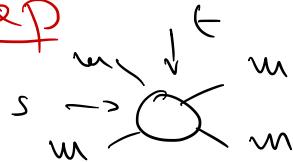


Day 7

## Week 1 Recap

Elastic Scattering



$$\cos \theta_s = 1 + \frac{2t}{s - 4m^2}$$

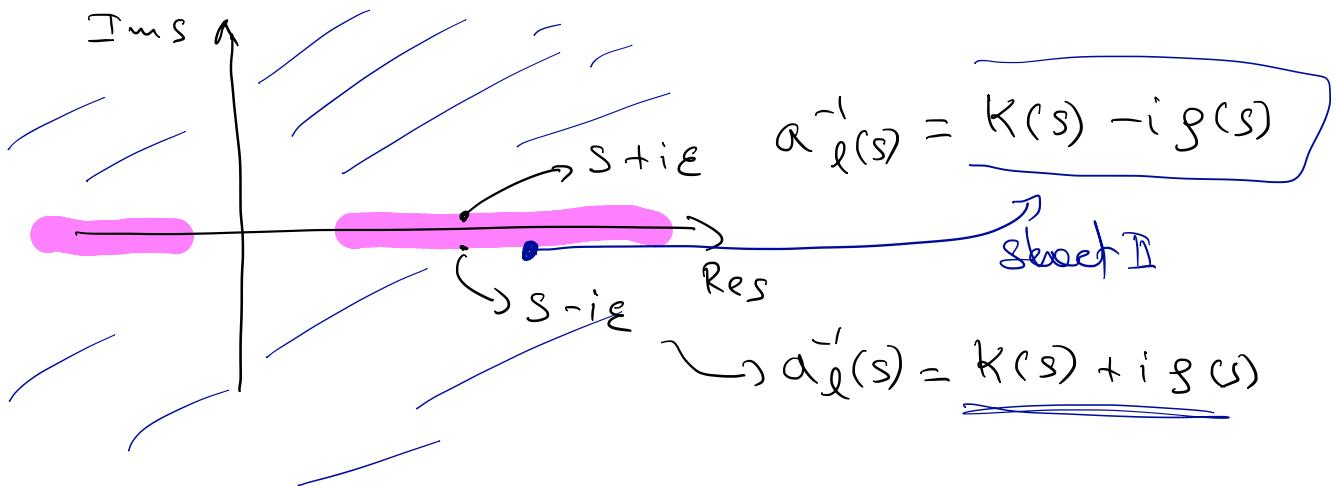
P. W. Expansion:  $A(s, t) = \sum_{\ell=0}^{\infty} (2\ell+1) a_\ell(s) P_\ell(\cos \theta_s)$

Unitarity relation:  $\text{Im } a_\ell^{-1}(s) = -g(s)$

$$g(s) = \frac{1}{2} \frac{1}{8\pi} \sqrt{1 - \frac{4m^2}{s}}$$

$$a_\ell(s \pm i\epsilon) = \frac{1}{K(s) \mp ig(s)}$$

real    ↴



$$g(s) = \frac{1}{16\pi} \sqrt{1 - \frac{4m^2}{s}} \Theta(s - 4m^2)$$

$$g_{CM}(s) = \frac{-i}{\pi} \int_{4m^2}^{\infty} \frac{g(s')}{s' - s} ds'$$

B-W

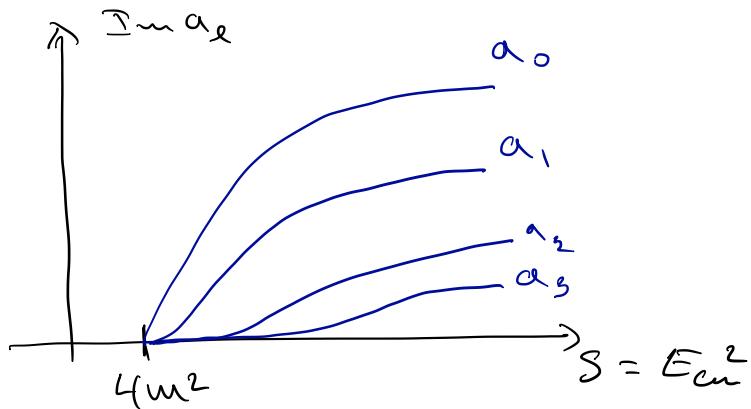
$$K(s) = \frac{m^2 - s}{m \Gamma}$$

$$A(s, t) = \sum_{\ell=0}^{\infty} (2\ell+1) a_\ell(s) P_\ell \left( 1 + \underbrace{\frac{2t}{s - 4m^2}}_{\text{blue bracket}} \right)$$

$A(s_1, t)$  is regular at threshold  $s = 4m^2$   $\forall t$

$$a_\ell(s) \sim (s - 4m^2)^\ell$$

$$a_\ell(s) = (s - 4m^2)^\ell \underbrace{f_\ell(s)}_{\text{blue bracket}} \quad \text{Im } f_\ell^{-1}(s) = -g(s)(s - 4m^2)$$



$$A(s, t) = \sum_{l=0}^{\infty} (2l+1) \underline{f_l(t)} P_l(\cos \theta_t) \quad \cos \theta_t = 1 + \frac{2s}{t - 4m^2}$$

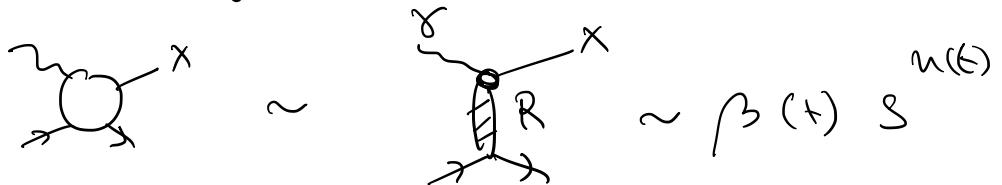
$$= \sum_{l=0}^{n(t)} (2l+1) f_l(t) P_l(\cos \theta_t)$$

