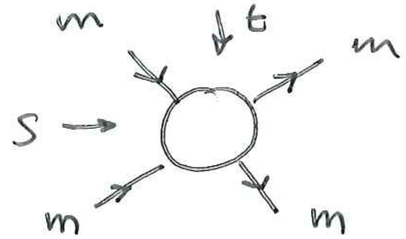
Week I Recap

Elastic scattering
of spinless particle of mass m



Partial wave expansion:

$$\cos\theta_s = 1 + \frac{2t}{s-4m^2}$$

$$A(s,t) = \sum_{\ell=0}^{\infty} (2\ell+1)\, a_\ell(s)\, P_\ell(\cos\theta_s)$$

Unitarity relation:

$$\operatorname{Im} a_\ell^{-1}(s) = -\rho(s) \qquad \rho(s) = \frac{1}{2}\,\frac{1}{8\pi}\sqrt{1-4m^2/s}\;\theta(s-4m^2)$$

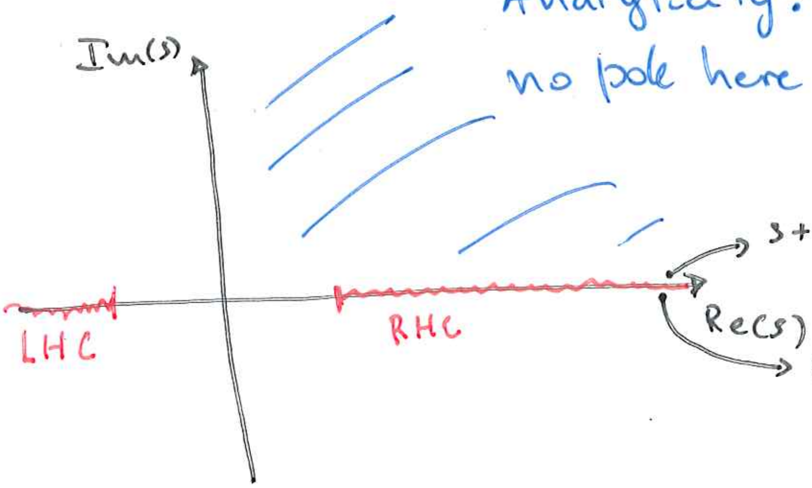
Solving unitarity:

$$a_\ell(s \pm i\epsilon) = \frac{1}{\underbrace{K(s)}_{\text{real}} \mp i\rho(s)}$$

$$K(s) = \frac{m^2 - s}{m\Gamma}$$

$$\uparrow$$

Analyticity:
no pole here $\rightarrow$ $K(s) - i\rho(s) \neq 0$



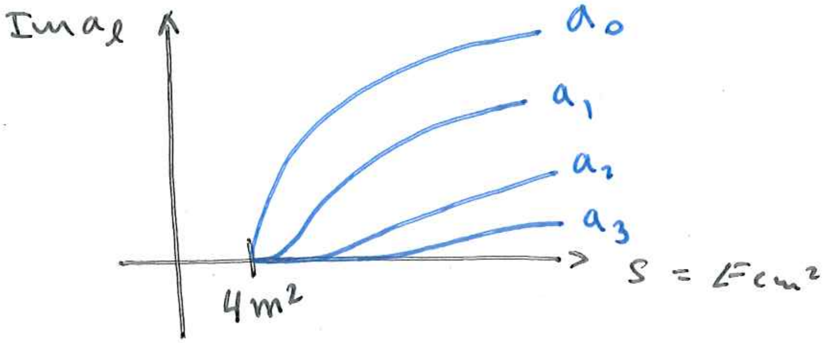
$s+i\epsilon \quad a_\ell^{-1}(s) = K(s) - i\rho(s)$

$s-i\epsilon \quad a_\ell^{-1}(s) = K(s) + i\rho(s)$

$K(s) - i\rho(s) = 0$ on sheet II $\Rightarrow$ poles on sheet II

Barrier factors: $\rho(s) \rightarrow \rho_\ell(s) = \rho(s) \cdot k^{2\ell} \sim k^{2\ell+1}$

k = break-up momentum.



relevant partial waves increases with energy.

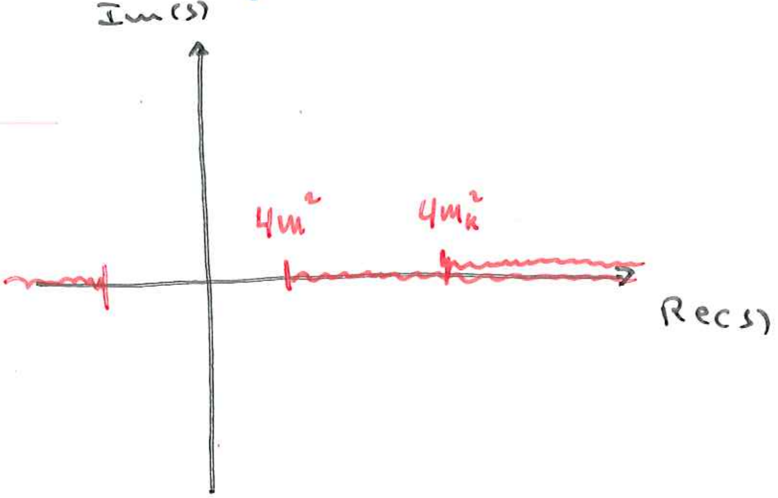
to find pole on sheet II use Chew-Mandelstam

$$\rho(s) = \frac{1}{2}\frac{1}{8\pi}\sqrt{1-4m^2/s}\;\underline{\Theta(s-4m^2)}$$

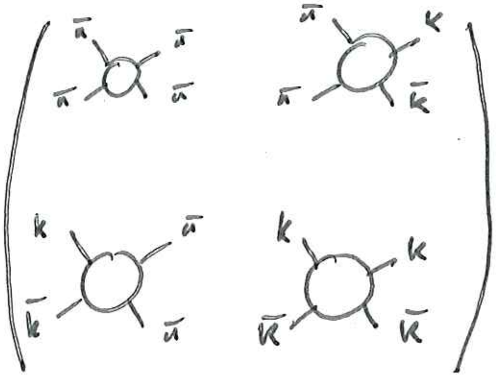
↳ on the real axis

$$\rho_{CM}(s) = \frac{1}{\pi}\int_{4m^2}^{\infty}\frac{\rho(s')\,ds'}{s'-s}$$

Above $K\bar{K}$ threshold, another unitarity cut.



