# Veneziano model and application to Dalitz plot analysis

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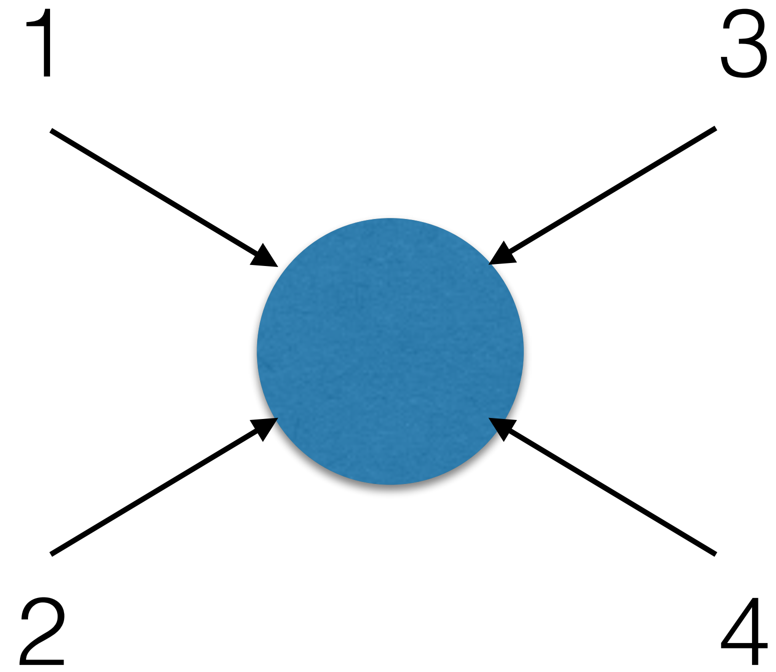
Motivation

Properties

Application to $J/\psi \to 3\pi$ decays

Generalizations (dual models)

# 2-to-2 kinematics
# e.g. $\pi^0 \pi^0 \to \pi^0 \pi^0$ : A(s,t,u)



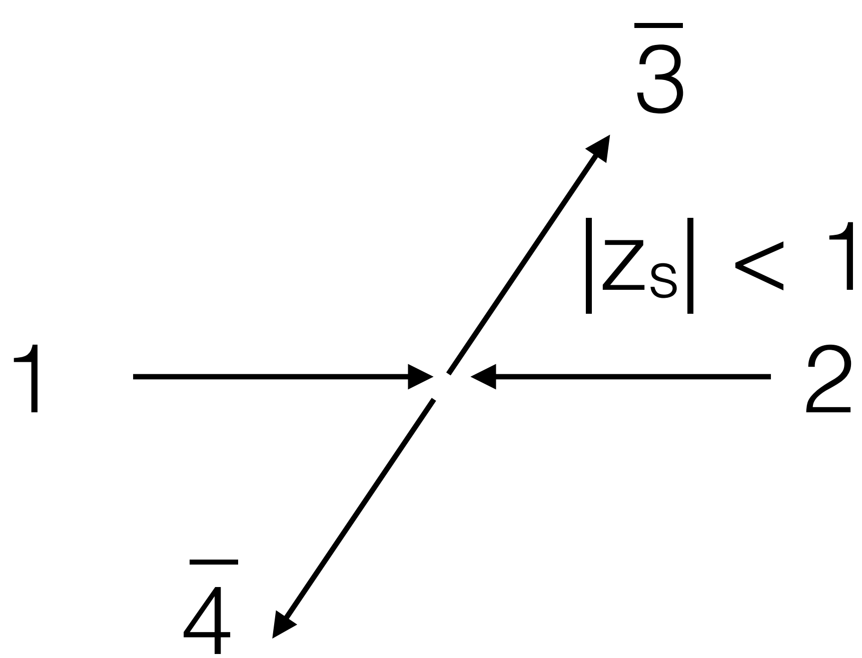

$$s = (1 + 2)^2 \qquad t = (1 + 3)^2 \qquad u = (1 + 4)^2$$

$$s + t + u = 4m^2$$

physical domain
in the s-channel

$$E_{\bar{3}} = -E_3 > 0$$

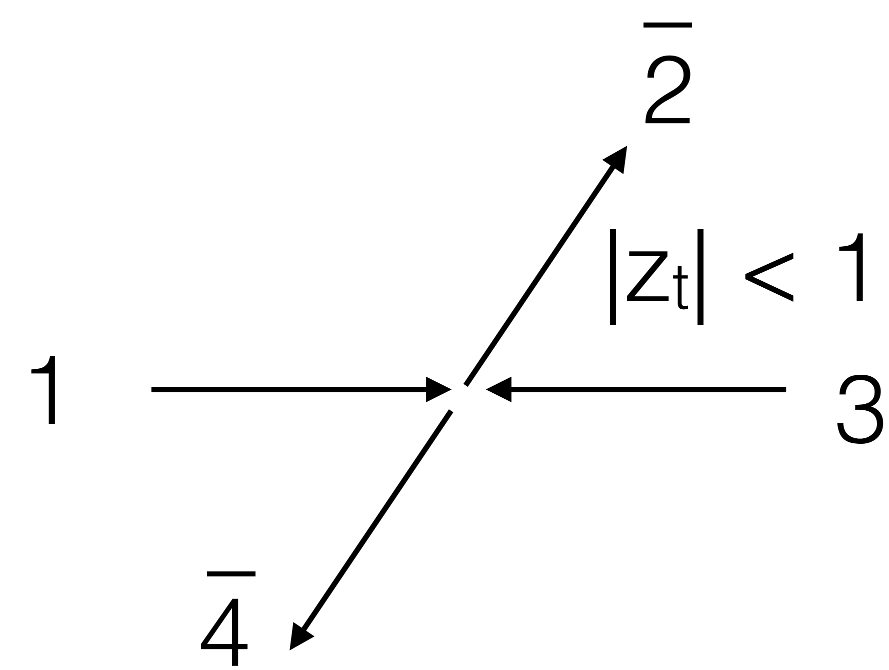
$$E_{\bar{4}} = -E_4 > 0$$

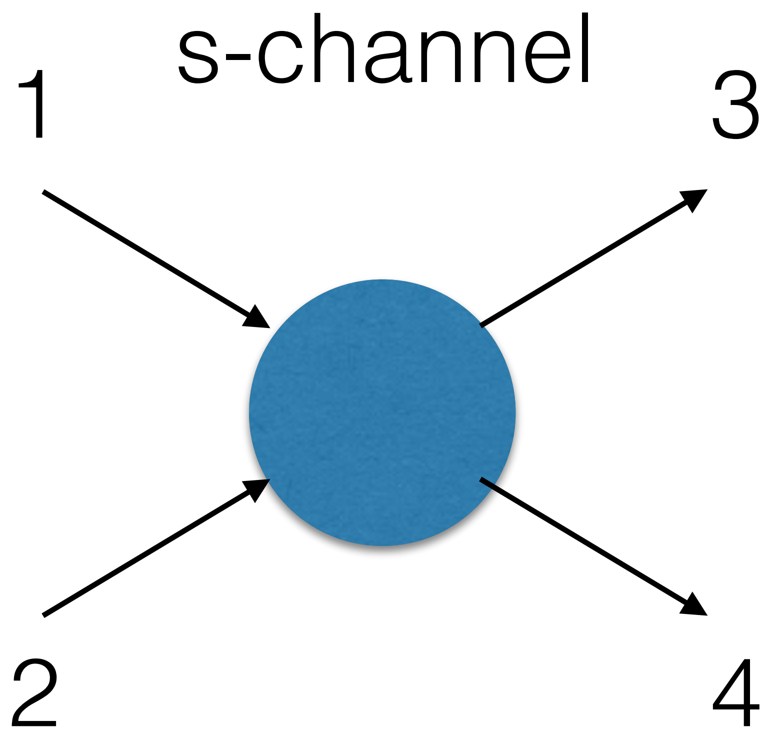



physical domain
in the t-channel

$$E_{\bar{2}} = -E_2 > 0$$

$$E_{\bar{4}} = -E_4 > 0$$

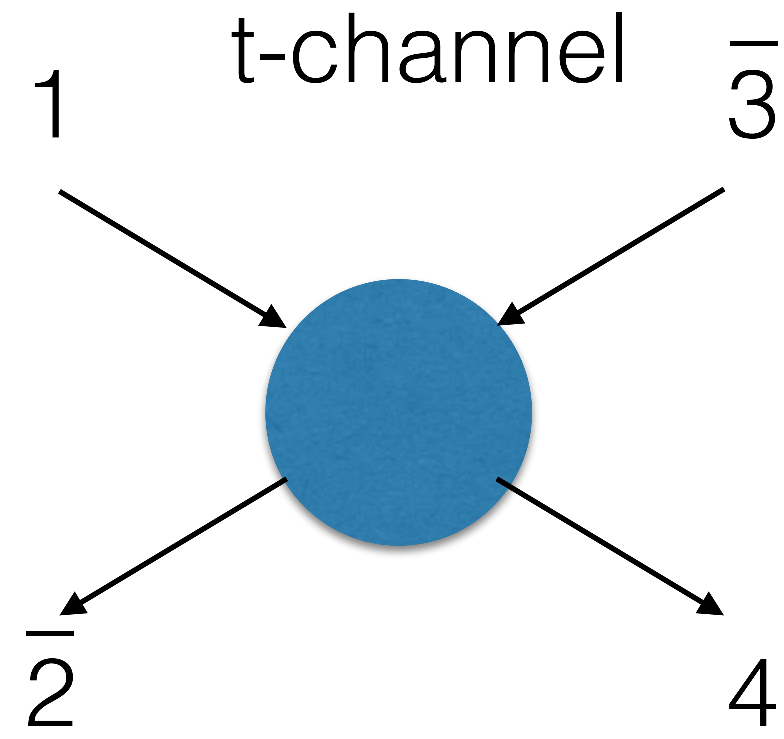





1 + 2 -> 3 + 4

peaks in s from resonances in (12)

$$A(s,t)$$




1 + $\bar{3}$ -> $\bar{2}$ + 4

peaks in t from resonances in (1$\bar{3}$)

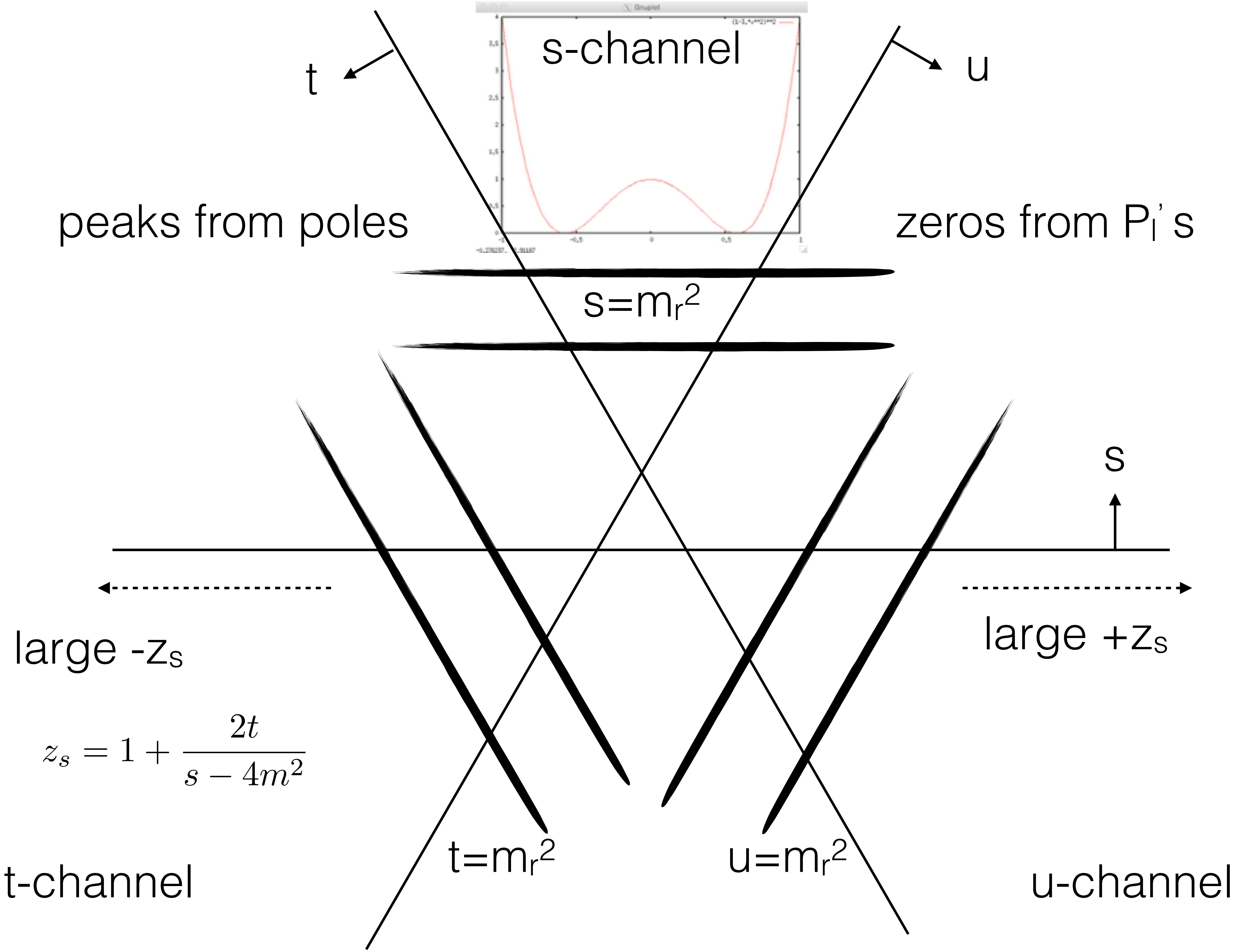
$$A(s,t) = \sum_{l=0}^{\infty} (2l+1) A_l(s) P_l(z_s)$$

