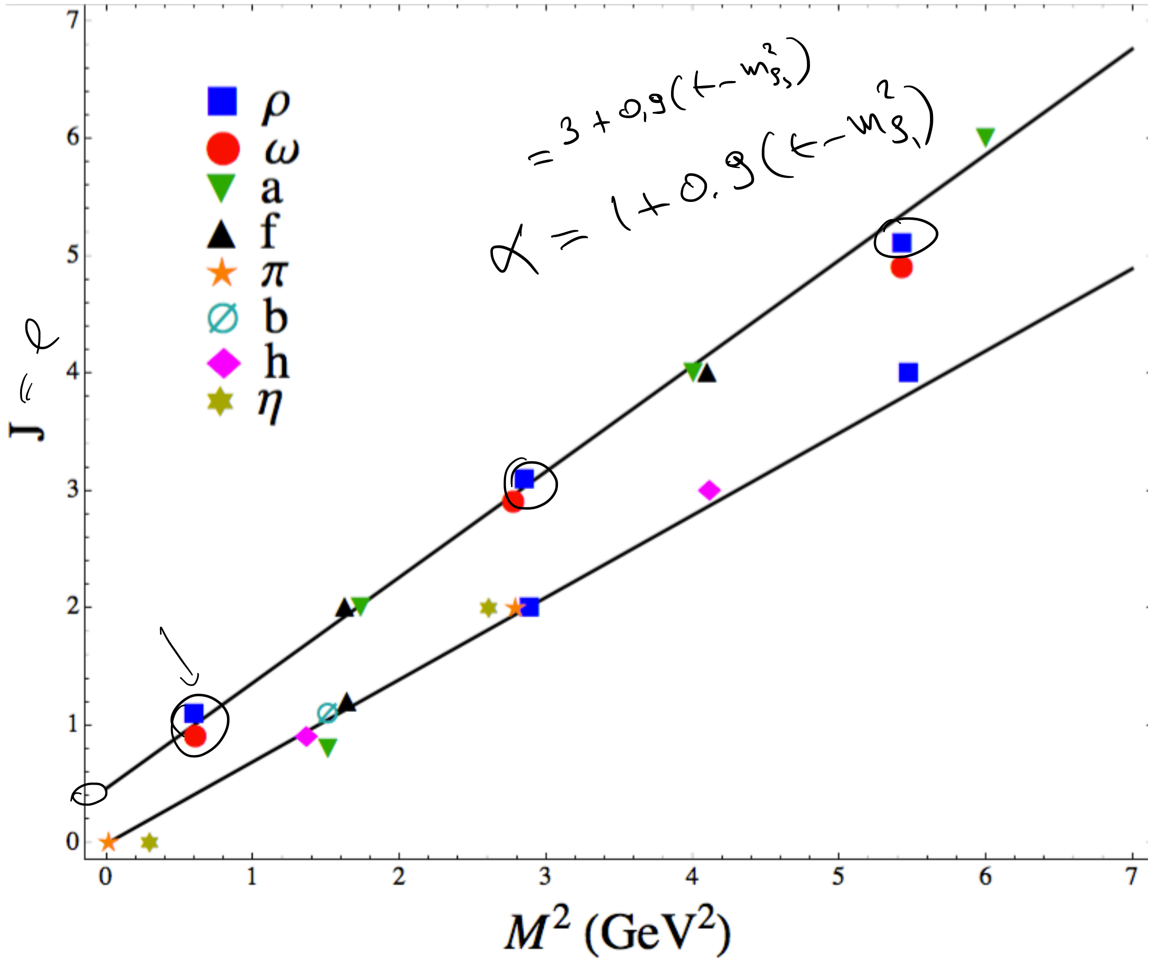
Day 7

Supplementary



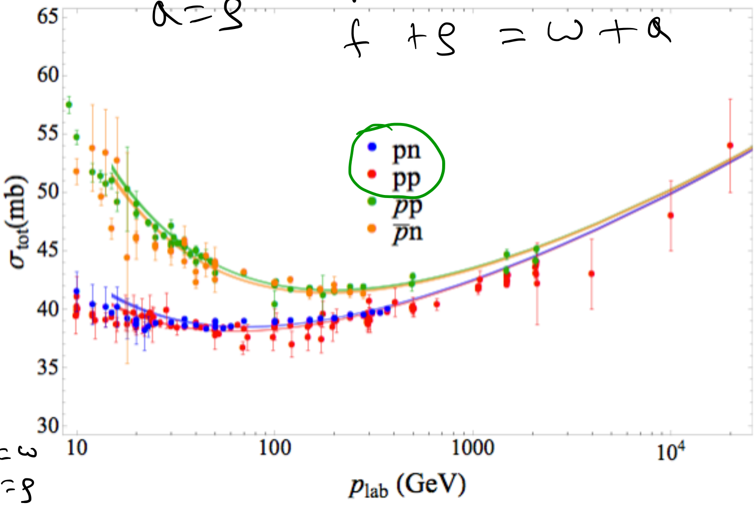
ρ
ω
a
f
π
b
h
η
$= 3 + 0{,}9(t - m_{\rho_1}^2)$
$\alpha = 1 + 0{,}9(t - m_{\rho_1}^2)$
J = ℓ
M^2 (GeV^2)

$f = \omega$

$a = \rho$

$f + a = \rho + \omega$

$f + \rho = \omega + a$



	pp	$\pi\pi$	KK
P	7.62	4.83	4.2
f	13.89	4.82	1.49
ρ	1.99	4.94	2.77
a	1.37	–	2.86
ω	8.11	–	2.7

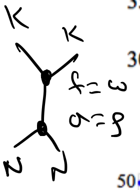
$$\pi^{\pm}p : P + f \mp \rho$$

$$K^{\pm}p : P + (f \mp \rho \mp \omega + a)$$

$$K^{\pm}n : P + (f \pm \rho \mp \omega - a)$$

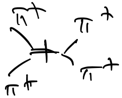
$$\overset{(-)}{p}p : P + f \mp \rho \mp \omega - a$$