$$+ \sum_{l=n+1}^{\infty} \dots \quad \dots$$

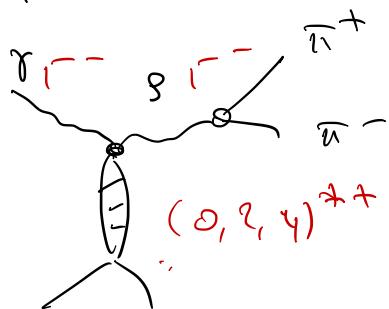
$$= \beta(t) (\cos \theta_t)$$



JLab  $E_\gamma \sim 2-12$  GeV

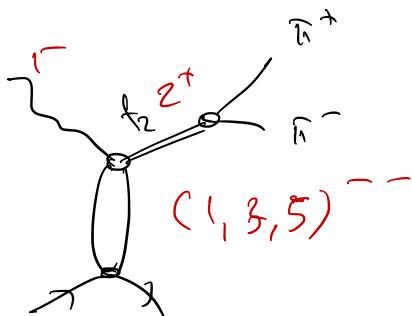


$\gamma p \rightarrow (\pi\pi) p$



$$A \sim S^1$$

$$\tau \sim cst$$



$$A \sim S^{0.5}$$

$$\tau \sim 1/E_{cm}$$

## Non-Relativistic Quantum Mechanics

$$\left[ -\frac{\hbar^2}{2m} \nabla_r^2 + \frac{l(l+1)}{r^2} + U(r) \right] \varphi_l(r) = E_l \varphi_l(r)$$

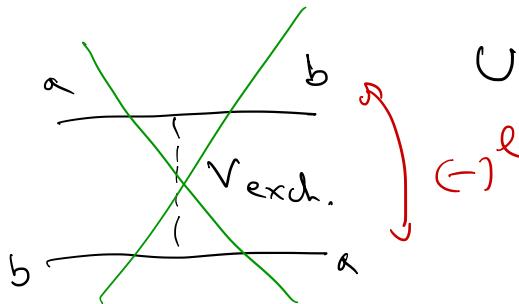
- a)  $l$  enters analyticity alley
- b) Symmetry  $l \rightarrow -l-1$  :  $l(l+) \leftrightarrow l(l+1)$

$$\varphi_l(r) \sim r^l$$

$$r \rightarrow 0$$

$$\varphi_l(r) \sim r^{-l-1}$$

$$\begin{array}{ccc} a & \xrightarrow{\quad V_{\text{dir}} \quad} & a \\ b & \xrightarrow{\quad V_{\text{exch.}} \quad} & b \end{array}$$



$$U = \begin{cases} V_{\text{dir}} + V_{\text{exch}} & \text{l even} \\ V_{\text{dir}} - V_{\text{exch}} & \text{l odd} \end{cases}$$

## Relativistic theory

$$z_t = \cos \theta_t = 1 + \frac{2s}{t - 4m^2}$$

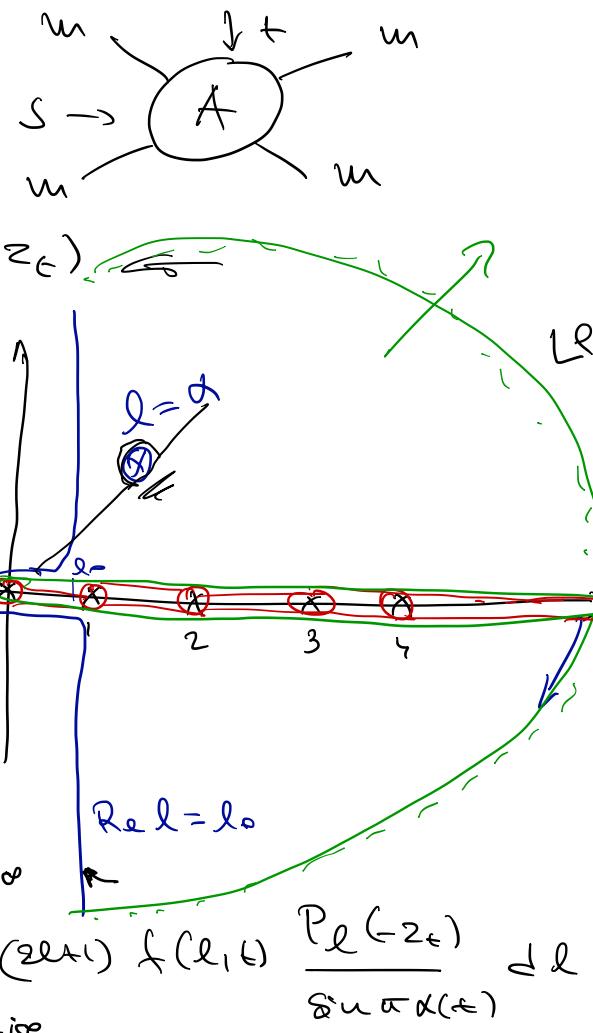
$$A(t, z_t) = \sum_{\ell=0}^{\infty} (2\ell+1) f_\ell(t) P_\ell(z_t)$$

$$= \frac{1}{2i} \oint_C (2\ell+1) f(\ell, t) P_\ell(-z_t) \frac{d\ell}{\sin \pi \ell}$$

$$\lim_{\ell \rightarrow \infty} \frac{\ell - n}{\sin \pi \ell} = \frac{(-1)^n}{\pi}$$

$n \in \mathbb{N}$

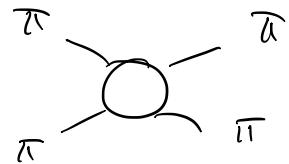
$$= -\pi (2\alpha(t)+1) \beta(t) \frac{P_{\alpha(t)}(-z_t)}{\sin \pi \alpha(t)} + \frac{1}{2i} \int_{\ell_0-i\infty}^{\ell_0+i\infty} (2\ell+1) f(\ell, t) \frac{P_\ell(-z_t)}{\sin \pi \alpha(t)} d\ell$$