$$z_s = 1 + \frac{2t}{s - 4m^2}$$

$$A(s,t) = \sum_{l=0}^{\infty} (2l+1) A_l(t) P_l(z_t)$$

$$z_t = 1 + \frac{2s}{t - 4m^2}$$

**to reproduce peaks in t (or s) need to continue the s (or t) channel p.w. sum outside its domain of convergence**


s-channel
t
u
peaks from poles
zeros from Pl's
s=mr2
s
large -zs
large +zs
$z_s = 1 + \frac{2t}{s - 4m^2}$
t=mr2
u=mr2
t-channel
u-channel

# Analytical continuation: simple example

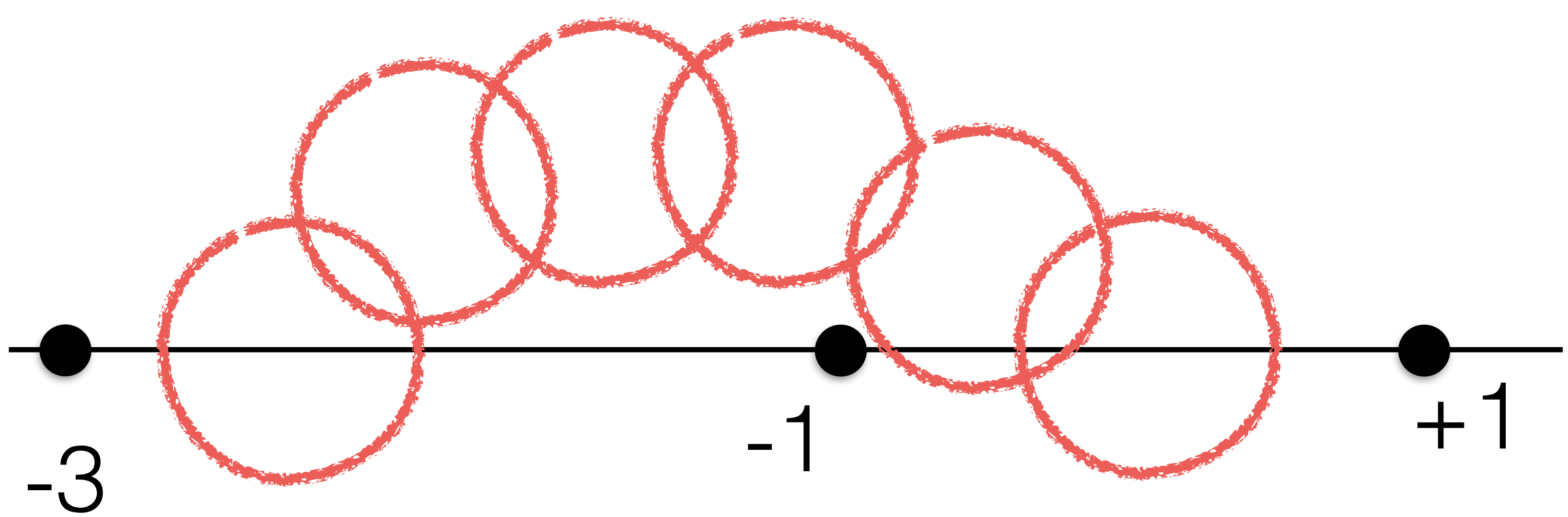
$$A(s,t) = A(s,t(s,z_s)) = \sum_{l=0}^{\infty}(2l+1)A_l(s)P_l(z_s)$$

$$t(s,z_s) = -(1-z_s)\frac{s-4m^2}{2}$$

$|z_s| < 1$ in the s-channel and $z_s < -1$ in the t-channel

$$A(z_s) = \sum_{l=0}^{\infty}(-z_s)^l = 1 - z_s + z_s^2 + \cdots$$

well defined for $|z_s| < 1$




$$A(z_s) = \frac{1}{1+z_s}$$

for large $-z_s$

$$A(z_s) = \frac{1}{1+z_s} = (z_s)^{\alpha}\left[1 + O(1/z_s)\right]$$

$$\alpha = -1$$

compare with starting point

$$A(s,t) = A(s,t(s,z_s)) = \sum_{l=0}^{\infty}(2l+1)A_l(s)P_l(z_s)$$

$$t(s,z_s) = -(1-z_s)\frac{s-4m^2}{2}$$

$$t \propto -z_s \text{ when } (t \to +\infty)$$

we derived the large +t behavior !

- **there is a pole in the physical t region (“resonance”)**
- **if α non-integer, tasks as a function of t the amplitude has a branch point: particle production at lathe t-channel energy**
- **the leading term at large t is “simple”: it must come from some specific property of the p.w. series**

**Exercise**: find the large-$z_s$ behavior using the Sommerfeld-Watson transform.

# Analytical continuation: realistic example

$$A(s,t) = \sum_{l=0}^{\infty}(2l+1)A_l(s)P_l(z_s) = \sum_{l=0}^{\infty} \frac{\beta(s)}{l-\alpha(s)} \frac{[P_l(z_s) - P_l(-z_s)]}{2} \quad (**)$$

$$\alpha(s) = \frac{1}{2} + s + i\gamma\rho(s) \qquad \rho(s)\beta(s) = Im\alpha(s) \quad (*)$$

$$\gamma \approx \Gamma_\rho m_\rho$$

- $A_1(s)$ ~ Breit-Wigner of the rho-meson
- $Re(\alpha(s))$ = 1/2 + s : linear Regge trajectory
- $Im(\alpha(s))$ = is related to resonance widths
- the relation between α (trajectory) and β (residue) follows from unitarity: $Im\, A_1(s) = |A_l(s)|^2 \rho(s)$
- Resonances with different spins in $A_1$, $A_3$, $A_5$, … are related by poles in l of the function $A_l$

$$\rho(s) = \sqrt{1 - 4m_\pi^2/s}$$

**Exercise**:show that

$$\sum_{l=0}^{\infty} \frac{z_s^l \pm (-z_s)^l}{l-\alpha} \propto \frac{1 \pm e^{i\pi\alpha}}{\sin \pi\alpha}(-z_s)^\alpha$$

**Exercise**:show that (*) follows from unitarity

**Exercise**:show that (**) has a Breit-Wigner form

- **the leading term at large t is "simple": it must come from some specific property of the p.w. series -> it comes from right most singularity of partial waves in the angular momentum plane**

$$\sum_{l=0}^{\infty} \frac{z_s^l \pm (-z_s)^l}{l-\alpha} \propto \frac{1 \pm e^{i\pi\alpha}}{\sin \pi\alpha} (-z_s)^{\alpha}$$

**Regge theory = origin and properties of singularities of p.w. in the angular momentum plane**

- **it is possible ("simple") to construct models of partial waves in one channel (e.g. s) which have Regge poles and produce right asymptotic behavior in another (e.g. t). It is not easy to do it simultaneously**

## s-channel p.w. : s-channel resonances

$$A(s,t) = \sum_{l}^{\infty} (2l+1) A_l(s) P_l(s), \;\; A_l(s) = \sum_{i=Regge\;\;poles} \frac{\beta_i(s)}{l - \alpha_i(s)}$$

**Q: Should you add t-channel resonances (interference model) ?**

**A: No. (resonances in t and s are dual not additive)**

**A finite number of t-channel resonances will break s-channel analyticity**

$$\sum_{l=0}^{L_{max}} (2l+1) A_l(t) P_l(z_t) \to s^{L_{max}}$$

**An infinite number of t-channel resonance -> double counting of A(s,t)**

**Veneziano model = has simultaneous resonances in s an channel and proper asymptotic behavior. To do this requi number of p.w/resonances (why?)**

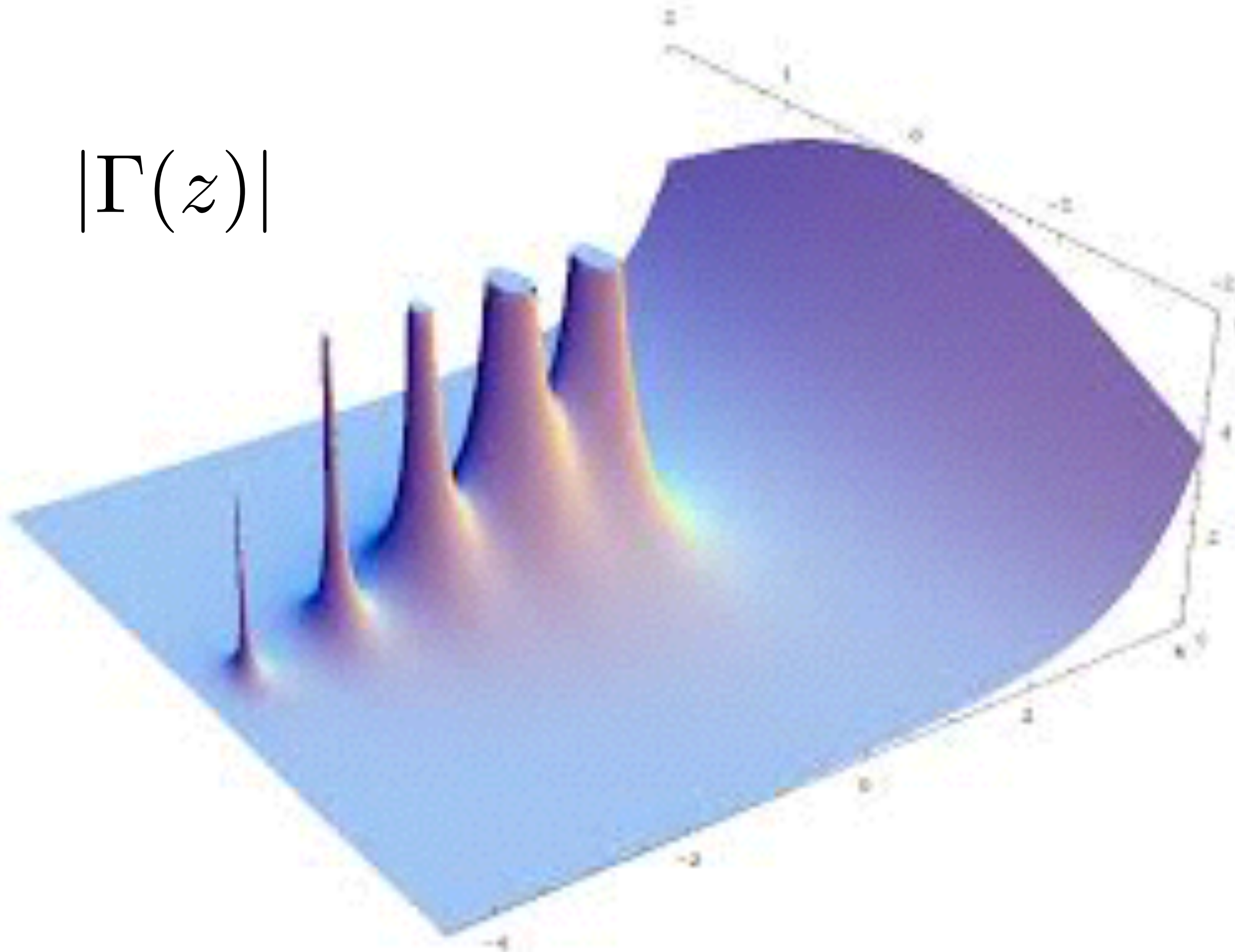
**What functions have an infinite number of poles (resonances) to be used to represent (model) the amplitude A(s,t) ?**