$A_{ij}(s) =$



Unitarity reads $\quad \mathrm{Im}(a_\ell)_{ij} = -\rho_i(s)\,\delta_{ij}$

$$\rho_i(s) = \frac{1}{2}\frac{1}{8\pi}\sqrt{1-4m_i^2/s}\;\Theta(s-4m_i^2)$$

Equivallently: in matrix form $\quad \mathrm{Im}\,\underline{a}_\ell = \underline{a}_\ell^* \cdot \underline{\rho} \cdot \underline{a}_\ell$

Solving unitarity: $\quad \underline{a}_\ell(s) = \dfrac{1}{\underline{K}(s) - i\underline{\rho}(s)}$

$\hookrightarrow$ real 2x2 matrix

At high energy, use t-channel unitarity:

$$A(s,t) = \sum_{\ell=0}^{n(t)} (2\ell+1)\, f_\ell(t)\, P_\ell(\cos\theta_t) + \sum_{\ell=n_t+1} \cdots \qquad \cos\theta_t = 1 + \frac{2s}{t-4m^2} = \frac{s-u}{t-4m^2}$$

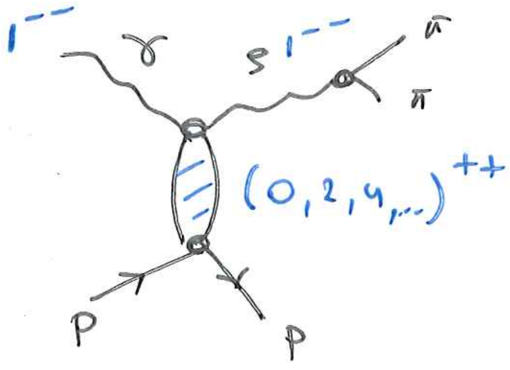
$$A(s,t) \sim \beta(t)\cdot(\cos\theta_t)^{n(t)} \sim \beta(t)\, s^{n(t)}$$

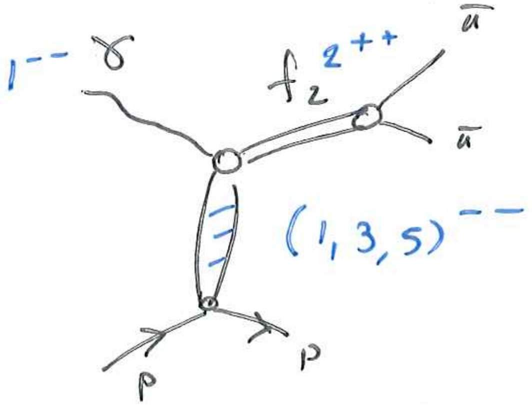
At JLab $E_\gamma \sim 2 - 12$ GeV

$\sim \beta(t)\, s^{\alpha(t)}$

$\beta(t)$, $\alpha(t)$ depend on the quantum numbers of R

$\gamma p \to (\pi\pi)\, p$ X = ρ or f_2 mesons





$A \sim s^{1}$

$\sigma \sim$ cst

$A \sim s^{0.5}$ $(s = E_{cm}^2)$

$\sigma \sim 1/E_{cm}$

Gribov: chapter 7

Start with non-relativistic quantum mechanics: (NRQM)

Schrödinger equation:

$$\left[-\frac{\hbar^2}{2m}\vec{\nabla}^2 + U(r)\right]\Psi(\vec{x},t) = i\hbar\frac{\partial}{\partial t}\Psi(\vec{x},t)$$

$$\Psi(\vec{x},t) = \sum_{n=0}^{\infty}\alpha_n\psi_n(\vec{x})\, e^{-iE_n/\hbar\, t}$$

$$\psi_n(\vec{x}) = \varphi_\ell(r)\, Y_{\ell m}(\Omega)$$

$$\left[-\frac{\hbar^2}{2m}\nabla_r^2 + \frac{\ell(\ell+1)}{r^2} + U(r)\right]\varphi_\ell(r) = \overset{\nearrow E(\ell)}{E_n^\ell}\,\varphi_\ell(r) \quad (*)$$

a) Symmetry $\ell \to -\ell-1 \Rightarrow \ell(\ell+1) \to \ell(\ell+1)$

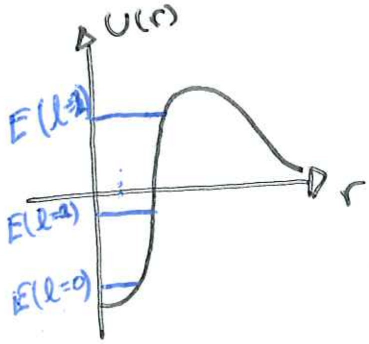
Pairs of solutions: $\varphi_\ell^{(1)}(r) \sim r^\ell$ as $r \to 0$

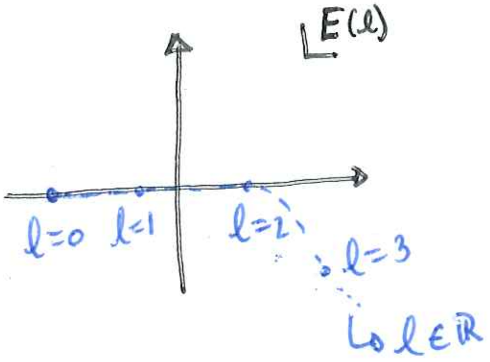
$\varphi_\ell^{(2)}(r) \sim r^{-\ell-1}$

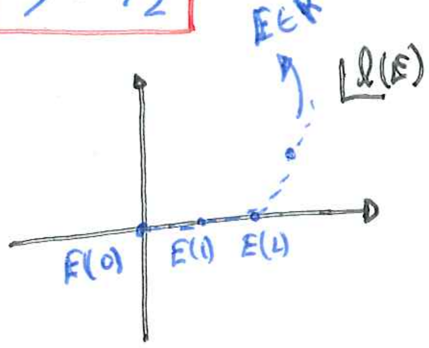
b) ℓ enters analytically in $(*)$.

ℓ can be any $\mathbb{C}$. $(*)$ defines $\varphi_\ell(r)$ for ℓ complex.

but need $\mathrm{Re}\,\ell > \mathrm{Re}(-\ell-1) \to \boxed{\mathrm{Re}(\ell) > -1/2}$



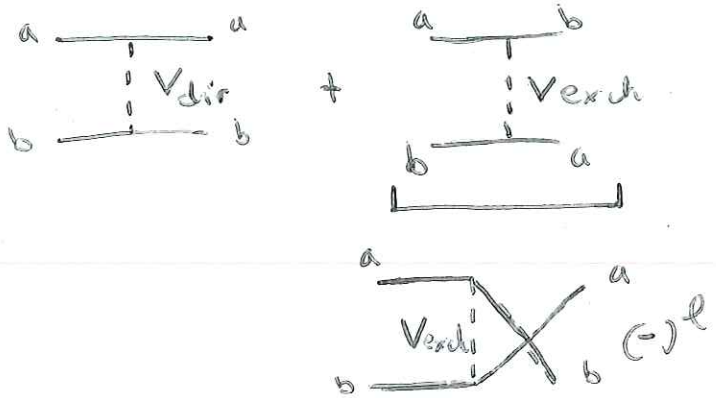




c) n dependence <u>non</u>-analytic
↳ radial quantum number

→ no Regge trajectory with "n".

