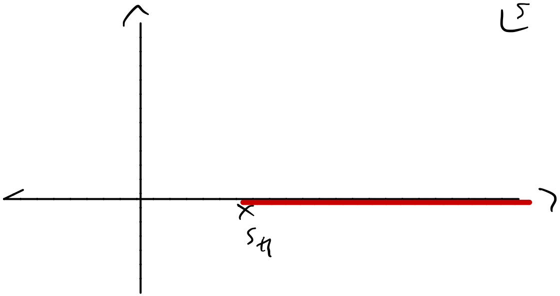
Day 4

$$\operatorname{Im} \hat{a}_\ell(s) = \rho_\ell(s)\, |\hat{a}_\ell(s)|^2$$

$$\Rightarrow \quad \operatorname{Im} \hat{a}_\ell^{-1}(s) = -\rho_\ell(s)\, \theta(s - s_{th})$$



$$\hat{a}_\ell^{-1}(s) = (\text{Real fcn}) - i\,\rho_\ell(s)$$

$$= K_\ell^{-1}(s) - i\,\rho_\ell(s)$$

$\uparrow$ Real

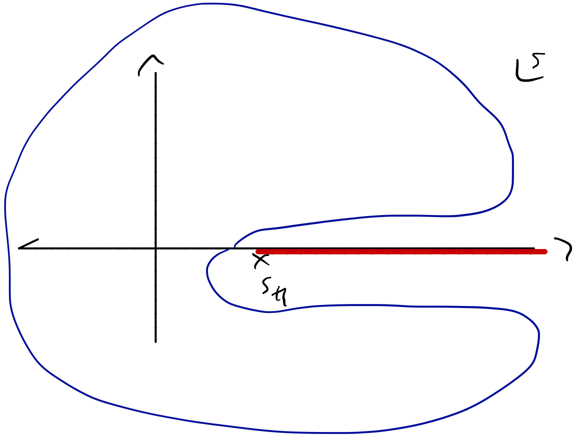
$$a_\ell(s) = |\vec{p}|^\ell\, |\vec{p}\,'|^\ell\, \hat{a}_\ell(s)$$

$$\rho_\ell(s) \sim \frac{2|\vec{k}|^{2\ell+1}}{\sqrt{s}}$$

Elastic: $|\vec{p}| = |\vec{p}\,'| = |\vec{k}|$

$$|\vec{k}| = \frac{\sqrt{s - s_{th}}\,\sqrt{s - s_{pa}}}{2\sqrt{s}}$$

$$K_\ell(s) = \sum_r \frac{g_r^2}{m_r^2 - s} + \sum_j \gamma_j s^j$$



$$\mathrm{Im}\, \hat{a}_\ell^{-1}(s) = -\rho_\ell(s)\, \theta(s - s_{th})$$

$$\hat{a}_\ell^{-1}(s) = (\mathrm{Recl}) + \frac{1}{\pi}\int_{s_{th}}^{\infty} ds' \frac{\mathrm{Im}\, \hat{a}_\ell^{-1}(s')}{s' - s}$$

$$= (\mathrm{Rcl}) - \frac{1}{\pi}\int_{s_{th}}^{\infty} ds' \frac{\rho_\ell(s')}{s' - s}$$

$$\hat{a}_\ell^{-1}(s) = \overline{K}_\ell^{-1}(s) - \frac{1}{\pi}\int_{s_{th}}^{\infty} ds' \frac{\rho_\ell(s')}{s' - s}$$

$$\frac{\Delta_s}{2i} = \mathrm{Im}$$

$$\Delta_s \frac{1}{\pi}\int_{s_{th}}^{\infty} ds' \frac{\rho_\ell(s')}{s' - s}, \quad \Delta_s \frac{1}{s' - s} = \pi\, \delta(s' - s)$$

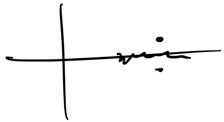
(K-matrix)

