Day 6

① Real analytic function (Hermitian analytic)

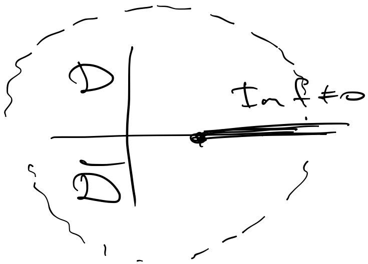
$\mathbb{C}$ Domain $D : \{ s, \operatorname{Im} s > 0 \}$

$f(z)$ is analytic, $\operatorname{Im} f(z) \neq 0$ at the border of the domain $z \in$ Real

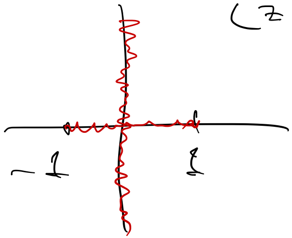
→ Schwarz reflection principle $f(z)$ is analytic in $D \cup \bar{D}$

$f(z^*) = f^*(z)$, ⟶ f is Hermitian analytic

if $f(z)$ has region $\operatorname{Im} f(x) \neq 0$, $x \in [a, b)$ in border

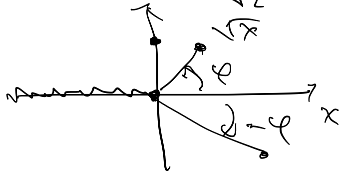


Example: $f(z) = \sqrt{z^2 - 1}$

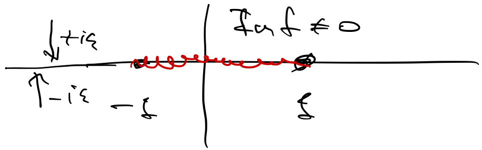


$\sqrt{1} = 1$, $\sqrt{i} = \frac{1}{\sqrt{2}}(1 + i)$

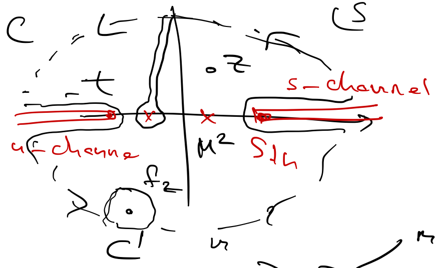
$\sqrt{-i} = \frac{1}{\sqrt{2}}(1 - i)$



b) $f(z) = \sqrt{z - 1}\,\sqrt{z + 1}$

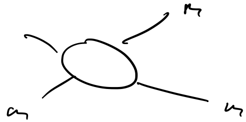


② Dispersion relation



$T(s,t)$ is scattering amplitude

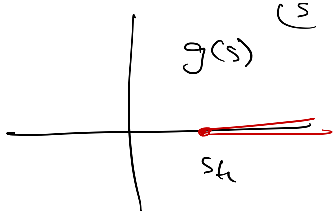
fix. t to be negative, in s-channel



$s_{th} = 4m^2$, t — ! $s + t + u = \sum m^2 = 4m^2$

$u_{th} = 4m^2 \Rightarrow s'_{th} = -t$

$T(s,t) = f(s)$ real analytic



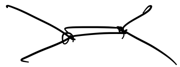
Cauchy theorem $\quad f(z) = \frac{1}{2\pi i}\int_{C'} \frac{f(z')}{z'-z}\, dz'$

$F(z',z) = \frac{f(z')}{z'-z}$.

$\frac{1}{2\pi i}\int_C F(z',z)\, dz = \sum_{z, z_p, z'_p} \operatorname{res} F(z',z)$ $\quad z_p = M^2$, bound state in s-channel

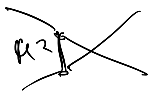
res in z_p in z'

$f(z) + \frac{g}{M^2-z} + \frac{g'}{4m^2-t-M^2-z} =$



$z'_p = 4m^2 - t - M^2 = s'_p$

$= \frac{1}{2\pi i}\left[\int_{\gamma_1} + \int_{\gamma_2} + \int_{C_\infty}\right] \frac{f(z')}{z'-z}\, dz'$

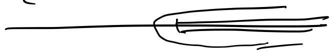


u - channel

if $|f(z)| \to 0$ when $z \to \infty$ then $\displaystyle\int_{C_\infty} \frac{f(z')}{z'-z}\, dz' = 0$

$$\Rightarrow \quad f(z) = \frac{g}{M^2 - z} + \frac{g'}{4m^2 - t - M^2 - z} + \frac{1}{2\pi i}\left[\int_{\gamma_1} + \int_{\gamma_2}\right] \frac{f(z')}{z'-z}\, dz'$$

$$\int_{\gamma_1} \frac{f(z')}{z'-z}\, dz = \int_{th}^{+\infty} dx\, \frac{f(x+i\varepsilon) - f(x-i\varepsilon)}{x-z} = \int_{th}^{+\infty} \frac{\mathrm{Disc}\, f(x)}{x-z}\, dx$$



$$\mathrm{Disc}\, f(x) = f(x+i\varepsilon) - f(x-i\varepsilon)$$

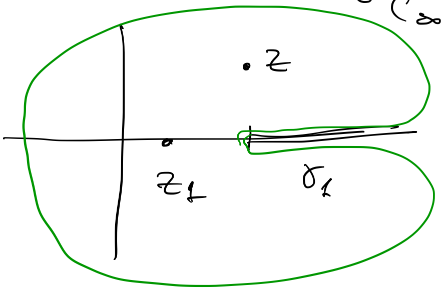
$f(z)$ is real analytic $\Rightarrow$ $\mathrm{Disc}\, f(x) = 2i\, \mathrm{Im}\, f(x)$

$$f(z) = \frac{g}{M^2 - z} + \frac{g'}{4m^2 - M^2 - t - z} + \frac{1}{\pi}\int_{th}^{+\infty} \frac{\mathrm{Im}\, f(x)}{x - z}\, dx + \frac{1}{\pi}\int_{-\infty}^{th'} \frac{\mathrm{Im}\, f(x)}{x-z}\, dx$$