$$\overset{(-)}{p}n : P + f \pm \rho \mp \omega + a$$

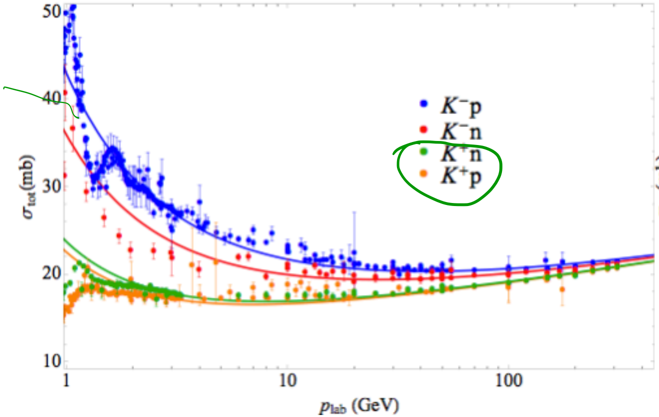


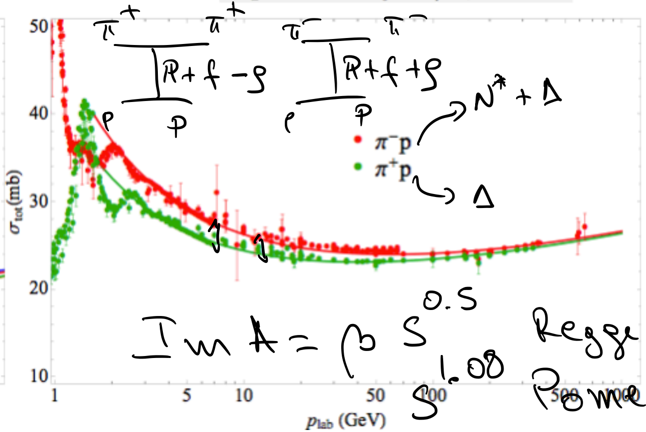
$f = \omega$

$a = \rho$



$\Rightarrow f = \rho$





$\mathrm{Im}\, A = \beta\, s^{0.5}$ Regge

$s^{1.08}$ Pomeron

Joined Physics Analysis Center

Summer Workshop on the Reaction Theory
Exercise sheet 7

Vincent Mathieu and Cesar Fernández-Ramírez

Contact: http://www.indiana.edu/~ssrt/index.html

June 12 – June 22 To be discussed on Tuesday of Week-II.

Classwork

1. Using $\int_0^\infty e^{-ax}dx = 1/a$, compute the series (and make the factor s^α appear)

$$F(s,\alpha) = \sum_{J=0}^{\infty} \frac{s^J}{J-\alpha}. \tag{1}$$

$$I_J \sim \frac{s^J}{J-\alpha}$$

$$I_{J=0} + I_{J=1} + \ldots$$

a) $\int_0^\infty e^{-(J-\alpha)x}\,dx = \frac{1}{J-\alpha}$; b) find a geometric series $\sum_{J=0}^\infty r^J = \frac{1}{1-r}$ $|r|<1$

c) change variable $t = se^{-x}$

Solution

1. Use the trick given and the change of variable $t = se^{-x}$ and obtain

$$F(s,\alpha) = \sum_{J}^{\infty} \int_0^{\infty} (se^{-x})^J e^{\alpha x} dx = \int_0^{\infty} \frac{e^{\alpha x} dx}{1 - se^{-x}} = s^{\alpha} \int_0^{s} \frac{t^{-\alpha-1}}{1-t} dt \tag{3}$$

2. Consider $\pi\pi \to \pi\pi$ with m being the pion mass. The reduce amplitude φ_ℓ is defined by removing the barrier factor $B_\ell = (s - 4m^2)^\ell$ from the (elastic) partial amplitude $t_\ell(s) = B_\ell(s)\varphi_\ell$. The phase space factor is $\rho(s) = (1/16\pi)\sqrt{1 - 4m^2/s}$. Use the unitarity equation $\mathrm{Im}\, t_\ell(s) = \rho(s)|t_\ell|^2$ to deduce the unitarity equation for the reduce amplitude. $\mathrm{Im}\, B\varphi = \rho B^2 |\varphi|^2 \to \mathrm{Im}\, \varphi = \rho B |\varphi|^2$

3. Consider $\pi\pi \to \pi\pi$ and $\pi\pi \to K\bar{K}$ with m_1 being the pion mass and m_2 being the kaon mass. Let us denote by 1 (2) the $\pi\pi$ $(K\bar{K})$ channel so that $t_\ell^{ij}(s)$ is the partial wave for the scattering $i \to j$. The reduce amplitude φ_ℓ^{ij} is defined by removing the barrier factors $B_\ell^i = (t - 4m_i^2)^\ell$ from the (elastic) partial amplitude $t_\ell^{ij}(s) = \sqrt{B_\ell^i(s)B_\ell^j(s)}\varphi_\ell^{ij}(s)$. Note that $t_\ell^{ji}(s) = t_\ell^{ij}(s)$. The phase space factors are $\rho_i(s) = (1/16\pi)\sqrt{1 - 4m_i^2/s}$. Use the unitarity equation $\mathrm{Im}\, t_\ell^{ij}(s) = \sum_{k=1,2} \rho_k(s) t_\ell^{ik*}(s) t_\ell^{kj}(s)$ or equivalently

or $\mathrm{Im}\, \varphi_\ell^{-1}(s) = -\rho(s) B_\ell(s)$

$\mathrm{Im}\, \varphi_\ell(s) = \rho(s) B_\ell(s) |\varphi_\ell|^2$

$$\mathrm{Im}\, t_\ell^{11}(s) = \rho_1(s)|t_\ell^{11}(s)|^2 + \rho_2(s)|t_\ell^{12}(s)|^2, \tag{2a}$$
$$\mathrm{Im}\, t_\ell^{12}(s) = \rho_1(s) t_\ell^{11*}(s) t_\ell^{12}(s) + \rho_2(s) t_\ell^{12*}(s) t_\ell^{22}(s), \tag{2b}$$
$$\mathrm{Im}\, t_\ell^{22}(s) = \rho_1(s)|t_\ell^{12}(s)|^2 + \rho_2(s)|t_\ell^{22}(s)|^2, \tag{2c}$$

$\varphi_\ell^{(s)} = \dfrac{\beta(s)}{\ell - \alpha(s)}$

$\to (\ell - \alpha) = \beta/\varphi \to \mathrm{Im}\,\alpha = -\beta\, \mathrm{Im}\,\varphi^{-1} = -\beta \rho B$

to derive the unitarity equations for the reduce amplitudes φ_ℓ^{ij}.

4. In the single channel case $\pi\pi \to \pi\pi$, assume the following form for the reduce amplitude $\varphi_\ell(s) = \beta(s)/(\ell - \alpha(s))$ and derive the unitarity equation for the Regge trajectory $\alpha(s)$ using the unitarity equation for the reduce amplitude, assuming the residue $\beta(s)$ is real. and ℓ is real

5. In the coupled channel case $\pi\pi \to \pi\pi, K\bar{K}$, assume the following form for the reduce amplitude $\varphi_\ell^{ij}(s) = \beta_{ij}(s)/(\ell - \alpha(s))$. Derive the unitarity equation for the Regge trajectory $\alpha(s)$ using the three unitarity equations for the reduce amplitudes $\varphi^{11}, \varphi^{12}$ and φ^{22}, assuming the residues $\beta_{ij}(s)$ are real (and $\beta_{12} = \beta_{21}$). Since these unitarity equations are equal, what are the resulting constraints on the residues $\beta_{ij}(s)$?

$\mathrm{Im}\,\alpha =$

(11)

(12)

(22)

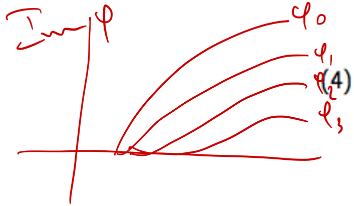
Use $\mathrm{Im}\, \varphi^{ij} = \sum_{k=1,2} \rho_k B_\ell^k \varphi^{ik*} \varphi^{kj}$

$$\varphi = \frac{\beta}{\ell - \alpha} \longrightarrow (\ell - \alpha) = \frac{\beta}{\varphi}$$

$$\operatorname{Im} \alpha = -\beta \operatorname{Im} \varphi^{-1} \qquad \operatorname{Im} \varphi^{-1} = -\rho B$$

$$= \rho B \beta$$

2. By remplacement we obtain



$$\operatorname{Im}\varphi_\ell(s) = \rho(s)B_\ell(s)|\varphi(s)|^2 \tag{4}$$

or $\operatorname{Im}\varphi_\ell^{-1}(s) = -\rho(s)B_\ell(s)$.