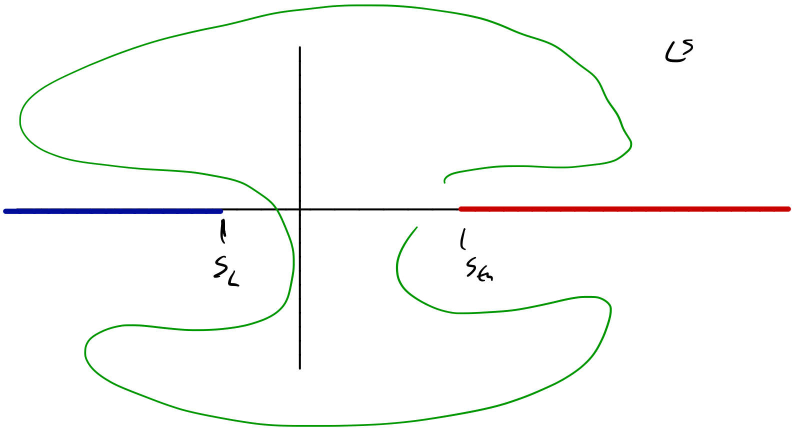
$$= \rho_\ell(s)$$

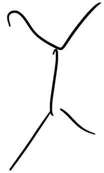
N/D

$$\Delta_z f(z) = f(z+i\epsilon) - f(z-i\epsilon)$$

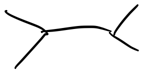
$$\frac{\Delta_z f(z)}{2i} = \mathrm{Im}\, f(z)$$







$\Rightarrow$ LHC



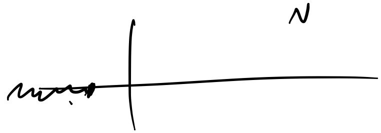
$\Rightarrow$ RHC

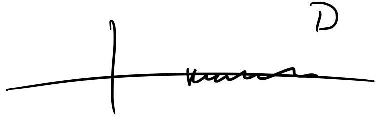
$$\hat a_\ell(s) = \frac{1}{2\pi i}\oint_C ds' \, \frac{\hat a_\ell(s')}{s'-s}$$

$$\mathrm{Im}_R\, \hat a_\ell(s) = \rho_\ell(s)\,|\hat a_\ell(s)|^2$$

$$\Rightarrow \quad \hat a_\ell(s) = \frac{1}{\pi}\int_{-\infty}^{s_L} ds' \, \frac{\mathrm{Im}_L\, \hat a_\ell(s')}{s'-s} + \frac{1}{\pi}\int_{s_{th}}^{\infty} ds' \, \frac{\mathrm{Im}_R\, \hat a_\ell(s')}{s'-s}$$

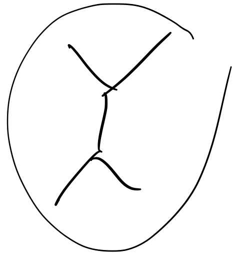
ld $\quad \hat{a}_\ell(s) = \dfrac{N_\ell(s)}{D_\ell(s)} \begin{array}{l} \to \text{LHC} \\ \to \text{RHC} \end{array}$





$$\operatorname{Im} D_\ell(s) = -\rho_\ell(s)\, N_\ell(s) \qquad s > s_{th}$$

$$\operatorname{Im} N_\ell(s) = \operatorname{Im}_L \hat{a}_\ell(s)\, D_\ell(s) \qquad s < s_L$$