③ Subtraction

What if $|f(z)| \not\to 0$ as $z\to\infty$, $|g(z)| \not\to 0$, $\quad F(z',z,z_1) = \dfrac{f(z')}{(z'-z)(z'-z_1)}$

$$G(z',z,z_1) = \frac{g(z')}{(z'-z)(z'-z_1)}$$



$$\frac{1}{2\pi i}\oint G(z',z,z_1)\,dz' = \sum \text{res}$$

$$\Rightarrow \text{Now } \left|\frac{g(z')}{z'-z_1}\right| \xrightarrow{z\to\infty} 0$$

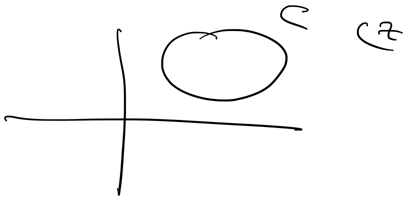
$$\frac{1}{\pi}\int_{th}^{+\infty} \frac{\operatorname{Im} g(x)}{(x-z)(x-z_1)}\,dx = \frac{g(z)}{z-z_1} + \frac{g(z_1)}{z_1-z} \Rightarrow$$

$$g(z) = g(z_1) + \frac{(z-z_1)}{\pi}\int_{th}^{+\infty} \frac{\operatorname{Im} g(x)}{(x-z)(x-z_1)}\,dx$$

why subtraction?

$$g(z) = \frac{1}{2i\pi}\int_C \frac{g(z')}{z'-z}\,dz'$$

$$g(z_1) = \frac{1}{2\pi i}\int_C \frac{g(z')}{z'-z_1}\,dz'$$



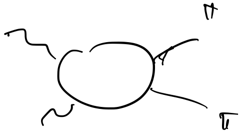
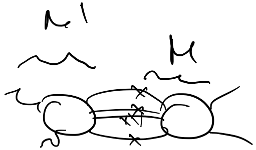
$$g(z) - g(z_1) = \frac{z-z_1}{2\pi i}\int_C \frac{g(z')}{(z'-z)(z'-z_1)}\,dz'$$

Multible subtraction $F(z', z, z_1 \ldots z_n) = \dfrac{f(z')}{(z'-z)(z'-z_1)\cdots}$

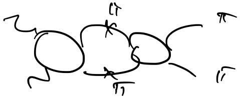
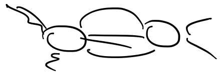
$$\left| \frac{f(z')}{(z'-z_1)\cdots(z'-z_n)} \right| \xrightarrow{|z| \to \infty} 0$$

$$g(z) = g(0) + z g'(0) + \frac{z^2}{\pi} \int \frac{g(z')}{(z'-z) z'^2} dz'$$

(4). Omnes functions

 $= A(s)$; $\Delta A = i$  $= i \sum_n \int d\Phi_n \, \Gamma^* \Gamma$

weakly coupled channel

$\Delta A = i$  $+ \; i \underbrace{\sum_{n \neq 2}}_{\text{neglect}}$ 

$$\Delta A = i \int T^* A \, d\Phi$$

partial waves $\Delta f_\ell = 2i \, t_\ell^* \rho \, f_\ell$, $\rho = \frac{1}{2} \Phi = \frac{1}{16\pi} \sqrt{1 - \frac{4m^2}{s}}$

$$T(s,t) = \sum (2\ell+1) \, t_\ell(s) \, P_\ell(\cos\theta)$$

← elastic PW amplitud

$$\begin{cases} \Delta t_\ell = 2i \, t_\ell^* \rho \, t_\ell \\ \Delta f = 2i \, t_\ell^* \rho \, f_\ell \end{cases} \longrightarrow f_\ell = a_\ell + b_\ell$$

b_ℓ ← only left hand cut

a_ℓ: has only right hand cut

$$\boxed{\Delta a_\ell = 2i \, t^* \rho \, (a_\ell + b_\ell)}$$ ← Omnès-like equation

$a_\ell = P_\ell \, \Omega_\ell$ ← Omnès function, defined by t

P_ℓ: defined by b_ℓ

$$\Delta a_\ell = 2i f^* \rho (a_\ell + b_\ell) \longrightarrow a^+ - a^- = 2i f^* \rho (a^+ + b)$$

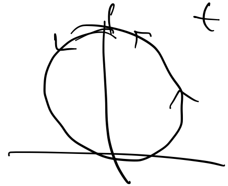
$$a_\ell(s+i\varepsilon) = a_\ell^+ , \quad a_\ell(s-i\varepsilon) = a_\ell^-$$

$$\boxed{\Delta\Omega = (\Omega_+ - \Omega_-) = 2i\rho f^* \Omega(+)}$$

$$\Rightarrow a^+(1 - 2i\rho f^*) = a^- + 2i\rho f^* b \leftarrow$$ inhomogeneous Omnes equation