$$f_1 = \frac{g_1^2}{m_{g_1}^2 - t}$$

$$f_3 = \frac{g_3^2}{m_{g_3}^2 - t}$$

⋮



$$f(l, \epsilon) = \frac{\beta(\epsilon)}{l - \alpha(\epsilon)}$$

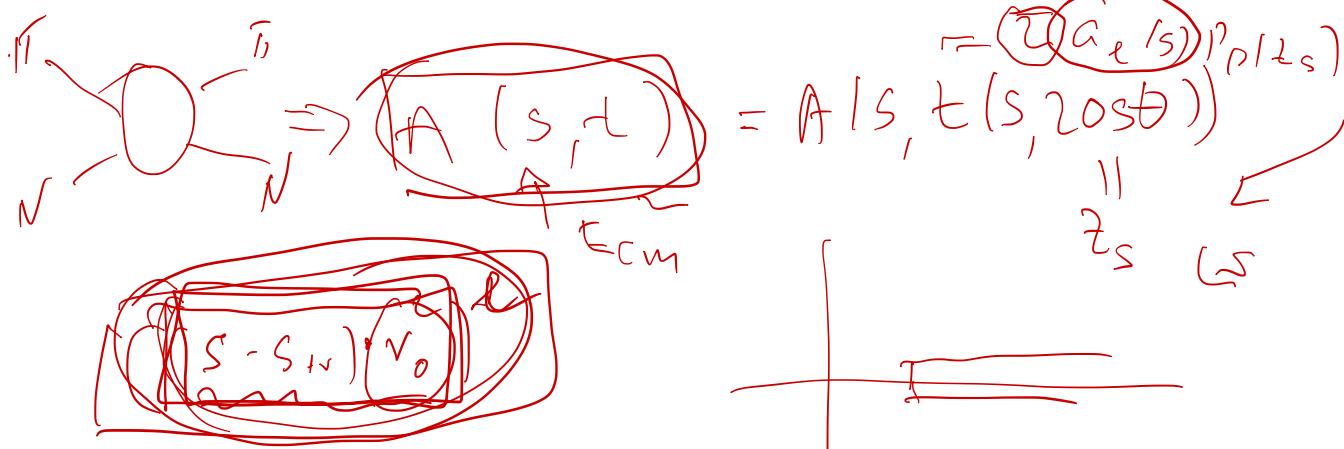
$$l=1 : f(1, \epsilon) = \frac{\beta(\epsilon)}{1 - \alpha(\epsilon)}$$

$$f_1 : t = m_{g_1}^2$$

$$\alpha(t = m_{g_1}^2) = 1$$

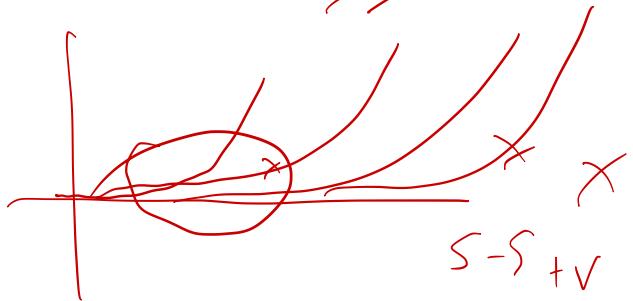
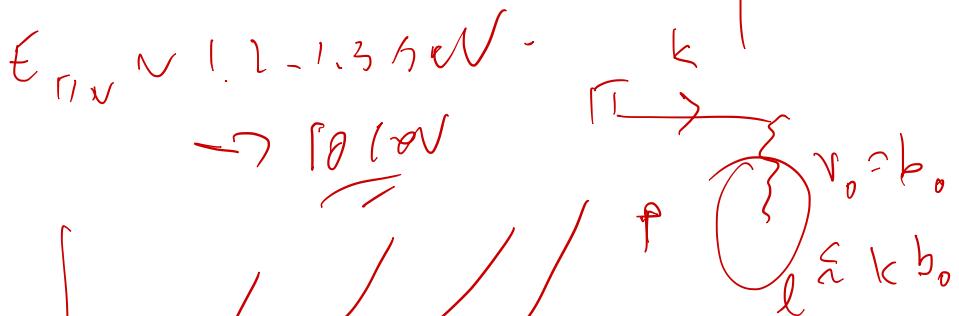
$$\alpha(t = m_{g_3}^2) = 3$$

$$f(3, \epsilon) = \frac{\beta(\epsilon)}{3 - \alpha(\epsilon)}$$

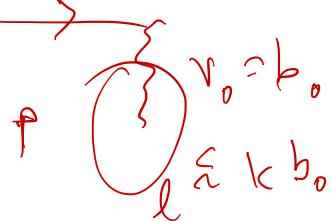


$$= A(s, t(s, 20s)) \stackrel{||}{z_s} \leftarrow$$

A red annotation above the equation shows a small circle with a checkmark and the text "A\_e (s) P(2s)". A red arrow points from this annotation to the term  $t(s, 20s)$ .



$S - S_{IN}$



$$10 \text{ fm} \times 0.5 \text{ fm}$$

$$\frac{5 \text{ fm} \times 10 \text{ fm}}{0.2 \text{ fm}} \approx 15$$

$$a_\ell(s) \sim \left[ (s - s_{\text{fr}}) v_2 \right]^\ell$$

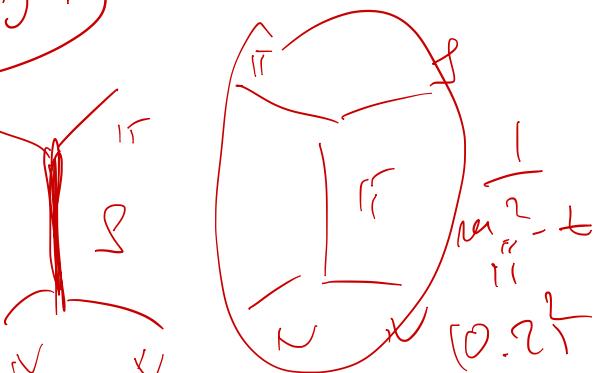
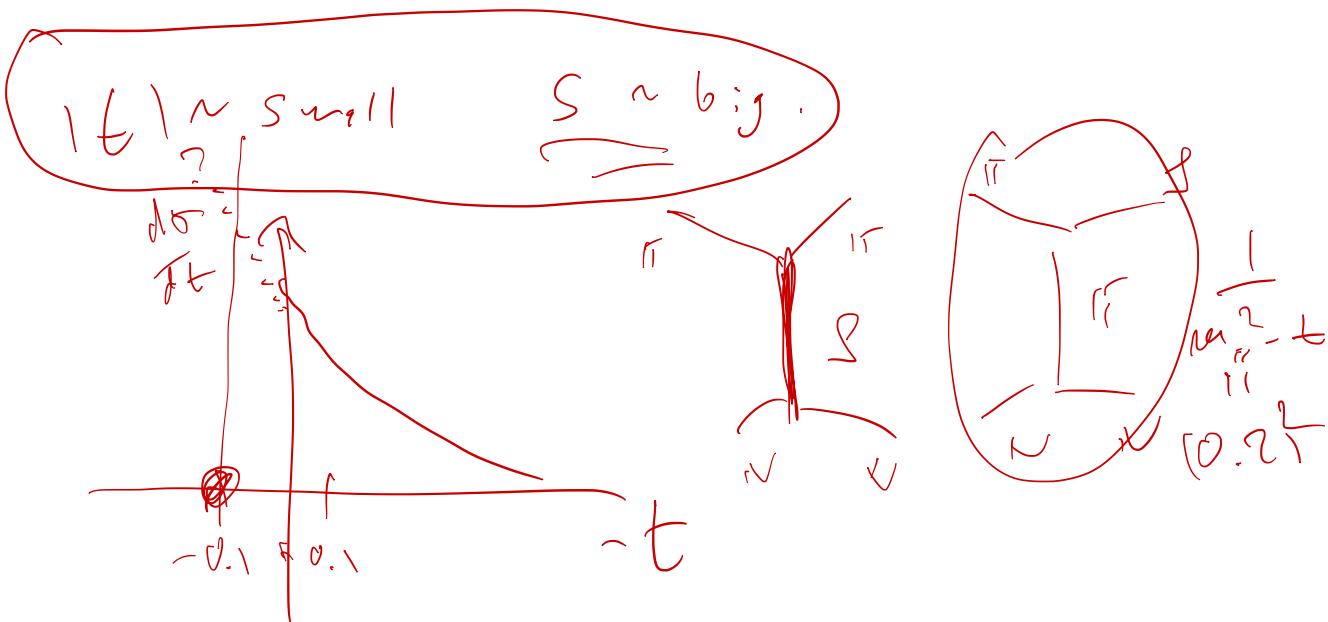
$$x < 1$$

