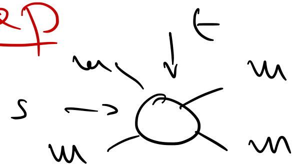
Day 7

Week I Recap

Elastic Scattering



$$\cos\theta_s = 1 + \frac{2t}{s-4m^2}$$

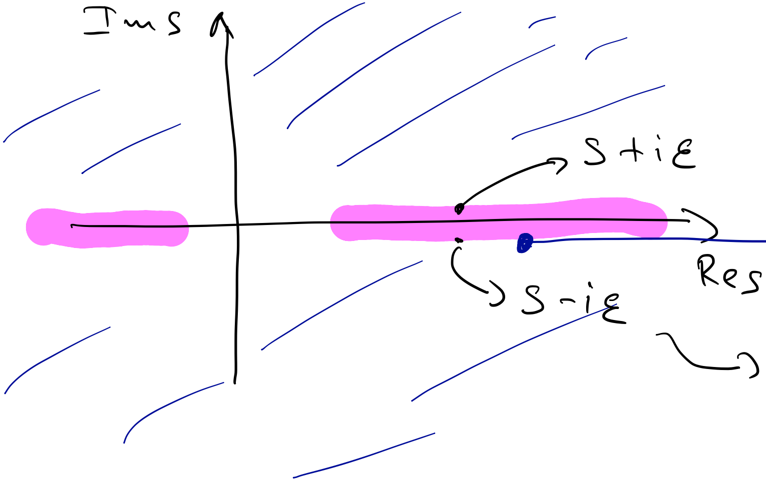
P. W. Expansion: $A(s,t) = \sum_{\ell=0}^{\infty} (2\ell+1)\, a_\ell(s)\, P_\ell(\cos\theta_s)$

Unitarity relation: $\mathrm{Im}\, a_\ell^{-1}(s) = -\rho(s)$

$$\rho(s) = \frac{1}{2}\,\frac{1}{8\pi}\sqrt{1-\frac{4m^2}{s}}$$

$$a_\ell(s \pm i\varepsilon) = \frac{1}{K(s) \mp i\rho(s)}$$

real



$$\boxed{a_\ell^{-1}(s) = K(s) - i\rho(s)}$$

sheet II

$$\underline{a_\ell^{-1}(s) = K(s) + i\rho(s)}$$

$$\rho(s) = \frac{1}{16\pi}\sqrt{1-\frac{4m^2}{s}}\;\theta(s-4m^2)$$

B-W

$$K(s) = \frac{m^2 - s}{m\Gamma}$$

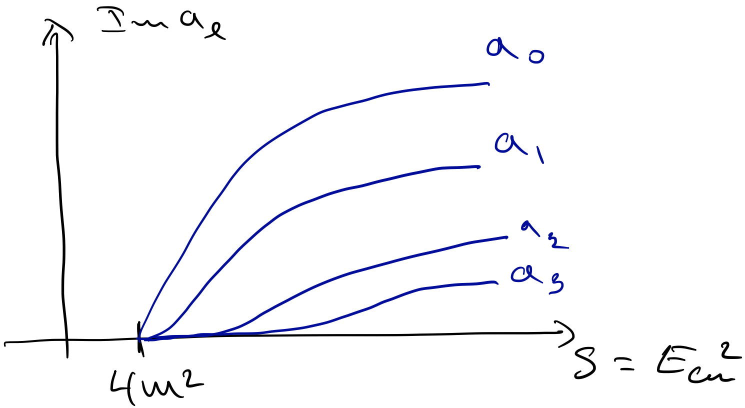
$$\rho_{CM}(s) = \frac{-i}{\pi}\int_{4m^2}^{\infty} \frac{\rho(s')\,ds'}{s'-s}$$

$$A(s,t) = \sum_{\ell=0}^{\infty} (2\ell+1)\, a_\ell(s)\, P_\ell\left(1 + \underbrace{\frac{2t}{s-4m^2}}\right)$$

$A(s,t)$ is regular at threshold $\underline{s = 4m^2}$ $\forall t$

$$a_\ell(s) \sim (s-4m^2)^\ell$$

$$a_\ell(s) = (s-4m^2)^\ell\, \underline{f_\ell(s)} \qquad \mathrm{Im}\, f_\ell^{-1}(s) = -\rho(s)(s-4m^2)^{\ell}$$



$$A(s,t) = \sum_{l=0}^{\infty} (2l+1)\, \underline{f_l}(t)\, P_l(\cos\theta_t) \qquad \cos\theta_t = 1 + \frac{2s}{t-4m^2}$$

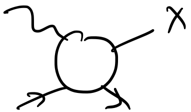
$$= \frac{s-4}{t-4m^2}$$

$$= \sum_{l=0}^{n(t)} (2l+1)\, f_l(t)\, P_l(\cos\theta_t)$$

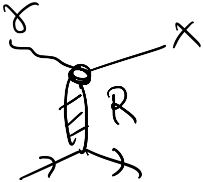
$$+ \sum_{l=n+1}^{\infty} \cdots$$

$$= \beta(t)\, (\cos\theta_t)^{n(t)} \sim \boxed{\underline{\beta(t)}\, s^{n(t)}}$$

