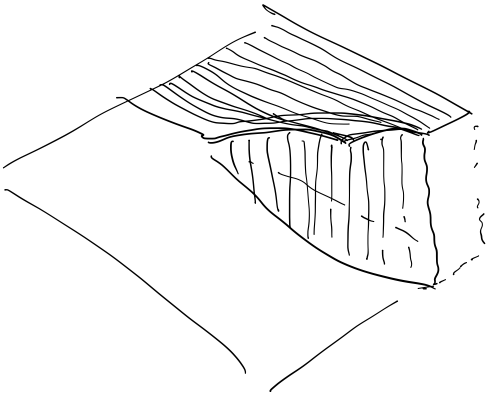
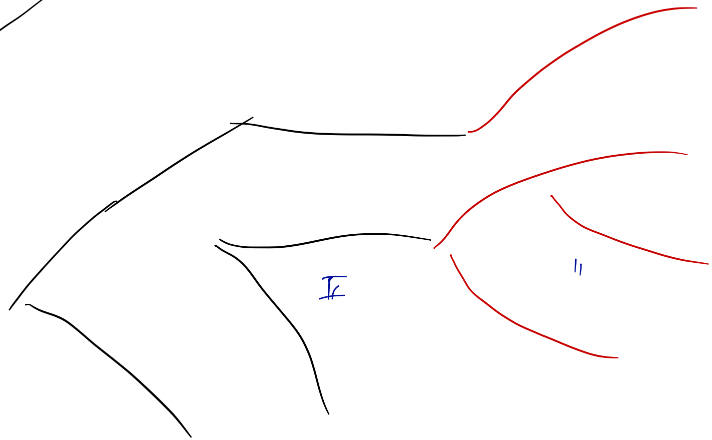
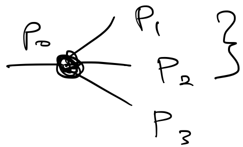
Day 4




I
II

# Three body problem

$0 \to 1 + 2 + 3$




$s = s_3$, $t = s_1$

## ① Kinematics

$A = A(\text{variables})$

4 par × 4 var = 16 deg. f. ; 4 cons. laws $E^2 - \vec{p}^{\,2} = m^2$
(E $\vec{p}$)

4 laws $E = \sum E$, $\vec{p}_0 = \sum \vec{p}_i$

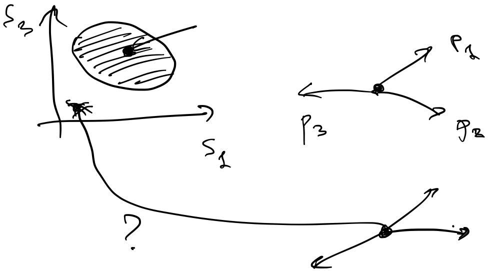
3 boosts, 3 rotations $\longrightarrow$ 2 remain d.f.

Amplitude depends on 2 variables, $(|p_1|, E_3)$, $(T_2 = E_2 - m_2, \cos\theta_{12})$

$\to$ good chose is $(s, t)$, why?

$$d\Gamma = \frac{1}{2M} |M|^2 d\Phi \; , \quad d\Phi = \underbrace{\frac{d^3p_1}{2E_1(2\pi)^3} \frac{d^3p_2}{(2\pi)^3 2E_2} \frac{d^3p_3}{(2\pi)^3 2E_3}}\; \underbrace{(2\pi)^4 \delta(P_0 - P_1 - P_2 - P_3)}_{4 \text{ constr.}}$$