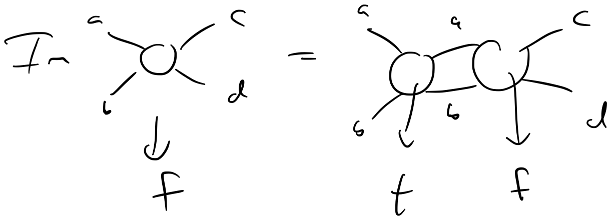


$$\Rightarrow \left.\begin{aligned} N_\ell(s) &= \frac{1}{\pi}\int_{-\infty}^{s_L} ds' \, \frac{\operatorname{Im}_L \hat{a}_\ell(s')\, D_\ell(s')}{s'-s} \\ D_\ell(s) &= -\frac{1}{\pi}\int_{s_{th}}^{\infty} ds' \, \frac{\rho_\ell(s')\, N_\ell(s')}{s'-s} \end{aligned}\right\}$$

$$\hookrightarrow \quad D_\ell(s) = D^0_\ell(s) - \frac{1}{\pi}\int_{s_{th}}^{\infty} ds' \, \frac{\rho_\ell(s')\, N_\ell(s')}{s'-s}$$

$$\downarrow_{\text{Red}} \quad D^0_\ell(s) = c_0 - c_1 s + \sum_r \frac{\gamma_r}{\alpha_r - s}$$

Omnes Problem



$$\mathrm{Im}\, \hat{f}_\ell(s) = \rho_\ell(s)\, \hat{t}^{*}_\ell(s)\, \hat{f}_\ell(s)$$

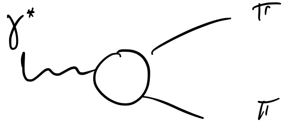
$\downarrow$

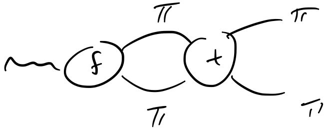
elastic

$$\hat{t}_\ell(s) = \frac{e^{i\delta_\ell(s)} \sin\delta_\ell(s)}{\rho_\ell(s)}$$

$$\rho_\ell(s)\, \hat{t}_\ell(s) = e^{i\delta_\ell(s)} \sin\delta_\ell(s) \qquad *$$

$$\Rightarrow \mathrm{Im}\, \hat{f}_\ell(s) = e^{-i\delta_\ell(s)} \sin\delta_\ell(s)\, \hat{f}_\ell(s)$$





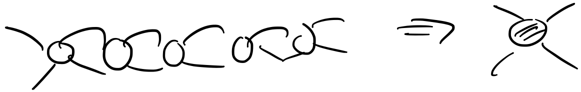
Watson's Final state theorem

$$\hat{f}_\ell(s) = |\hat{f}_\ell(s)|\, e^{i\delta_\ell(s)}$$

$$\mathrm{Im}\,\hat{f}_\ell(s) = e^{-i\delta_\ell(s)}\,\sin\delta_\ell(s)\,\hat{f}_\ell(s)$$

$$\mathrm{Im} = \frac{\Delta_s}{2i}$$

$$\hat{f}_\ell(s) = \underset{\substack{\uparrow \\ \text{Poly}}}{P_\ell(s)}\;\underset{\substack{\uparrow \\ \text{RHC}}}{\Omega_\ell(s)}$$



$$\vdots$$

$$\frac{1}{2i}\left(\Omega(s+i\epsilon) - \Omega(s-i\epsilon)\right) = \Omega(s+i\epsilon)\,\sin\delta_\ell(s)\,e^{-i\delta_\ell(s)}$$

$$\vdots$$

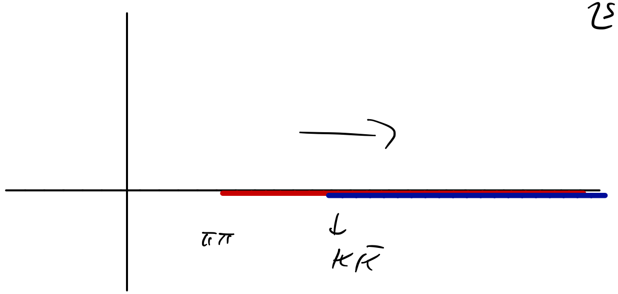
$$\Rightarrow \quad \Delta_s\left(\ln\Omega(s)\right) = 2i\,\delta_\ell(s) \quad \nearrow \quad \underline{\text{Omnes function}}$$

$$\Rightarrow \quad \Omega(s) = \exp\left(\frac{1}{\pi}\int_{s_{th}}^{\infty} ds' \,\frac{\delta_\ell(s')}{s'-s}\right), \quad \Omega(0) = 1$$