JLab $E_\gamma \sim 2-12$ GeV

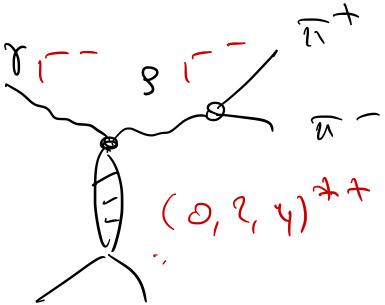


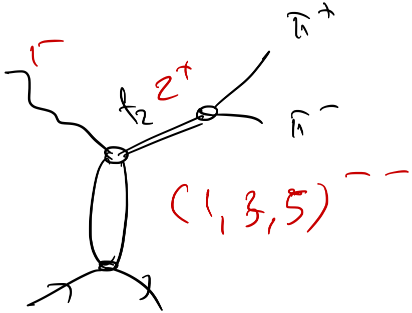
$\sim$



$\sim \beta(t)\, s^{\alpha(t)}$

$\gamma p \to (\pi\pi)\, p$





$A \sim s^1$

$\sigma \sim cst$

$A \sim s^{0.5}$

$\sigma \sim 1/E_{cm}$

Non-Relativistic Quantum Mechanics

$$\left[-\frac{\hbar^2}{2m}\nabla_r^2 + \frac{\ell(\ell+1)}{r^2} + U(r)\right]\varphi_\ell(r) = E_\ell\,\varphi_\ell(r)$$

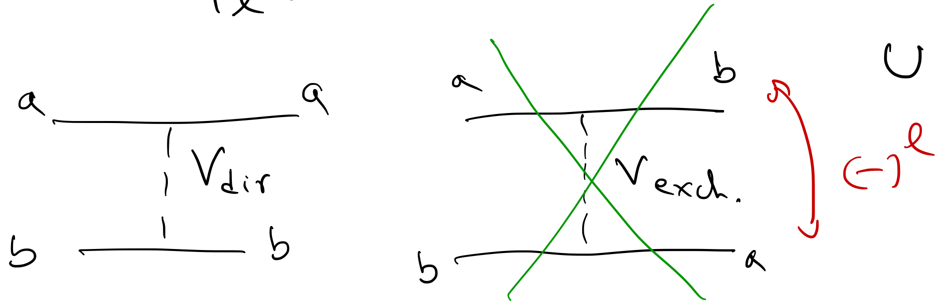
a) ℓ enters analytically

b) Symmetry $\ell \to -\ell-1$: $\ell(\ell+1) \leftrightarrow \ell(\ell+1)$

$$\varphi_\ell(r) \sim r^{\ell}$$

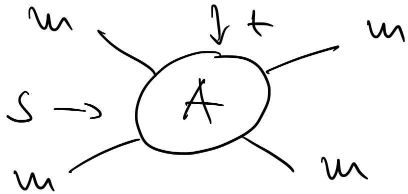
$$r \to 0$$

$$\varphi_\ell(r) \sim r^{-\ell-1}$$

c) 

$$U = \begin{cases} V_{dir} + V_{exch} & \ell \text{ even} \\ V_{dir} - V_{exch} & \ell \text{ odd} \end{cases}$$

Relativistic theory

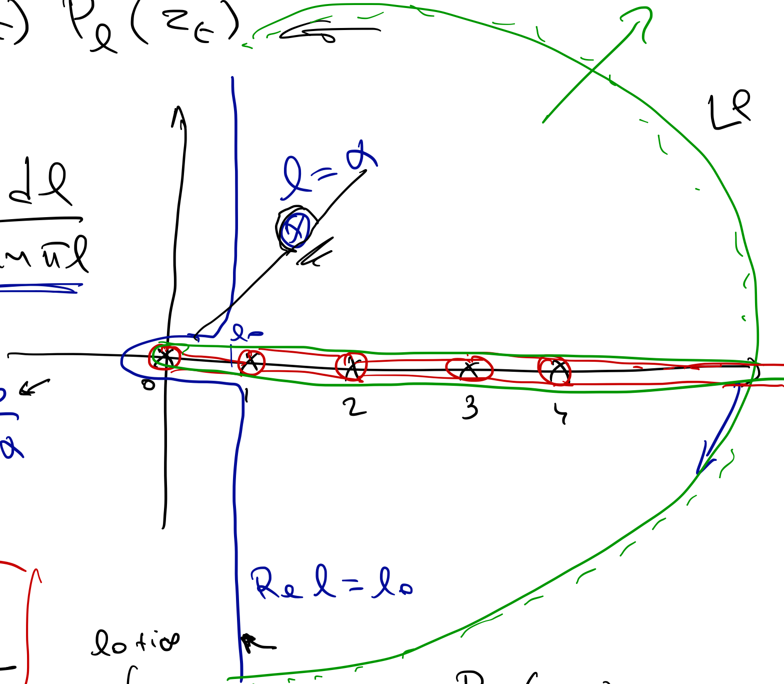


$$z_t = \cos\theta_t = 1 + \frac{2s}{t-4m^2}$$

$$A(t, z_t) = \sum_{\ell=0}^{\infty} (2\ell+1)\, f_\ell(t)\, P_\ell(z_t)$$

$$= \frac{1}{2i} \oint_C (2\ell+1)\, f(\ell,t)\, P_\ell(-z_t) \frac{d\ell}{\sin\pi\ell}$$

$$f(\ell,t) = \frac{\beta}{\ell-\alpha}$$



$$\lim_{\substack{\ell\to n \\ n\in\mathbb{N}}} \frac{\ell-n}{\sin\pi\ell} = \frac{(-1)^n}{\pi}$$

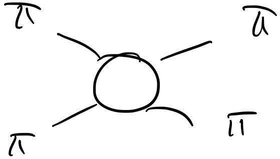
$$= -\pi\,(2\alpha(t)+1)\,\beta(t)\,\frac{P_{\alpha(t)}(-z_t)}{\sin\alpha(t)\pi} + \frac{1}{2i}\int_{\ell_0-i\infty}^{\ell_0+i\infty} (2\ell+1)\, f(\ell,t)\, \frac{P_\ell(-z_t)}{\sin\pi\alpha(t)}\, d\ell$$