**The Gamma function !!!**

$$\Gamma(z) \sim \frac{(-1)^n}{\Gamma(n+1)} \frac{1}{n+z}$$

z ~ -n

$|\Gamma(z)|$



so we want something like

$$A(s,t) \sim \Gamma(-t)\Gamma(-s)$$

$$A(s,t) \sim \Gamma(-t)\Gamma(-s)$$

- **to connect poles at t (or s) = 1,2,3 … with physical masses use the Regge trajectory**

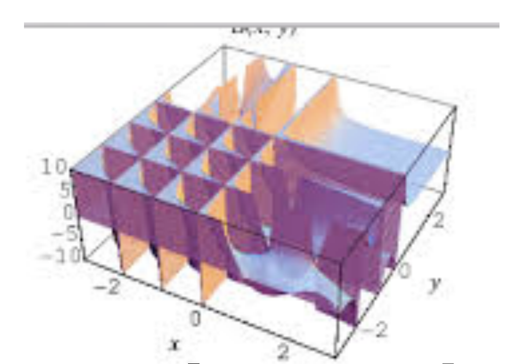
$$A(s,t) \sim \Gamma(n-\alpha(s))\Gamma(n-\alpha(t))$$

- **n,m determine location of first poles, e.g. Γ(-α(s)) has, for α(s) = 0.5 + s, the first pole at s=-1/2 i.e. particle with imaginary mass. But Γ(1-α(s)) has the first pole at s=+1/2 i.e. the rho-meson**

- **simultaneous poles in s and t ("overlapping channels") are unexpected. To remove them use**

$$A(s,t) \sim \frac{\Gamma(n-\alpha(s))\Gamma(n-\alpha(t))}{\Gamma(n+m-\alpha(s)-\alpha(t))}$$

$n \geq m \geq 1$



**it has poles and zeros!**

- **what is missing is kinematical (spin) and symmetry (e.g. isospin) factors**

Examples:

**π π -> π π**

- after isospin decomposition there are three scalar amplitudes A(s,t,u), B(s,t,u), C(s,t,u) -> Veneziano

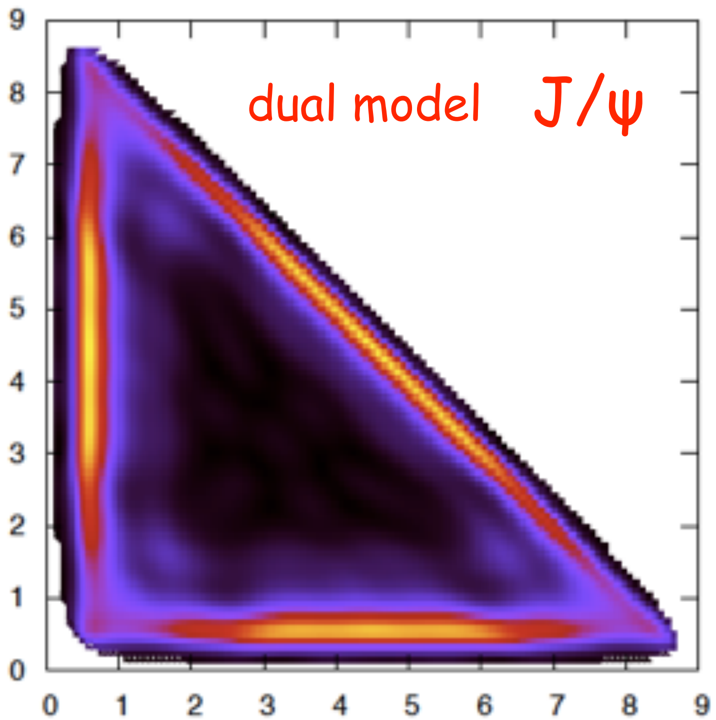
**π N -> π N**

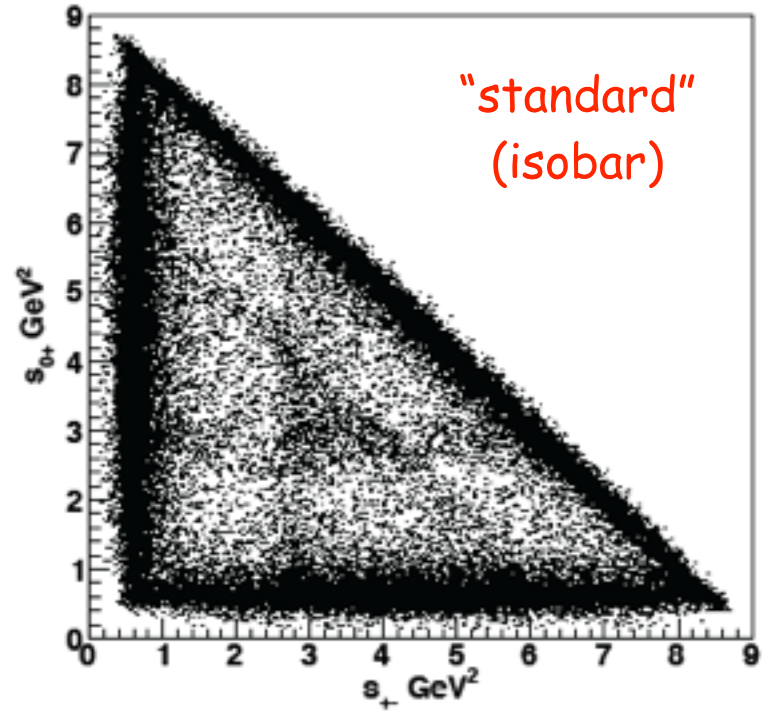
- there are two scalar amplitudes A(s,t), B(s,t)

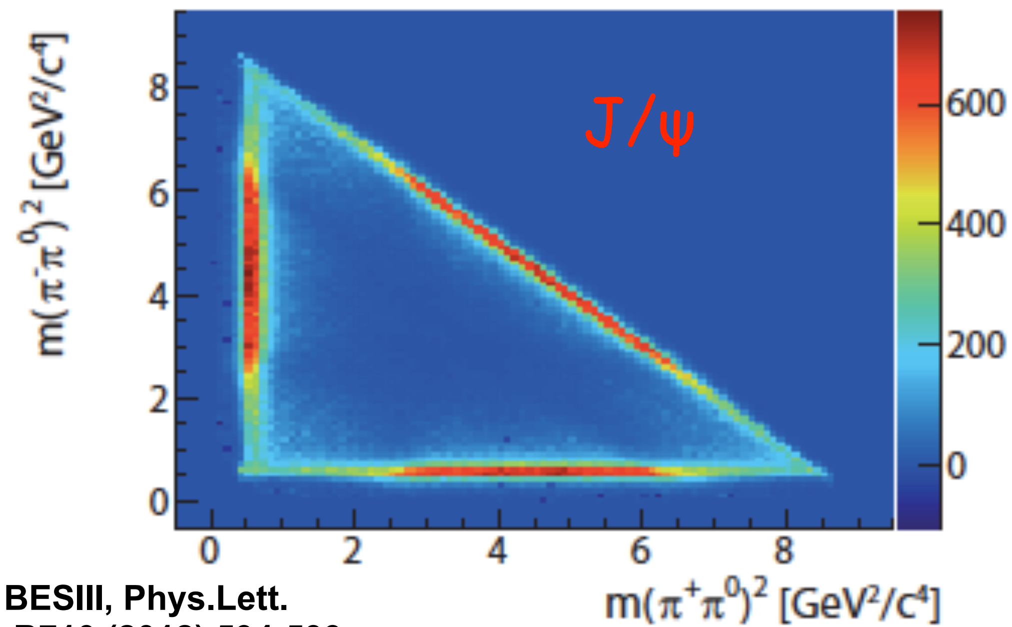
**V π -> π π**

- there is one scalar amplitude (*)

**Exercise**: verify (*)

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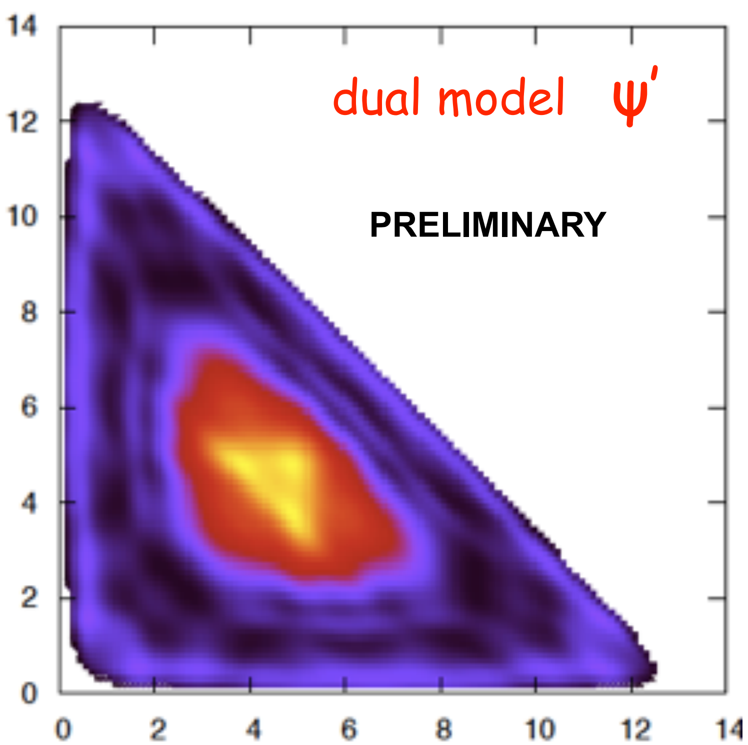
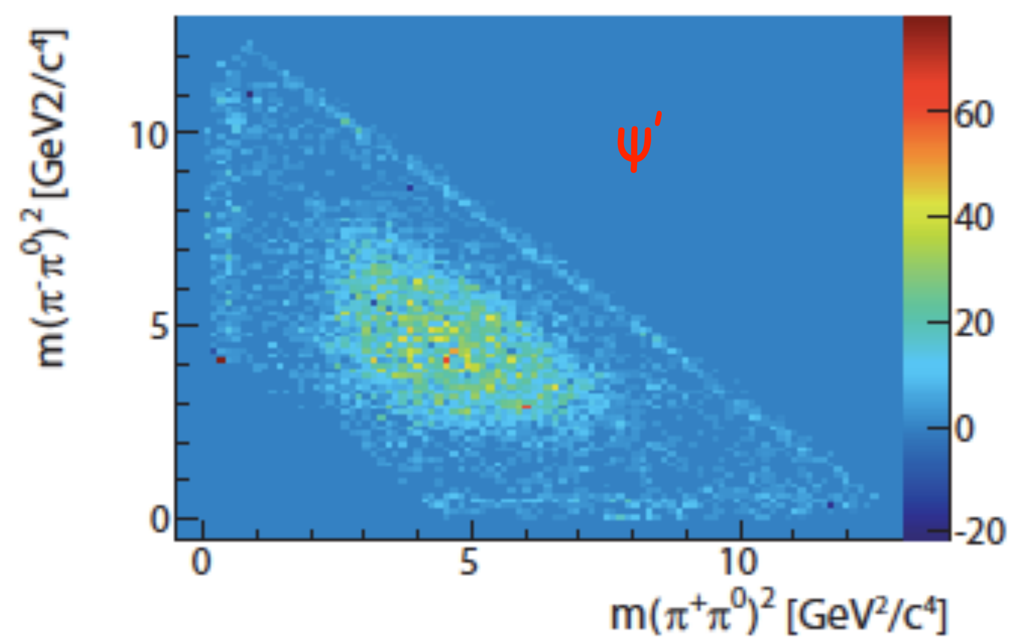
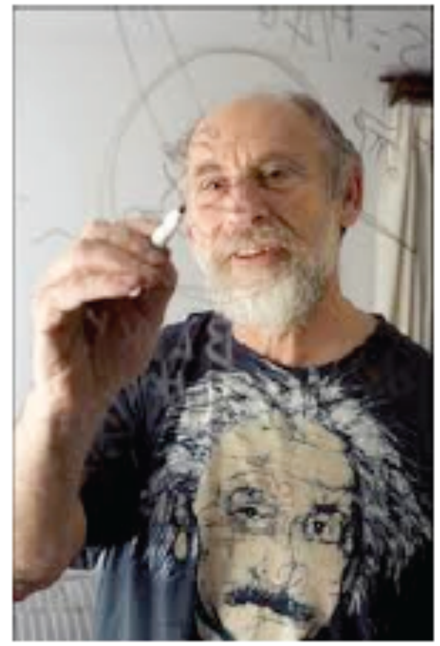