3. By remplacement we obtain

$$\operatorname{Im} t_\ell^{ij}(s) = \rho_1(s)t_\ell^{i1*}(s)t_\ell^{1j}(s) + \rho_2(s)t_\ell^{i2*}(s)t_\ell^{2j}(s) \tag{5a}$$

$$\begin{aligned}\operatorname{Im}\sqrt{B_\ell^i(s)B_\ell^j(s)}\varphi_\ell^{ij}(s) &= \rho_1(s)\sqrt{B_\ell^i(s)B_\ell^1(s)}\varphi_\ell^{i1*}(s)\sqrt{B_\ell^1(s)B_\ell^j(s)}\varphi_\ell^{1j}(s)\\ &\quad+\rho_2(s)\sqrt{B_\ell^i(s)B_\ell^2(s)}\varphi_\ell^{i2*}(s)\sqrt{B_\ell^2(s)B_\ell^j(s)}\varphi_\ell^{2j}(s)\end{aligned} \tag{5b}$$

$$\operatorname{Im}\varphi_\ell^{ij}(s) = \rho_1(s)B_\ell^1(s)\varphi_\ell^{i1*}(s)\varphi_\ell^{1j}(s) + \rho_2(s)B_\ell^2(s)\varphi_\ell^{i2*}(s)\varphi_\ell^{2j}(s) \tag{5c}$$

$$\operatorname{Im}\varphi_\ell^{ij}(s) = \sum_{k=1,2}\rho_k(s)B_\ell^k(s)\varphi_\ell^{ik*}(s)\varphi_\ell^{kj}(s) \tag{5d}$$

We can equivalently perform the same derivation in a matrix form. Let us define the matrices $(\boldsymbol{t}_\ell)_{ij} = t_\ell^{ij}(s)$, $(\boldsymbol{\varphi}_\ell)_{ij} = \varphi_\ell^{ij}(s)$, $(\boldsymbol{\rho})_{ij} = \rho_i(s)\delta_{ij}$ and $\left(\boldsymbol{B}_\ell^{1/2}\right)_{ij} = \sqrt{B_\ell^i(s)}\delta_{ij}$. Note that $\boldsymbol{t}_\ell^T = \boldsymbol{t}_\ell$ and $\boldsymbol{B}_\ell^{-1} = \boldsymbol{B}_\ell$. The unitarity equations read $\operatorname{Im}\boldsymbol{t}_\ell = \boldsymbol{t}_\ell^\dagger\boldsymbol{\rho}\boldsymbol{t}_\ell$ or $\operatorname{Im}\boldsymbol{t}_\ell^{-1} = -\boldsymbol{\rho}$ (by writing $\operatorname{Im}\boldsymbol{t}_\ell = (1/2i)(\boldsymbol{t}_\ell^\dagger - \boldsymbol{t}_\ell)$ and multiplying to left by $(\boldsymbol{t}_\ell^\dagger)^{-1}$ and to the right by $\boldsymbol{t}_\ell^{-1}$). Since $\boldsymbol{t}_\ell = \boldsymbol{B}_\ell^{1/2}\boldsymbol{\varphi}_\ell\boldsymbol{B}_\ell^{1/2}$, we obtain $\operatorname{Im}\boldsymbol{\varphi}_\ell = \boldsymbol{\varphi}_\ell^\dagger\boldsymbol{B}_\ell^{1/2}\boldsymbol{\rho}\boldsymbol{B}_\ell^{1/2}\boldsymbol{\varphi}_\ell$ or $\operatorname{Im}\boldsymbol{\varphi}_\ell^{-1} = -\boldsymbol{B}_\ell^{1/2}\boldsymbol{\rho}\boldsymbol{B}_\ell^{1/2} = -\boldsymbol{\rho}\boldsymbol{B}_\ell$.

4. The trajectory is $\alpha(s) = \ell - \beta(s)/\varphi_\ell(s)$. We obtain

$$\operatorname{Im}\alpha(s) = -\operatorname{Im}\frac{\beta(s)}{\varphi_\ell(s)} = \beta(s)\frac{\operatorname{Im}\varphi_\ell(s)}{|\varphi_\ell(s)|^2} = \rho(s)B_\ell(s)\beta(s), \tag{6}$$

as expected since $\operatorname{Im}\varphi^{-1} = -\operatorname{Im}\alpha/\beta = -\rho B_\ell$.

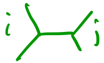
$ij = 11;\ 22;\ 12$

5. The trajectory is $\alpha(s) = \ell - \beta_{ij}(s)/\varphi_\ell^{ij}(s)$. We obtain

$$\operatorname{Im}\alpha(s) = \beta_{ij}(s)\frac{\operatorname{Im}\varphi_\ell^{ij}(s)}{|\varphi_\ell^{ij}(s)|^2} = \beta_{ij}^{-1}(s)\sum_{k=1,2}\rho_k(s)B_\ell^k(s)\beta_{ik}(s)\beta_{kj}(s) \tag{7}$$

More explicitly the three equations are ($ij = \{11, 12, 22\}$)

$\varphi_{ij}(s) = \dfrac{\beta_{ij}(s)}{\ell-\alpha}$

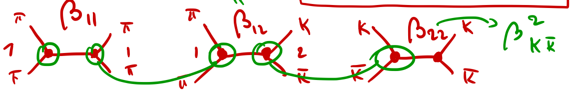


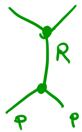
$$\operatorname{Im}\alpha(s) = \left[\rho_1(s)B_\ell^1(s)\beta_{11}^2(s) + \rho_2(s)B_\ell^2(s)\beta_{12}^2(s)\right]/\beta_{11}(s) \tag{8a}$$
$$= \rho_1(s)B_\ell^1(s)\beta_{11}(s) + \rho_2(s)B_\ell^2(s)\beta_{22}(s) \tag{8b}$$
$$= \left[\rho_1(s)B_\ell^1(s)\beta_{12}^2(s) + \rho_2(s)B_\ell^2(s)\beta_{22}^2(s)\right]/\beta_{22}(s) \tag{8c}$$