Homogeneous equation

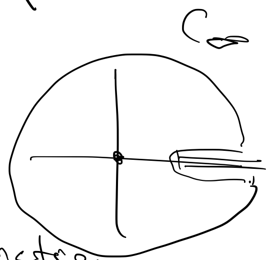
$$a^+(1-2i\rho f^*) = a^- , \quad f = \frac{1}{2\rho i}\left[e^{2i\delta} - 1\right]$$



$$a^+ e^{-2i\delta} = a^- \leftarrow$$

$$\ln a^+ - 2i\delta = \ln a^- \Rightarrow \Delta \ln a^+ = 2i\delta$$

$$\Rightarrow \ln a = +\frac{s}{\pi}\int_{th}^{+\infty} \frac{\delta(s')}{(s'-s)s'}\, ds' + \underbrace{\ln a(0)}_{\text{subtraction const}}$$



$$a = a(0)\exp\left(+\frac{s}{\pi}\int_{th}^{+\infty}\frac{\delta(s')}{s'-s}\, ds'\right) \leftarrow$$ Omnès function

$$\Omega = \Omega(0)\exp\left(+\frac{s}{\pi}\int_{th}^{+\infty}\frac{\delta(s')}{(s'-s)s'}\, ds'\right) \leftarrow$$

- same phase as f
- no LHC

Solution of inhomogeneous equation

$$a^+(1-2i\rho f^*) = a^- + 2i\rho b f^* , \quad a = P\,\Omega$$

↑ function

$$P^+\underbrace{\Omega^+(1-2i\rho f^*)} = P^-\Omega^- + 2i\rho b f^*$$

$$\Rightarrow \Omega^- P^+ = P^+\Omega^- + 2i\rho b f^* \Rightarrow$$

$$\Rightarrow \Delta P = 2i\rho b f^*/\Omega^- \Rightarrow P = \frac{1}{\pi}\int \frac{b f^* \rho}{\Omega(s'-s)}\,ds +$$

Solution $\boxed{a(s) = \Omega(s)\left[\text{regular function} + \frac{1}{\pi}\int \frac{b f^* \rho}{\Omega^-(s'-s)}\,ds'\right]}$ + regular function

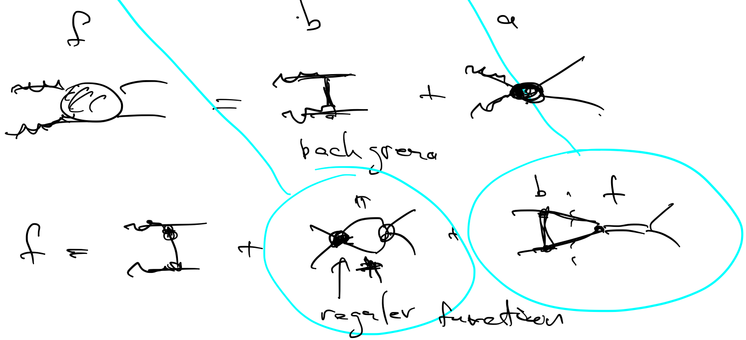
if b is not present $\Rightarrow$ $a(s) = \Omega(s)\cdot$ regular function

$\sqrt{\text{LHC}}$

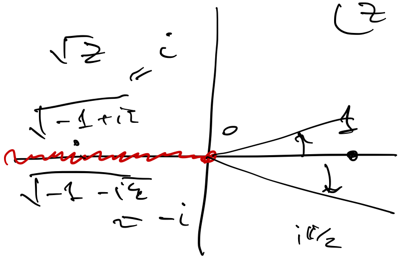
$f(s) = a(s) + b(s)$

↑ only RHC

Unitarization →

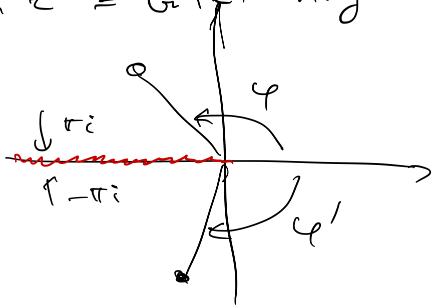


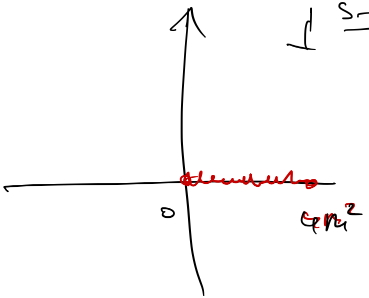
Solution 6.1



$$\sqrt{-a+i\varepsilon} = \sqrt{|a|}\cdot i = e^{i\pi/2}$$

$$\ln z = \ln|z| + i\,\mathrm{Arg}\, z$$





$$\sqrt{x} = 0 + \frac{x}{2i\pi}\int_{-\infty}^{0} \frac{\Delta\sqrt{x'}}{(x'-x)x'}\,dx'$$

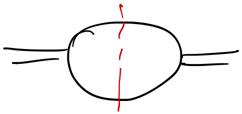
$$\Delta\sqrt{x} = 2\sqrt{x}$$

$$\sqrt{x} = 0 + \frac{x}{\pi}\int_{-\infty}^{0} \frac{\sqrt{-x'}}{(x'-x)x'}\,dx'$$

$$\Delta \ln x = 2\pi i$$

$$\ln x = \ln 1 + \frac{(x-1)}{2i\pi}\int_{-\infty}^{0} \frac{2\pi i\,dx'}{(x'-x)(x'-1)}$$