Coupled Channels

$$a_{jk} = \begin{pmatrix} \pi\pi \to \pi\pi & \pi\pi \to K\bar{K} \\ K\bar{K} \to \pi\pi & K\bar{K} \to K\bar{K} \end{pmatrix}$$

2-channels



$2 \to 2$, inelastic

$$\mathrm{Im}\, \hat{a}_{jk}(s) = \sum_n \hat{a}^*_{kn}(s)\, \rho_n(s)\, \hat{a}_{nj}(s)$$

j-channel k-channel

n-intermediate state

Strong interactions: $a_{jk} = a_{kj}$

$$\Rightarrow \quad \mathrm{Im}\left[\hat{a}^{-1}(s)\right]_{jk} = -\rho_k(s)\,\delta_{jk}$$

K-matrix :

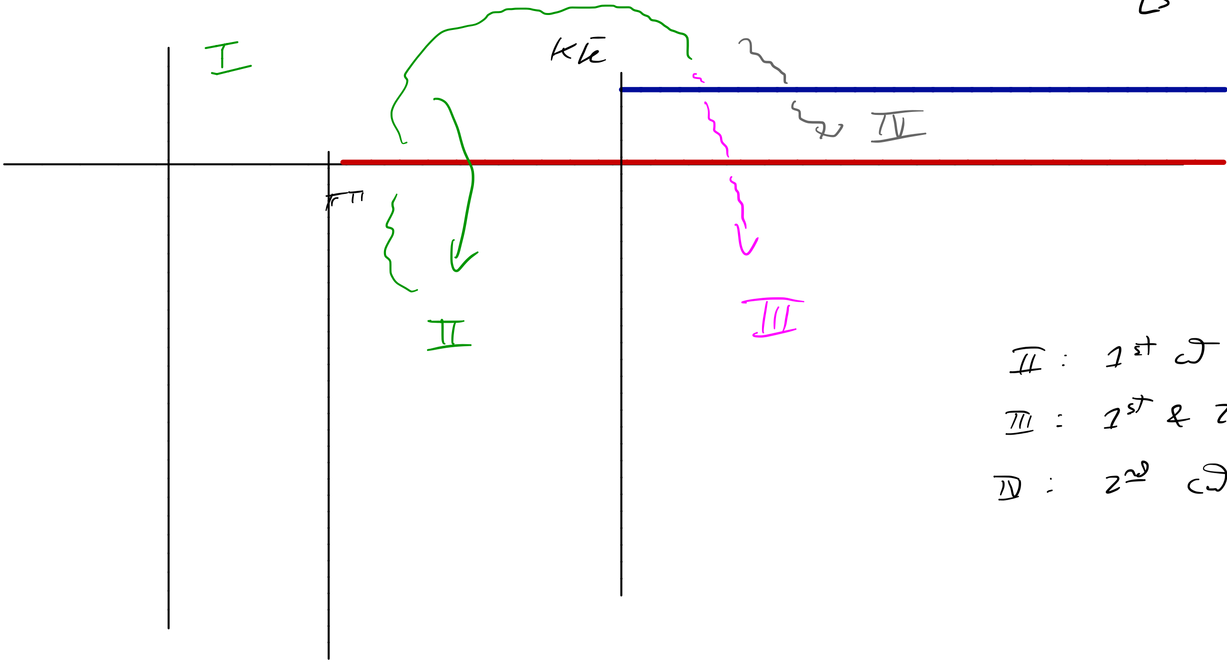
$$\left[\hat{a}^{-1}\right]_{jk} = \left[k^{-1}\right]_{jk} - \frac{1}{\pi}\int_{s_{th}^{(k)}}^{\infty} ds' \,\frac{\rho_k(s')\,\delta_{jk}}{s'-s}$$