We then derive the factorization of residues

$\beta_{\pi\pi}^2$ $\qquad$ $\beta_{\pi\pi}\cdot\beta_{KK}$

$$\boxed{\beta_{12}^2(s) = \beta_{11}(s)\beta_{22}(s)} \tag{9}$$





Joined Physics Analysis Center

Summer Workshop on the Reaction Theory
Exercise sheet 8

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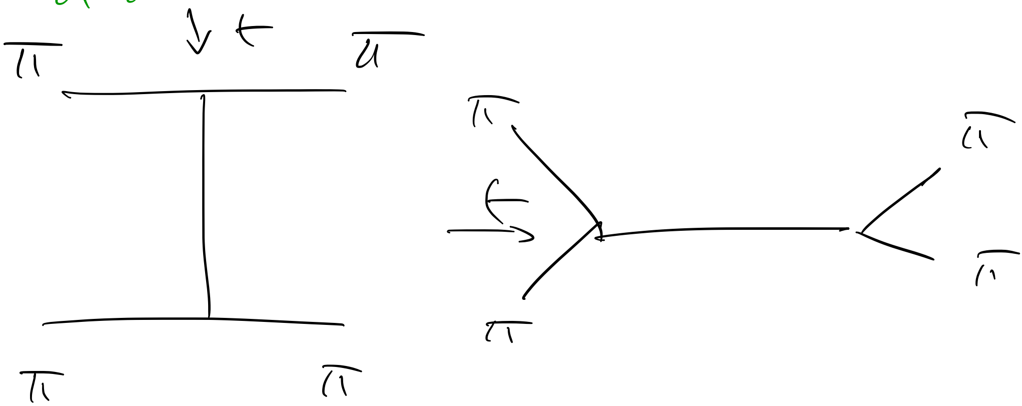
June 12 – June 22 To be discussed on Tuesday of Week-II.

Classwork

1. Derive all the quantum numbers $I^G J^{PC}$ in the $t-$channel of the following reactions
   (a) $\pi\pi \to \pi\pi$ and $K\bar{K} \to K\bar{K}$
   (b) $\pi N \to \pi N$, $\pi N \to \eta N$ and $KN \to KN$
   (c) $\gamma N \to \eta N$ and $\gamma N \to \pi N$
   (d) $\pi\rho \to \rho\pi$

   Notation: $\pi = (\pi^+, \pi^-, \pi^0)$; $\rho = (\rho^+, \rho^-, \rho^0)$; $K = (K^+, K^0)$; $N = (p, n)$.

$\pi\pi \to \pi\pi$



$I=0 \quad C=+ \quad$ "f_+" $\; f_0, f_2, f_4 \ldots$

$I=1 \quad C=- \quad$ "ρ_-" $\; \rho_1, \rho_3, \rho_5$

$$G = P(-)^I = +$$

$$f_+ : \; I^G J^{PC} = 0^+ (0, 2, 4, \ldots)^{PC} \quad \tau=+ \;\; \eta=+$$

$$\rho_- : \; I^G J^{PC} = 1^+ (1, 3, 5 \ldots)^{PC} \quad \tau=- \;\; \eta=+$$

$\tau = (-)^J$

$\eta = P(-)^J$ naturality

$I^{G\tau\eta}$

$\pi : 0^{-+}$

$\eta : 0^{-+}$

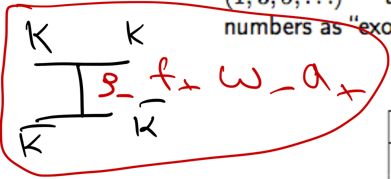
$q\bar{q} ; N\bar{N}$ $\quad P = (-)^{L+1}$

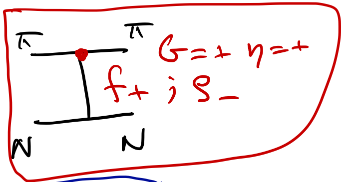
$C = (-)^{L+S}$

$I = 0, 1 \quad G = \pm ; \eta = \pm ; \tau = \pm$

Solution

1. The list of exchanges having only $I = 0, 1$ is presented on Table 1. Notation: signature $\tau = (-1)^J$ and naturality $\eta = P(-1)^J$. In the quark model, $P = (-1)^{\ell+1}$ and $C = (-1)^{\ell+S}$, hence 0^{--}, $(1,3,5,\ldots)^{-+}$ and $(0,2,4,\ldots)^{+-}$ are forbidden in the quark model. Let's refer to these quantum numbers as "exotic".





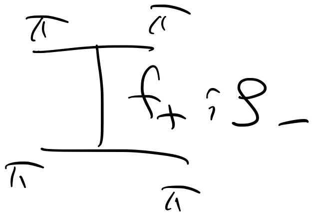
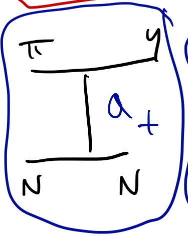
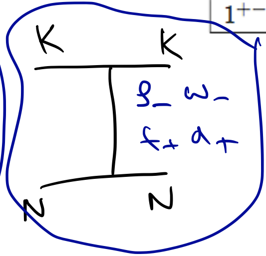
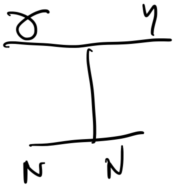


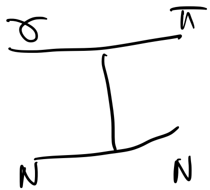
Table 1: Regge Trajectories

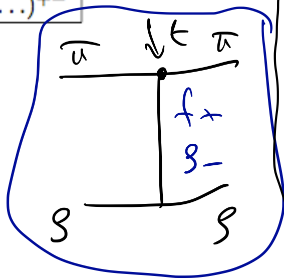
$I^{G\tau\eta}$		J^{PC}	$I^{G\tau\eta}$		J^{PC}
0^{+++}	f_+	$(0,2,4,\ldots)^{++}$	0^{+--}	f_-	$(1,3,5,\ldots)^{++}$
0^{--+}	ω_-	$(1,3,5,\ldots)^{--}$	0^{-+-}	ω_+	$(0,2,4,\ldots)^{--}$
1^{-++}	a_+	$(0,2,4,\ldots)^{++}$	1^{---}	a_-	$(1,3,5,\ldots)^{++}$
1^{+-+}	ρ_-	$(1,3,5,\ldots)^{--}$	1^{++-}	ρ_+	$(0,2,4,\ldots)^{--}$
0^{++-}	η_+	$(0,2,4,\ldots)^{-+}$	0^{+-+}	η_-	$(1,3,5,\ldots)^{-+}$
0^{---}	h_-	$(1,3,5,\ldots)^{+-}$	0^{-++}	h_+	$(0,2,4,\ldots)^{+-}$
1^{-+-}	π_+	$(0,2,4,\ldots)^{-+}$	1^{--+}	π_-	$(1,3,5,\ldots)^{-+}$
1^{+--}	b_-	$(1,3,5,\ldots)^{+-}$	1^{+++}	b_+	$(0,2,4,\ldots)^{+-}$

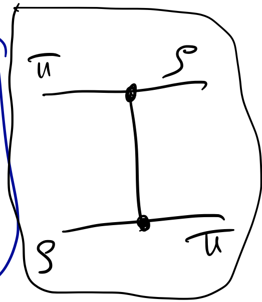






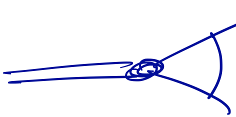






0^- $\qquad$ $0^- = J^P$

π, K, η



0^+

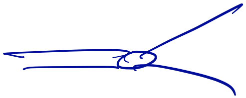
$0^+ \quad 1^- \quad 2^+ \quad 3^- \quad \ldots$