$$A(s,t) = \frac{\Gamma(-J(s))\Gamma(-J(t))}{\Gamma(-J(s)-J(t))}$$

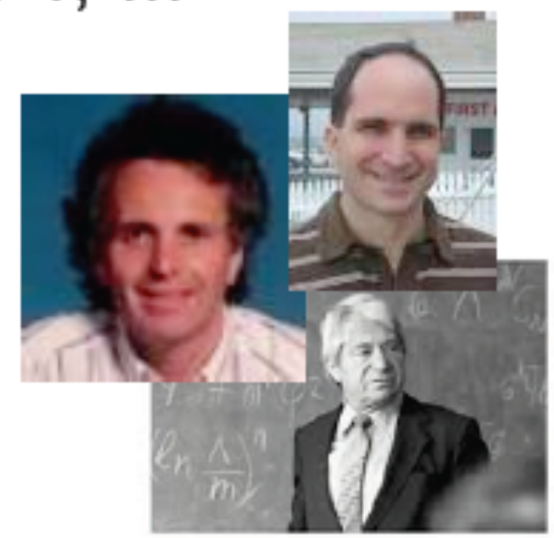
$\omega \to 3\pi$

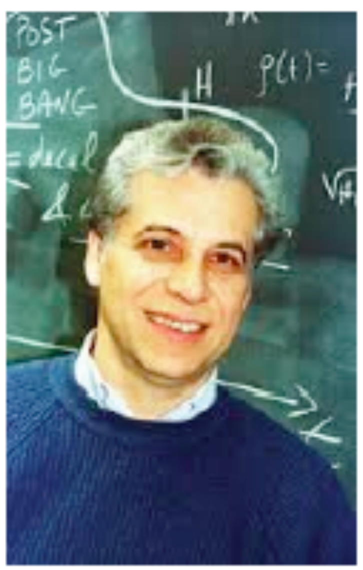
relativistic h.o.



string of relativistic oscillators

QCD, loop representation, large-$N_c$, AdS/CFT, ...





$\omega \to 3\pi$


string revolution

$$A(s,t) = \frac{\Gamma(-J(s))\Gamma(-J(t))}{\Gamma(-J(s)-J(t))}$$

# Veneziano amplitude: “compact” expression for the full amplitude

$$A(s,t) = \frac{\Gamma(-\alpha(s))\Gamma(-\alpha(t))}{\Gamma(-\alpha(s)-\alpha(t))} \qquad \alpha(s) = a + bs$$

A(s,t) can be written as sum over resonances in ether channel.

$$A(s,t) = \sum_k \frac{\beta_k(t)}{k-\alpha(s)} = \sum_k \frac{\beta_k(s)}{k-\alpha(t)}$$

Note: in V-model resonance couplings, β, are fixed!   (*)

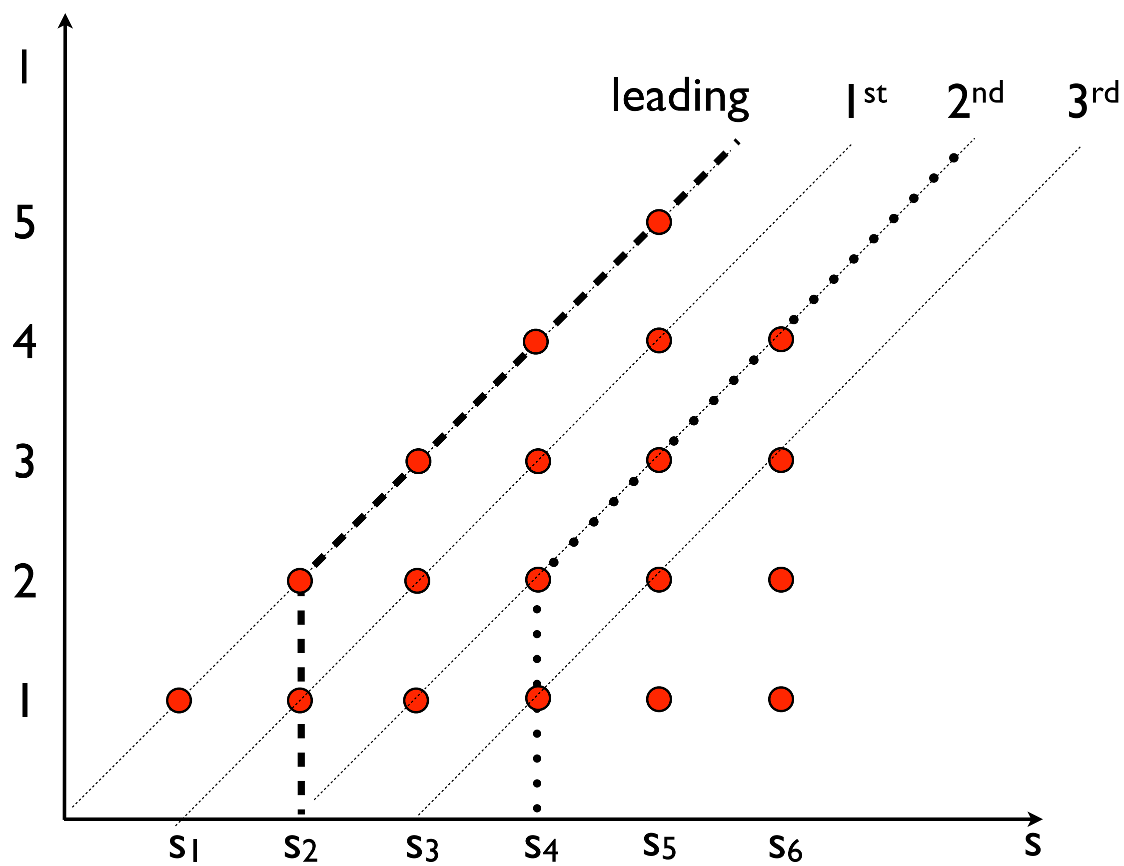
$$\beta_k(t) \propto (1+\alpha(t))(2+\alpha(t))\cdots(k+\alpha(t))$$

**Exercise**: verify (*)

$$V(p,\lambda) \to \pi^i(p_1)\pi^j(p_2)\pi^k(p_3)$$

$$A(s,t,u) = \epsilon_{ijk}\epsilon_{\mu\nu\alpha\beta}\epsilon_\mu(p,\lambda)p_1^\nu p_2^\alpha p_3^\beta$$
$$\times[A_{n,m}(s,t) + A_{n,m}(s,u) + A_{n,m}(t,u)]$$

$$A_{n,m}(s,t) \equiv \frac{\Gamma(n-\alpha_s)\Gamma(n-\alpha_t)}{\Gamma(n+m-\alpha_s-\alpha_t)}.$$




$$\alpha(s) = \alpha_0 + \alpha' s.$$

$$\alpha(s) = \frac{1}{2} + s$$

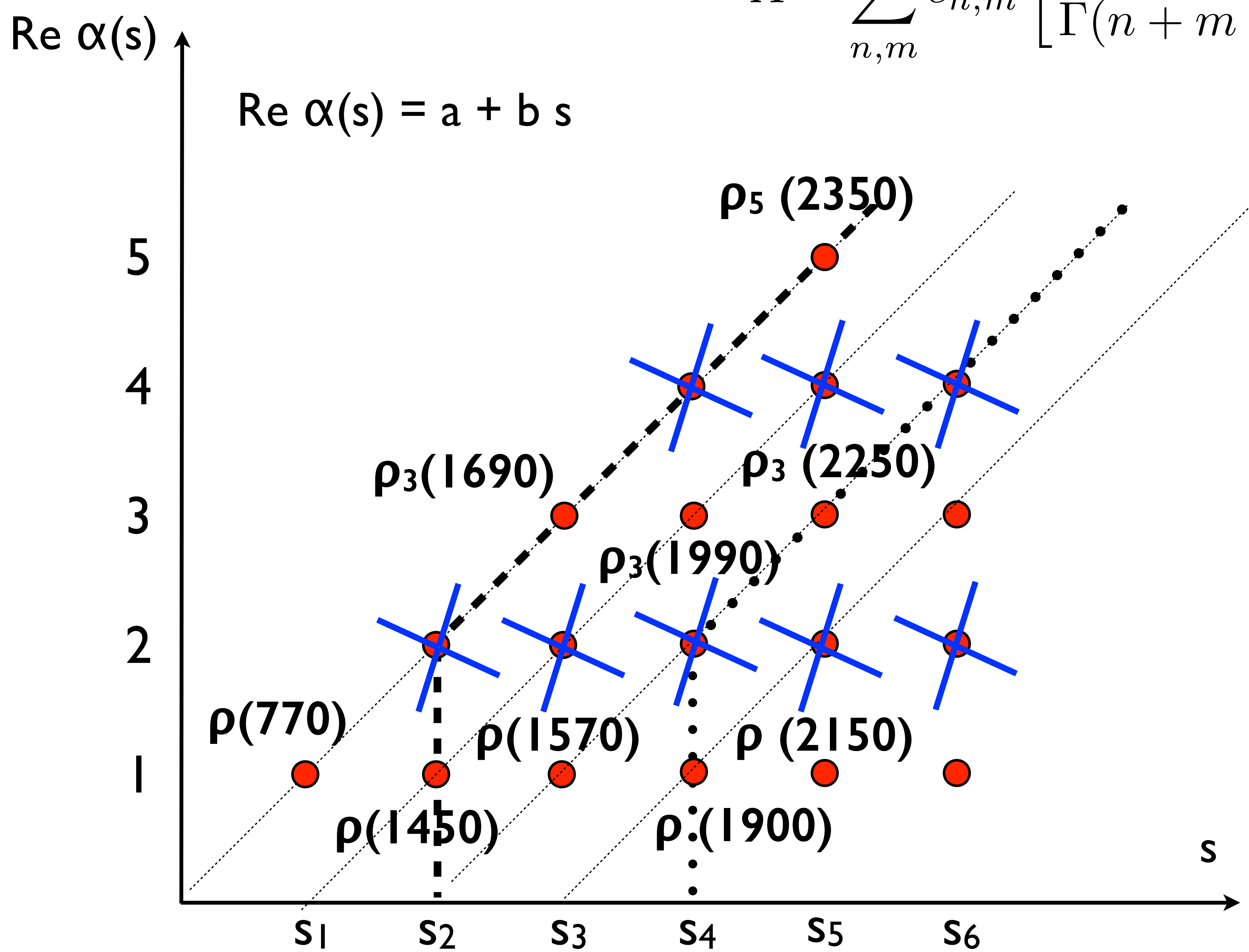
**no-double poles**
**correct asymptotic**
**limit** **n ≥ m ≥ 1** (*)

**Exercise**: verify (*)

# Resonances couplings, β, should depend on final state particles: a linear superposition of Veneziano amplitudes can be used to suppress or enhance individual resonances or trajectories