(n)

$$A(s) = \sum_{\ell} \underbrace{(2\ell+1) a_\ell(s) P_\ell(\cos s)}_{\sim}$$

$$\hookrightarrow = \dots$$

$$\begin{aligned} S &\gtrsim (10 \alpha \nu)^2 \\ \text{small } |t| &< \\ 0.5 C |t| &< 1 \alpha \nu^2 \end{aligned}$$



$$A(t, z_t) = \sum_{l=0}^{\infty} (1_l + 1) a_l(t) P_l(z_t)$$

Diagram illustrating the decomposition of the function  $A(t, z_t)$  into a sum of terms involving coefficients  $a_l(t)$  and polynomials  $P_l(z_t)$ . The diagram shows three main components:

- A large oval containing the equation  $A(t, z_t) = \sum_{l=0}^{\infty} (1_l + 1) a_l(t) P_l(z_t)$ .
- An oval labeled  $|A(t)| < 1$  positioned above the first oval.
- An oval labeled  $|t| > 0$  positioned above the second oval.

Annotations to the right of the ovals include:

- $t < 0$
- $s \sim \text{big}$
- A question mark (?)
- A checkmark ( $\checkmark$ )

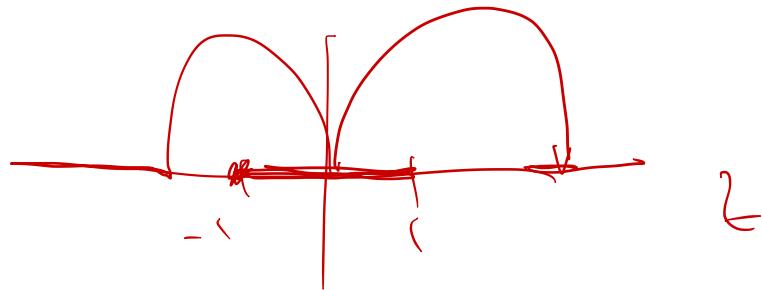
$$|z_x| < 1 \xrightarrow{s} |z_x| > 1$$

~~$\sum$~~   ~~$a_x(t)$~~   ~~$P_x(t_x)$~~  = dinge

$$P_x(t_n) = \dots \cdot t \cdot \frac{t^{n-1}}{t}$$

$$f(z) \equiv 1 + z + z^2 + z^3 + \dots = \sum_{n=0}^{\infty} z^n$$

$$|z| < 1$$



$$f(z) = \frac{1}{z-1}$$

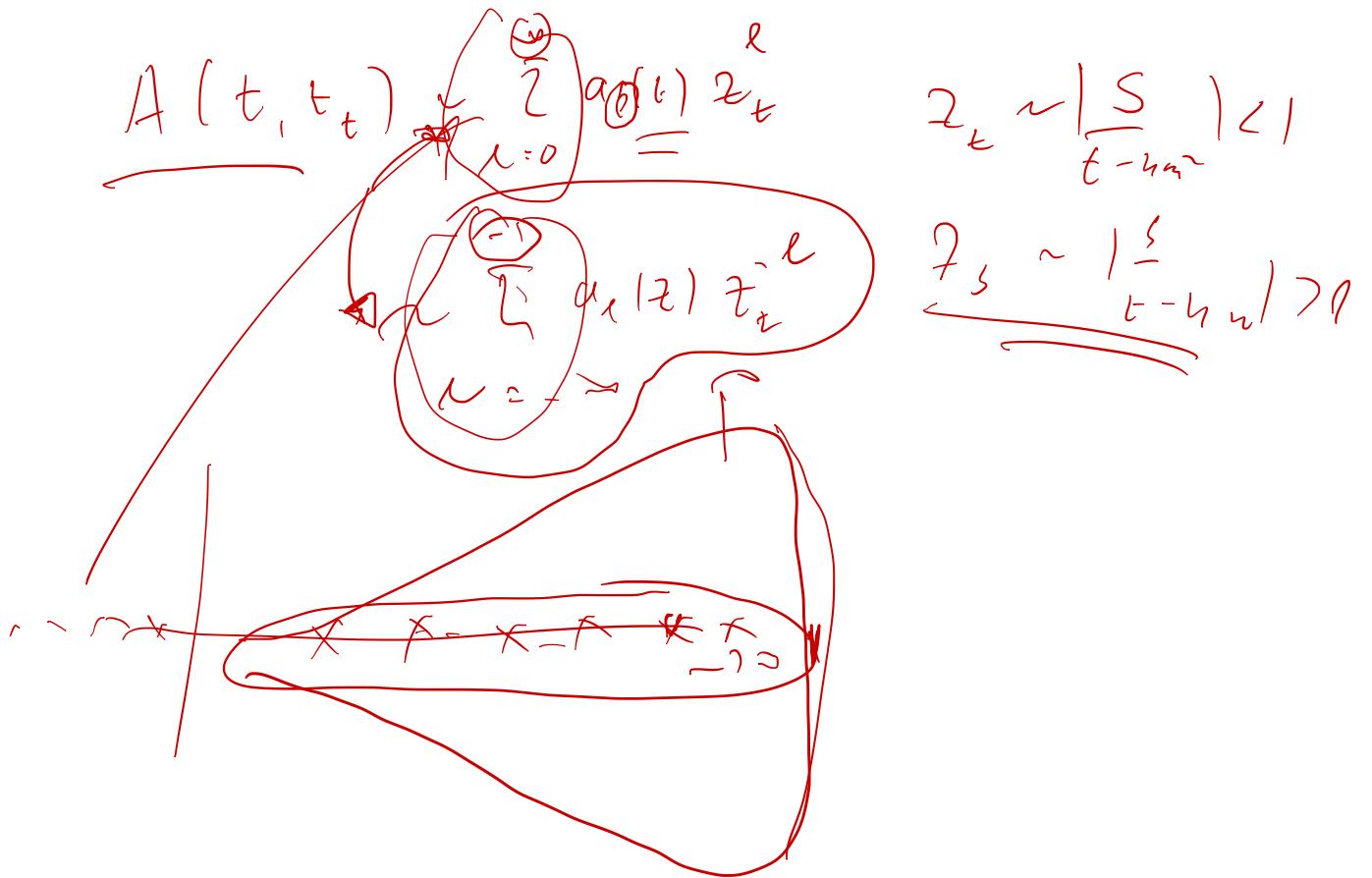
$$z = 1$$

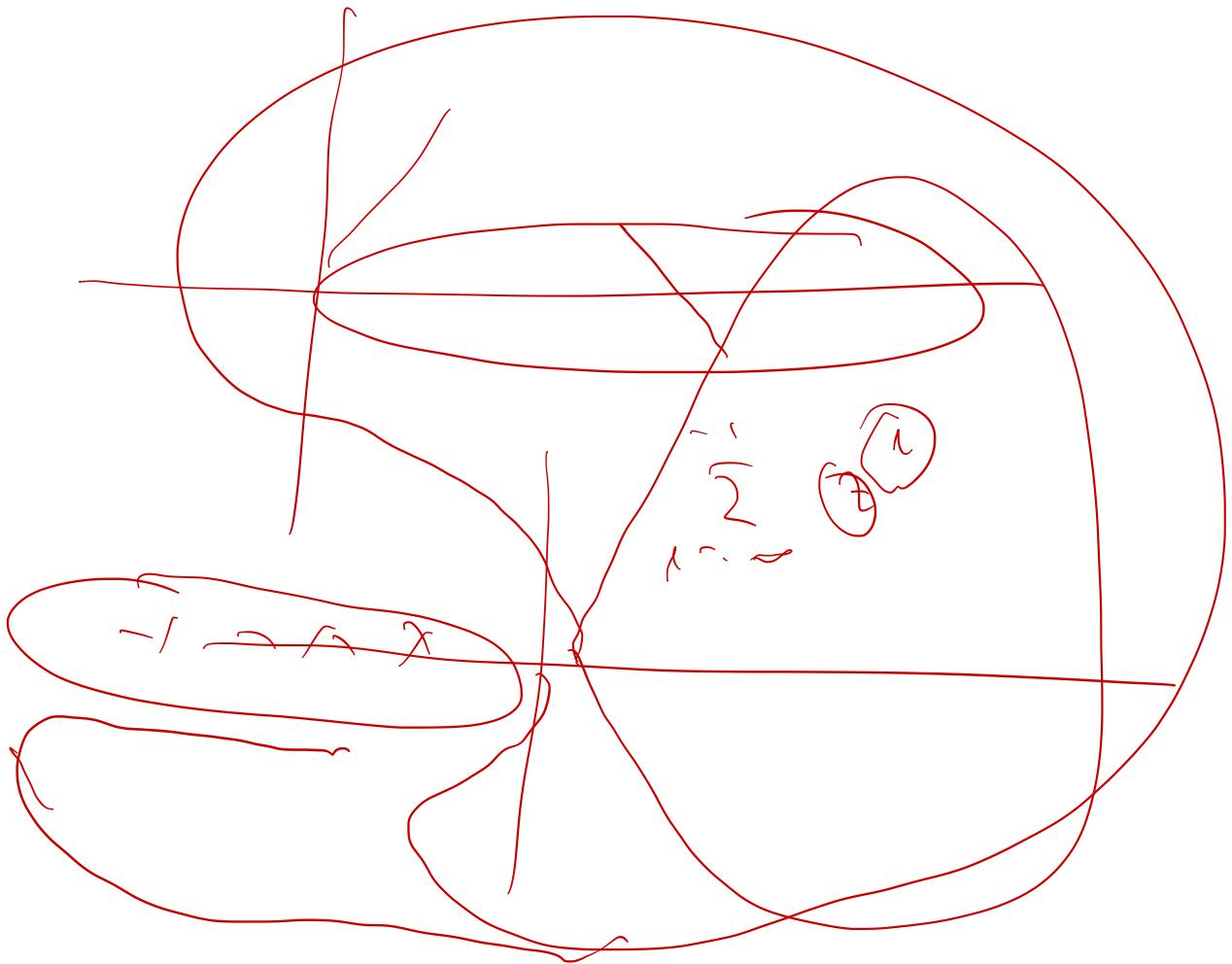
$$z = 2 f(z) = 1$$

$$\begin{aligned} &= -\frac{1}{z(1-\frac{1}{z})} = -\frac{1}{z} \left( 1 + \frac{1}{z} + \frac{1}{z^2} + \dots \right) \\ &= -\frac{1}{z} (z^0) \end{aligned}$$