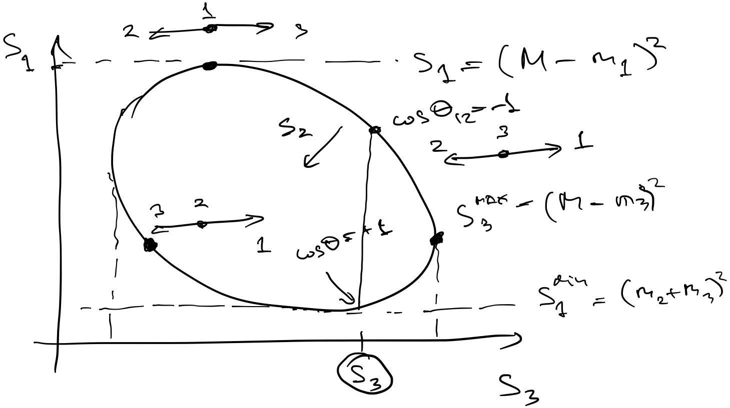
$$d\Phi \sim ds_1 ds_3 \to \frac{d\Gamma}{ds_1 ds_3} \sim |M|^2 \quad \text{g integr} \; , \quad M = M(s_1, s_3)$$




$E_1 + E_2 + E_3 = M \qquad s_1 = (p_2 + p_3)^2$

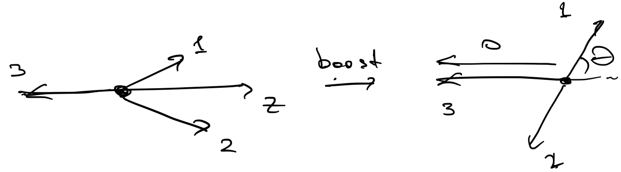
$\vec{p}_1 + \vec{p}_2 + \vec{p}_3 = \vec{0} \qquad s_3 = (p_1 + p_2)^2$

$\longrightarrow$ border of allowed region




$$s_1 + s_2 + s_3 = \text{const} = \sum_{i=1}^{3} m_i^2 + M^2$$

$s_3$ is fixed




$$t = s_1 = (p_2+p_3)^2 = t(s, \cos\theta_{12})$$

$$= m_2^2 + m_3^2 + 2E_2^{(12)} E_3^{(12)} - 2p_2^{(12)} p_3^{(12)} \cos\theta_{12}$$

$$E_2^{(12)} = \frac{s_3 + m_2^2 - m_1^2}{2\sqrt{s}} \; ; \quad s_3 = (p_1+p_2)^2, \quad p_2 = (P_1+P_2) - p_2 \longleftarrow \text{square}$$

$\uparrow$ — $(\sqrt{s_3}, 0, 0, 0)$

$$E_3^{(12)} = \frac{M^2 - s_3 - m_3^2}{2\sqrt{s_3}} \; ; \quad p_2^{(12)} = \frac{\lambda^{\frac{1}{2}}(s_3, m_2^2, m_1^2)}{2\sqrt{s_3}} \; ; \quad p_3^{(12)} = \frac{\lambda^{\frac{1}{2}}(M^2, s_3, m_3^2)}{2\sqrt{s_3}}$$

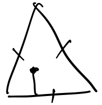
Dalitz plot = representation of decay kinematics, which reflects only dynamics, $\sim |M|^2$

$$\frac{d\Gamma}{ds_1 ds_3} \sim |M|^2$$ . Q: are there other variables to keep DP const

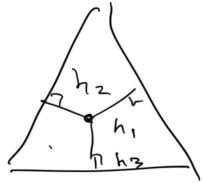
$$\frac{d\Gamma}{dE_1 dE_3} \overset{\text{number}}{\sim} |M|^2, \quad \frac{d\Gamma}{dT_1 dT_3} \sim |M|^2 \; ; \quad \frac{d\Gamma}{dE_1 dE_3} = c \cdot |M|^2, \; c \text{ is just } \frac{1}{128\pi^3} \frac{1}{M^3}$$

# 1.2 Triangle representation

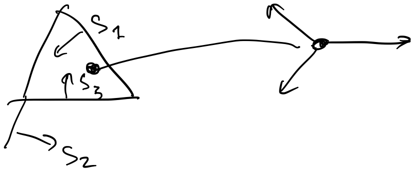
Theorem from math:

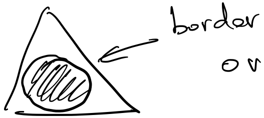


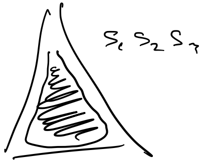
equilateral, $\forall p$ inside $h_1+h_2+h_3 = \text{const} = \frac{\sqrt{3}}{2}a^2$




$S_1+S_2+S_3 = \text{const}$



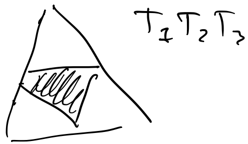



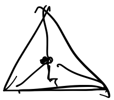

non relat. case

$K \to 3\pi$

relativistic

$J/\psi \to 3\pi$

$$S = S_1+S_2+S_3 = a(h_1+h_2+h_3)\frac{1}{2}$$

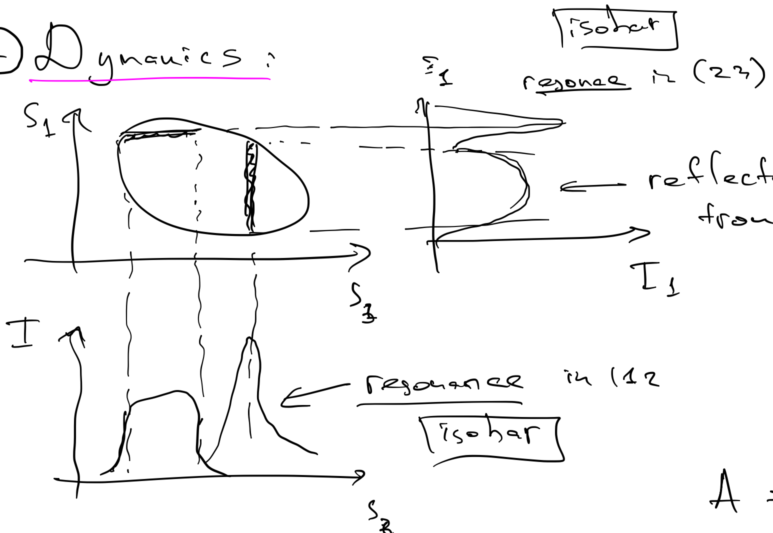
Exercise:

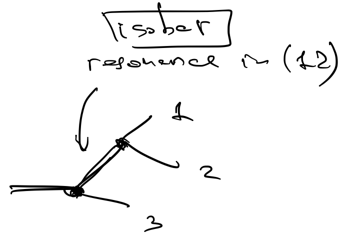
$\longrightarrow$ prove for point outside of triangle

$h_1+h_2+h_3 = \text{const}$

## ② Dynamics:

$$I_3 = \frac{d\Gamma}{ds_3} \; ; \; I_1 = \frac{d\Gamma}{ds_1}$$




$$A = A(s,t)$$

$$A = g \cdot g' \frac{1}{m^2 - S_3 - im\Gamma}$$

2 resonances first approximation

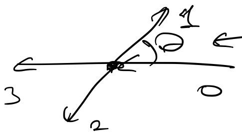
$$A = \frac{c_1}{m_1^2 - s_3 - im_1\Gamma_1} + \frac{c_3}{m_3^2 - s_3 - im_3\Gamma_3}$$

← approximately for scalars with scalar isobar

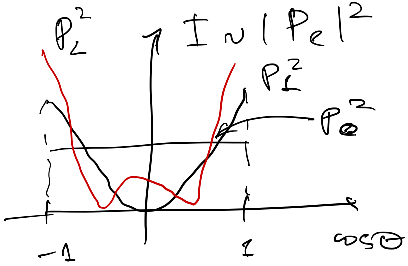
## 2.2 Angular dependence

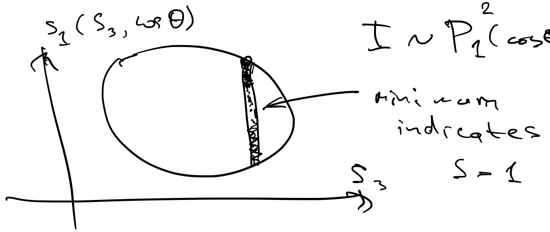
expansion in (12), $s = s_3$

$$A(s,t) = \sum (2l+1) f_l(s) P_l(z_s) = \sum (2l+1) \boxed{f_l}(s_3) P_l(\cos\theta_{12})$$