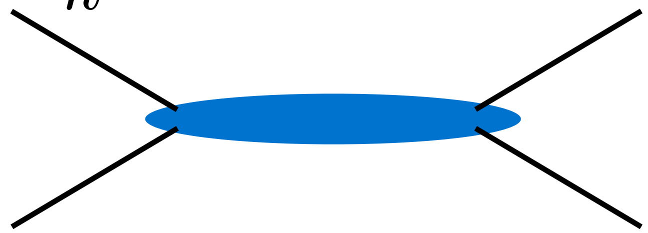
$$M = \epsilon_{\mu\nu\alpha\beta} p_1^\mu p_2^\nu p_3^\alpha \epsilon^\beta A(s,t,u)$$

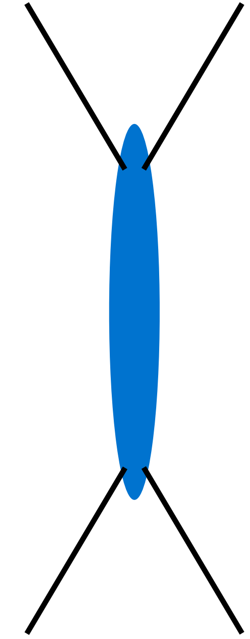
$$A = \sum_{n,m} c_{n,m} \left[ \frac{\Gamma(n-\alpha(s))\Gamma(n-\alpha(t))}{\Gamma(n+m-\alpha(s)-\alpha(t))} + (s,u) + (t,u) \right]$$

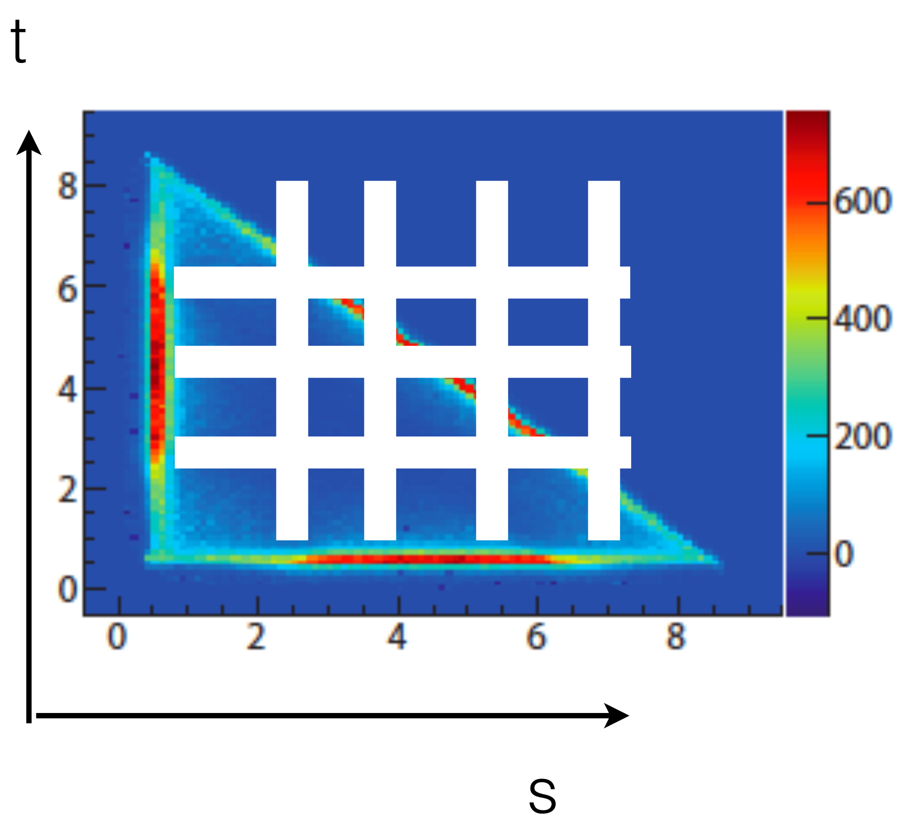



- even-spin ρ's do not couple to π π and should decouple in J/ψ→3 π
- coupling of odd-spin ρ's depend of can depend vary depending on trajectory

$$A_{n,m}(s,t) = \sum_k^{\infty} \frac{\beta(t)}{k - \alpha(s)} = \sum_k^{\infty} \frac{\beta(s)}{k - \alpha(t)}$$








**How to isolate individual poles ?**

$n \geq m \geq 1$

$$A_{1,1} = \frac{\Gamma(1-\alpha_s)\Gamma(1-\alpha_t)}{\Gamma(2-\alpha_s-\alpha_t)}$$ has poles at $\alpha_s$=1,2,3,...

$$A_{2,1} = \frac{\Gamma(2-\alpha_s)\Gamma(2-\alpha_t)}{\Gamma(3-\alpha_s-\alpha_t)}$$ has poles at $\alpha_s$=2,3,4,...

$$A_{2,2} = \frac{\Gamma(2-\alpha_s)\Gamma(2-\alpha_t)}{\Gamma(4-\alpha_s-\alpha_t)}$$ has poles at $\alpha_s$=2,3,4,...

$A_{3,1}, A_{3,2}, A_{3,3}$ have poles at $\alpha_s$=3,4,5,...

$A_{4,1}, A_{4,2}, A_{4,3}, A_{4,4}$ have poles at $\alpha_s$=4,5,6,...

Use a linear combination of $A_{2,1}$ and $A_{2,2}$ to remove pole at $\alpha_s$ =2

Use a linear combination of $A_{3,1}$, $A_{3,2}$ ,$A_{3,3}$, to remove pole at $\alpha_s$ =3,

etc.

$$A_{n,m}(s,t) \to \mathcal{A}(s,t) = \sum_{n\geq 1, n\leq m\leq 1} c_{n,m} A_{n,m}(s,t)$$

remove all poles but the one at α=1

$$c_{n,1} = \frac{c_{1,1}}{\Gamma(n)},\ c_{n,2} = -\frac{c_{1,1}}{\Gamma(n-1)},\ c_{n,m} = 0 \text{ for } m > 2,$$

$$\mathcal{A}_1(s,t) = c_{1,1}\frac{2-\alpha_s-\alpha_t}{(1-\alpha_s)(1-\alpha_t)}.$$

... but the Regge limit is now lost !

remove all poles between N ≥ α ≥ 2

$$\begin{aligned}\mathcal{A}_1(s,t;N) &= c_{1,1}\frac{2-\alpha_s-\alpha_t}{(1-\alpha_s)(1-\alpha_t)} \\ &\times \frac{\Gamma(N+1-\alpha_s)\Gamma(N+1-\alpha_t)}{\Gamma(N)\Gamma(N+2-\alpha_s-\alpha_t)}\end{aligned}$$

has Regge limit is for s > N

In the past this was done by choosing an arbitrary set of n,m and fitting c(n,m) to the data (e.g. Lovelace, Phys. Lett. B28, 265 (1968),Altarelli, Rubinstein, Phys. Rev. 183, 1469 (1969))

The "new" model does this in a systematic way. In addition it allows for imaginary non-linear (and complex) trajectories without introducing "ancestors"

Application of the Veneziano Model in Charmonium Dalitz Plot Analysis

Adam P. Szczepaniak[1,2,3] and M.R. Pennington[2]

$$\mathcal{A}_n(s,t;N) = \frac{2n-\alpha_s-\alpha_t}{(n-\alpha_s)(n-\alpha_t)} \sum_{i=1}^{n} a_{n,i}(-\alpha_s-\alpha_t)^{i-1} \times \frac{\Gamma(N+1-\alpha_s)\Gamma(N+1-\alpha_t)}{\Gamma(N+1-n)\Gamma(N+n+1-\alpha_s-\alpha_t)}.$$

n: number of Regge trajectories

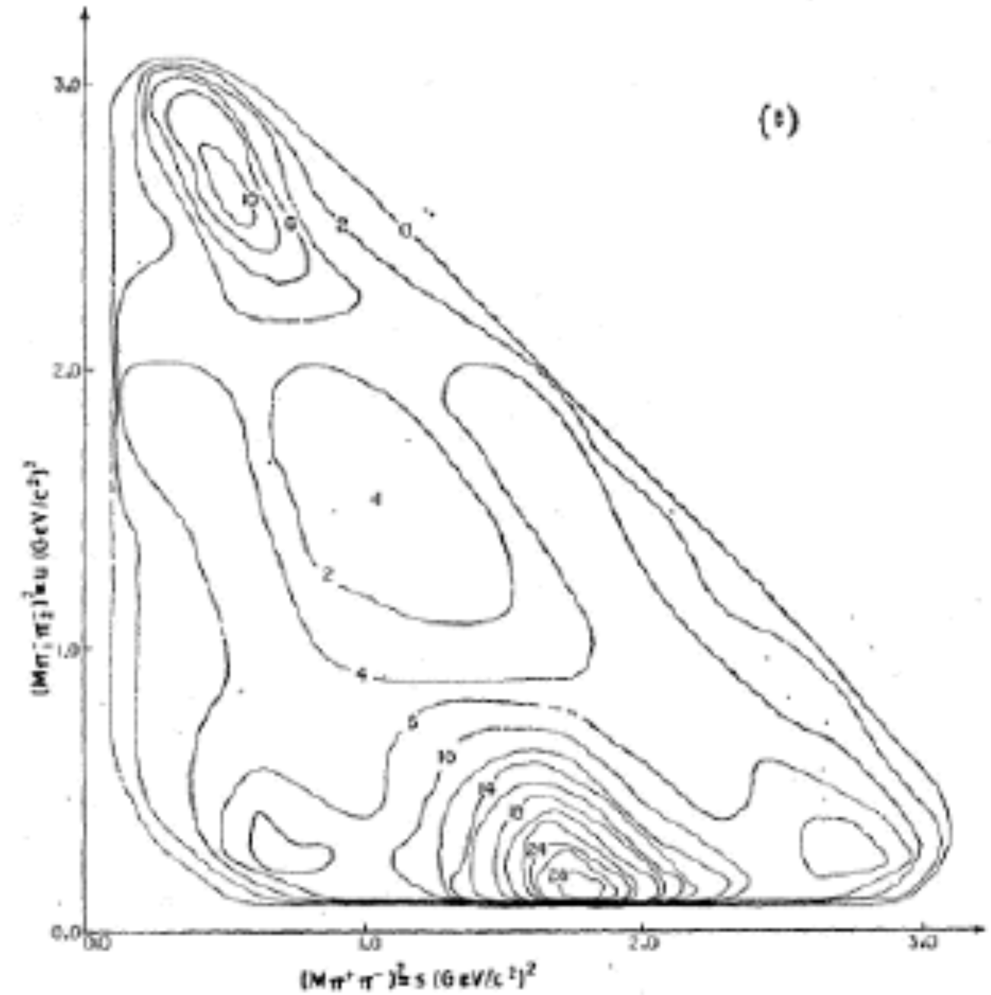
$a_{n,i}$: determine resonance couplings

N: determines the onset of Regge behavior

$\alpha(s)$, $\alpha(t)$ = Re $\alpha$ + i Im $\alpha$: with Im $\alpha$ related to resonance widths

Different authors employed the Veneziano model for the analysis of the at-rest annihilation $N\overline{N} \rightarrow 3\pi$, using a finite number of Veneziano terms.