$$\begin{aligned} &\sum_{k=0}^{\infty} z^k \\ &(k=0) = \end{aligned}$$

$$(k=-\infty) =$$





$$\frac{1}{s} = \zeta$$



$\tilde{z}_t$

$\tilde{z}_{t+1}$

$\tilde{z}_{t+1}$

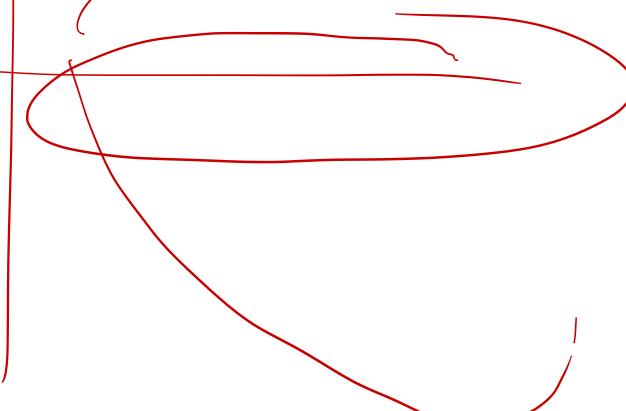
$\tilde{z} \Rightarrow$

$\tilde{z} \alpha_{\epsilon}(t) P_{\epsilon}(z_t)$

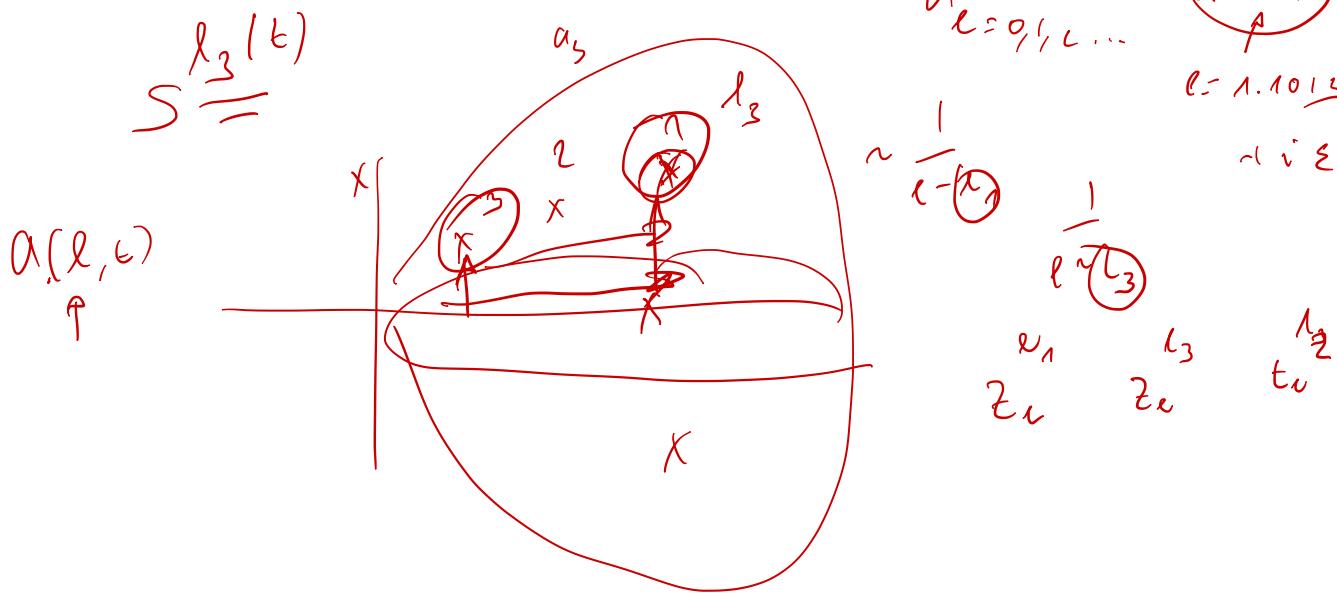
$\Rightarrow \underset{\epsilon}{\text{f}} \alpha(\epsilon, t) P_{\epsilon}(z)$

$x$

$-1$



$$\begin{aligned}
 & \text{Diagram showing } S_2 \text{ with boundary } \Gamma \text{ and interior } \mathbb{D}_2. \\
 & \text{Equation: } \underline{a_1(\epsilon)} = \frac{\infty}{\epsilon} \underset{\epsilon \rightarrow 0}{\alpha_1 P_1(\mathbb{D}_2)} = \alpha_1(\epsilon) P_1(z_2) \\
 & \text{Equation: } \underline{a_2} = \alpha_{\ell=0, 1, \dots} \rightarrow \underline{a(\ell)} \\
 & \text{Equation: } \ell = 1, 10, \dots \\
 & \text{Equation: } \alpha \sim \epsilon^{-\frac{1}{2}}
 \end{aligned}$$



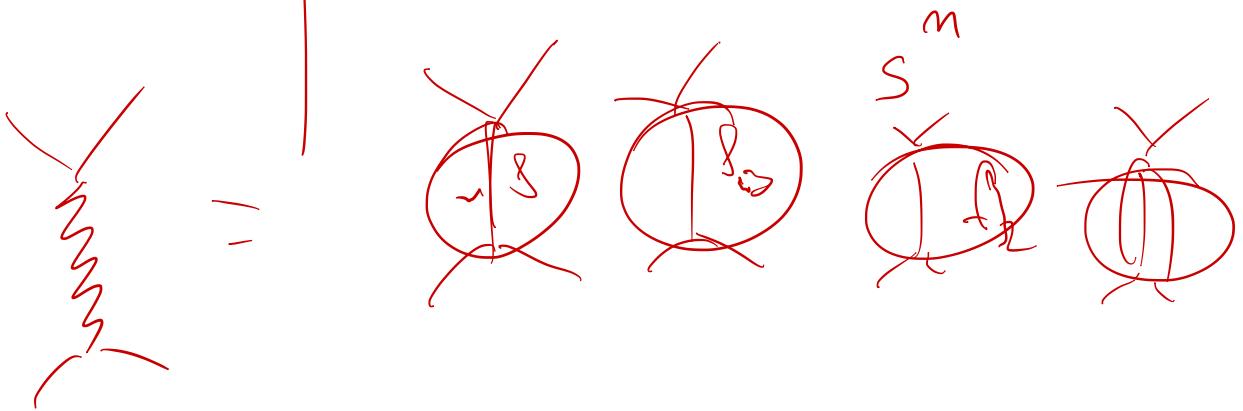
$$\int dL \frac{t^{(l)}}{\sim} \frac{a(l)}{\sin r_l(l)} \sim \frac{1}{l + l_{1,2,3}} t^{l_1} \sim S^{\frac{l_1}{l_1}}$$

$$t^{l_3} \sim S^{\frac{l_3}{l_3}}$$

$$\int d\ell \frac{1}{\sin(\ell)} a(\ell, \dots)$$

$$a(\ell) \sim \frac{1}{\ell - l_3(t)}$$

$$l_3 = 4$$
$$\sim S \sim$$
$$t < 0$$



$$A(s, \lambda) = \sum_{l=0}^{\infty} (1l+1)(a_l(s)P_l(\cos \theta))$$

$$-\frac{1}{r_0} \sin \theta$$

$$\left[ l \approx k l_0 \right]$$

$$\begin{aligned} & \text{Lmax} \approx l_0 \\ & \sum (1l+1) 1 \times 1 \approx \frac{l_{\max}}{2} \quad k \approx l_0 \\ & l = 0 \quad S^{\rightarrow} \end{aligned}$$