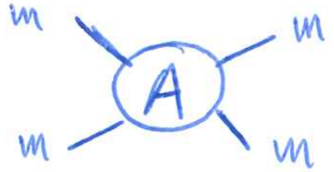
d) U is the sum of a direct and exchange potentials



$$\Rightarrow U = \begin{cases} V_{dir} + V_{exch} & \ell \text{ even} \\ V_{dir} - V_{exch} & \ell \text{ odd} \end{cases}$$

→ need to separate ℓ even and odd

Relativistic theory



$A(s,t,u)$

$s+t+u=4m^2$

Partial wave expansion in the t-channel:

$$A(t,z_t) = \sum_{\ell=0}^{\infty} (2\ell+1)\, f_\ell(t)\, P_\ell(z_t)$$

$$\longrightarrow f(\ell,t) \quad \ell \in \mathbb{C}$$

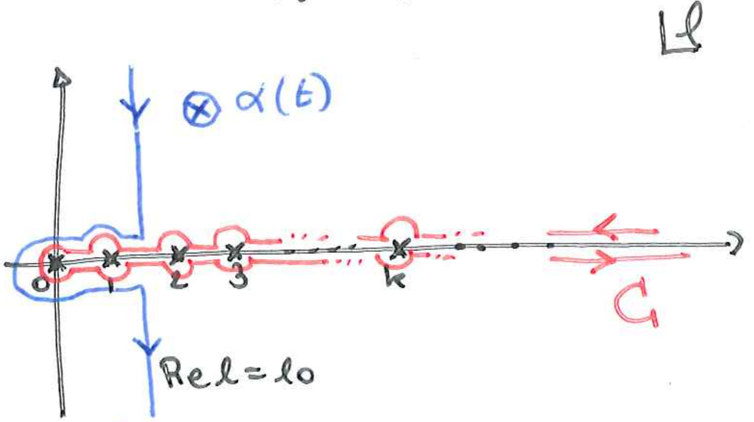
$$z_t = 1 + \frac{2s}{t-4m^2}$$

Sommerfeld-Watson representation:

$$A(t,z_t) = \frac{1}{2i} \oint_C (2\ell+1)\, f(\ell,t)\, P_\ell(-z_t)\, \frac{d\ell}{\sin \pi \ell}$$

$$\lim_{\ell \to n} \frac{(\ell-n)}{\sin \pi \ell} = \frac{(-1)^n}{\pi}$$

$n \in \mathbb{N}$



Consider $\pi\pi \to \pi\pi$ with ρ in t-channel:

$$\left.\begin{aligned} f_1(t) &= \frac{g_1^2}{m_{\rho_1}^2 - t} \\ f_3(t) &= \frac{g_3^2}{m_{\rho_3}^2 - t} \end{aligned}\right\} f(\ell,t) = \frac{\beta(t)}{\ell-\alpha(t)}$$

$\beta(t), \alpha(t)$ known for $t = m_{\rho_k}^2$ k odd

Deform contour and get:

$$A(t,z_t) = -\pi\,(2\alpha(t)+1)\,\beta(t)\, \frac{\overset{\to s^{\alpha(t)}}{P_{\alpha(t)}(-z_t)}}{\sin \pi\alpha(t)} + \frac{1}{2i} \int_{\ell_0 - i\infty}^{\ell_0 + i\infty} (2\ell+1)\, f(\ell,t)\, P_\ell(-z_t)\, \frac{d\ell}{\sin \pi \ell}$$

7.4

The background integral goes at <u>best</u> as

$$\int_{l_0-i\infty}^{l_0+i\infty} \frac{(2l+1)}{\sin\pi l} f(l,t)\, P_l(-z_t)\, dl \xrightarrow[\substack{z_t\to\infty \\ \text{or } s\to\infty}]{} s^{-1/2} \quad \text{if } l_0 = -1/2$$

because

$$P_l(z) \xrightarrow[z\to\infty]{} z^l \quad \text{if } \mathrm{Re}\, l \geq -1/2$$

$$\xrightarrow[z\to\infty]{} z^{-l-1} \quad \text{if } \mathrm{Re}\, l \leq -1/2$$

In order to neglect the circle at infinity, we need to know the large l behavior of $f(l,t)$

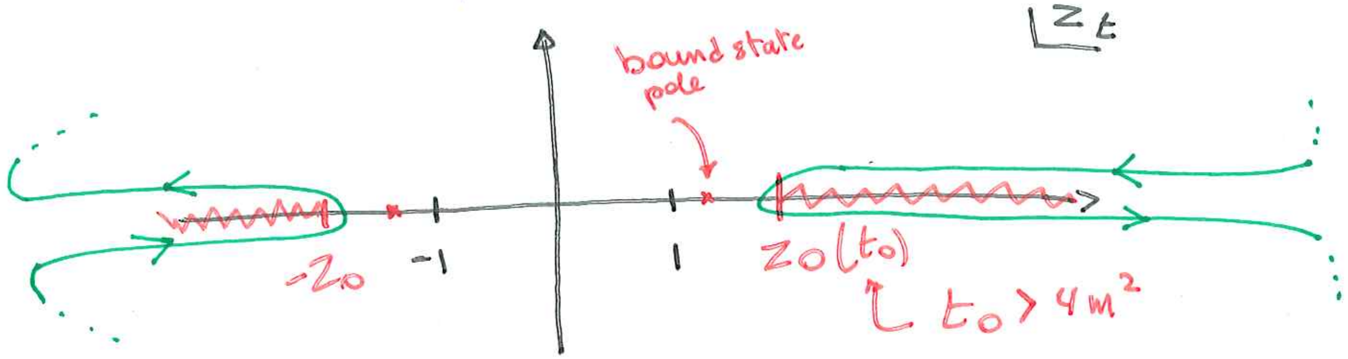
If we define $f(l,t)$ as

$$f(l,t) = \frac{1}{2}\int_{-1}^{1} A(z_t,t)\, P_l(z_t)\, dz_t$$

P_l grow too fast as $|l| \to \infty$

$$P_l(z) \xrightarrow[l\to\infty]{} e^{|\mathrm{Im}\, l|\theta} \quad \text{for } -1<z<1 \quad z=\cos\theta$$

Analatic structure of $A(t,z_t)$ in z_t plane; $t=t_0$



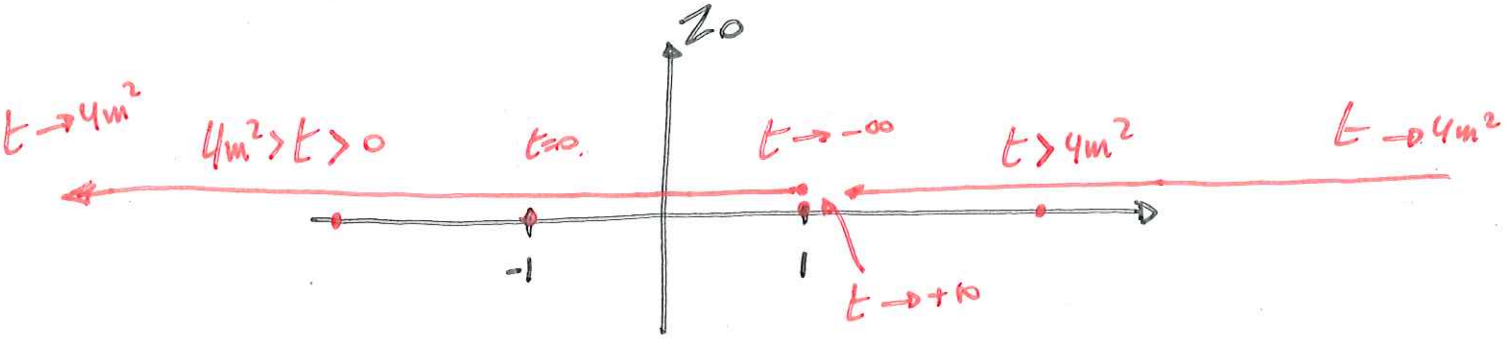
cuts starting at

$s > 4m^2 \longrightarrow z_t > z_0 = 1 + \frac{8m^2}{t_0 - 4m^2}$

$u > 4m^2 \longrightarrow z_t < -z_0$

$$z_t = 1 + \frac{2s}{t-4m^2} = -1 - \frac{2u}{t-4m^2}$$

Movement of $z_0(t)$:



Froissart-Gribov representation: (Poles & substractions omitted)

$$A(z_t,t) = \frac{1}{2\pi i}\int_{z_0(t)}^{\infty} \frac{A(z'+i\epsilon,t) - A(z'-i\epsilon,t)}{z'-z_t}\,dz' + \frac{1}{2\pi i}\int_{-\infty}^{-z_0(t)} \frac{A(z'-i\epsilon,t) - A(z'+i\epsilon,t)}{z'-z_t}\,dz'$$

$$= \frac{1}{\pi}\int_{z_0(t)}^{\infty} \frac{D_s A(z',t)}{z'-z_t}\,dz' + \frac{1}{\pi}\int_{z_0}^{\infty} \frac{D_u A(-z'',t)}{z''+z_t}\,dz''$$

with the definition:

$$D_s A(z,t) = \frac{1}{2i}\Big[\underbrace{A(z+i\epsilon,t)}_{A(s+i\epsilon,t)} - A(z-i\epsilon,t)\Big]; \quad D_u A(-z,t) = \frac{1}{2i}\Big[\underbrace{A(-z+i\epsilon,t)}_{A(u+i\epsilon,t)} - A(-z-i\epsilon,t)\Big]$$

Plugging the inverse P.W. expansion:

$$f_\ell(t) = \frac{1}{2}\int_{-1}^{1} A(z_t, t)\, P_\ell(z_t)\, dz_t \qquad (1)$$

with the definition

$$Q_\ell(z) = \frac{1}{2}\int_{-1}^{1} \frac{P_\ell(z_t)}{z - z_t}\, dz_t$$

we obtain:

$$\boxed{f_\ell(t) = +\frac{1}{\pi}\int_{z_0(t)}^{\infty} D_s A(z',t)\, Q_\ell(z')\, dz' - \frac{1}{\pi}\int_{z_0}^{\infty} D_u A(-z',t)\, Q_\ell(-z')\, dz' \qquad (2)}$$

The domain of integration in (1) and (2) are different. Permutting the $\int$ makes sense if they converge.

$$Q_\ell(z) \xrightarrow[z\to\infty]{} z^{-\ell-1}$$

if $D_{s,u}\, A(\pm z, t) \xrightarrow[z\to\infty]{} z^N$, (2) is defined for $\ell > N$

but $N \lesssim 1$ by the Froissart bound.

the integration domain of (1) in s is

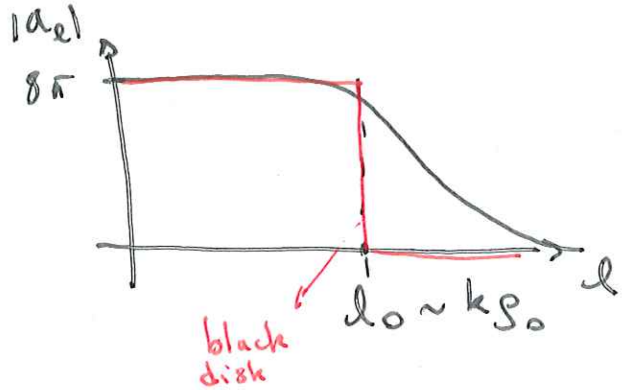
$t > 4m^2$ $\quad s \in [4m^2 - t, 0]$ $\quad$ physical region

$t < 0$ $\quad s \in [0, 4m^2 - t]$ $\quad$ unphysical region $[0, 4m^2]$

Reminder

Low energy : # PW << unitary → resonances

High energy : # PW >> $|a_\ell| \sim 8\pi f_\ell \sim 8\pi$



$$a_\ell = \frac{1}{2i\rho}\left[\eta_\ell e^{2i\delta_\ell} - 1\right]$$

$$\eta_\ell \sim 0 \to a_\ell \sim \frac{1}{2i\rho} \xrightarrow[s\gg]{} 8\pi \;;$$

$$\rho(s) = \frac{1}{8\pi}\frac{k(s)}{\sqrt{s}} \;;\; k(s) \to \frac{\sqrt{s}}{2}$$

S-channel P.W. expansion: (for black disk)

$$A(s, z_s) \sim \sum_{\ell=0}^{\ell_0} (2\ell+1)\, a_\ell(s)\, P_\ell(z_s)$$

Optical theorem gives: $\sigma_{tot}(s) \sim \frac{1}{s} 8\pi \ell_0^2 \sim 2\pi \rho_0^2$

t-channel P.W. expansion:

$$A(t, z_t) = \sum_{\ell}^{\infty} (2\ell+1)\, f_\ell(t)\, P_\ell(z_t) \qquad z_t = 1 + \frac{2s}{t-4m^2}$$

$|t| \ll \to$ # P.W. <<

$$A(t, z_t) \sim \sum_{\ell}^{n_0(t)} (2\ell+1)\, f_\ell(t)\, P_\ell(z_t) \underset{s\to\infty}{\sim} z_t^{\,n_0(t)} \sim s^{\overset{?}{\boxed{n_0(t)}}} \quad t<0$$

Qualitative argument → need to formalize.

Schrödinger Equation

$\boxed{7.11}$

$$\left[-\frac{\hbar^2}{2m}\nabla_r^2+\frac{l(l+1)}{r^2}+U(r)\right]\psi_l(r)=E_l\,\psi_l(r)$$

a) analytic on l: $E_l \to E(l)$

b) symmetry $l \to -l-1$: ok for $\mathrm{Re}(l) \geqslant -1/2$

c) QFT → QM: $U(r) = V_{dir} + (-)^l V_{exch}$

Sommerfeld-Watson transformation

$$A(t,z_t) = \sum_{l=0}^{\infty}(2l+1)f_l(t)P_l(z_t)$$

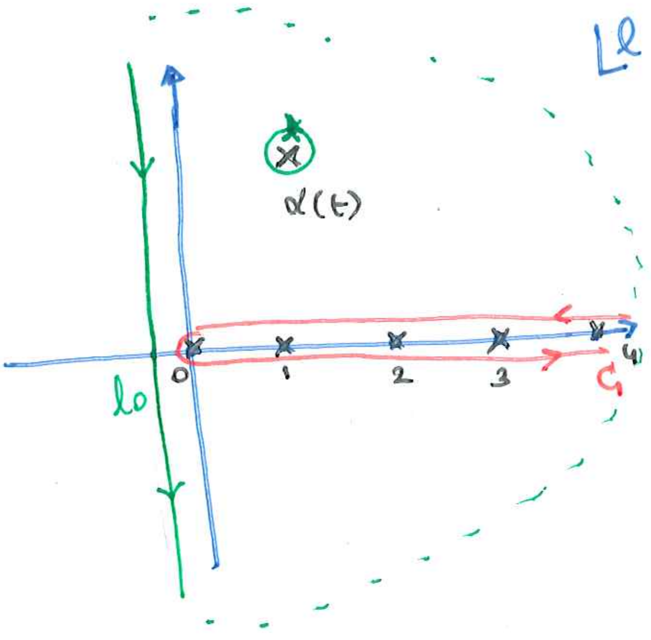
$$= \frac{1}{2i}\oint_{C}(2l+1)\,f(l,t)\,P_l(-z_t)\,\frac{dl}{\sin\pi l}$$

$$= -\pi\left[2\alpha(t)+1\right]\beta(t)\,\frac{P_{\alpha(t)}(-z_t)}{\sin\pi\alpha(t)}$$

$$+\frac{1}{2i}\int_{l_0-i\infty}^{l_0+i\infty}(2l+1)\,f(l,t)\,\frac{P_l(-z_t)}{\sin\pi l}\,dl$$

for $f(l,t) = \dfrac{\beta(t)}{l-\alpha(t)}$

and $l_0 \geqslant -1/2$



Need to know info about large $|\ell|$ behavior of $f(\ell, t)$ to drop the contour at ∞.