$$-\pi(2\alpha+1)\beta \frac{s^\alpha}{\sin\pi\alpha(t)}$$

$$f_1 = \frac{g_1^2}{m_{\rho_1}^2 - t}$$

$$f_3 = \frac{g_3^2}{m_{\rho_3}^2 - t}$$

$\vdots$



$$\boxed{f(l,t) = \frac{\beta(t)}{l - \alpha(t)}}$$

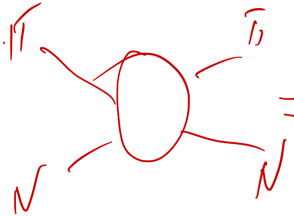
$l = 1$: $$f(1,t) = \frac{\beta(t)}{1 - \alpha(t)}$$

$$f(3,t) = \frac{\beta(t)}{3 - \alpha(t)}$$

f_1 : $t = m_{\rho_1}^2$

$$\alpha(t = m_{\rho_1}^2) = 1$$

$$\alpha(t = m_{\rho_3}^2) = 3$$



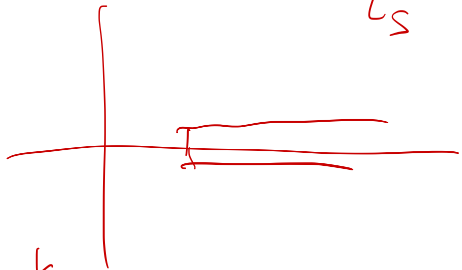
$$\Rightarrow A(s,t) = A(s, t(s, \cos\theta)) = \sum (a_\ell(s)) P_\ell(z_s)$$

$$\cos\theta = z_s$$

E_{cm}

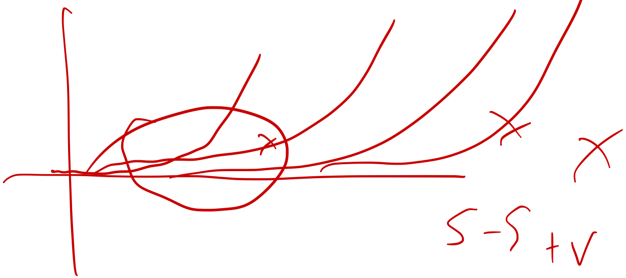
$$(s - s_{thr})^\ell \, \nu_0$$

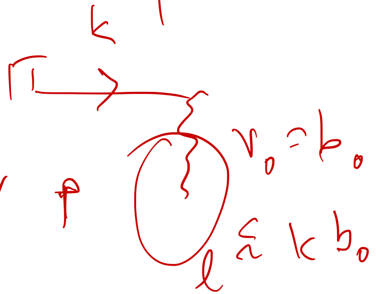
ℓ



$E_{thr} \sim 1.2 - 1.3\ \mathrm{GeV}$.

$\to 10\ \mathrm{GeV}$





$10\ \mathrm{GeV} \times 0.5\ \mathrm{fm}$

$$\frac{5\ \mathrm{GeV\,fm}}{0.2\ \mathrm{GeV\,fm}} \sim 25$$

$$a_\ell(s) \sim \left[(s - s_{thr})\,\nu^2\right]^{\ell}$$

$x < 1$

$$A(s,t) = \sum_{\ell}^{n_0} \underbrace{(2\ell+1)\, a_\ell(s)\, P_\ell(z_s)}$$

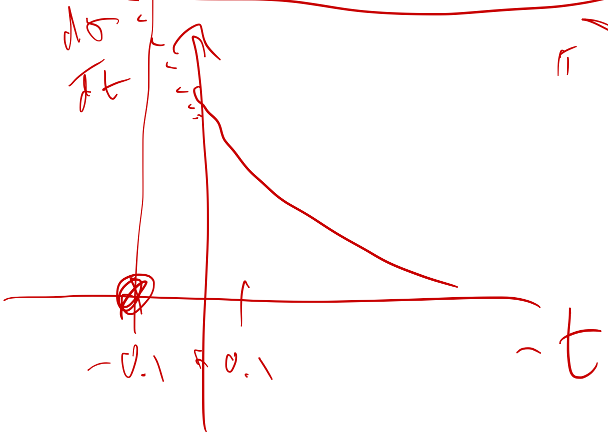
$= ? ? ? ?$

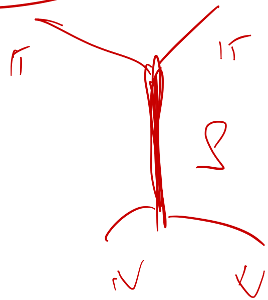
$s \approx (10\ \mathrm{GeV})^2$

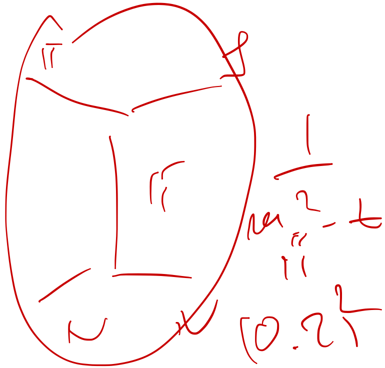
small $|t| <$

$0.1 < |t| < 1\ \mathrm{GeV}^2$

|t| ~ small $\qquad$ S ~ big.







$$A(t, z_t) = \sum_\ell (2\ell+1)\, a_\ell(t)\, P_\ell(z_t)$$

$|z_t| < 1$

$|t| \gg 0$

$t < 0$

$s \sim$ big

?

