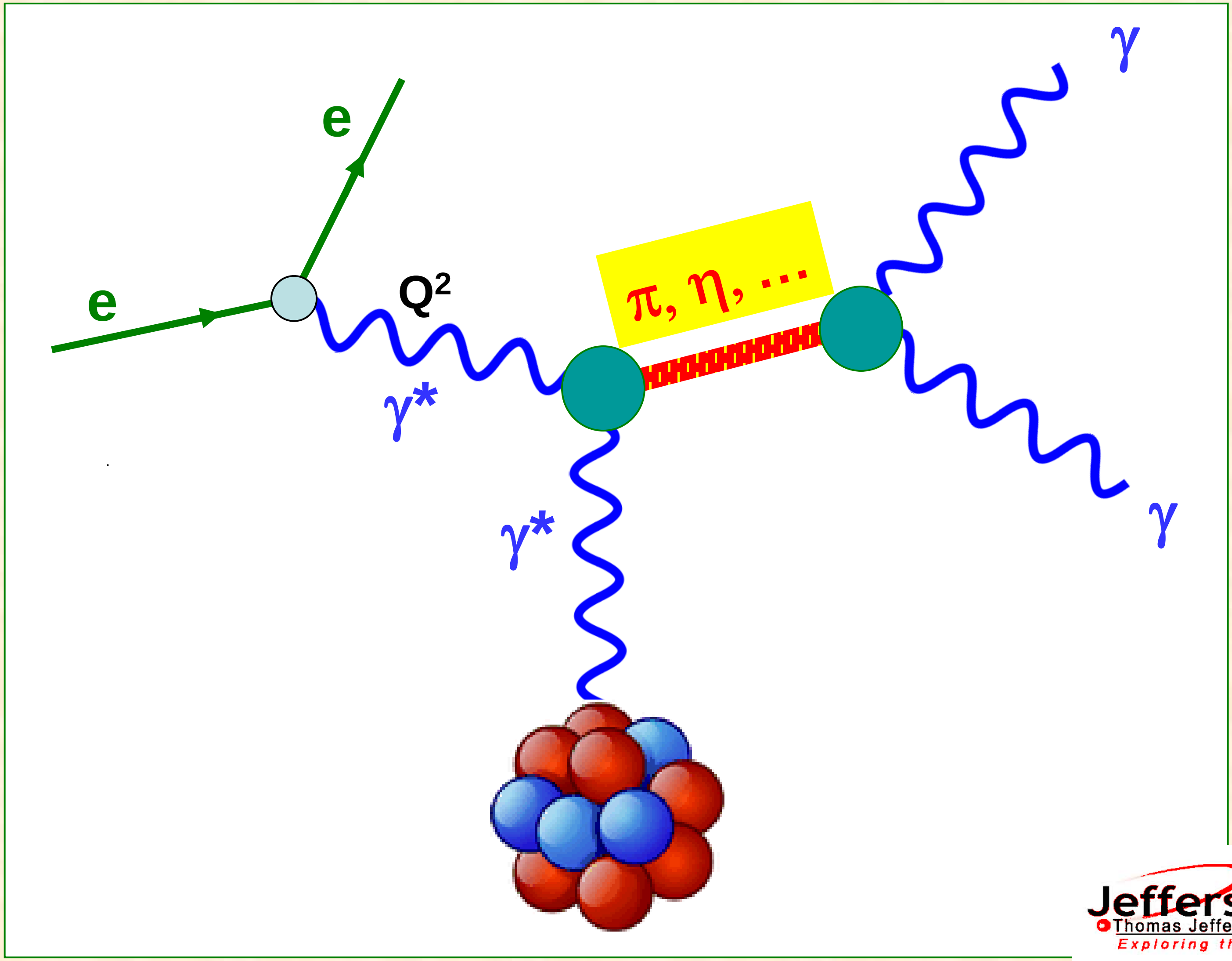


Light by Light



Light* by Light

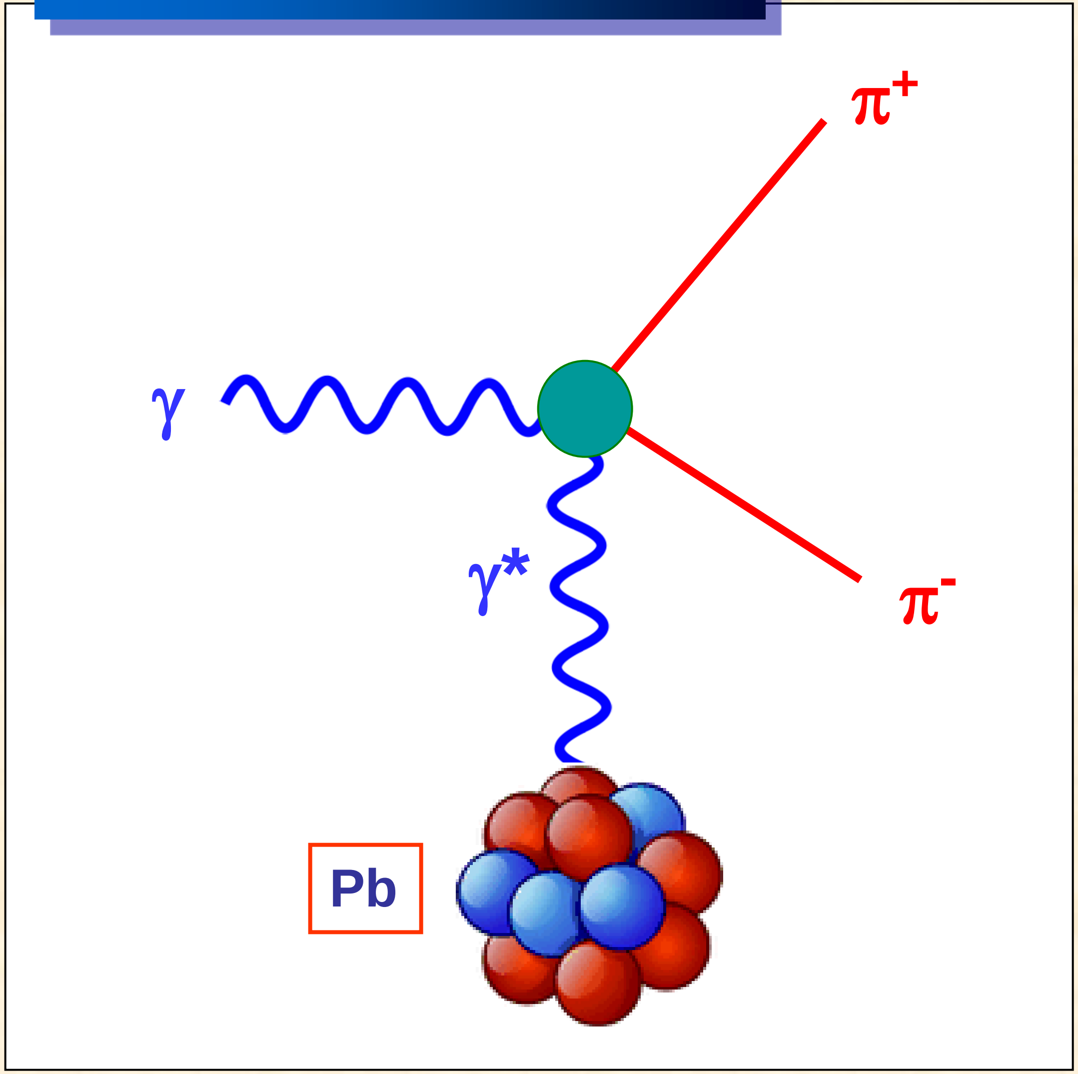
PRIMEX: E12-10-011

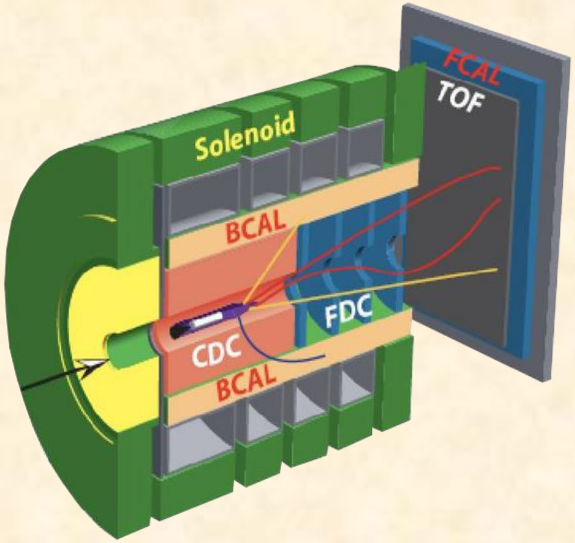


Jefferson Lab
Thomas Jefferson National Accelerator Facility
Exploring the Nature of Matter