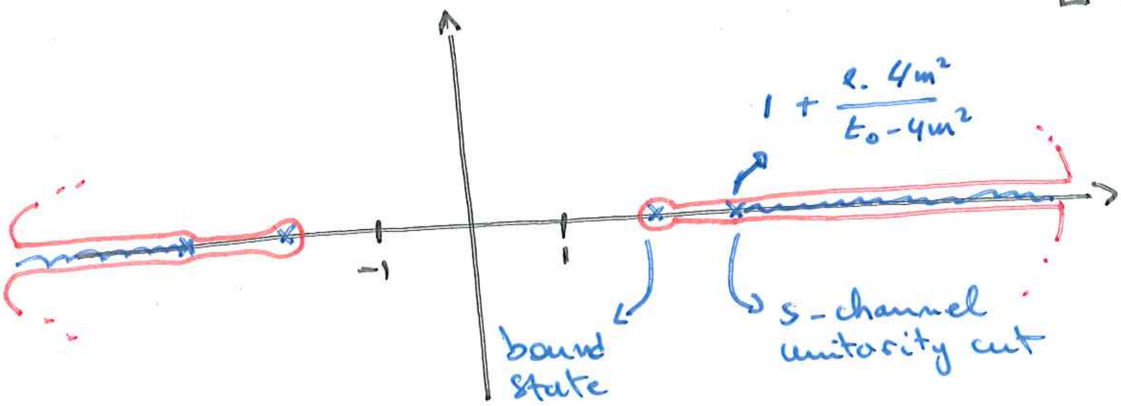
Natural definition:

$$f_\ell(t) = \frac{1}{2}\int_{-1}^{1} A(t, z_t)\, P_\ell(z_t) \qquad \text{(IPW)}$$

Better definition: Froissart - Gribov projection

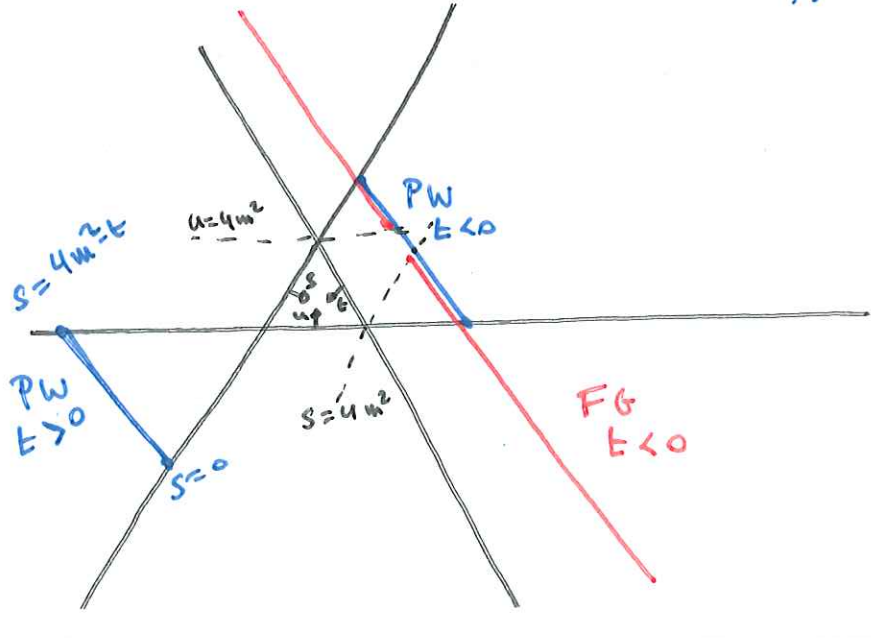
$$f(\ell, t) = \frac{1}{\pi}\int_{z_0(t)}^{\infty} D_s A(z', t)\, Q_\ell(z')\, dz' - \frac{1}{\pi}\int_{z_0(t)}^{\infty} D_u A(-z', t)\, Q_\ell(-z')\, dz' \qquad \text{(FG)}$$

Let $t = t_0$



Region of integration in (IPW) and (FG) are different.

Mandelstam plane:



Can we use (2) to define $f(\ell,t)$?

If (2) is use in Sommerfeld-Watson, I need the large ℓ behavior.

$$Q_\ell(z) \xrightarrow[\ell\to\infty]{} e^{-(\ell+1/2)\,\xi(z)} \qquad \xi(z) = \mathrm{Log}\left[z+\sqrt{z^2-1}\right]$$

So the first and second terms of (2) go like

$$f_\ell^R(t) \sim e^{-\ell\,\xi(z_0)}$$

$$f_\ell^L(t) \sim e^{-\ell\,\xi(z_0)}\,\underbrace{e^{-i\pi\ell}}_{(-)^\ell}$$

Need to define P.W with signature.

$$\boxed{\begin{aligned} f^{\pm}(\ell,t) &= \frac{1}{\pi}\int_{z_0(t)}^{\infty} D_s^{\pm} A(z',t)\, Q_\ell(z')\, dz' \\ D_s^{\pm} A(z,t) &= D_s A(z,t) \pm D_u A(-z,t) \end{aligned}}$$

where $f^+(\ell,t)$ for ℓ even matches f_ℓ

$f^-(\ell,t)$ is for ℓ odd matches f_ℓ

because $Q_\ell(-z) = (-)^{\ell+1} Q_\ell(z)$

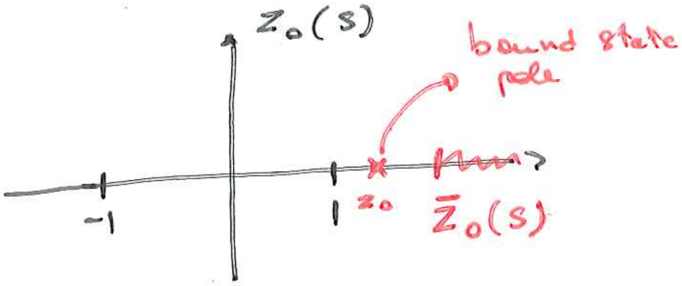
To derive the $\boxed{\text{Froissart bound}}$,
Apply F-G representation in s-channel