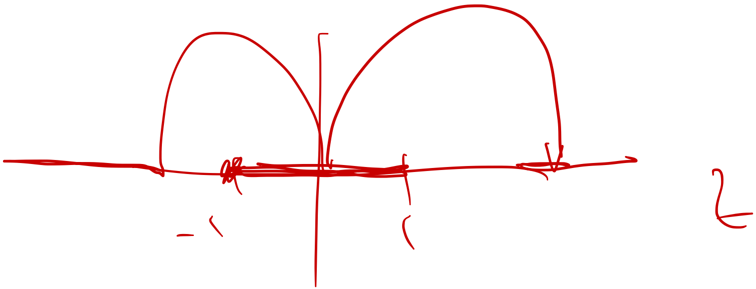
$$|z_t| < 1 \xrightarrow{s} |z_t| > 1$$

$$\sum a_\ell(t)\, P_\ell(z_t) = \text{diverges}$$

$$P_\ell(t_n) = \dots z^{\ell-1} \dots t^\ell$$

$$f(z) = 1 + z + z^2 + z^3 + \dots = \sum_{\ell=0}^{\infty} z^\ell$$

$$|z| < 1$$



$$f(z) = \frac{1}{1-z} \qquad z = 1$$

$$z = 2f(z) - 1$$

$$= \frac{1}{-z\left(1 - \frac{1}{z}\right)} = -\frac{1}{z}\left(1 + \frac{1}{z} + \frac{1}{z^2} + \dots\right) \qquad |z| > 1$$