Light by Light*

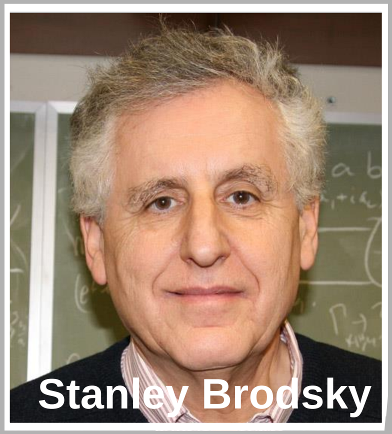
GlueX: E12-13-008



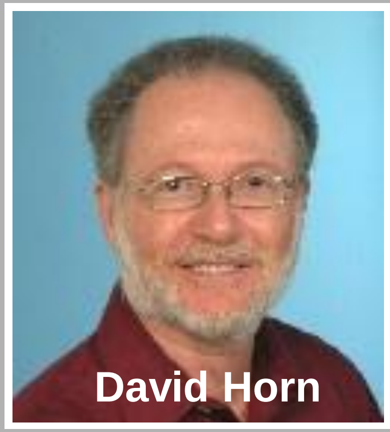


Hall D@JLab


Edmond Berger


Stanley Brodsky


Geoffrey Fox


David Horn


Jonathon Rosner


Michael Pennington


Timothy Londergan


Peter Landshoff