- Lovelace[2]: a single term amplitude, $n = m = 1$, $\alpha_s = 0.483 + 0.885s + 0.28i\sqrt{s - 4m^2}$
- Altarelli[3]: 5 terms with $n + m \leq 3$ (to reproduce the zero at $\alpha_s + \alpha_t \simeq 3$)
- Gopal[4]: 5 terms with $n + m \leq 3$, $\alpha_s = 0.483 + 0.885s + iA(s - 4m^2)^B$, $B < 1$



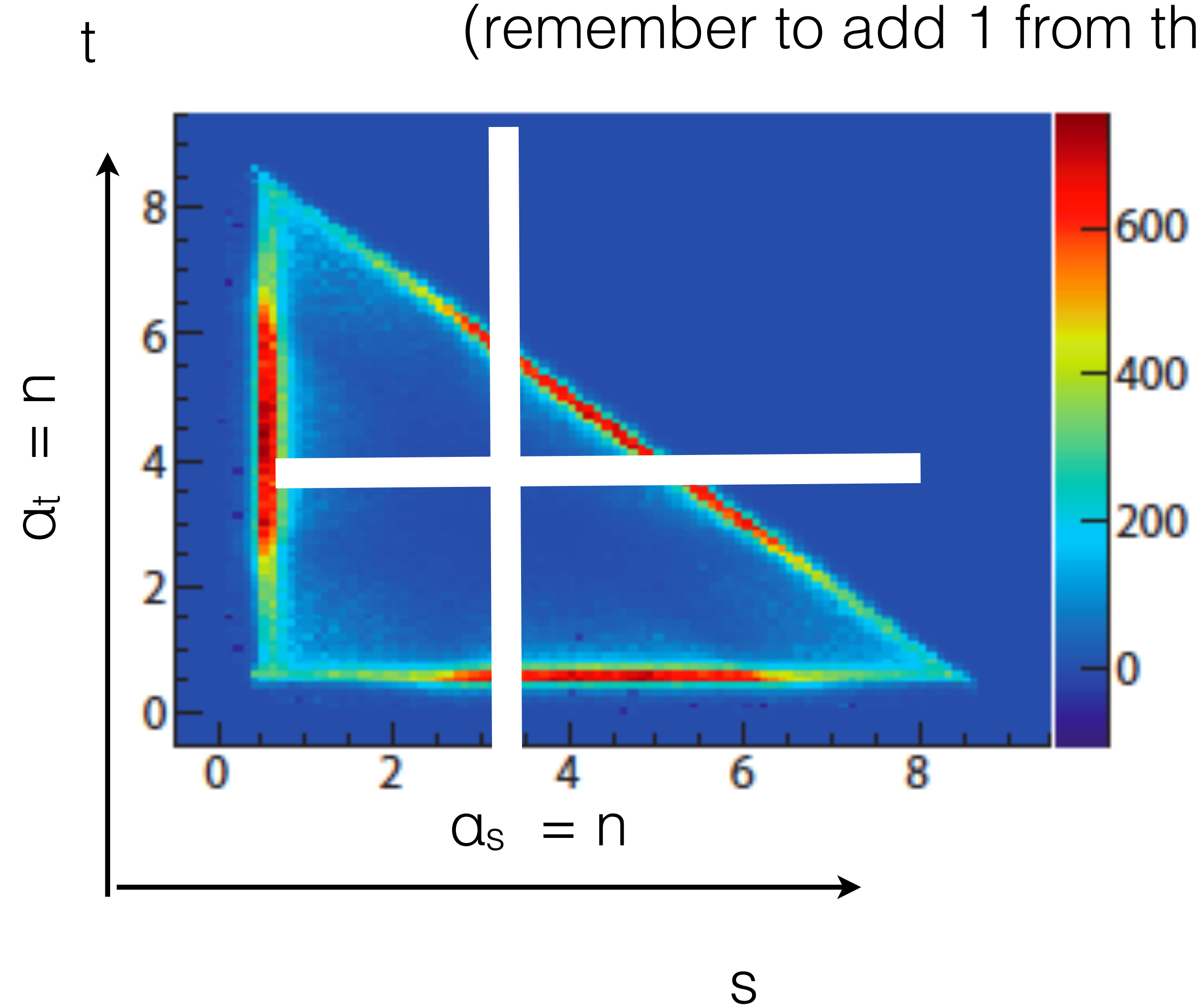

[2] C. Lovelace, Phys. Lett. 25B (1968), 264
[3] G. Altarelli, Phys. Rev. 183 (1969), 1469
[4] G. P. Gopal, Phys. Rev. D 3 (1971), 2262

from A.Celentano

All poles below α = N except at α = n

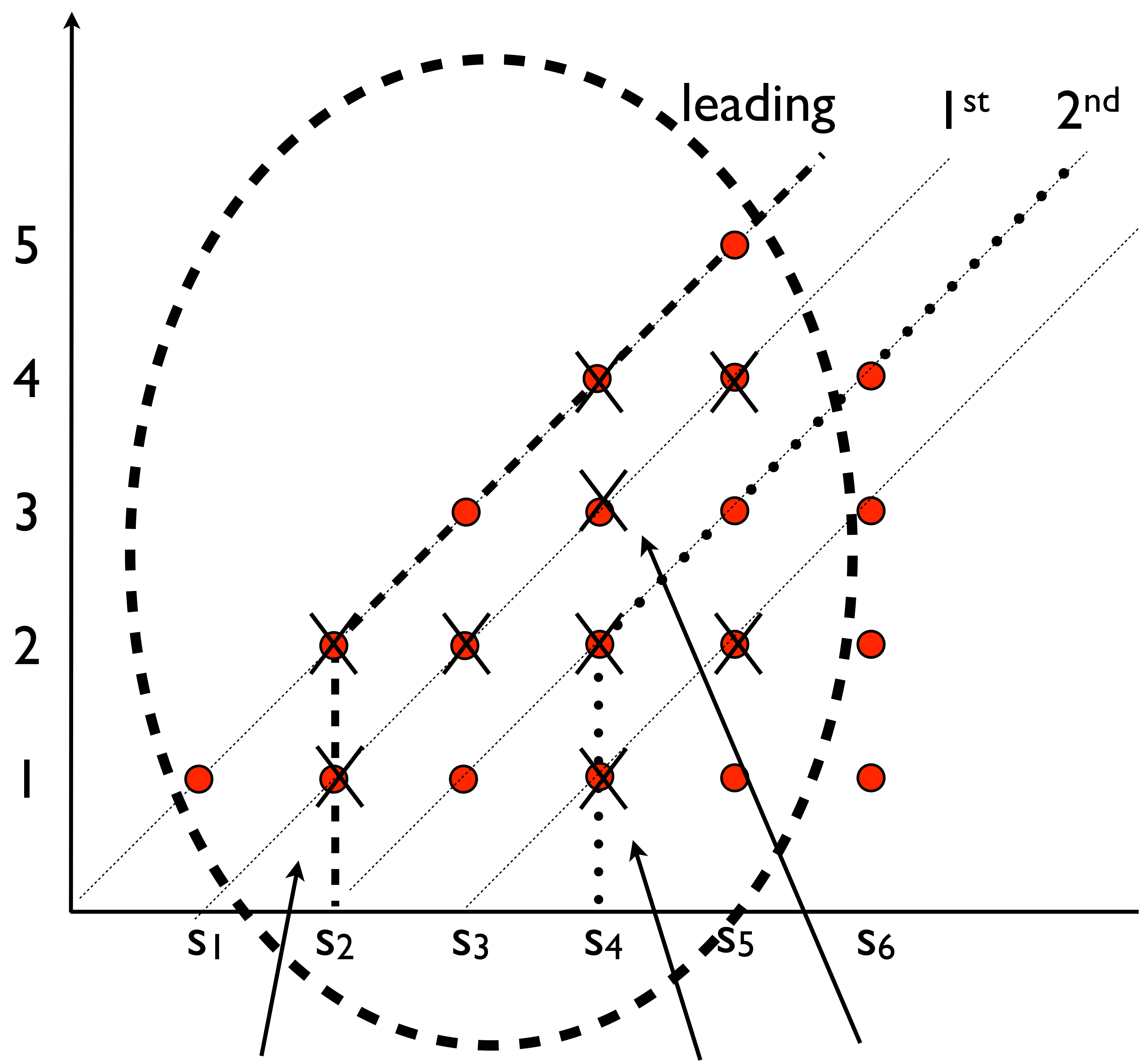
$$\mathcal{A}_n(s,t;N) = a_{n,0}\frac{2n-\alpha_s-\alpha_t}{(n-\alpha_s)(n-\alpha_t)}\left[\Pi_{i=1}^{n-1}(a_{n,i}-\alpha_s-\alpha_t)\right] \times \frac{\Gamma(N+1-\alpha_s)\Gamma(N+1-\alpha_t)}{\Gamma(N+1-n)\Gamma(N+n+1-\alpha_s-\alpha_t)}$$

at $\alpha_s = n$ residue is a polynomial in t of order n-1
(remember to add 1 from the Levi-Civita tensor)

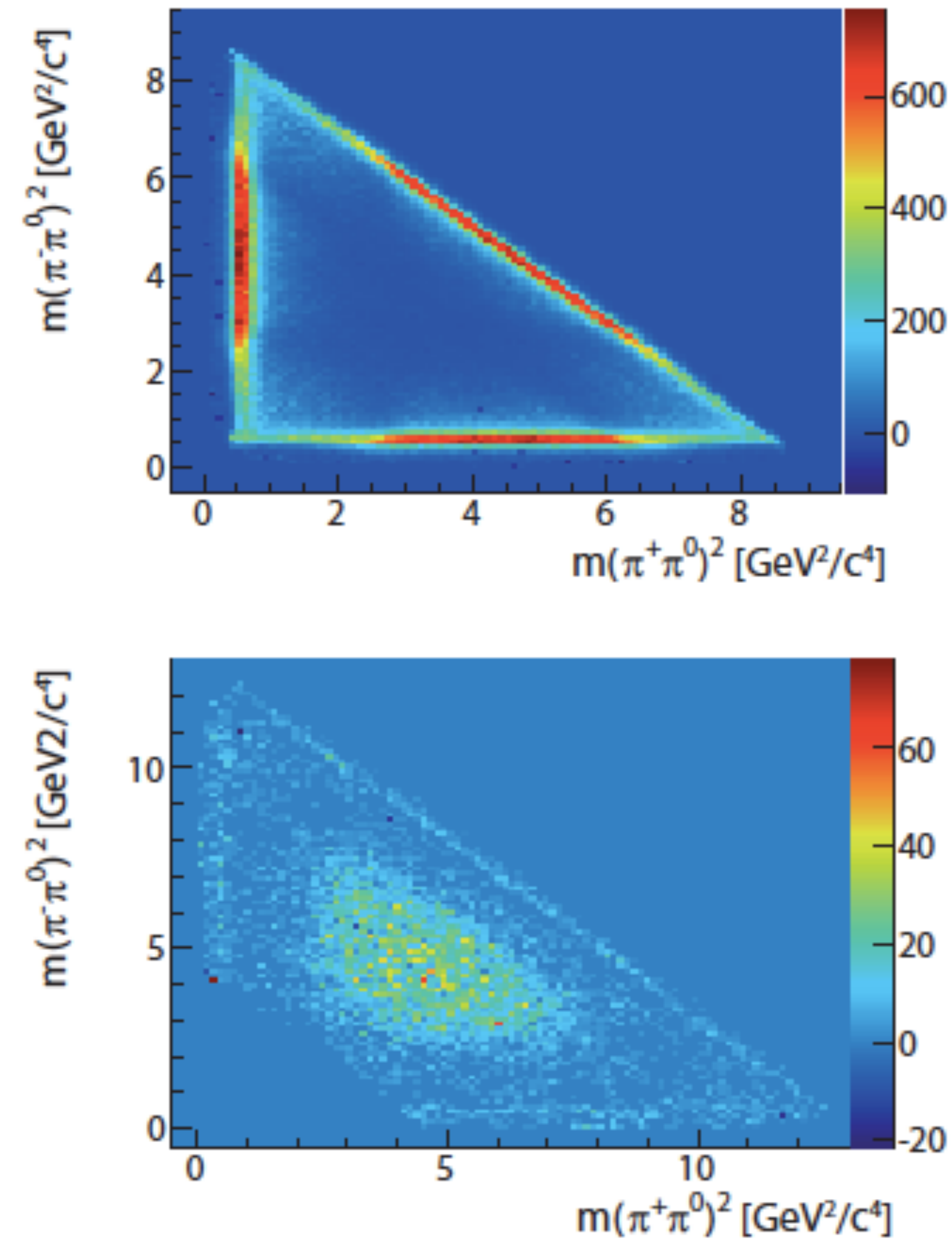



$$A(s,t,u) = \epsilon_{ijk}\epsilon_{\mu\nu\alpha\beta}\epsilon_\mu(p,\lambda)p_1^\nu p_2^\alpha p_3^\beta$$
$$\times[A_{n,m}(s,t) + A_{n,m}(s,u) + A_{n,m}(t,u)]$$

$A_1$ has ρ(770)
$A_3$ has ρ(1700), $\rho_3$(1690)
$A_5$ has ρ''(2150), $\rho_3$(2250), $\rho_5$(2350)