$$= -\sum_{l=-\infty}^{-1} \left(z^l\right) =$$

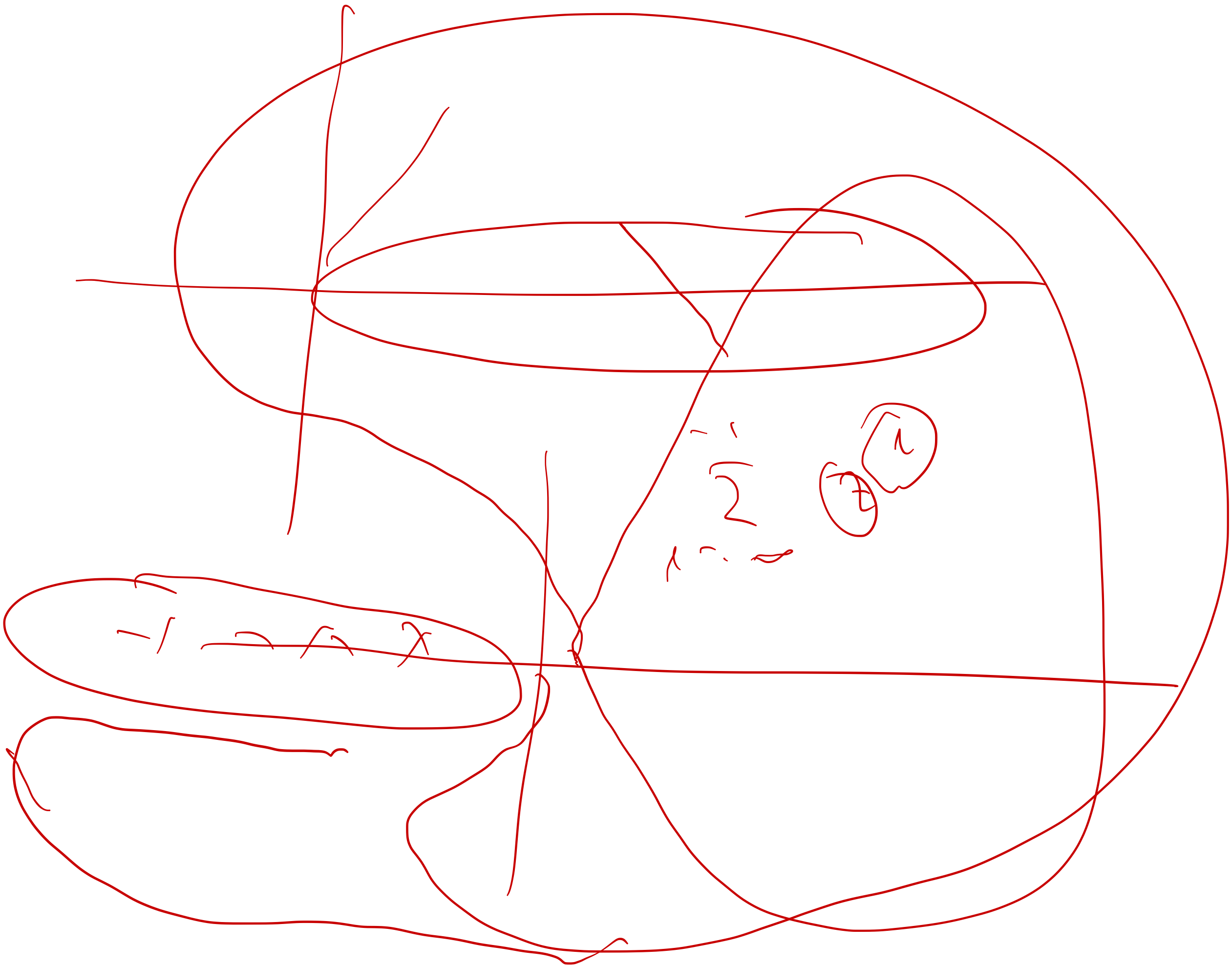
$$\sum_{l=0}^{\infty} z^l =$$

$A(t, t_t) = \sum_{l=0}^{\infty} a_l(t)\, z_t^{l}$

$z_t \sim \left| \frac{S}{t - 4m^2} \right| < 1$

$= \sum_{l=-\infty}^{-1} a_l(t)\, z_t^{-l}$

$z_s \sim \left| \frac{s}{t - 4u} \right| > 1$



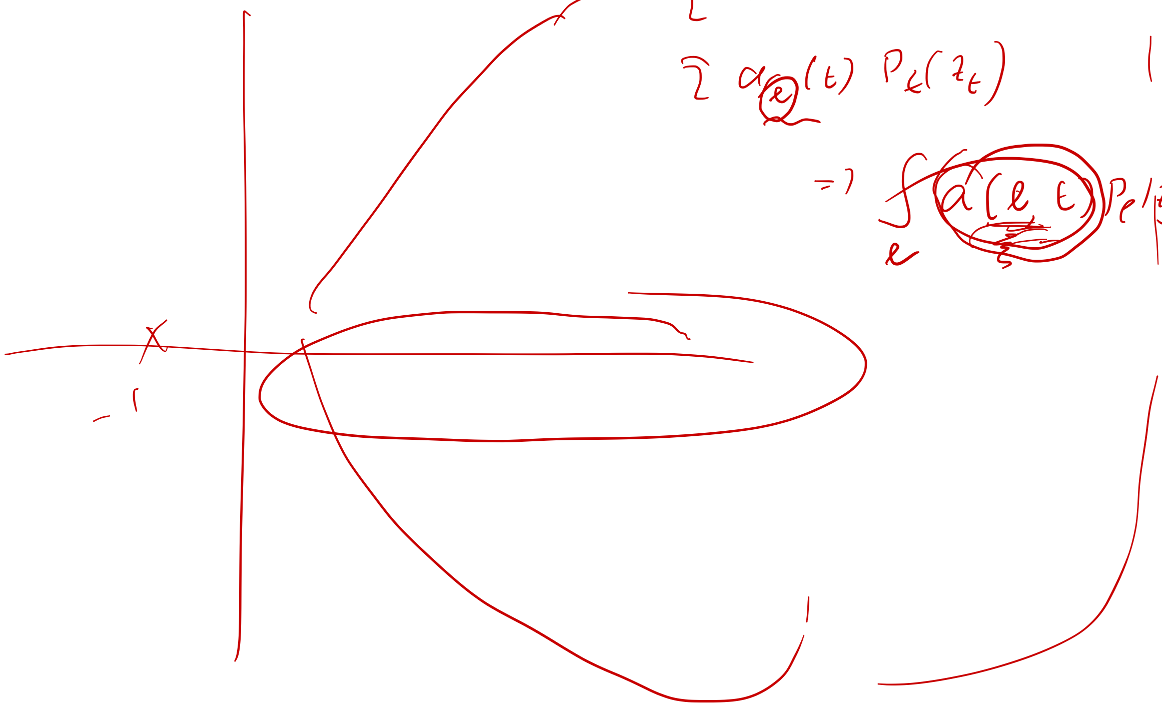
$\frac{1}{5} = 5^{-1}$

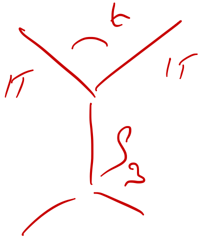
Σz $\quad$ $\ln z$ $\quad$ $|z|$

$\Sigma \Rightarrow$

$\Sigma\, a_{\ell}(t)\, p_{\ell}(z_t)$

$\Rightarrow \int_{\ell} a(\ell, t)\, p_{\ell}(z)$





$\Rightarrow$ $\underline{a_1(t)} = \sum_{\ell=0}^{\infty} a_\ell P_\ell(z_t) = a_1(t) P_1(z_s)$

a_2

a_3

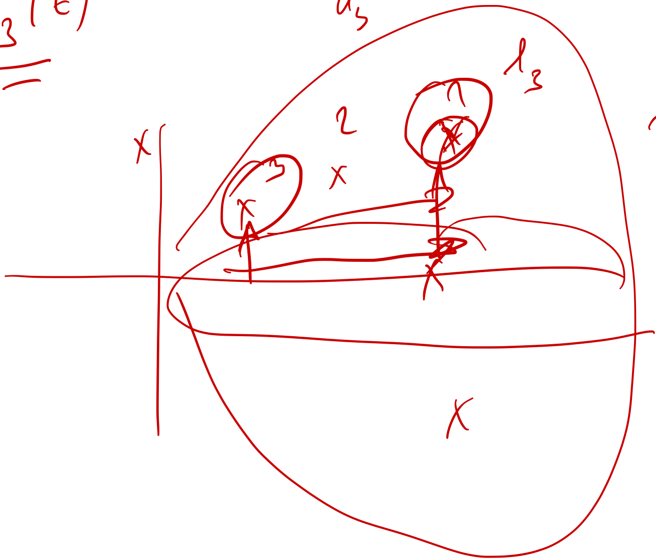
$= a_{\ell=0,1,2\ldots} \Rightarrow a(\ell)$