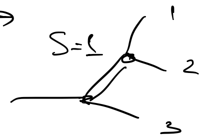


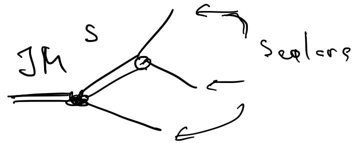

$\theta_{12}$ for isobar with spin $S$ $\longrightarrow P_S(\cos\theta_{12})$




$I \sim P_1^2(\cos\Theta) \longrightarrow$



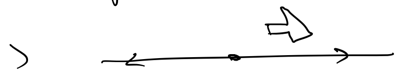

## Reminders:

$|jm\rangle$ — state wit spin $j$, projection $m$

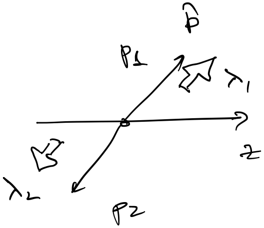
$R|jm\rangle = D^j_{m'm}(\Omega)\,|jm'\rangle$

$D^j_{m'm}(\Omega) = e^{-im'\varphi} \cdot d^j_{m'm}(\theta) \cdot e^{-im\gamma}$

$\varphi, \theta, \gamma$




$|\hat{p}, \lambda_1 \lambda_2\rangle \leftarrow$ state of 2 particles with helicity $\lambda_1 \lambda_2$ going to direction $\hat{p}$



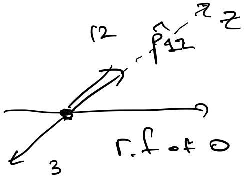

$= D^S_{\lambda\,\lambda_1-\lambda_2}(\Omega)$

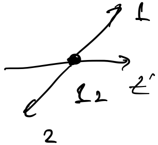
$$|\hat{p}, \lambda_1\lambda_2\rangle = \sum_{S\lambda} |S\lambda\rangle\langle S\lambda|\hat{p}, \lambda_1, \lambda_2\rangle =$$

$$= \sum_{S\lambda} D^S_{\lambda\,\lambda_1-\lambda_2}(\Omega)\,|S\lambda\rangle \left(\frac{2S+1}{4\pi}\right)^{\frac{1}{2}}$$

$$|\hat{p}, \lambda_1\lambda_2\rangle = R(\hat{p})\,|\vec{0}, \lambda_1, \lambda_2$$

rest frame of (12)




Boost to mother p. rest frame $\Lambda_z(|\vec p_{12}|)$

Rotation $R(\hat{\vec p}_{12})$

$$\Lambda_z(|p_{12}|)\,|\hat p\,\lambda_1\lambda_2\rangle = \Lambda \sum_{s\lambda} D^{s}_{\lambda,\lambda_1-\lambda_2}(\Omega_{12})\,|s\lambda;\lambda_1\lambda_2\rangle \left(\frac{2s+1}{4\pi}\right)^{\frac{1}{2}}$$

boost $\longleftarrow$ $\lambda$ $p$

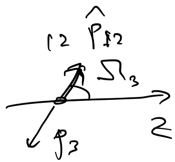
$$= \Lambda_z(|\vec p|)\,|j\lambda\rangle = \sum_{\lambda'} d_{\lambda'\lambda}(\Theta_\nu)\,|j\lambda'\rangle$$