ρ(1400) ρ(1900), ρ3(1990),




$$m_2 = \sqrt{s_2} = \sqrt{2 - \frac{1}{2}} = 1.23 \qquad m_4 = \sqrt{s_4} = \sqrt{4 - \frac{1}{2}} = 1.87$$

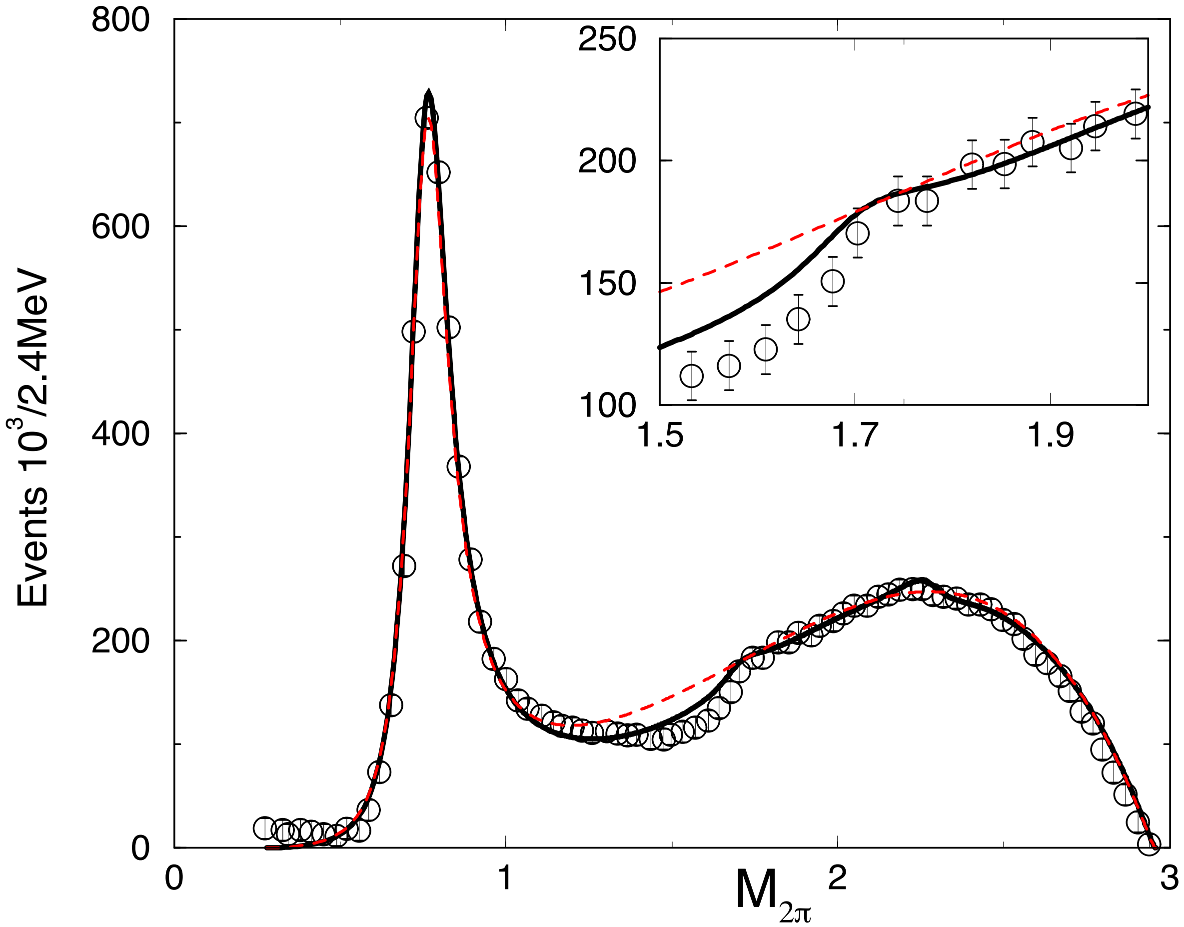



FIG. 2: Dalitz plot projection of the di-pion mass distribution from $J/\psi$ decay. The solid is the result of the fit with three amplitudes and the dashed line with the amplitude $\mathcal{A}_1$ alone. The insert shows the mass region of the $\rho_3$ and its contribution from the fit with the full set of amplitudes (solid line) as compared. Absence of the structure at 1.7GeV from the fit with the $\mathcal{A}_1$ amplitude is indicated by the dashed line.

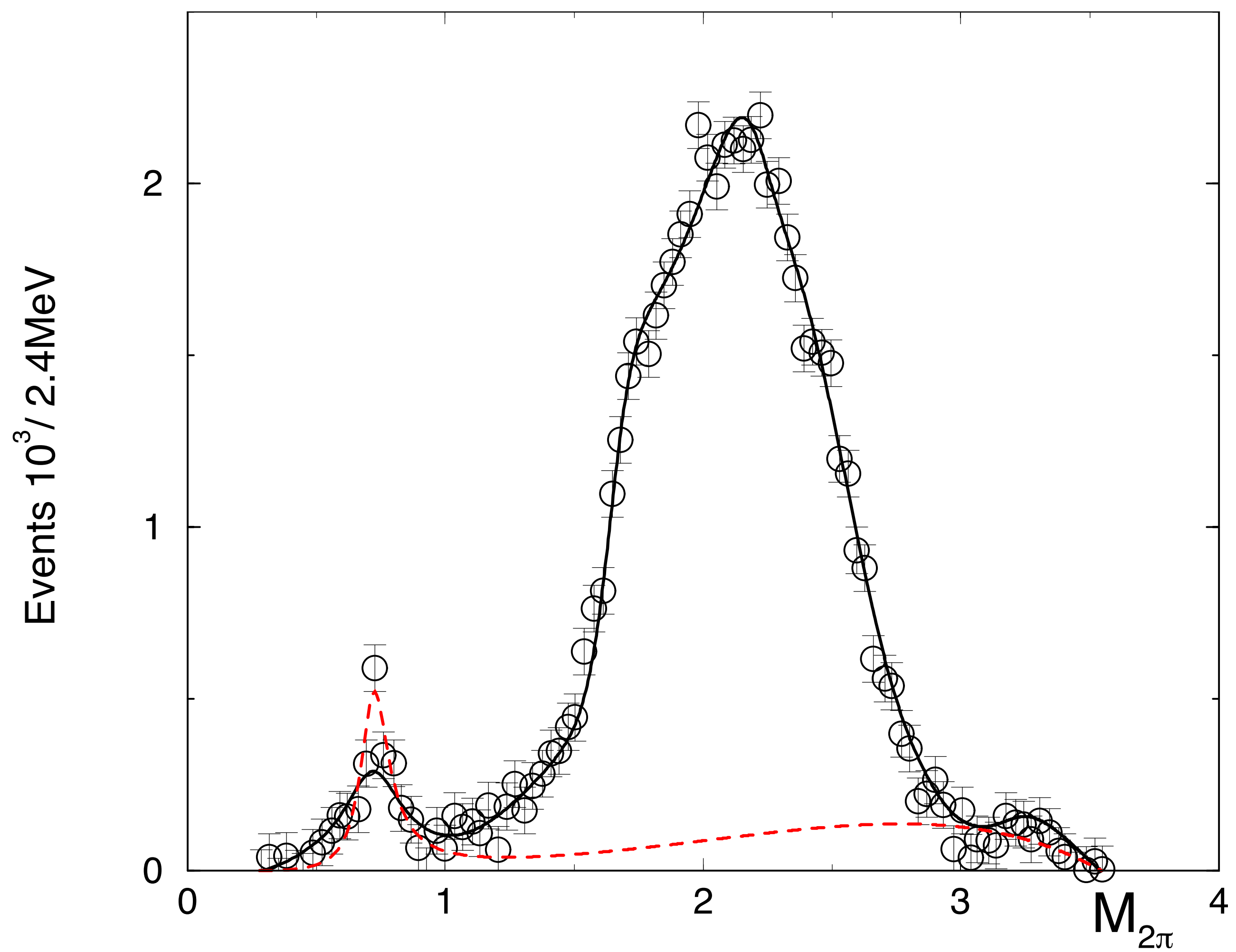
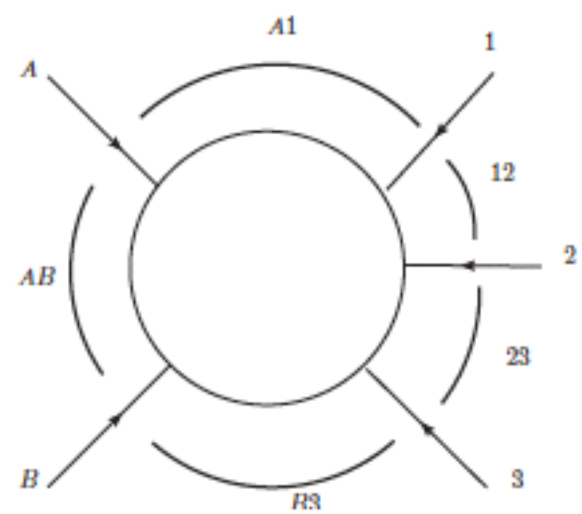
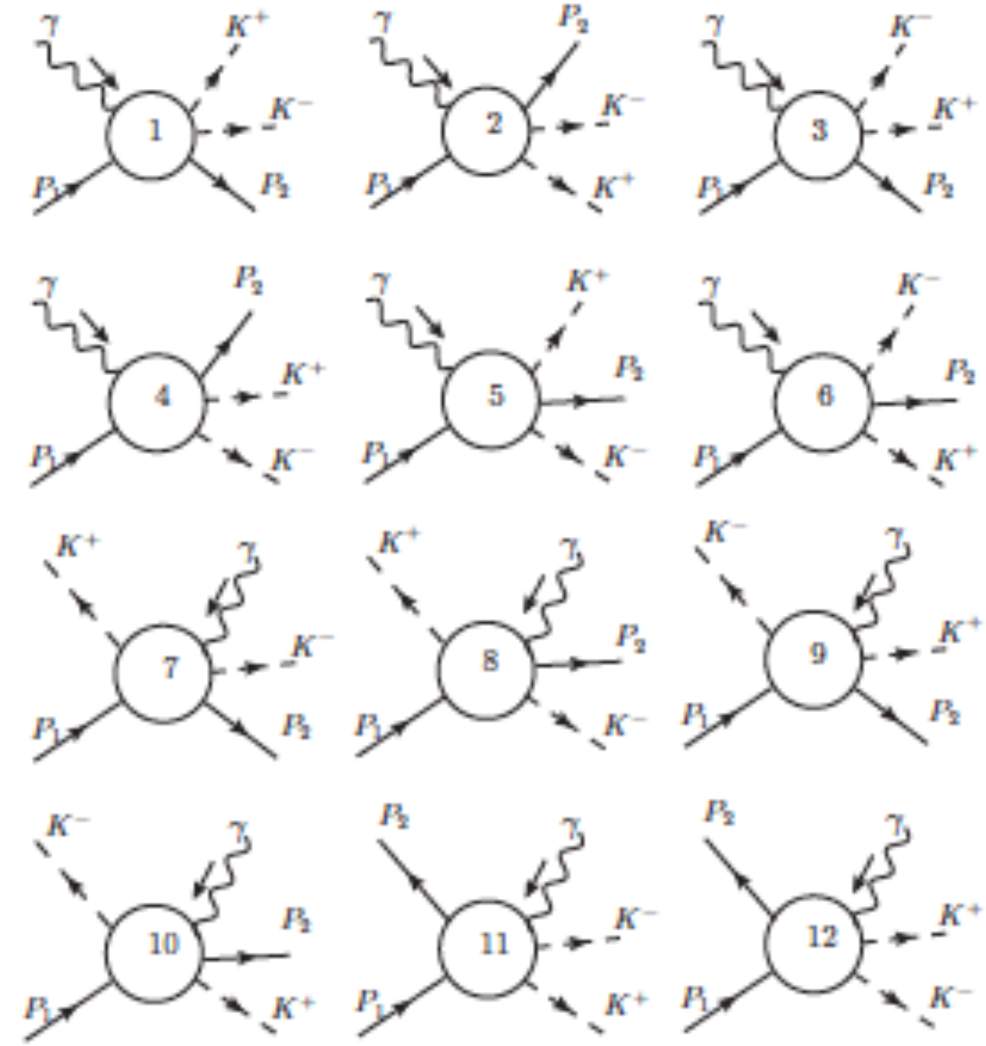