$\ell = 1.1013$

$\sim i\varepsilon$

$S = \ell_3(t)$

$a(\ell, t)$



$\sim \frac{1}{\ell - \ell_1}$

$\frac{1}{\ell - \ell_3}$

ℓ_1 ℓ_3 ℓ_2

z_t z_t t_t

$$t^{\ell_1} \sim s^{\ell_1}$$

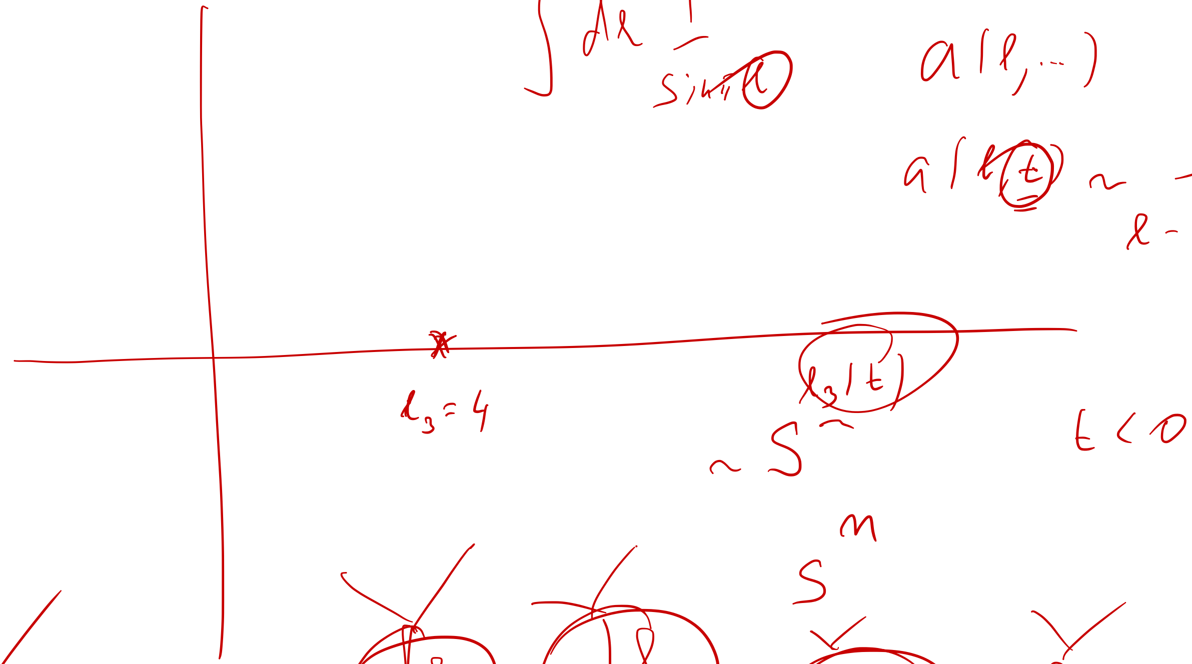
$$t^{\ell_3} \sim s^{\ell_3}$$

$$\int dt\, t^{t} \, \frac{\langle a(\ell)\ldots\rangle}{\sin \pi \ell} \sim \frac{1}{\ell - \ell_{1,2,3}}$$

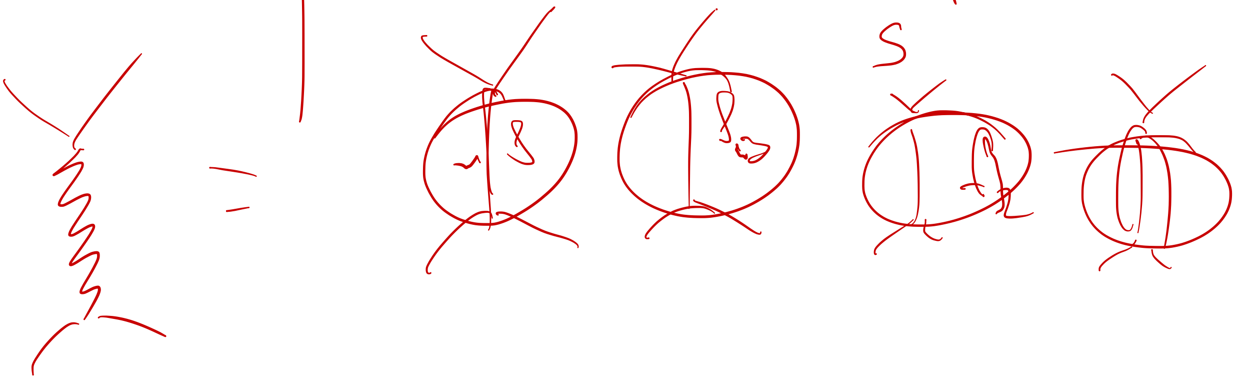
$$\int d\ell \, \frac{1}{\sin \pi \ell}$$

$$a(\ell, \dots)$$

$$a(\ell(t)) \sim \frac{1}{\ell - \ell_3(t)}$$



S^n



$$A(s,t) = \sum_{\ell=0}^{\infty} (2\ell+1)\, a_\ell(s)\, P_\ell(\cos\theta)$$

ρ

$\ell \lesssim k\, r_0$

r_0

$\ell_{max} \sim \sqrt{s}$

$k \sim \sqrt{s}$

$$\sum_{\ell=0} (2\ell+1)\, 1 \times 1 \sim \frac{\ell_m^2}{2} \sim \frac{s}{2}$$

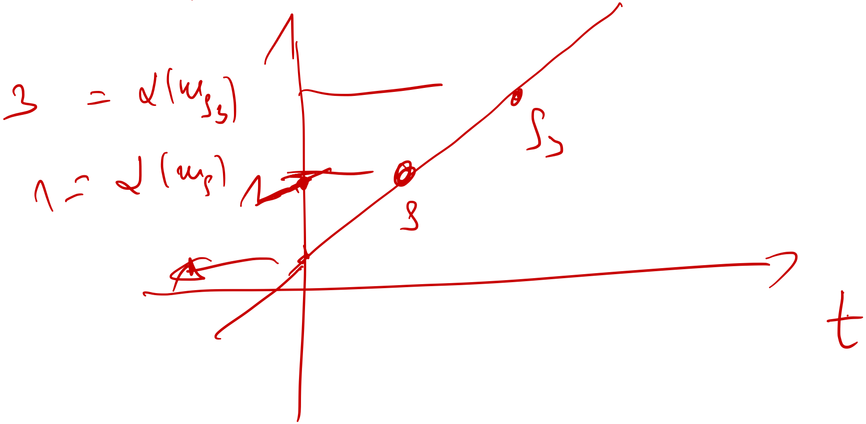
s^{+0}

e^{s}

$F_V \Rightarrow \quad s^1 \quad \Rightarrow \quad s^{\omega} \quad s^{\alpha(t)}$

in s physics $\quad s \sim 10\,GeV^2$

$\alpha(t) \qquad t < 0 \qquad \alpha(t) < 1$



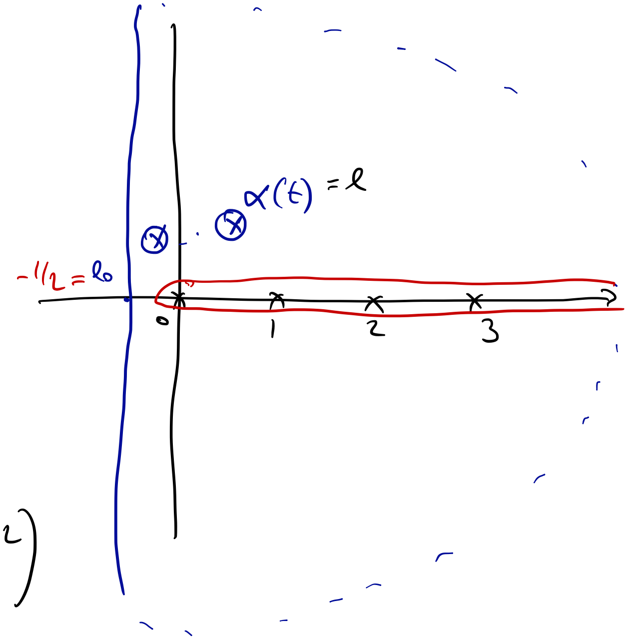
$$\underline{A(s,t)} = \sum_{\ell=0}^{\infty} (2\ell+1)\, f_\ell(t)\, P_\ell(z_t) \qquad z_t = 1 + \frac{2s}{t-4m^2}$$