Scalar loop



$$a = \int \frac{d^4k}{(2\pi)^4} \frac{1}{(m^2-k^2)(m^2-(p-k)^2)}$$

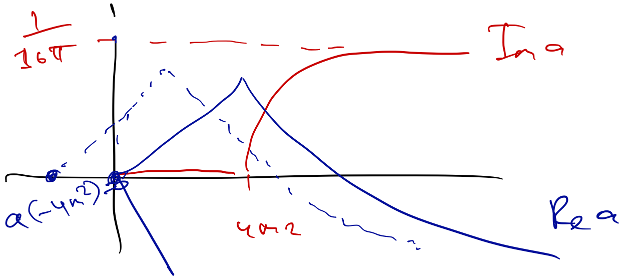
integral diverges

$$\mathrm{Im}\, a = \frac{1}{2}\,\Phi = \frac{1}{16\pi}\sqrt{1-\frac{4m^2}{s}}\,, \quad s > 4m^2$$

$$= 0\,, \quad \text{when} \quad s \leq 4m^2$$

$$a = a(0) + \frac{s}{\pi}\int \frac{\rho(s')}{(s'-s)\,s'}\,ds' = \Sigma(s)$$

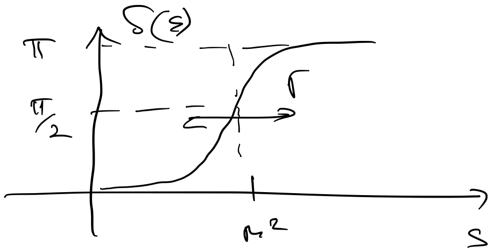
renormalization



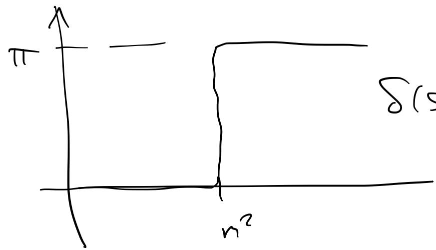
shift is determined by subtraction constant

6.2. Solution

$$F_{BW} = \frac{g^2}{m^2 - s - im\Gamma(s)}, \qquad \arg F_{BW} = \arctan\frac{\mathrm{Im}\, F_{BW}}{\mathrm{Re}\, F_{BW}} = \arctan\frac{m\Gamma}{m^2 - s}$$



$$\xrightarrow{\Gamma \to 0}$$



$$\delta(s) \approx \pi \cdot \Theta(s - m^2)$$

$$\Omega = \Omega(0)\exp\left(-\frac{\pi s}{\pi}\int \frac{\Theta(s'-m^2)}{s'(s'-s)}\, ds'\right) =$$

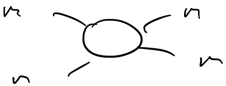
$$= \Omega(0)\exp\left(-s\int_{m^2}^{+\infty} \frac{ds'}{(s'-s)s'}\right) = \frac{1}{m^2 - s}$$

$$\simeq \log(m^2 - s)$$

→ We have got Breit–Wigner back

→ Omnès is just t without LHC.

Lecture: High energy for partial waves

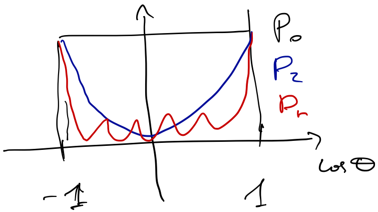


$$A(s,t) = \sum (2\ell+1)\, f_\ell\, P_\ell(z_s), \quad z = \cos\Theta$$

take s to be large, t is small and fixed

$$\cos\Theta = 1 + \frac{2t}{s-4m^2} \longrightarrow 1 \text{ when } s \to \infty$$

1) P_ℓ for high ℓ — ?



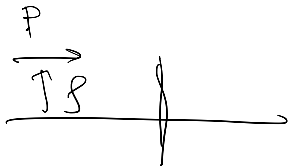
$P_n(\cos\Theta)$ has peaks forward and backward

$$\to \quad \Theta \sim \frac{1}{\ell_{max}}$$

Impact parameter representation

$$P_\ell(\cos\Theta) \longrightarrow J_0(\Theta\ell) \text{ for small } \Theta$$

$$A(s,t) = \sum (2\ell+1)\, f_\ell\, P_\ell(\cos\Theta) \longrightarrow \int \ell\, d\ell\; f_\ell(s)\, J_0(\ell\Theta)$$

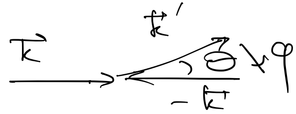


$f_\ell(s) = f(\rho, s)$ pw amplitude

$$\rho \, \frac{\sqrt{s}}{2} = \ell \Rightarrow \rho = \ell/k, \quad k = \frac{\sqrt{s}}{2}$$

$f(\rho, s)$ is impact parameter representation of PW

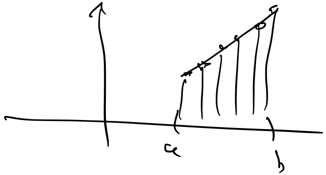
$$A(s,t) = 2\int_{\ell=0}^{+\infty} d\ell \cdot \ell \, f(\ell, s)\, J_0(\ell \Theta) \Rightarrow$$