$l = 0 \quad 1 \quad 2 \quad 3$

$\boxed{0^+ \; 1^- \; 2^+ \; 3^-}$

$J^P : P(-)^J = +$

$K^+ K^-$



$K^+ : (0, 2, 4)^+$ $\quad I=0 \; f_+$ $\quad =1 \; a_+$ $\qquad \eta = +$

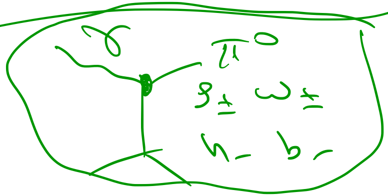
$K^- : (1, 3, 5 \ldots)^-$ $\quad I=1 \; \rho_-$ $\; (-)^J$

$I = 0 \; \omega_-$

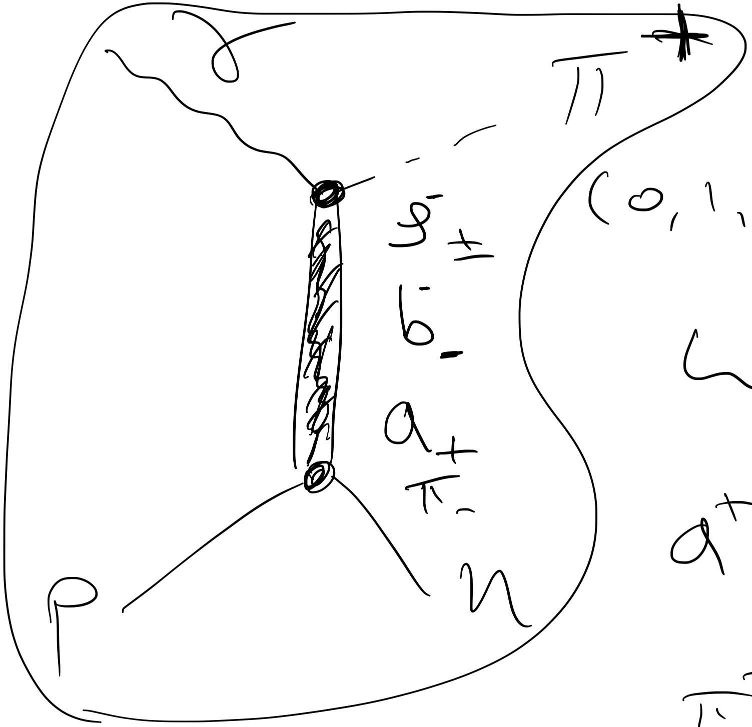
$\boxed{C = -}$

$a_0 : 0^{++} \quad I = 1$

$f_0 : 0^{++} \quad I = 0$



h^0



$\cancel{\gamma}^{--}$

$(0, 1, 2, 3, \ldots)$

$$\omega \longrightarrow \gamma \pi^+$$

$$a^+ \longrightarrow \gamma \pi^+$$

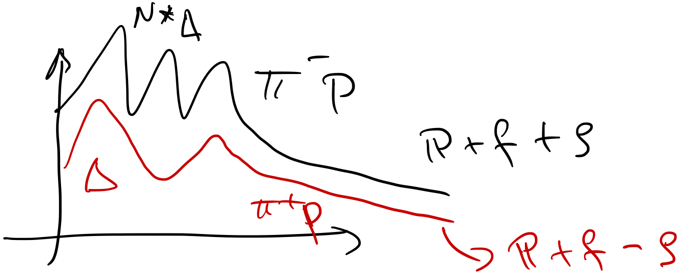
$$\pi^+ \longrightarrow \gamma \pi^+$$

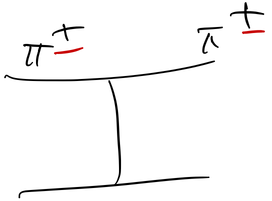
$$\pi^{\mp} p : \mathbb{P} + f \pm \rho$$

$$K^{\mp} p : \mathbb{P} + f \pm \rho + a \pm \omega$$

$$K^{\mp} n : \mathbb{P} + f \mp \rho - a \pm \omega$$

$$\left.\begin{array}{l} \overset{(-)}{p}\, p : \mathbb{P} + f \mp \rho + a \mp \omega \\[2ex] \overset{(-)}{p}\, n : \mathbb{P} + f \pm \rho - a \mp \omega \end{array}\right\} \mathrm{I}$$





(a) for $\pi\pi$: $G = +$, $\eta = +$ and $\eta(-1)^I = +$ (Bose symmetry) $\Rightarrow$ f_+ and ρ_-.
for $K\bar{K}$: $\eta = +$ and $\eta(-1)^I = + \Rightarrow f_+$, a_+, ω_- and ρ_-.

(b) for $\pi\eta$: $G = -$, $\eta = +$, $I = 1$; for NN: $I = 0, 1$ and no exotic $\Rightarrow a_+$.
for $K\bar{K}$: $\eta = +$; for NN: $I = 0, 1$ and no exotic $\Rightarrow f_+$, a_+, ω_- and ρ_-.

(c) for $\gamma\eta$ and $\gamma\pi^0$: $C = -$; for NN: $I = 0, 1$ and no exotic. $\Rightarrow \omega_\pm$, $\rho_\pm$, b_- and h_-.
for $\gamma\pi^+$: $I = 1$; for NN: $I = 0, 1$ and no exotic. $\Rightarrow a_\pm$, $\rho_\pm$, b_- and π_-.

(d) for $\pi\rho$: $G = - \Rightarrow a_\pm$, $\pi_\pm$, $\omega_\pm$ and $h_\pm$

Table 2: Exchanges

(a)	$\pi^+\pi^\mp \to \pi^+\pi^\mp$	$f_+ \pm \rho_-$
	$\pi^0\pi^0 \to \pi^0\pi^0$	f_+
	$K^+K^\mp \to K^+K^\mp$	$f_+ \pm \omega_- + a_+ \pm \rho_-$
	$K^+K^0 \to K^0K^+$	$a_+ - \rho_-$
(b)	$\pi^-p \to \eta n$	a_+
	$\pi^-p \to \pi^0 n$	$\sqrt{2}\rho_+$
	$\pi^\mp p \to \pi^\mp p$	$f_+ \pm \rho_+$
	$\pi^\mp n \to \pi^\mp n$	$f_+ \mp \rho_+$
	$K^-p \to \bar{K}^0 p$	$\sqrt{2}\,(\rho_- + a_+)$
	$K^+n \to K^0 p$	$\sqrt{2}\,(\rho_- - a_+)$
	$K^\mp p \to K^\mp p$	$f_+ \pm \rho_- + a_+ \pm \omega_-$
	$K^\mp n \to K^\mp n$	$f_+ \mp \rho_- - a_+ \pm \omega_-$
(c)	$\gamma p \to \eta p$	$(\omega_- + \rho_-) + (h_- + b_- + \omega_+ + \rho_+)$
	$\gamma p \to \pi^0 p$	$(\omega_- + \rho_-) + (h_- + b_- + \omega_+ + \rho_+)$
	$\gamma p \to \pi^+ n$	$(\rho_- + a_+) + (b_- + \pi_+ + \rho_+ + a_-)$
	$\gamma n \to \pi^- p$	$(\rho_- - a_+) + b_- - \pi_+ + \rho_+ - a_-)$
(d)	$\pi^+\rho^0 \to \rho^0\pi^+$	$(a_+ + \pi_-) + (a_- + \pi_+)$
	$\pi^+\rho^+ \to \rho^+\pi^+$	$(\omega_- - a_+ + h_+ - \pi_-) + (\omega_+ - a_- + h_- - \pi_+)$
	$\pi^+\rho^+ \to \pi^+\rho^+$	$f_+ - \rho_-$

2. Assume that the Regge exchange form a $SU(3)$ octet and a $SU(3)$ singlet with the coupling for the octet and the singlet being different. Consider a vector and a tensor nonet (octet plus singlet). From the duality hypothesis and the absence of double charge meson, find the combination of octet-singlet tensor that decouples from $\pi\pi$. Use the SU(3) Clebsch-Gordan coefficients from Rev.Mod.Phys. 36 (1964) 1005. What are the quark content and the $K\bar{K}$ couplings of these states?