$$e^{-s}$$

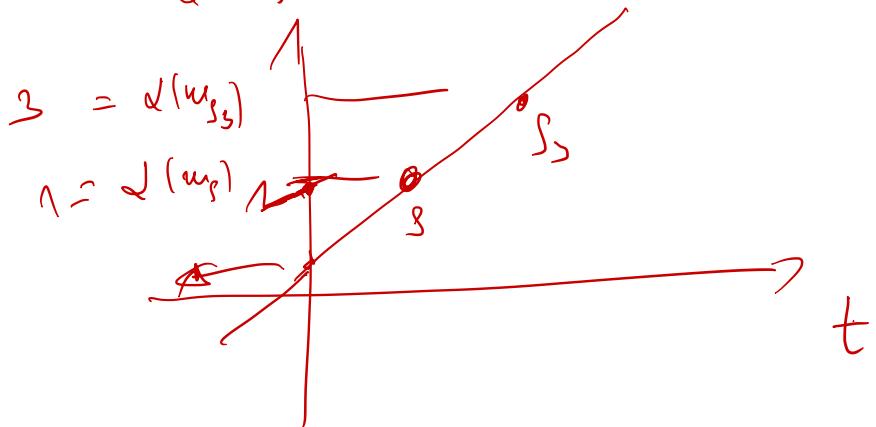
$$F_v \Rightarrow S^A = \zeta \omega S^{\alpha(t)}$$

in  $S$  physical  $s \sim 10 \text{ GeV}^2$

$$\alpha(t)$$

$$t < 0$$

$$\sqrt{|t|} < 1$$



$$A(s, t) = \sum_{l=0}^{\infty} (2l+1) f_l(t) P_l(z_t) \quad z_t = 1 + \frac{2s}{t - 4m^2}$$

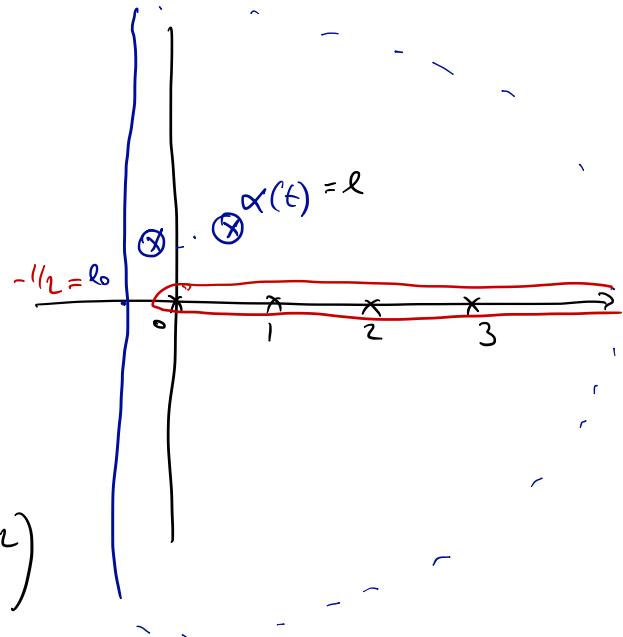
$$= \frac{1}{2i} \oint_C (2l+1) \underbrace{f(l, t)}_{P_l(-z_t)} \frac{dl}{\sin \pi l}$$

$$f(l, t) = \frac{\beta(t)}{l - \alpha(t)}$$

$$= -\pi(2\alpha+1) \beta(t) \frac{P_{\alpha(t)}(-z_t)}{\sin \pi \alpha(t)}$$

$$+ \frac{1}{2i} \int_{l_0-i\infty}^{l_0+i\infty} (2l+1) f(l, t) \frac{P_l(-z_t)}{\sin \pi l} dl$$

$$= \sum_{\alpha > -l_0} -\pi(2\alpha+1) \beta \frac{(s/s_0)^{\alpha}}{\sin \pi \alpha} + O(s^{-l_0})$$



$$P_l(z) \xrightarrow[z \rightarrow \infty]{} z^l \quad \text{if } \operatorname{Re} l > -1/2$$

$$\qquad\qquad\qquad l = -1/2$$

$$\xrightarrow{} z^{-l-1} \quad \text{if } \operatorname{Re} l \leq -1/2$$

$$l(l+1)$$

$$\underline{l} \rightarrow \underline{-l-1}$$

$$f(l, t) = \frac{1}{2} \int_{-1}^1 \underbrace{A(t, z_t)}_{\begin{array}{l} | \operatorname{Im} l | \Theta \\ \cos \theta = z \end{array}} P_l(z_t) dz_t$$

$$\int_{-1}^1 P_\alpha(x) P_\beta(x) dx = \frac{2\pi}{\alpha - \beta} \frac{\sin \alpha}{\beta + \alpha + 1}$$

$$P_l(z) \xrightarrow[l \rightarrow \infty]{} e^{i \operatorname{Im} l \Theta} \quad \alpha \in \mathbb{C}; l \in \mathbb{N}$$

$$-1 \leq z \leq 1$$

$$\begin{aligned} f(l, t) &= \frac{1}{2} \int_{-1}^1 \sum_{l' \in \mathbb{N}}^{\infty} (2l'+1) f_{l'}(t) P_{l'}(z_t) P_l(z_t) dz \\ &\stackrel{l \in \mathbb{C}}{=} \frac{(-)^l}{\pi} \sin \pi l \sum_{l' \in \mathbb{N}} \frac{2l'+1}{l+l'+1} \frac{1}{l-l'} f_{l'}(t) \\ &= f_l(t) S_{ll} \quad l \in \mathbb{N} \end{aligned}$$