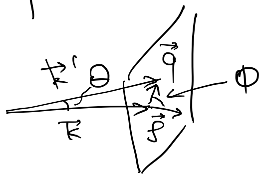


$$= 2k^2 \int d\rho\, \rho\, f(\rho, s)\, J_0(q\rho)$$

$$\vec{q} = \vec{k} - \vec{k}'$$
$$|\vec{q}| = |\vec{k}| \cdot \Theta$$

$$\int_a^b f(x)\, dx = \sum f(x_i)\, \Delta x$$





$$J_0(q\rho) = \int e^{+i\vec{q}\vec{\rho}} \frac{d\phi}{2\pi}$$

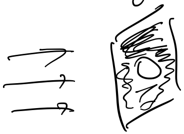
$$t = -q^2$$

$$A(s,t) = k^2 \int 2\rho\, d\rho\, f(\rho, s) \int \frac{d\varphi}{2\pi} e^{+i\vec{\rho}\vec{q}} =$$

$$= \boxed{\frac{k^2}{\pi} \int d\rho_x\, d\rho_y\, f(\rho, s)\, e^{+i\vec{\rho}\vec{q}} = A(s, -q^2)}$$

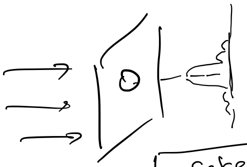
$$|\vec{k}| = \frac{\sqrt{s}}{2}$$

Very similar to optics. Fraunhofer scattering

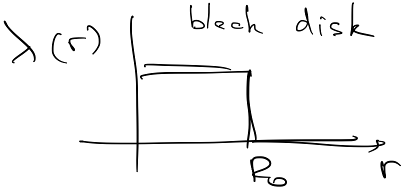


$$A = \int_S dx\, dy\, e^{iqr} = \int \lambda(x,y)\, e^{+i\vec{q}\vec{r}}\, dx\, dy$$

density of screen

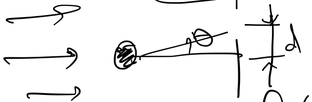


$$I_{\text{Hole}}(q) = I_{0\,\text{disk}}(q)$$

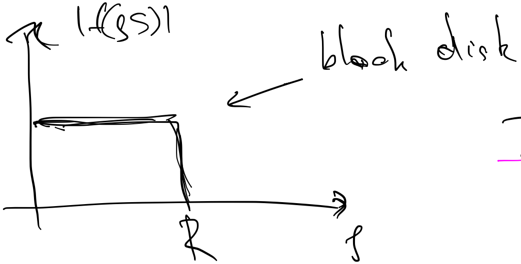


Screen far away

$$A_{H}(\vec{q}) + A_0(\vec{q}) = 0 \Rightarrow A(\vec{q}) = -A(\vec{q})$$



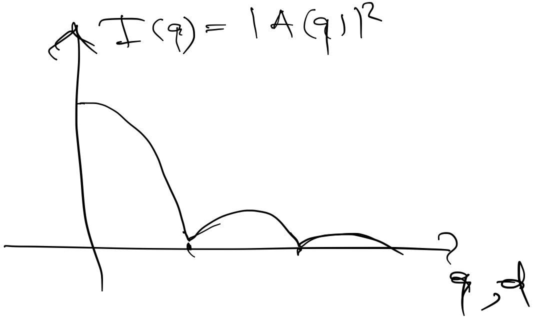
$f(\rho, S)$ can be viewed as density just behind the target.



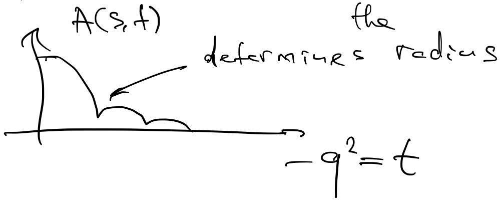
Interference picture.

a) optics $\displaystyle\int_0^R dx\,dy\; e^{irq} = \frac{R J_0(Rq)}{q}$

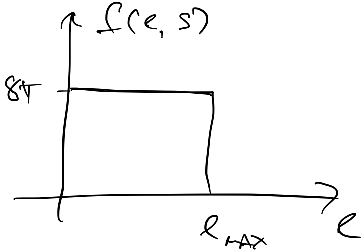
$$|q| = |k|\cdot\Theta \Rightarrow \theta = \frac{|\vec{q}|}{|\vec{k}|}, \quad d = L\cdot\Theta$$



b) at particle scattering



Cross-section



$$A(s,t) = \sum_{0}^{l_{MAX}} (2l+1)\, f_l\, P_l(\theta)$$

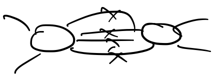
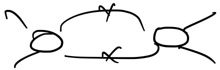
$t=0,\ \cos\theta = 1$

$$\sigma = \frac{1}{s}\,\mathrm{Im}\,A(s,0)\ ,\quad A(s,0) = \sum f_l\,(2l+1)$$

$$\sigma = \frac{1}{s}\cdot 8\pi\cdot \underbrace{\sum_{l=0}^{l_{MAX}} (2l+1)} =$$

$$= \frac{1}{s}\, 8\pi\, (l_{MAX}+1)^2\ ,\quad l_{MAX} = R\cdot\frac{\sqrt{s}}{2}$$

$$= \frac{1}{s}\cdot 8\pi\,\frac{s}{4}R^2 = \boxed{2\pi R^2}$$ ← Twice area of the disc

$$\mathrm{Im}\,A = \frac{1}{2}\sum_{n}$$  $= \frac{1}{2}$  $+\ \frac{1}{2}\sum_{n\geq 2}$ 