$$= \frac{1}{2i} \oint_C (2\ell+1)\, \underline{f(\ell,t)}\, P_\ell(-z_t) \frac{d\ell}{\sin \pi \ell} \qquad f(\ell,t) = \frac{\beta(t)}{\ell - \alpha(t)}$$

$$= -\pi(2\alpha+1)\beta(t) \frac{P_{\alpha(t)}(-z_t)}{\sin \pi\alpha(t)}$$

$$+ \frac{1}{2i} \int_{\ell_0 - i\infty}^{\ell_0 + i\infty} (2\ell+1)\, f(\ell,t) \frac{P_\ell(-z_t)}{\sin \pi \ell}\, d\ell$$

$$= \sum_{\alpha > -1/2} -\pi(2\alpha+1)\beta \frac{(s/s_0)^{\alpha}}{\sin \pi\alpha} + O\left(s^{-1/2}\right)$$



$$P_\ell(z) \xrightarrow[z\to\infty]{} z^{\ell} \quad \text{if } \mathrm{Re}\,\ell \geq -1/2$$

$$\ell = -1/2$$

$$\longrightarrow z^{-\ell-1} \quad \text{if } \mathrm{Re}\,\ell \leq -1/2$$

$$\ell(\ell+1)$$

$$\underline{\ell} \to \underline{-\ell-1}$$

$$f(\ell, t) = \frac{1}{2} \int_{-1}^{1} \underline{A(t, z_t)}\, P_\ell(z_t)\, dz_t$$

$$\int_{-1}^{1} P_\alpha(x)\, P_\ell(x)\, dx = (-)^\alpha \frac{2/\pi}{\alpha - \ell} \frac{\sin \pi\alpha}{\ell + \alpha + 1}$$

$\alpha \in \mathbb{C}$; $\ell \in \mathbb{N}$

$$P_\ell(z) \xrightarrow[\ell \to \infty]{} e^{|\mathrm{Im}\, \ell|\, \theta} \qquad \cos\theta = z$$

$$-1 \leq z \leq 1$$

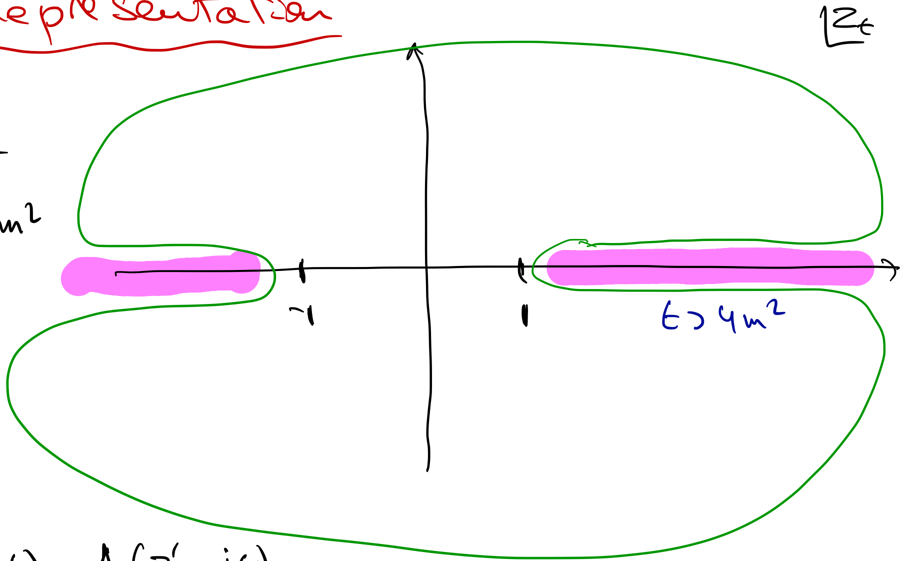
$$f(\ell, t) = \frac{1}{2} \int_{-1}^{1} \sum_{\ell' \in \mathbb{N}}^{\infty} (2\ell' + 1)\, f_{\ell'}(t)\, P_{\ell'}(z_t)\, P_\ell(z_t)\, dz$$

$\ell \in \mathbb{C}$

$$= \frac{(-)^\ell}{\pi} \sin \pi\ell \sum_{\ell' \in \mathbb{N}'} \frac{2\ell' + 1}{\ell + \ell' + 1} \frac{1}{\ell - \ell'} f_{\ell'}(t)$$

$$= f_\ell(t)\, \delta_{\ell\ell'} \qquad \ell \in \mathbb{N}$$

Froissart-Gribov Representation



unitarity cuts

$s > 4m^2 \to z_t > z_0 = 1 + \frac{8m^2}{t-4m^2}$

$z_t = 1 + \frac{2s}{t-4m^2}$

$= -1 - \frac{2u}{t-4m^2}$

$u > 4m^2 \to z_t < -z_0$

$$A(z_t, t) = \frac{1}{2\pi i}\int_{z_0}^{\infty} \frac{A(z'_t+i\varepsilon,t) - A(z'_t-i\varepsilon,t)}{z'_t - z_t}\, dz'_t \to \frac{1}{\pi}\int \frac{D_s A(z',t)}{z'-z_t}\, dz'$$

$$+ \frac{1}{2\pi i}\int_{-\infty}^{-z_0} \frac{A(z'_t+i\varepsilon,t) - A(z'_t-i\varepsilon,t)}{z'_t - z_t}\, dz'_t \to \frac{1}{\pi}\int \frac{D_u A(z',t)}{z'-z}\, dz'$$