$\boxed{\text{tricky}}$

Scalars at final state

$$|\hat p,\lambda_1\lambda_2\rangle = |\hat p,00\rangle = \sum |s\lambda\rangle\, D^{s}_{\lambda 0}(\Omega_{12}) \left(\frac{2s+1}{4\pi}\right)^{\frac{1}{2}}$$

$$R\Lambda_z|\hat p,00\rangle = \sum_{s\lambda} |p_{12}, s\lambda,\dots\rangle\, D^{s}_{\lambda 0}(\Omega) \left(\frac{2s+1}{4\pi}\right)^{\frac{1}{2}}$$




skip 3 lines

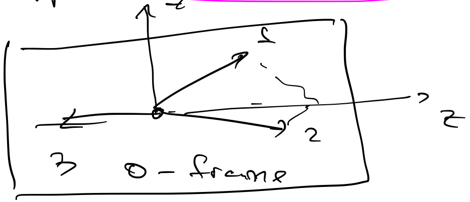
$$|p_1p_2p_3\rangle = \sum_{JMs\lambda} |JMs\lambda\rangle\, D^{J}_{M\lambda}(\Omega_3)\left(\frac{2J+1}{4\pi}\right)^{\frac{1}{2}}\cdot D^{s}_{\lambda 0}(\Omega_{12})\left(\frac{2s+1}{4\pi}\right)^{\frac{1}{2}}$$

LS-recoupling

$$A = \frac{gg'}{m^2-s_3-im\Gamma}\, A^{JM}_{LS},$$

$$\boxed{A^{JM}_{LS} = \left(\frac{2L+1}{2J+1}\right)^{\frac{1}{2}} \sum_{\lambda} \langle L0S\lambda|J\lambda\rangle \left(\frac{2J+1}{4\pi}\right)^{\frac{1}{2}} D^{J*}_{M\lambda}(\Omega_1)\cdot\left(\frac{2S+1}{4\pi}\right)^{\frac{1}{2}} D^{S*}_{\lambda 0}(\Omega_{23})}$$

## Application $0 \to 1,2,3$

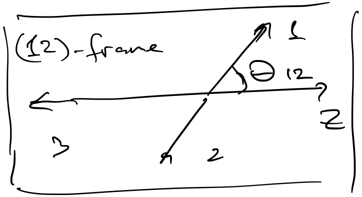



$$A = A(s,t) = A(s, \theta_{12})$$

$\Omega_3 \leftarrow$ direction of Isobar 12

$\Omega_3 = (0,0)$ ; $D^J_{M\lambda}(\Omega_1) = \delta_{M\lambda}$

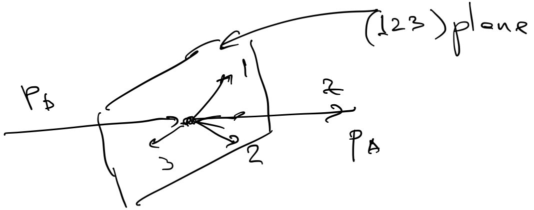
$\Omega_{12} = (\theta_{12}, 0)$

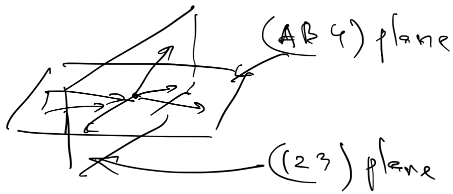



$$\Rightarrow A^{JM}_{LS} = \frac{1}{4\pi} \langle L0SM|JM\rangle \, d^S_{M0}(\theta_{12}) \sqrt{(2S+1)(2L+1)}$$

$$\boxed{A^J_M(s,t) = \sum_{JMLS} A^{JM}_{LS}(z) \, f^{JM}_{LS}(s)}$$

## Application to $A+B \to 1,2,3$ and $A+B \to (123)\,4$

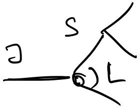



$\Omega_3 = (\theta_3, 0)$, $\Omega_{12} = (\theta_{12}, \varphi_{12})$

$\Omega_3 = (\theta_3 \varphi_3)$ ; $\Omega_{12} = (\theta_{12} \varphi_{12})$

# threshold behavior




$$A = \sum_{J\eta} f_{LS}^{J\eta} A_{LS}^{J\eta}, \quad f_{LS} \sim \underline{|\vec{p}_3|}^L \cdot |\vec{p}_1^{\,*}|^S$$