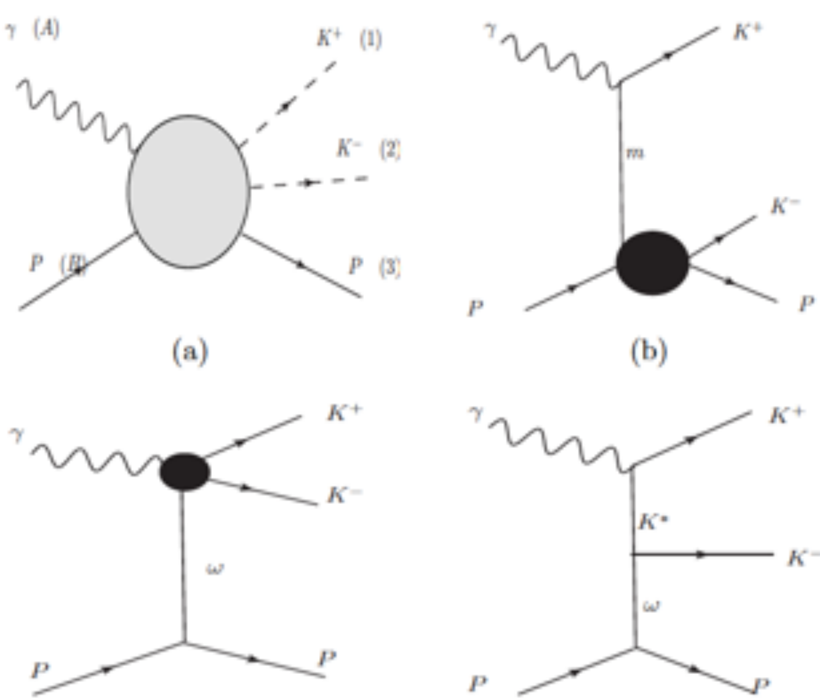

FIG. 3: Dalitz plot projection of the di-pion mass distribution from $\psi'$ decay. The solid is the result of the fit with three amplitudes and the dashed line is the fit with $\mathcal{A}_1$ alone.

# $B_5$ amplitude:

Reggeons/
Resonances in
all 5 channels




(a) (b)

(c) (d)

**Double-Regge Exchange Limit for the $\gamma p \to K^+K^-p$ Reaction**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$$B_5(s_{AB}, s_{A1}, s_{12}, s_{23}, s_{B3}) = B_4(-\alpha_{12}, -\alpha_{A1})\, B_4(-\alpha_{23}, -\alpha_{B3}) \times {}_3F_2(\alpha_{AB} - \alpha_{12} - \alpha_{23}, -\alpha_{A1}, -\alpha_{B3}; -\alpha_{12} - \alpha_{A1}, -\alpha_{23} - \alpha_{B3}).$$

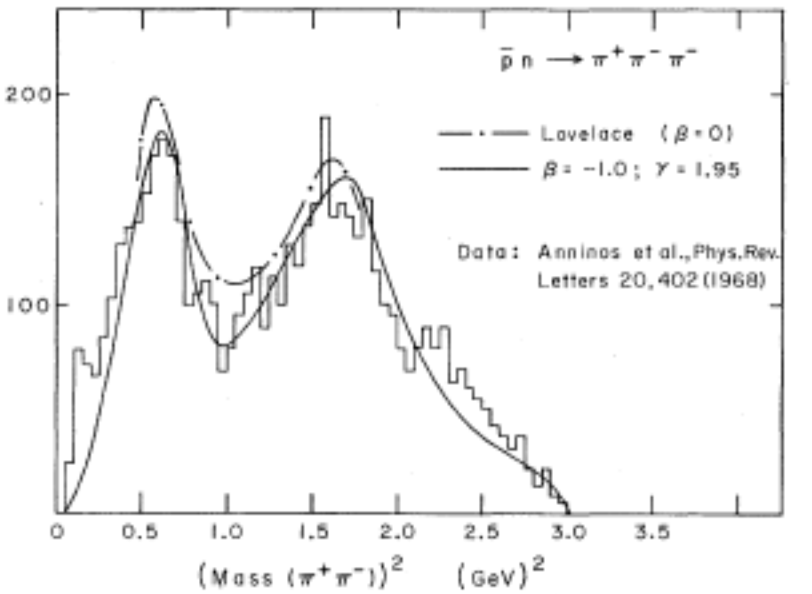
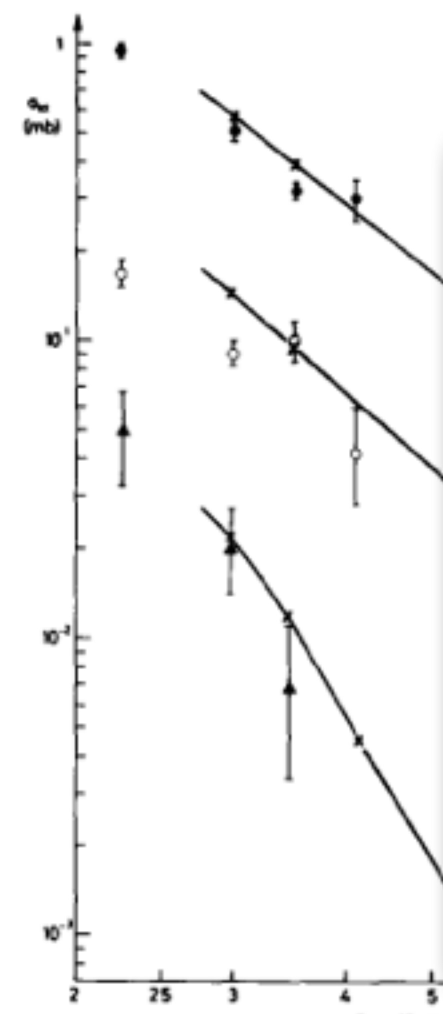
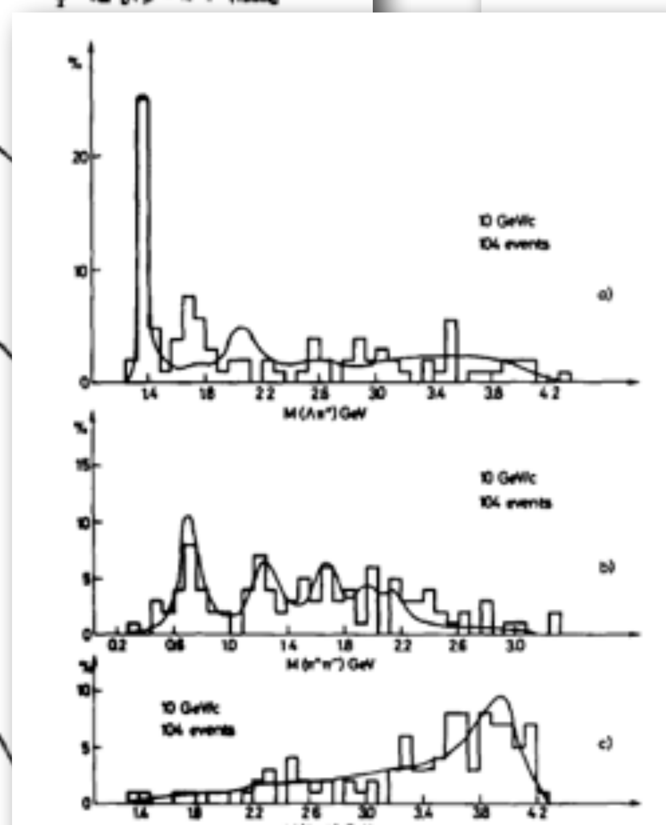


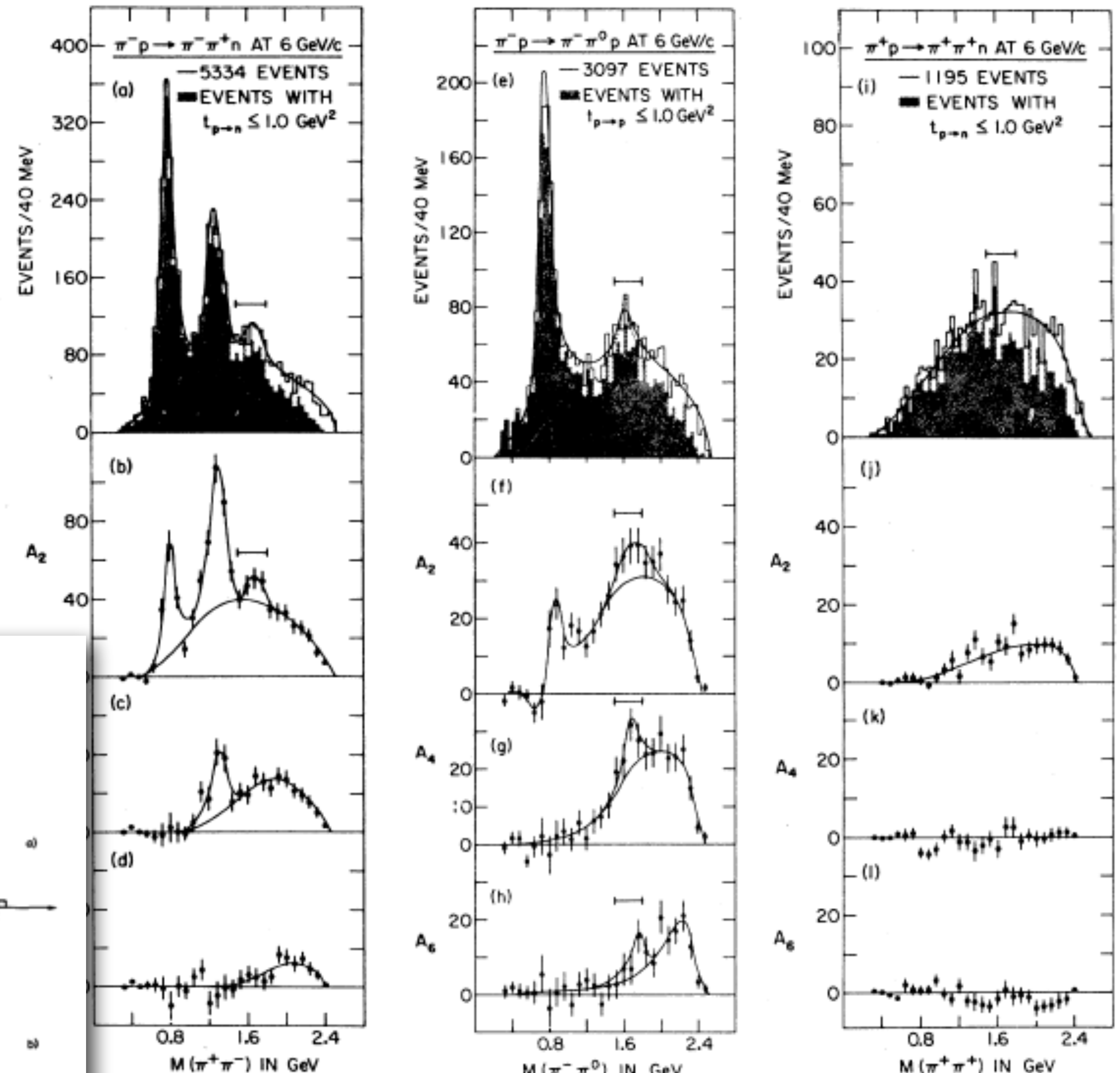

FIG. 10.3. Invariant mass distribution for $\pi^+\pi^-$ from $\bar{p}n \to \pi^+\pi^-\pi^-$. Data taken from Anninos *et al.* (1968). Theoretical curves are those of Lovelace (1969b) and Berger (1969a).

$K^- p \to \pi^- \pi^+ \Lambda$ IN VENEZIANO MODEL






...ion–pion mass distributions and Legendre polynomial coefficients for $\pi N \to \pi\pi N$ at 6 GeV/$c$ from Crennel *et al.* (1968). The columns are, from left to right, for $\pi^- p \to \pi^+\pi^- n$, $\pi^- p \to \pi^-\pi^0 p$, $\pi^+ p \to \pi^+\pi^+ n$.