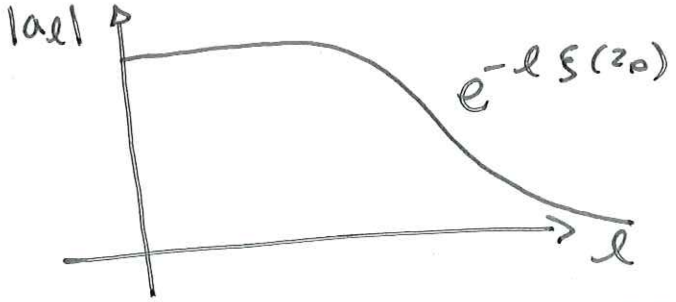
$$(*) \quad A(s,t) = \sum_{l=0}^{\infty} (2l+1)\, a_l(s)\, P_l(z_s) \qquad z_s = 1 + \frac{2t}{s-4m^2}$$

$$a_l(s) = \frac{1}{\pi} \int_{z_0(s)}^{\infty} D_t A(z', s)\, Q_l(z')\, dz' - \frac{1}{\pi} \int_{z_0(s)}^{\infty} D_u A(-z', s)\, Q_l(-z')\, dz'$$

In the s-channel region $s \geq 4m^2$:

$$a_l(s) \underset{l\to\infty}{\sim} e^{-l \underbrace{\xi(z_0)}_{\text{closest singularity}}}$$





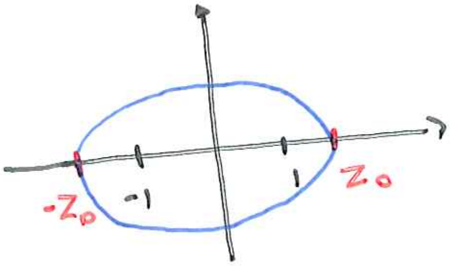
We also have

$$P_l(z_s) \sim e^{lx} \qquad z_s = \cosh x \qquad z_s > 1$$

So (*) converges if $\operatorname{arch}(z_s) < \xi(z_0)$ $\qquad \xi(x) = \operatorname{Log}\left[x + \sqrt{x^2-1}\right] = \operatorname{arch}(x)$

$$z_s < z_0$$

the s-channel P.W. converges in the Lehmann ellipse



For large s :

$$a_l(s) \underset{l\to\infty}{\sim} \underbrace{g(s)\, e^{-l\,\xi(z_0)}}_{e^{-l\xi(z_0)+\log[g(s)]}}$$

$$\xi(z_0) = \text{arch}(z_0)$$

$$z_0 = 1 + \frac{2t_0}{s-4m^2}$$

t_0 = closest singularity.

$$\xi(z_0) \underset{s\to\infty}{\sim} 2\sqrt{t_0/s}$$

Only the wave $l \lesssim l_M \neq \xi^{-1}(z_0)$ contribute to the amplitudes

$$l_M \sim \frac{1}{\xi(z_0)}\left[1 + \log[\underbrace{g(s)}]\right]$$

g(s) is polynomial

$$\sim \sqrt{s/t_0}\,\log(s)$$

Partial wave expansion becomes

$$A(s,t) \sim \sum_{l=0}^{l_M} (2l+1)\cdot 1\cdot 1 \sim l_M^2 \sim s\,\text{Log}^2 s$$

At high energy, the interaction radius is

$$l_M \sim k\,\rho_0 \sim \frac{\sqrt{s}}{2}\rho_0 \to A(s,t) \sim \frac{s}{2}\rho_0^2 \to \rho_0 \sim \log(s)$$

Also $|A(s,t)| < s^N \quad N \lesssim 1$

Froissart-Gribov with signature:

$$f^{\pm}(\ell,t) = \frac{1}{\pi}\int_{z_0(t)}^{\infty} D_s^{\pm} A(z',t)\, Q_\ell(z')\, dz' \qquad (*)$$

with the definitions:

$$D_s^{\pm} A(z_t,t) = D_s A(z_t,t) \pm D_u A(-z_t,t)$$

$$D_s A(z_t,t) = \frac{1}{2i}\Big[\underbrace{A(z_t+i\epsilon,t)}_{\hookrightarrow\, \equiv A(s+i\epsilon,t)} - A(z_t-i\epsilon,t)\Big]$$

$$D_u A(-z_t,t) = \frac{1}{2i}\Big[\underbrace{A(-z_t+i\epsilon,t)}_{\hookrightarrow\, \equiv A(u+i\epsilon,t)} - A(-z_t-i\epsilon,t)\Big]$$

because $z_t = 1 + \frac{2s}{t-4m^2} = -\left(1 + \frac{2u}{t-4m^2}\right)$

The inverse relation of (*) is:

$$\boxed{D_s^{\pm} A(z_t,t) = \frac{1}{2i}\oint_{C} (2\ell+1)\, f^{\pm}(\ell,t)\, P_\ell(z_t)\, d\ell \quad C \rightarrow \text{all positive integers.}}$$

Proof: $f^{\pm}(\ell,t) = \frac{1}{2\pi i}\int_{z_0}^{\infty}\oint_C (2\ell'+1)\, f^{\pm}(\ell',t)\, P_{\ell'}(z_t)\, d\ell'\, Q_\ell\, dz'$

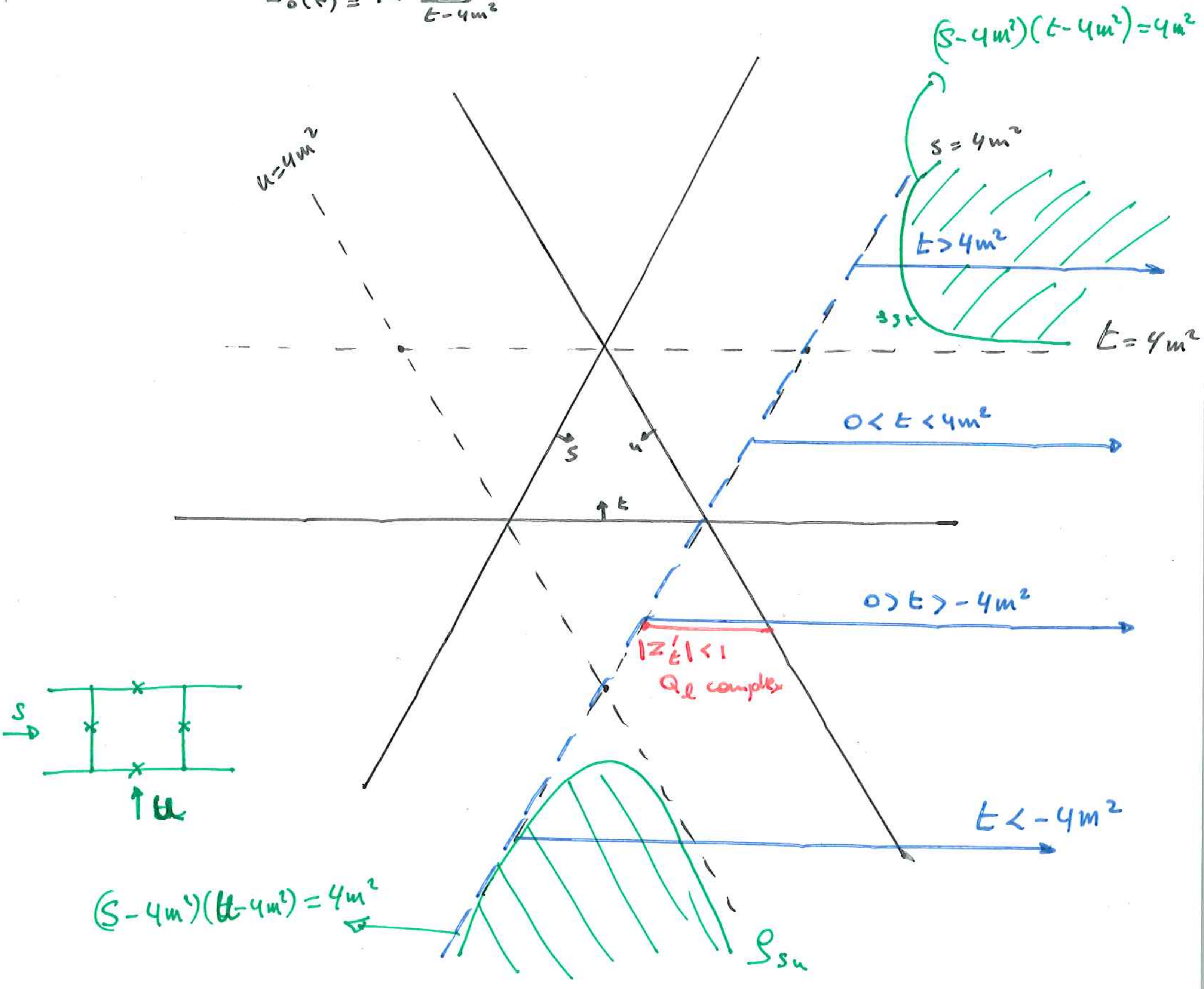
$$= \frac{1}{2\pi i}\oint_C (2\ell'+1)\, f^{\pm}(\ell',t)\, \frac{d\ell}{\ell-\ell'} \cdot \frac{1}{\ell+\ell'+1} = f^{\pm}(\ell,t)$$