$|\vec{p}_3| \leftarrow$ is break up momentum at 0-part. frame

$$|\vec{p}_3| = \frac{\lambda^{\frac{1}{2}}(M^2, s_3, m^2)}{2\sqrt{M^2}}$$

$$\boxed{h_L(p) = \left[\frac{p^2R^2}{1+p^2R^2}\right]^{L/2}}, \quad h_L \sim p^L R^L$$

$\uparrow$

$R = 1$ fm $\sim$ $\sim 5\,\mathrm{GeV}^{-1}$

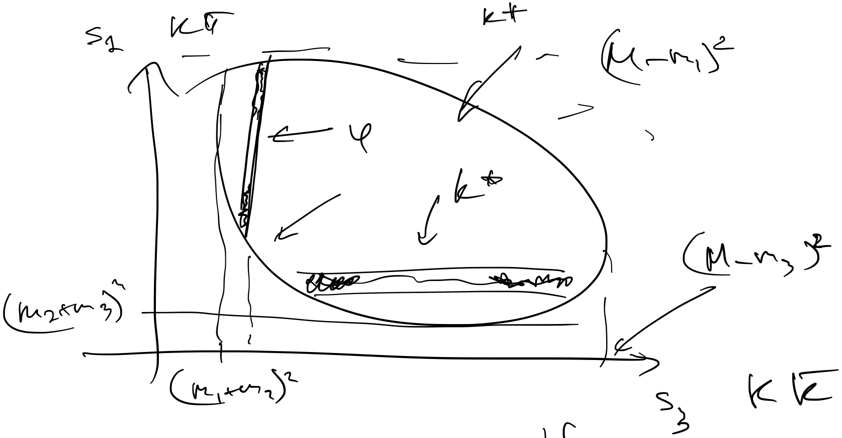
$|\vec{p}_1^{\,*}|$ is break up momentum at (12) frame

$$|\vec{p}_1^{\,*}| = \frac{\lambda^{\frac{1}{2}}(s_3, m_1^2, m_2^2)}{2\sqrt{s_3}}$$

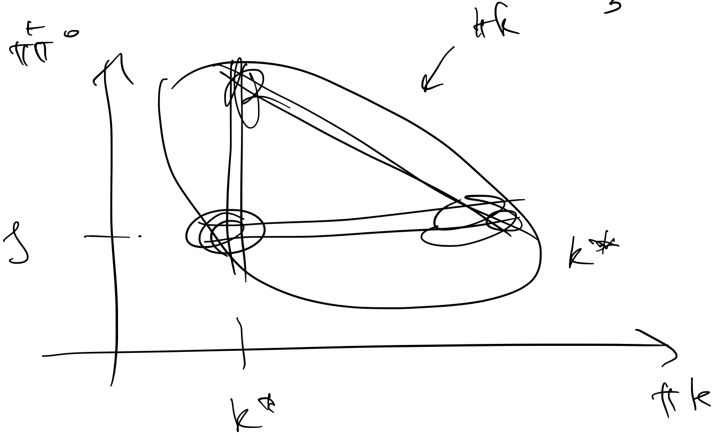
$$f_{LS}^{J\eta} = \frac{h_L(|\vec{p}_3|)\, h_S(|\vec{p}_1^{\,*}|)}{m^2 - s_3 - im\Gamma}$$

$h_L$ introduces left hand singularity

$h_L$ are Blatt-Weisskopf-<u>like</u> factors




$D_s \to KK\pi$




$D^0 \to K^- \pi^+ \pi^0$