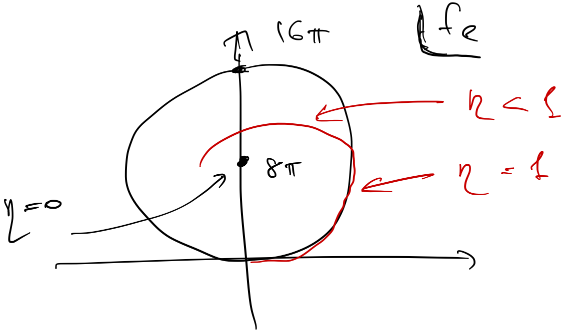
elastic

PW expansion ← $|f_l|^2 = \mathrm{Im}^2 f + (\mathrm{Re}\, f_l)^2$

$$\mathrm{Im}\, f_l = \rho\, f_l^* f_l + \Delta_l\ ,\quad \frac{1}{2}\sum_{n>0}$$  $= \sum \Delta_l\, P_l\,(2l+1)$

$\Delta_l > 0$

$$\longrightarrow\quad f_l = \frac{\eta e^{2i\delta} - 1}{2i\rho}\ ,\quad \eta \text{ is elasticity}\quad \eta^2 = 1 - 4\rho\Delta$$



black disk $\longrightarrow \eta = 0$

$$\mathrm{Im}\, f_\ell = \rho\, \underbrace{f_\ell^2}_{\text{elastic}} + \underbrace{\Delta_\ell}_{\text{inelastic}}$$

$$\sigma = \frac{1}{s}\,\mathrm{Im}\,A = \frac{1}{s}\sum (2\ell+1)\,\mathrm{Im}\, f_\ell$$

$$\sigma = \sigma_{el} + \sigma_{inel} \quad ; \quad \sigma_{elastic} = \frac{1}{s}\sum(2\ell+1)\,\mathrm{Im}\, f_\ell =$$

$$= \frac{1}{s}\sum_{\ell=0}^{\ell_{max}}(2\ell+1)\,\rho\,|f_\ell|^2 =$$

$$\sigma_{inelastic} = \frac{1}{s}\sum_{\ell=0}^{\ell_{max}}(2\ell+1)\,\Delta_\ell = \frac{1}{s}\sum_{\ell=0}^{\ell_{max}}(2\ell+1)\left[1-\eta^2\right]\frac{1}{4\rho}$$

$$\sigma_{elastic} = \frac{1}{s}\sum(2\ell+1)\left[1+\eta^2-2\cos\delta\,\eta\right]\frac{1}{4\rho}$$

$$\eta = 0 \longrightarrow \sigma_{el} = \sigma_{in} = \pi R^2 \implies \sigma_{TOT} = 2\pi R^2$$