using: $\int_1^{\infty} Q_\ell(z)\, P_{\ell'}(z)\, dz = \frac{1}{\ell-\ell'} \cdot \frac{1}{\ell+\ell'+1}$

7.14

Region of integration in F G:

$$f^{\pm}(\ell,t) = \frac{1}{\pi}\int_{z_0(t) = 1 + \frac{2\cdot 4m^2}{t-4m^2}}^{\infty} D_s^{\pm} A(z'_t, t)\, Q_\ell(z'_t)\, dz'_t$$



$Q_\ell(z)$ complex for $|z|<1$ or $z_t = 1 + \frac{2s}{t-4m^2} = -1 - \frac{2u}{t-4m^2}$

$z_t = 1 \to s = 0$

$z_t = -1 \to u = 0$ or $s = 4m^2 - t$

In the green regions, $s > 4m^2$ so $D_s A(s,t) \neq 0$

if $u > 4m^2$ then $D_s A(s,t)$ is imaginary

7.15

Define the signature amplitude:

$$A^{\pm}(t,z_t) = \sum_{l=0}^{\infty} (2l+1)\, f_l^{\pm}(t)\, P_l(z_t)$$

$$\left.\begin{array}{ll} f_l^{+}(t) = f_l(t) & l \text{ even} \\ f_l^{-}(t) = f_l(t) & l \text{ odd} \end{array}\right\} \tfrac{1}{2}[1+(-)^l] f_l^{+} + \tfrac{1}{2}[1-(-)^l] f_l^{-} = f_l$$

l integer

This yields:

$$A(t,z_t) = \frac{1}{2}\left[A^{+}(t,z_t) + A^{+}(t,-z_t) + A^{-}(t,z_t) - A^{-}(t,-z_t) \right]$$
$$= \sum_{l=0}^{\infty} (2l+1)\left[f_l^{+}(t) P_l^{+}(z_t) + f_l^{-}(t) P_l^{-}(t) \right] \qquad P_l^{\pm}(z) \equiv \tfrac{1}{2}[P_l(z) \pm P_l(-z)]$$

$A^{\pm}$ admit a S-W representation and the contour at ∞ vanishes:

$$A^{\pm}(t,z_t) = \frac{1}{2i}\oint_C (2l+1)\, f^{\pm}(l,t)\, \frac{P_l(-z_t)}{\sin \pi l}\, dl$$

$f^{\pm}(l,t)$ has poles for $t>0$ and l $\begin{smallmatrix}\text{even}\\ \text{odd}\end{smallmatrix}$

$$f^{\pm}(l,t) = \frac{\beta^{\pm}(t)}{l-\alpha^{\pm}(t)}$$

$\Rightarrow$ Regge poles have a signature.

Analyticity in l connect $\rho(770)$, $\rho_3(1690)$, ... J^{--}

$f_2(1270)$, $f_4(2050)$, ... J^{++}

$f_1(1285)$, $f_3(\text{xxx})$, ... J^{++}

Problem: $P_\ell(z) \sim z^{-1/2}$ at best for large z

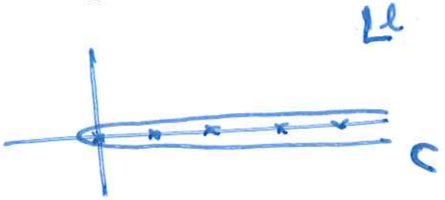
7.16

Solution: $$\frac{P_\ell(z)}{\sin\pi\ell} - \frac{Q_\ell(z)}{\pi\cos\pi\ell} = -\frac{Q_{-\ell-1}(z)}{\pi\cos\pi\ell}$$

Add and substract what is needed:

$$A^{\pm}(t,z_t) = \frac{1}{2i}\oint_C (2\ell+1)\, f^{\pm}(\ell,t)\left[\frac{P_\ell(-z)}{\sin\pi\ell} - \frac{Q_\ell(-z)}{\pi\cos\pi\ell}\right] d\ell$$

$$+ \frac{1}{2i\pi}\oint_C (2\ell+1)\, f^{\pm}(\ell,t)\, \frac{Q_\ell(-z)}{\cos\pi\ell}\, d\ell$$