3. Assuming ideal mixing for the vector and tensor, derive the exchange degeneracy relations coming duality and the absence of resonance in the following reactions

   (a) $\pi\pi \to \pi\pi$

   (b) $K\bar{K} \to K\bar{K}$

   (c) $KN \to KN$

   (d) $\pi\rho \to \rho\pi$ (and $\pi\pi \to \rho\rho$)

4. Derive a Lorentz-covariant basis, the isospin decomposition and the crossing properties for the following reactions

   (a) $\pi N \to \pi N$ and $KN \to KN$

   (b) $NN \to NN$

   (c) $\omega \to \pi\pi\pi$ and $B \to J/\psi K\pi$

   (d) $\pi\rho \to \pi\rho$

   (e) $\gamma N \to \pi N$ and $\gamma^* N \to \pi N$ (use $F^{\mu\nu} = \epsilon^\mu k^\nu - \epsilon^\nu k^\mu$)

$\underline{8} \otimes \underline{8}$

	$\underline{27}$ 0 2 2
0 1 1 0 1 1	1

	$\underline{27}$ 0 2 1	$\underline{27}$ 0 1 1	$\underline{10}$ 0 1 1	$\overline{\underline{10}}$ 0 1 1	$\underline{8}$ 0 1 1	$\underline{8}'$ 0 1 1
$1\ \frac{1}{2}\ \frac{1}{2}$ $-1\ \frac{1}{2}\ \frac{1}{2}$	0	$\sqrt{\frac{1}{5}}$	$-\sqrt{\frac{1}{6}}$	$\sqrt{\frac{1}{6}}$	$-\sqrt{\frac{3}{10}}$	$\sqrt{\frac{1}{6}}$
0 1 1 0 1 0	$\sqrt{\frac{1}{2}}$	0	$\sqrt{\frac{1}{12}}$	$-\sqrt{\frac{1}{12}}$	0	$\sqrt{\frac{1}{3}}$
0 1 1 0 0 0	0	$\sqrt{\frac{3}{10}}$	$\sqrt{\frac{1}{4}}$	$\sqrt{\frac{1}{4}}$	$\sqrt{\frac{1}{5}}$	0
0 1 0 0 1 1	$\sqrt{\frac{1}{2}}$	0	$-\sqrt{\frac{1}{12}}$	$\sqrt{\frac{1}{12}}$	0	$-\sqrt{\frac{1}{3}}$
0 0 0 0 1 1	0	$\sqrt{\frac{3}{10}}$	$-\sqrt{\frac{1}{4}}$	$-\sqrt{\frac{1}{4}}$	$\sqrt{\frac{1}{5}}$	0
$-1\ \frac{1}{2}\ \frac{1}{2}$ $1\ \frac{1}{2}\ \frac{1}{2}$	0	$\sqrt{\frac{1}{5}}$	$\sqrt{\frac{1}{6}}$	$-\sqrt{\frac{1}{6}}$	$-\sqrt{\frac{3}{10}}$	$-\sqrt{\frac{1}{6}}$

	$\underline{27}$ 0 2 0	$\underline{27}$ 0 1 0	$\underline{10}$ 0 1 0	$\overline{\underline{10}}$ 0 1 0	$\underline{8}$ 0 1 0	$\underline{8}'$ 0 1 0	$\underline{27}$ 0 0 0	$\underline{8}$ 0 0 0	$\underline{8}'$ 0 0 0	$\underline{1}$ 0 0 0
$1\ \frac{1}{2}\ \frac{1}{2}$ $-1\ \frac{1}{2}\ \frac{-1}{2}$	0	$\sqrt{\frac{1}{10}}$	$-\sqrt{\frac{1}{12}}$	$\sqrt{\frac{1}{12}}$	$-\sqrt{\frac{3}{20}}$	$\sqrt{\frac{1}{12}}$	$\sqrt{\frac{3}{40}}$	$\sqrt{\frac{1}{20}}$	$\sqrt{\frac{1}{4}}$	$\sqrt{\frac{1}{8}}$
$1\ \frac{1}{2}\ \frac{-1}{2}$ $-1\ \frac{1}{2}\ \frac{1}{2}$	0	$\sqrt{\frac{1}{10}}$	$-\sqrt{\frac{1}{12}}$	$\sqrt{\frac{1}{12}}$	$-\sqrt{\frac{3}{20}}$	$\sqrt{\frac{1}{12}}$	$-\sqrt{\frac{3}{40}}$	$-\sqrt{\frac{1}{20}}$	$-\sqrt{\frac{1}{4}}$	$-\sqrt{\frac{1}{8}}$
0 1 1 0 1 -1	$\sqrt{\frac{1}{6}}$	0	$\sqrt{\frac{1}{12}}$	$-\sqrt{\frac{1}{12}}$	0	$\sqrt{\frac{1}{3}}$	$-\sqrt{\frac{1}{120}}$	$-\sqrt{\frac{1}{5}}$	0	$\sqrt{\frac{1}{8}}$
0 1 0 0 1 0	$\sqrt{\frac{2}{3}}$	0	0	0	0	0	$\sqrt{\frac{1}{120}}$	$\sqrt{\frac{1}{5}}$	0	$-\sqrt{\frac{1}{8}}$
0 1 0 0 0 0	0	$\sqrt{\frac{3}{10}}$	$\sqrt{\frac{1}{4}}$	$\sqrt{\frac{1}{4}}$	$\sqrt{\frac{1}{5}}$	0	0	0	0	0
0 1 -1 0 1 1	$\sqrt{\frac{1}{6}}$	0	$-\sqrt{\frac{1}{12}}$	$\sqrt{\frac{1}{12}}$	0	$-\sqrt{\frac{1}{3}}$	$-\sqrt{\frac{1}{120}}$	$-\sqrt{\frac{1}{5}}$	0	$\sqrt{\frac{1}{8}}$
0 0 0 0 1 0	0	$\sqrt{\frac{3}{10}}$	$-\sqrt{\frac{1}{4}}$	$-\sqrt{\frac{1}{4}}$	$\sqrt{\frac{1}{5}}$	0	0	0	0	0
0 0 0 0 0 0	0	0	0	0	0	0	$\sqrt{\frac{27}{40}}$	$-\sqrt{\frac{1}{5}}$	0	$-\sqrt{\frac{1}{8}}$
$-1\ \frac{1}{2}\ \frac{1}{2}$ $1\ \frac{1}{2}\ \frac{-1}{2}$	0	$\sqrt{\frac{1}{10}}$	$\sqrt{\frac{1}{12}}$	$-\sqrt{\frac{1}{12}}$	$-\sqrt{\frac{3}{20}}$	$-\sqrt{\frac{1}{12}}$	$-\sqrt{\frac{3}{40}}$	$-\sqrt{\frac{1}{20}}$	$\sqrt{\frac{1}{4}}$	$-\sqrt{\frac{1}{8}}$
$-1\ \frac{1}{2}\ \frac{-1}{2}$ $1\ \frac{1}{2}\ \frac{1}{2}$	0	$\sqrt{\frac{1}{10}}$	$\sqrt{\frac{1}{12}}$	$-\sqrt{\frac{1}{12}}$	$-\sqrt{\frac{3}{20}}$	$-\sqrt{\frac{1}{12}}$	$\sqrt{\frac{3}{40}}$	$\sqrt{\frac{1}{20}}$	$-\sqrt{\frac{1}{4}}$	$\sqrt{\frac{1}{8}}$