$$\boxed{\Omega^+ - \Omega^- = 2i\, t^* \rho\, \Omega^+}$$

$t(s)$ $\quad\Omega(s)$

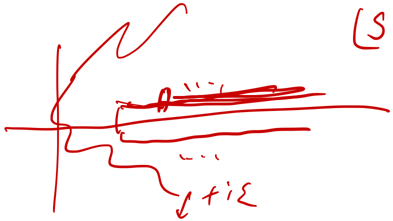
$t^* = t(s - i\epsilon)$

$\Omega^\pm = \Omega(s \pm i\epsilon)$

$\Omega^*(s+i\epsilon) = \Omega^*(s_+) = \Omega(s_-)$

$\Omega = P$

$\Omega(s)$



$t^* = |t| e^{i\delta(s)}$

$\Omega(s+i\epsilon) = |\Omega| e^{i\delta(s)}$

$\frac{\Omega^+ - \Omega^-}{2i} = \mathrm{Im}\,\Omega(s) =$ Real #

$e^{i\delta(s)}$ $=1$ const

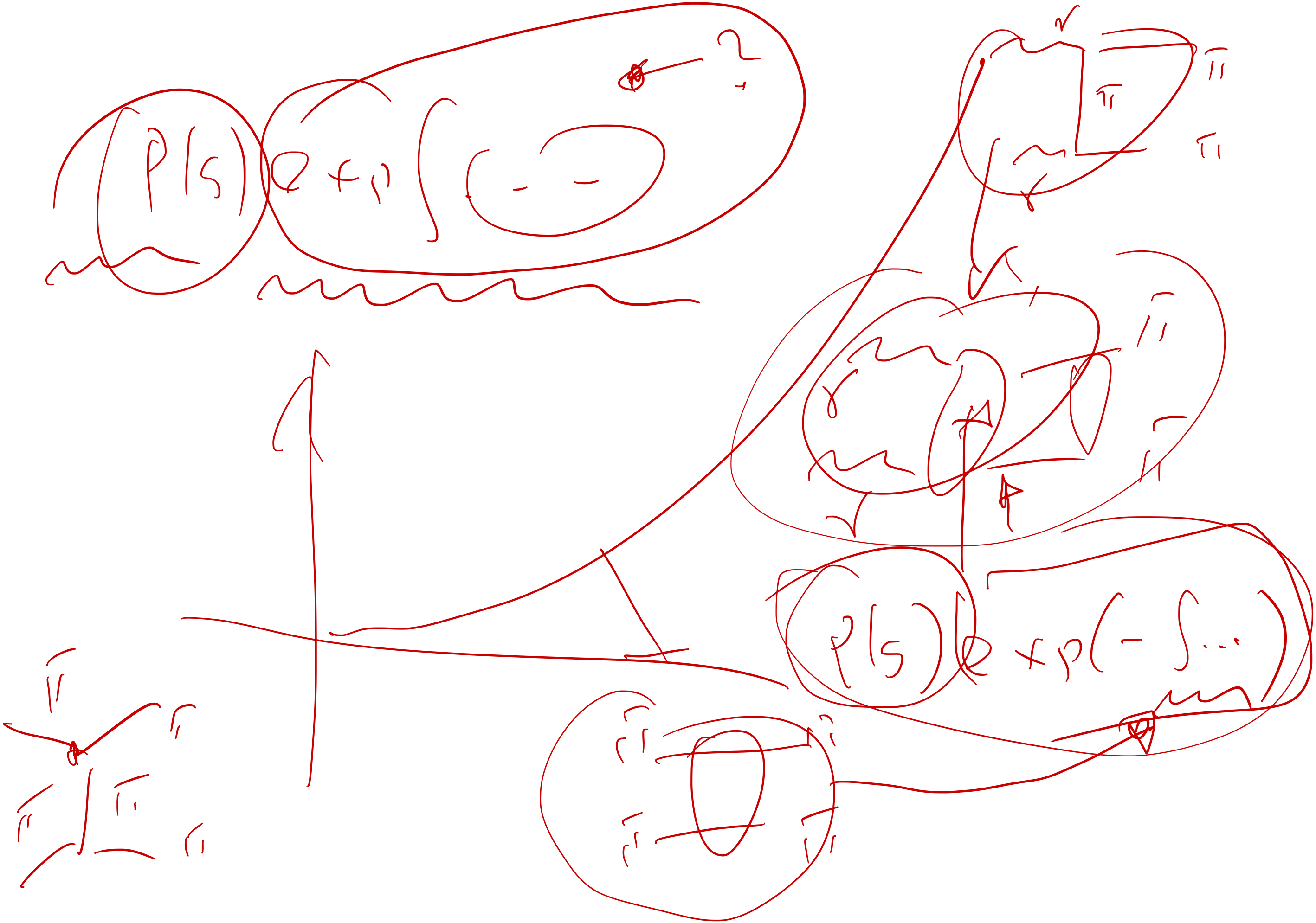
$$e^{\frac{1}{\pi}\int_{4m^2}^{\infty} ds' \frac{\delta(s')}{s'-s}}$$

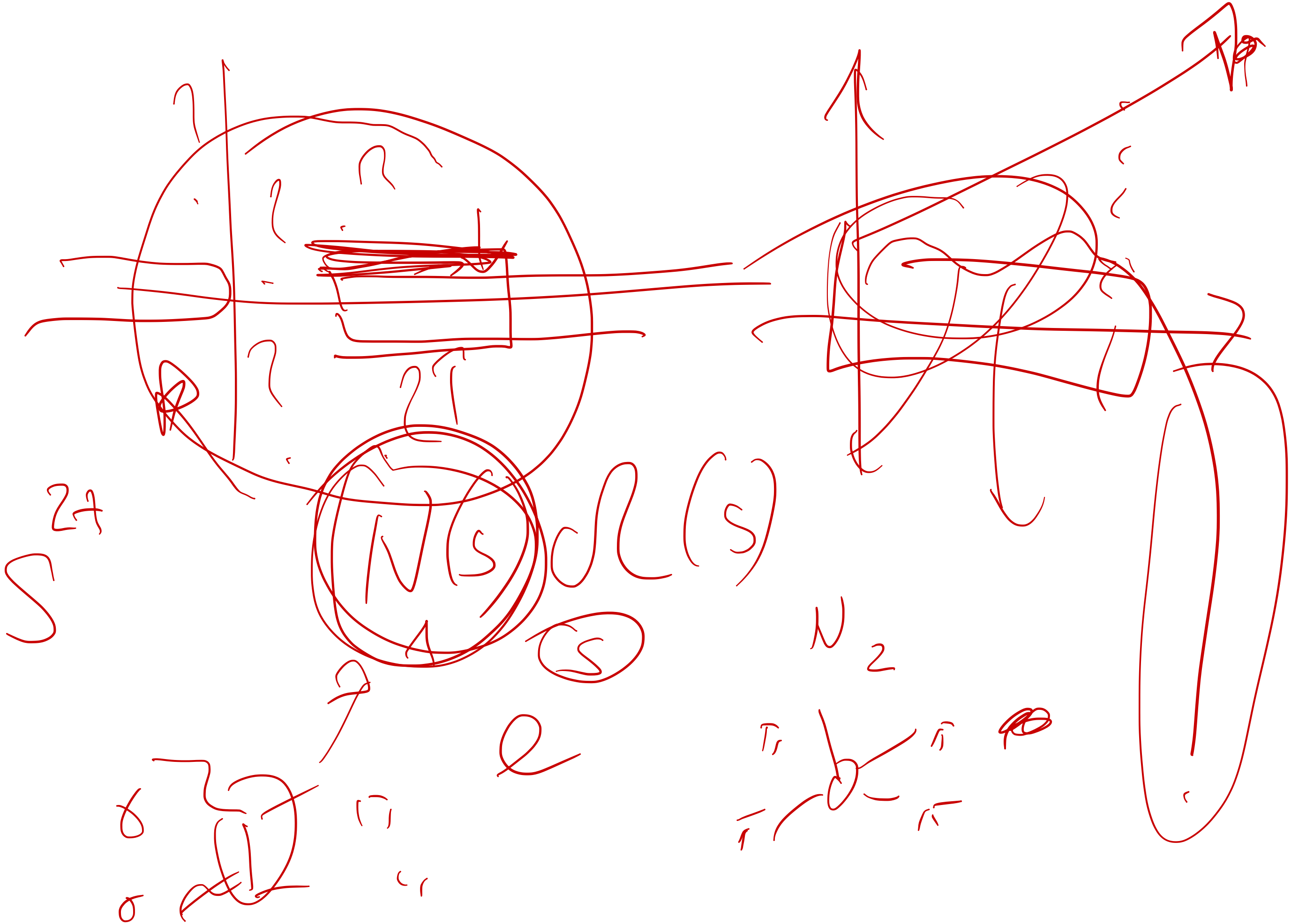
$$\frac{1}{x \pm i\epsilon} = P.V.\frac{1}{x} \mp i\pi\delta(x)$$