$D_s A = \frac{1}{2i}\left[A(s+i\varepsilon,t) - A(s-i\varepsilon,t)\right]$

$$f_\ell(t) = \frac{1}{2}\int_{-1}^{1} \underline{A(z,t)}\, P_\ell(z)\, dz$$

$$f(\ell,t) = \frac{1}{2\pi}\int_{z_0}^{\infty}\int_{-1}^{1} D_s A(z',t)\, \frac{P_\ell(\hat{z})}{z'-z}\, dz' + \frac{1}{2\pi}\int_{-\infty}^{-z_0}\int_{-1}^{1} D_u A(z',t)\, \frac{P_\ell(\hat{z}')}{z'-z}\, dz'$$

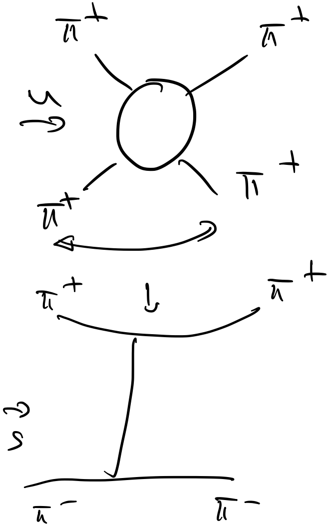
$$Q_\ell(-z) = (-)^{\ell+1} Q_\ell(z) \qquad Q_\ell(z) = \frac{1}{2}\int_{-1}^{1} \frac{P_\ell(z_t)}{z-z_t}\, dz_t$$

$$\boxed{f^{\pm}_{\text{Res}}(\ell,t) = \frac{1}{\pi}\int_{z_0}^{\infty}\left[D_s A^{\text{Rs}}(z',t) \;\underset{e^{-i\pi\ell}}{\pm}\; D_u A^{\text{Res}}(z',t)\right] Q_\ell(z')\, dz'}$$

$f^{+}(\ell,t)$ matches $f_\ell(t)$ ℓ even $\qquad (-)^\ell \equiv \tau$ signature

$f^{-}(\ell,t)$ " " " odd

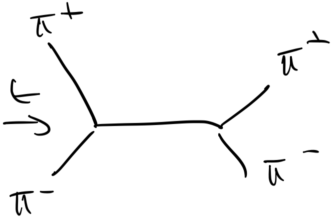
$D_s A = \mathrm{Im}\, A = \mathrm{Im}\, A_{bckg} + \mathrm{Im}\, A_{res}$



$: \quad \mathrm{Im}\, A_{res} \sim 0 \Rightarrow \underbrace{f^{+}(\ell,t)} = \underbrace{f^{-}(\ell,t)}$

$$\boxed{\begin{array}{c} \beta^{f}_{\pi\pi}(t) = \beta^{\rho}_{\pi\pi}(t) \\ \alpha_f(t) = \alpha_\rho(t) \end{array}} \Leftarrow \frac{\beta^{f}_{\pi\pi}(t)}{\ell - \alpha_f(t)} = \frac{\beta^{\rho}_{\pi\pi}(t)}{\ell - \alpha_\rho(t)}$$

Exchange Degeneracy (EXD)



$: \quad$ "ρ" $\quad I = 1 \quad C = -\quad I^G J^{PC} = 1^+ (1,3,5,\ldots)^{--}$

"f" $\quad I = 0 \quad C = + \quad I^G J^{PC} = 0^+ (0,2,4\ldots)^{++}$

$\pi^+ p$. Δ^+ not exotic

~~$\pi^- p$~~

$K^+ p$

~~$K^- p$~~

$K^+ n$

~~$K^- n$~~

$p\ p$

~~$\bar{p}\ p$~~ not exotic

$p\ n$

~~$\bar{p}\ n$~~

π^+ $u\bar{d}$

π^- $\bar{u}d$

K^+ $u\bar{s}$

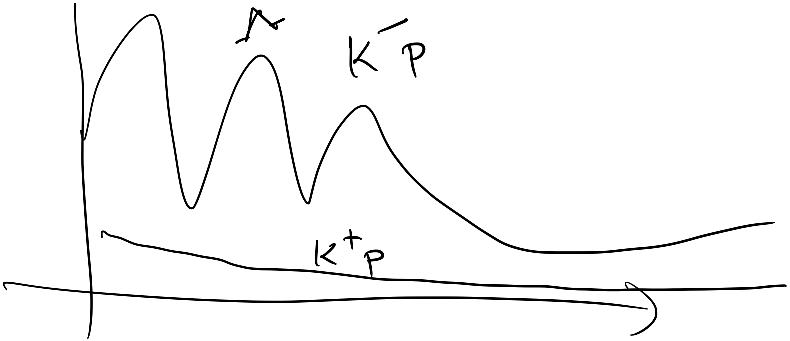
K^- $\bar{u}s$

$\bar{p}$ $\bar{u}\bar{u}\bar{d}$

n udd

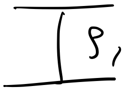
$$\boxed{\begin{array}{ll} K^+p\,; & K^+n \\ pp & pn \end{array}}$$

$$\sigma_{tot} = \frac{1}{s}\,\mathrm{Im}\,A(s, t=0)$$



$$\mathrm{Im}\,A = \mathrm{Im}\,A^{Res} + \mathrm{Im}\,A^{bck}$$

$$\mathrm{Im}\,A^{Res} \to \mathrm{Im}\,A^{Regge} \qquad \mathrm{Im}\,A^{bck} \to \mathrm{Im}\,A^{Pom}$$



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