	$\underline{27}$ 0 2 -1	$\underline{27}$ 0 1 -1	$\underline{10}$ 0 1 -1	$\overline{\underline{10}}$ 0 1 -1	$\underline{8}$ 0 1 -1	$\underline{8}'$ 0 1 -1
$1\ \frac{1}{2}\ \frac{-1}{2}$ $-1\ \frac{1}{2}\ \frac{-1}{2}$	0	$\sqrt{\frac{1}{5}}$	$-\sqrt{\frac{1}{6}}$	$\sqrt{\frac{1}{6}}$	$-\sqrt{\frac{3}{10}}$	$\sqrt{\frac{1}{6}}$
0 1 0 0 1 -1	$\sqrt{\frac{1}{2}}$	0	$\sqrt{\frac{1}{12}}$	$-\sqrt{\frac{1}{12}}$	0	$\sqrt{\frac{1}{3}}$

$\mathbf{8}\otimes\mathbf{8}$

		$\mathbf{27}$	$\mathbf{10}$
		1	1
		$\frac{3}{2}$	$\frac{3}{2}$
		$\frac{3}{2}$	$\frac{3}{2}$
$1\ \frac{1}{2}\ \frac{1}{2}$	$0\ 1\ 1$	$\sqrt{\frac{1}{2}}$	$-\sqrt{\frac{1}{2}}$
$0\ 1\ 1$	$1\ \frac{1}{2}\ \frac{1}{2}$	$\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{2}}$

		$\mathbf{27}$	$\mathbf{10}$	$\mathbf{27}$	$\overline{\mathbf{10}}$	$\mathbf{8}$	$\mathbf{8}'$
		1	1	1	1	1	1
		$\frac{3}{2}$	$\frac{3}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
		$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$1\ \frac{1}{2}\ \frac{1}{2}$	$0\ 1\ 0$	$\sqrt{\frac{1}{3}}$	$-\sqrt{\frac{1}{3}}$	$\sqrt{\frac{1}{60}}$	$-\sqrt{\frac{1}{12}}$	$\sqrt{\frac{3}{20}}$	$\sqrt{\frac{1}{12}}$
$1\ \frac{1}{2}\ \frac{1}{2}$	$0\ 0\ 0$	0	0	$\sqrt{\frac{9}{20}}$	$\sqrt{\frac{1}{4}}$	$-\sqrt{\frac{1}{20}}$	$\sqrt{\frac{1}{4}}$
$1\ \frac{1}{2}\ -\frac{1}{2}$	$0\ 1\ 1$	$\sqrt{\frac{1}{6}}$	$-\sqrt{\frac{1}{6}}$	$-\sqrt{\frac{1}{30}}$	$\sqrt{\frac{1}{6}}$	$-\sqrt{\frac{3}{10}}$	$-\sqrt{\frac{1}{6}}$
$0\ 1\ 1$	$1\ \frac{1}{2}\ -\frac{1}{2}$	$\sqrt{\frac{1}{6}}$	$\sqrt{\frac{1}{6}}$	$-\sqrt{\frac{1}{30}}$	$-\sqrt{\frac{1}{6}}$	$-\sqrt{\frac{3}{10}}$	$\sqrt{\frac{1}{6}}$
$0\ 1\ 0$	$1\ \frac{1}{2}\ \frac{1}{2}$	$\sqrt{\frac{1}{3}}$	$\sqrt{\frac{1}{3}}$	$\sqrt{\frac{1}{60}}$	$\sqrt{\frac{1}{12}}$	$\sqrt{\frac{3}{20}}$	$-\sqrt{\frac{1}{12}}$
$0\ 0\ 0$	$1\ \frac{1}{2}\ \frac{1}{2}$	0	0	$\sqrt{\frac{9}{20}}$	$-\sqrt{\frac{1}{4}}$	$-\sqrt{\frac{1}{20}}$	$-\sqrt{\frac{1}{4}}$

		$\mathbf{27}$	$\mathbf{10}$	$\mathbf{27}$	$\overline{\mathbf{10}}$	$\mathbf{8}$	$\mathbf{8}'$
		1	1	1	1	1	1
		$\frac{3}{2}$	$\frac{3}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
		$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
$1\ \frac{1}{2}\ \frac{1}{2}$	$0\ 1\ -1$	$\sqrt{\frac{1}{6}}$	$-\sqrt{\frac{1}{6}}$	$\sqrt{\frac{1}{30}}$	$-\sqrt{\frac{1}{6}}$	$\sqrt{\frac{3}{10}}$	$\sqrt{\frac{1}{6}}$
$1\ \frac{1}{2}\ -\frac{1}{2}$	$0\ 1\ 0$	$\sqrt{\frac{1}{3}}$	$-\sqrt{\frac{1}{3}}$	$-\sqrt{\frac{1}{60}}$	$\sqrt{\frac{1}{12}}$	$-\sqrt{\frac{3}{20}}$	$-\sqrt{\frac{1}{12}}$
$1\ \frac{1}{2}\ -\frac{1}{2}$	$0\ 0\ 0$	0	0	$\sqrt{\frac{9}{20}}$	$\sqrt{\frac{1}{4}}$	$-\sqrt{\frac{1}{20}}$	$\sqrt{\frac{1}{4}}$
$0\ 1\ 0$	$1\ \frac{1}{2}\ -\frac{1}{2}$	$\sqrt{\frac{1}{3}}$	$\sqrt{\frac{1}{3}}$	$-\sqrt{\frac{1}{60}}$	$-\sqrt{\frac{1}{12}}$	$-\sqrt{\frac{3}{20}}$	$\sqrt{\frac{1}{12}}$
$0\ 1\ -1$	$1\ \frac{1}{2}\ \frac{1}{2}$	$\sqrt{\frac{1}{6}}$	$\sqrt{\frac{1}{6}}$	$\sqrt{\frac{1}{30}}$	$\sqrt{\frac{1}{6}}$	$\sqrt{\frac{3}{10}}$	$-\sqrt{\frac{1}{6}}$
$0\ 0\ 0$	$1\ \frac{1}{2}\ -\frac{1}{2}$	0	0	$\sqrt{\frac{9}{20}}$	$-\sqrt{\frac{1}{4}}$	$-\sqrt{\frac{1}{20}}$	$-\sqrt{\frac{1}{4}}$