$$\Omega(s) = \exp\left(\frac{1}{T_1}\int \partial s' \, \frac{\partial \Sigma(s')}{s'-s}\right)$$

$\Omega(s) \to$ (Fubinas(S T_0 $\to$?(dx

$$\Sigma_4(s) = \frac{e^{4} - e^{-s}}{i}$$

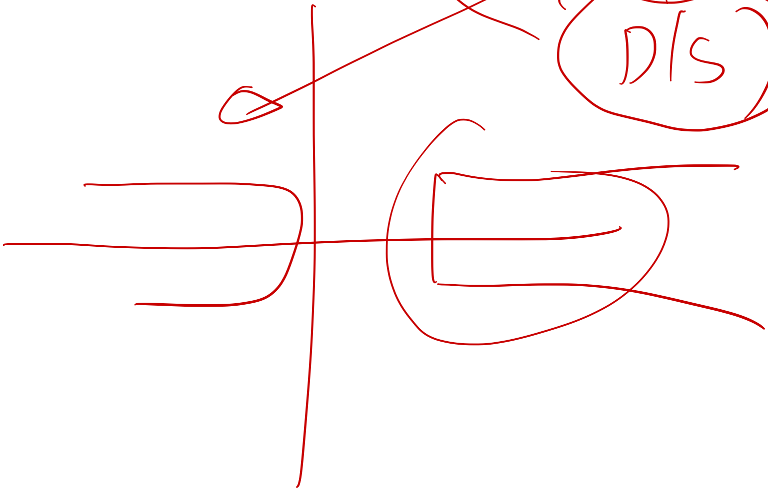

p(s) exp(
p(s) exp(-∫...)



$P(s)\,\Omega(s)$

$d(s) \Rightarrow \frac{1}{\Omega(s)} = \frac{1}{D(s)}$

$\frac{N(s)}{D(s)}$



$\ln \quad S^\dagger S = 1$

$$\ln T = i\,(T - T^\dagger)$$

$$T = A(s, \cos\theta)$$

$$\to \sum_\ell a_\ell(s)\, P_\ell(\cos\theta)$$

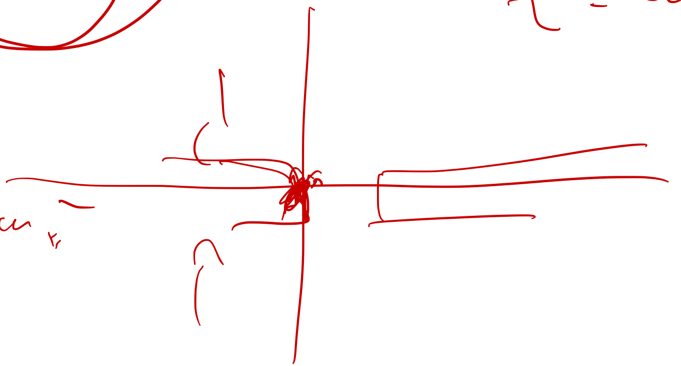
$$\ln a_\ell(s) = |a_\ell(s)|^2 \rho(s)$$

$$a_\ell(s) = \frac{1}{\pi}\int ds' \, \frac{\text{Im}\, a_\ell(s')}{s'-s} \qquad |a|^2$$

$$a_\ell(s) = \frac{N(s)}{D(s)}$$

$$t = m_\pi^2$$

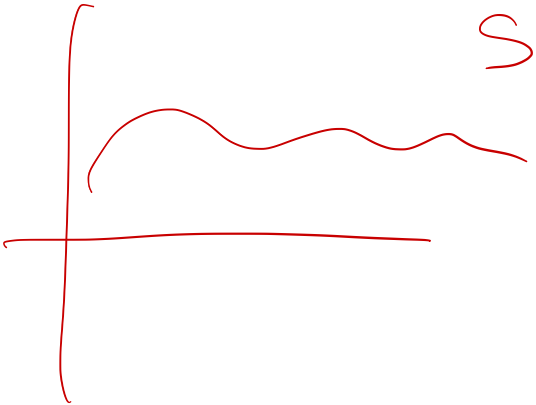
$$s = -u - t + \sum m_\pi^2$$



$N(s)$ = some function

that has a branch point

~ as from OPE



$s = s_0$

$$\frac{\sqrt{s-s_0}-\sqrt{B}}{\sqrt{s-s_0}+\sqrt{B}}$$