$\downarrow$

Real, symmetric matrix

$$\left[K^{-1}(s)\right]_{jk} = \sum_r \frac{g_j^r\, g_k^r}{m_r^2 - s} + \sum_\alpha \gamma_{jk}^\alpha\, s^\alpha$$

2s

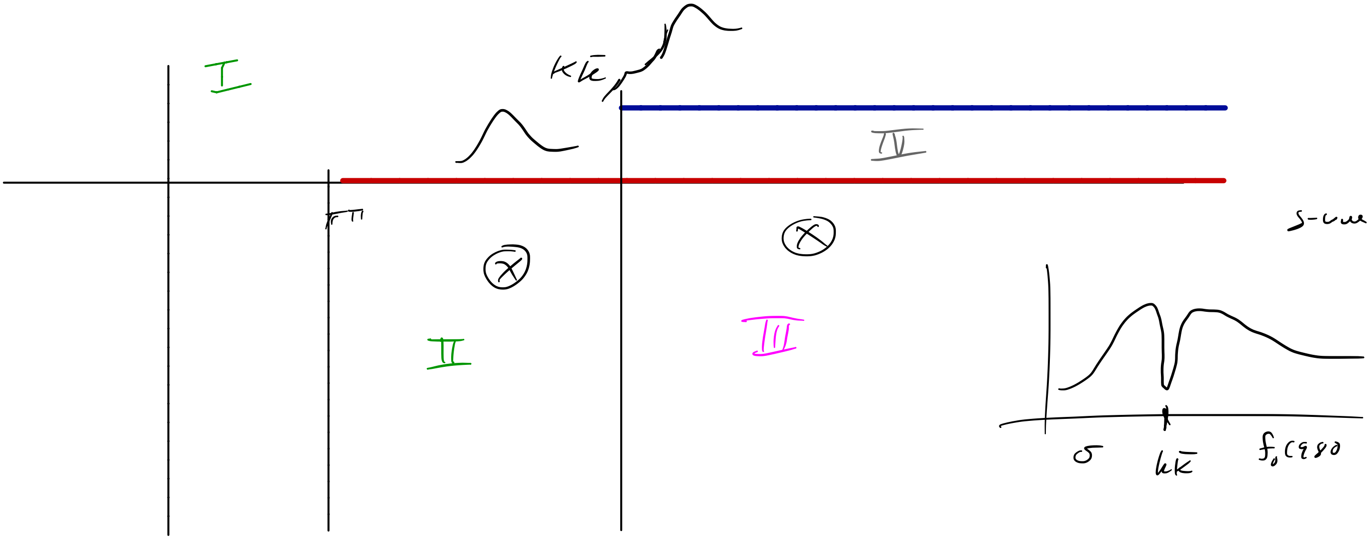


II : 1st cut

III : 1st & 2nd cut

IV : 2nd cut

$$a_{jk} = \begin{pmatrix} \pi\pi \to \pi\pi & \pi\pi \to K\bar{K} \\ & \\ K\bar{K} \to \pi\pi & K\bar{K} \to K\bar{K} \end{pmatrix}$$



$$\left.\begin{array}{l} \rho_1 \to -\rho_1 \\ \rho_2 \to \rho_2 \end{array}\right\} \Rightarrow \text{II}$$

$$\left.\begin{array}{l} \rho_1 \to -\rho_1 \\ \rho_2 \to -\rho_2 \end{array}\right\} \Rightarrow \text{III}$$

$$\left.\begin{array}{l} \rho_1 \to \rho_1 \\ \rho_2 \to -\rho_2 \end{array}\right\} \Rightarrow \text{IV}$$

$$\begin{pmatrix} \rho_1 & 0 \\ 0 & \rho_2 \end{pmatrix}$$