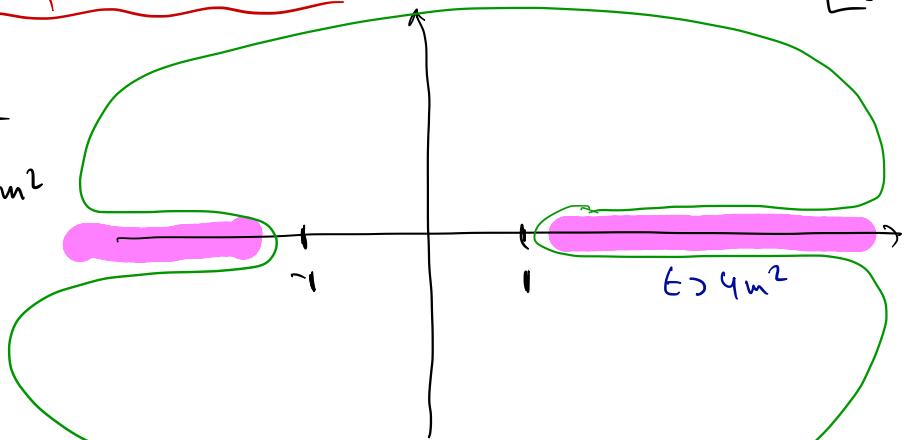
# Froissart-Gribov Representation

Unitarity cuts

$$S > 4m^2 \rightarrow z_t > z_0 = 1 + \frac{8m^2}{t - 4m^2}$$

$$z_t = 1 + \frac{2S}{t - 4m^2} \\ = -1 - \frac{2u}{t - 4m^2}$$

$$u > 4m^2 \rightarrow z_t < -z_0$$



$$A(z_{+1}, t) = \frac{1}{2\pi i} \int_{z_t}^{z_0} \frac{A(z'_t + i\varepsilon, t) - A(z'_t - i\varepsilon, t)}{z'_t - z_t} dz'_t \xrightarrow{\text{Residue Theorem}} \frac{1}{\pi} \int \frac{D_S A(z'_t, t)}{z'_t - z_t} dz'_t$$

$$+ \frac{1}{2\pi i} \int_{-\infty}^{z_0} \frac{A(z'_t + i\varepsilon, t) - A(z'_t - i\varepsilon, t)}{z'_t - z_t} dz'_t \xrightarrow{\text{Residue Theorem}} \frac{1}{\pi} \int \frac{D_u A(z'_t, t)}{z'_t - z} dz'_t$$

$$D_S A = \frac{1}{2i} [A(s+i\varepsilon, t) - A(s-i\varepsilon, t)]$$

$$f_l(t) = \frac{1}{2} \int \underline{A(z, t)} P_l(z) dz$$

$$f(l, t) = \frac{1}{2\pi} \int_{z_0}^{\infty} \int_{-1}^1 D_s A(z', t) \frac{P_l(z)}{z' - z} dz' + \frac{1}{2\pi} \int_{-\infty}^{-z_0} \int_{-1}^1 D_u A(z', t) \frac{P_l(z')}{z' - z} dz'$$

$$Q_l(-z) = (-)^{l+1} Q_l(z)$$

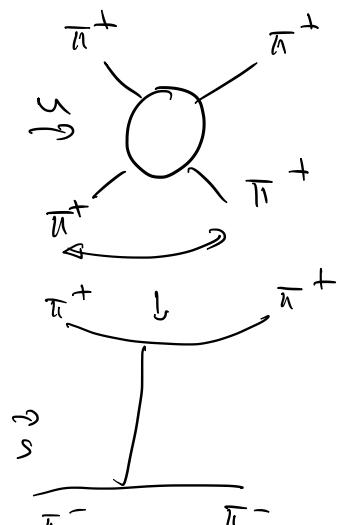
$$Q_l(z) = \frac{1}{2} \int_{-1}^1 \frac{P_l(z_+)}{z - z_+} dz_+$$

$$f_{\text{Res}}^{(l,t)} = \frac{1}{\pi} \int_{z_0}^{\infty} \left[ D_s \overset{\text{Res}}{A}(z', t) \pm D_u \overset{\text{Res}}{A}(z', t) \right] Q_l(z') dz'$$

$f_{\text{Res}}^{(l,t)}$  matches  $f_l(t)$   $l$  even  
 $f_{\text{Res}}^{(l,t)}$  u u odd

$(-)^l \equiv \epsilon$  signature

$$\mathbb{D}_s A = \text{Im } A = \text{Im } A_{\text{bek}} + \text{Im } A_{\text{res}}$$

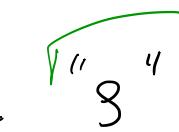


:  $\text{Im } A_{\text{res}} \sim 0 \Rightarrow f(\ell, \epsilon) = \bar{f}(\bar{\ell}, \epsilon)$

$\beta_{\pi\pi}^f(\epsilon) = \beta_{\pi\pi}^g(+)$   
 $\alpha_f(+) = \alpha_g(+)$

$$\Leftrightarrow \frac{\beta_{\pi\pi}^f(\epsilon)}{\ell - \alpha_f(\epsilon)} = \frac{\beta_{\pi\pi}^g(\epsilon)}{\ell - \alpha_g(\epsilon)}$$

Exchange Degeneracy END

:   $I=1$      $C = -I^G J^{PC} = + (1, 3, 5, \dots) -$

  $I=0$      $C = +$      $I^G J^{PC} = + (0, 2, 4, \dots) ++$

$\pi^+ p$ ,  $\Delta^+$  not exotic

~~$\pi^- p$~~

( $K^+ p$ )

~~$K^- p$~~

( $K^+ n$ )

~~$K^- n$~~

( $p p$ )

~~$\bar{p} p$~~

( $p n$ )

~~$\bar{p} n$~~

not exotic

$\pi^+ u \bar{d}$

$\pi^- \bar{u} d$  ↗

$K^+ u \bar{s}$

$K^- \bar{u} s$

$\bar{p} \bar{u} \bar{u} \bar{d}$  ↙

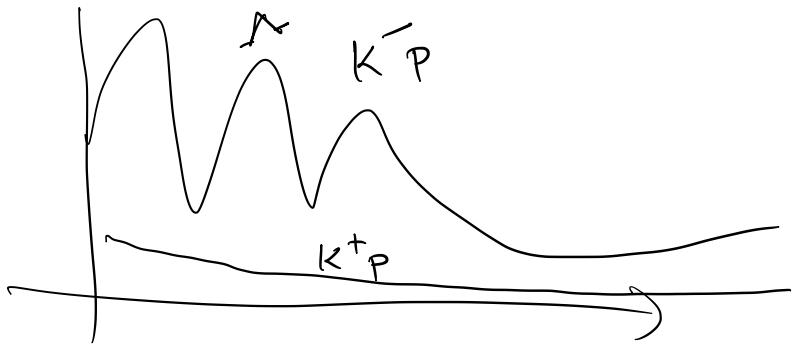
$n u \bar{d} \bar{d}$

$K^+ p ; K^+ n$

$p p \quad p n$

$K^+ p ; K^+ n$   
 $p p \quad p n$

$$\sigma_{\text{tot}} = \frac{1}{S} \text{Im } A(s, t=0)$$



$$\text{Im } A = \text{Im } A^{\text{Res}} + \text{Im } A^{\text{bek}}$$

$\downarrow$                        $\downarrow$   
 $\text{Im } A^{\text{Regge}}$      $\text{Im } A^{\text{Don}}$

I<sub>s</sub>, f