		$\mathbf{27}$	$\mathbf{10}$
		1	1
		$\frac{3}{2}$	$\frac{3}{2}$
		$-\frac{3}{2}$	$-\frac{3}{2}$
$1\ \frac{1}{2}\ -\frac{1}{2}$	$0\ 1\ -1$	$\sqrt{\frac{1}{2}}$	$-\sqrt{\frac{1}{2}}$
$0\ 1\ -1$	$1\ \frac{1}{2}\ -\frac{1}{2}$	$\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{2}}$

$8 \otimes 8$

		27	$\overline{10}$
		-1	-1
		$\frac{3}{2}$	$\frac{3}{2}$
		$\frac{3}{2}$	$\frac{3}{2}$
$0\ 1\ 1$	$-1\ \frac{1}{2}\ \frac{1}{2}$	$\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{2}}$
$-1\ \frac{1}{2}\ \frac{1}{2}$	$0\ 1\ 1$	$\sqrt{\frac{1}{2}}$	$-\sqrt{\frac{1}{2}}$

		27	$\overline{10}$	27	10	8	$8'$
		-1	-1	-1	-1	-1	-1
		$\frac{3}{2}$	$\frac{3}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
		$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$0\ 1\ 1$	$-1\ \frac{1}{2}\ -\frac{1}{2}$	$\sqrt{\frac{1}{6}}$	$\sqrt{\frac{1}{6}}$	$\sqrt{\frac{1}{30}}$	$\sqrt{\frac{1}{6}}$	$\sqrt{\frac{3}{10}}$	$\sqrt{\frac{1}{6}}$
$0\ 1\ 0$	$-1\ \frac{1}{2}\ \frac{1}{2}$	$\sqrt{\frac{1}{3}}$	$\sqrt{\frac{1}{3}}$	$-\sqrt{\frac{1}{60}}$	$-\sqrt{\frac{1}{12}}$	$-\sqrt{\frac{3}{20}}$	$-\sqrt{\frac{1}{12}}$
$0\ 0\ 0$	$-1\ \frac{1}{2}\ \frac{1}{2}$	0	0	$\sqrt{\frac{9}{20}}$	$-\sqrt{\frac{1}{4}}$	$-\sqrt{\frac{1}{20}}$	$\sqrt{\frac{1}{4}}$
$-1\ \frac{1}{2}\ \frac{1}{2}$	$0\ 1\ 0$	$\sqrt{\frac{1}{3}}$	$-\sqrt{\frac{1}{3}}$	$-\sqrt{\frac{1}{60}}$	$\sqrt{\frac{1}{12}}$	$-\sqrt{\frac{3}{20}}$	$\sqrt{\frac{1}{12}}$
$-1\ \frac{1}{2}\ \frac{1}{2}$	$0\ 0\ 0$	0	0	$\sqrt{\frac{9}{20}}$	$\sqrt{\frac{1}{4}}$	$-\sqrt{\frac{1}{20}}$	$-\sqrt{\frac{1}{4}}$
$-1\ \frac{1}{2}\ -\frac{1}{2}$	$0\ 1\ 1$	$\sqrt{\frac{1}{6}}$	$-\sqrt{\frac{1}{6}}$	$\sqrt{\frac{1}{30}}$	$-\sqrt{\frac{1}{6}}$	$\sqrt{\frac{3}{10}}$	$-\sqrt{\frac{1}{6}}$

		27	$\overline{10}$	27	10	8	$8'$
		-1	-1	-1	-1	-1	-1
		$\frac{3}{2}$	$\frac{3}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
		$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
$0\ 1\ 0$	$-1\ \frac{1}{2}\ -\frac{1}{2}$	$\sqrt{\frac{1}{3}}$	$\sqrt{\frac{1}{3}}$	$\sqrt{\frac{1}{60}}$	$\sqrt{\frac{1}{12}}$	$\sqrt{\frac{3}{20}}$	$\sqrt{\frac{1}{12}}$
$0\ 1\ -1$	$-1\ \frac{1}{2}\ \frac{1}{2}$	$\sqrt{\frac{1}{6}}$	$\sqrt{\frac{1}{6}}$	$-\sqrt{\frac{1}{30}}$	$-\sqrt{\frac{1}{6}}$	$-\sqrt{\frac{3}{10}}$	$-\sqrt{\frac{1}{6}}$
$0\ 0\ 0$	$-1\ \frac{1}{2}\ -\frac{1}{2}$	0	0	$\sqrt{\frac{9}{20}}$	$-\sqrt{\frac{1}{4}}$	$-\sqrt{\frac{1}{20}}$	$\sqrt{\frac{1}{4}}$
$-1\ \frac{1}{2}\ \frac{1}{2}$	$0\ 1\ -1$	$\sqrt{\frac{1}{6}}$	$-\sqrt{\frac{1}{6}}$	$-\sqrt{\frac{1}{30}}$	$\sqrt{\frac{1}{6}}$	$-\sqrt{\frac{3}{10}}$	$\sqrt{\frac{1}{6}}$
$-1\ \frac{1}{2}\ -\frac{1}{2}$	$0\ 1\ 0$	$\sqrt{\frac{1}{3}}$	$-\sqrt{\frac{1}{3}}$	$\sqrt{\frac{1}{60}}$	$-\sqrt{\frac{1}{12}}$	$\sqrt{\frac{3}{20}}$	$-\sqrt{\frac{1}{12}}$
$-1\ \frac{1}{2}\ -\frac{1}{2}$	$0\ 0\ 0$	0	0	$\sqrt{\frac{9}{20}}$	$\sqrt{\frac{1}{4}}$	$-\sqrt{\frac{1}{20}}$	$-\sqrt{\frac{1}{4}}$

		27	$\overline{10}$
		-1	-1
		$\frac{3}{2}$	$\frac{3}{2}$
		$-\frac{3}{2}$	$-\frac{3}{2}$
$0\ 1\ -1$	$-1\ \frac{1}{2}\ -\frac{1}{2}$	$\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{2}}$
$-1\ \frac{1}{2}\ -\frac{1}{2}$	$0\ 1\ -1$	$\sqrt{\frac{1}{2}}$	$-\sqrt{\frac{1}{2}}$

$\underset{\sim}{8} \otimes \underset{\sim}{8}$		$\underset{\sim}{27}$ 2 1 1	$\underset{\sim}{27}$ 2 1 0	$\underset{\sim}{\overline{10}}$ 2 0 0	$\underset{\sim}{27}$ 2 1 -1
1 $\frac{1}{2}$ $\frac{1}{2}$	1 $\frac{1}{2}$ $\frac{1}{2}$	1			
1 $\frac{1}{2}$ $\frac{1}{2}$	1 $\frac{1}{2}$ $-\frac{1}{2}$		$\sqrt{\frac{1}{2}}$	$-\sqrt{\frac{1}{2}}$	
1 $\frac{1}{2}$ $-\frac{1}{2}$	1 $\frac{1}{2}$ $\frac{1}{2}$		$\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{2}}$	
1 $\frac{1}{2}$ $-\frac{1}{2}$	1 $\frac{1}{2}$ $-\frac{1}{2}$				1