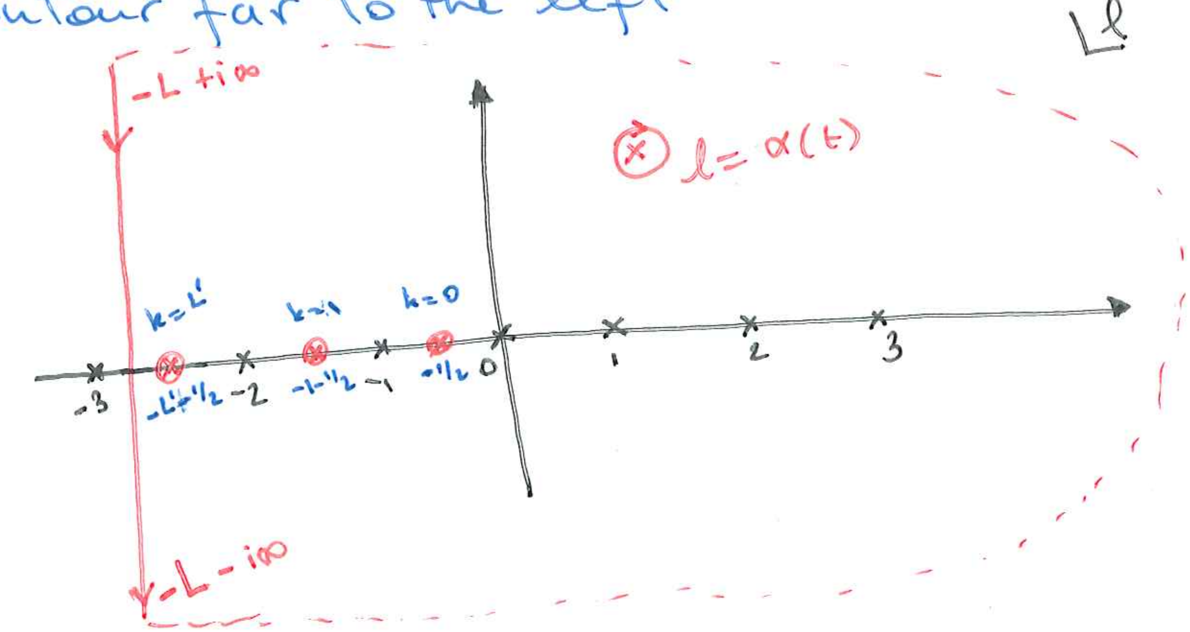
The contour is around the real axis

Pick up pole for $\ell = n - 1/2$ $\quad n \in \mathbb{N} \geq 1$

$$A^{\pm}(t,z_t) = \frac{-1}{2\pi i}\oint_C (2\ell+1)\, f^{\pm}(\ell,t)\, \frac{Q_{-\ell-1}(-z_t)}{\cos\pi\ell}\, d\ell$$

$$+ \sum_{n=1}^{\infty} 2n \frac{(-)^n}{\pi} f^{\pm}(n-1/2,t)\, Q_{n-1/2}(-z_t)$$

Move the contour far to the left



Split contributions:

7.17

$$A^{\pm}(t,z_t) = \left(2\alpha^{\pm}(t)+1\right)\beta^{\pm}(t)\,\frac{Q_{-\ell-1}(-z_t)}{\cos\pi\alpha^{\pm}}$$

$$-\frac{1}{2i\pi}\oint_{-L-i\infty}^{-L+i\infty}(2\ell+1)\,f^{\pm}(\ell,t)\,\frac{Q_{-\ell-1}(-z_t)}{\cos\pi\ell}\,d\ell$$

$$+\sum_{k=\cancel{0}\,1}^{L'}(-2k)\,\frac{(-)^k}{\pi}\,f^{\pm}(-k-1/2,t)\,Q_{k-1/2}(-z_t)$$

$$+\sum_{n=1}^{\infty}(2n)\,\frac{(-)^n}{\pi}\,f^{\pm}(n-1/2,t)\,Q_{n-1/2}(-z_t)$$

Use symmetry of F-G representation: $f(\ell,t) = f(-\ell-1,t)$

or $f(n-1/2,t) = f(-n-1/2,t)$

$$A^{\pm}(t,z_t) = \left[2\alpha^{\pm}(t)+1\right]\beta^{\pm}(t)\,\frac{Q_{-\alpha^{\pm}-1}(-z_t)}{\cos\pi\alpha^{\pm}} \qquad \to s^{\alpha^{\pm}(t)}$$

$$-\frac{1}{2\pi i}\oint_{-L-i\infty}^{-L+i\infty}(2\ell+1)\,f^{\pm}(\ell,t)\,\frac{Q_{-\ell-1}(-z_t)}{\cos\pi\ell}\,d\ell \qquad \to s^{-L}$$

$$+\sum_{n=L'}^{\infty}(2n)\,\frac{(-)^n}{\pi}\,f^{\pm}(n-1/2,t)\,Q_{n-1/2}(-z_t) \qquad \to s^{-L'-1/2}$$

Keeping only the leading term and recombining everything:

$$A(t, z_t) = \sum_{\alpha^+} \tilde{\beta}^+(t) \frac{1 + e^{-i\pi\alpha^+(t)}}{2\cos\pi\alpha^+(t)} \left(-\frac{s-u}{t-4m^2}\right)^{\alpha^+(t)}$$

$$+ \sum_{\alpha^-} \tilde{\beta}^-(t) \frac{1 - e^{-i\pi\alpha^-(t)}}{2\cos\pi\alpha^-(t)} \left(-\frac{s-u}{t-4m^2}\right)^{\alpha^-(t)}$$

with $\tilde{\beta} \equiv (2\alpha+1)\beta$ and $z_t = 1 + \frac{2s}{t-4m^2} = \frac{s-u}{t-4m^2}$

Regge formula :

$$A(s,t) = -\sum_{\alpha^\pm > -1/2} \beta^\pm(t) \frac{\pm 1 + e^{-i\pi\alpha^\pm(t)}}{2\sin\pi\alpha^\pm(t)} P_{\alpha^\pm(t)}(z_t)$$

$$+ O(s^{-1/2}) \quad ; \quad z_t = 1 + \frac{2s}{t-4m^2} \to s/s_0$$

At fixed t, large s : pick up pole $\alpha(t) > -1/2$

One way to see how emerge Regge pole is to start with the inverse p.w.e. :

$$f(l,t) = \frac{1}{2}\int_{-1}^{1} A(z_t,t)\, P_l(z_t)\, dz_t$$

$$= \frac{1}{2}\int_{-1}^{1} \sum_{l'\in N} (2l'+1)\, f_{l'}(t)\, \underbrace{P_{l'}(z_t)\, P_l(z_t)}_{\text{orthogonal only if } l' \text{ is integer!}}\, dz_t$$

use $\int_{-1}^{1} P_\alpha(x)\, P_l(x)\, dx = \frac{2/\pi}{\alpha - l} \cdot \frac{\sin\pi\alpha}{l+\alpha+1} \cdot (-)^\alpha$ for l integer, α anything.

$$\boxed{f(l,t) = \frac{(-)^l}{\pi} \sin\pi l \sum_{l'\in N} \frac{2l'+1}{l+l'+1} \cdot \frac{1}{l-l'}\, f_{l'}(t)}$$

$\to f_l(t)\, \delta_{l,l'}$ for l' integer

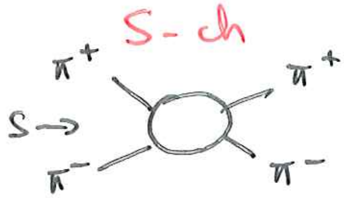
but $f(l,t)$ involves all $f_{l'}(t)$ when l is not an integer!

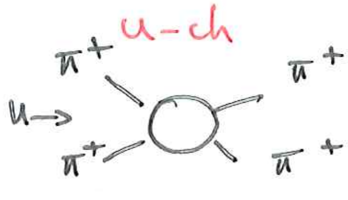
F-G representation:

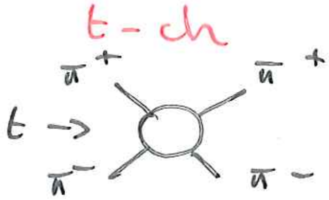
$$f^{\pm}(\ell,t) = \frac{1}{\pi}\int_{z_0(t)}^{\infty}\left[D_s A(z,t) \pm D_u A(-z,t)\right] Q_\ell(z)\, dz$$

$f^{\pm}(\ell,t)$ corresponds to $f_\ell(t)$ for ℓ $\genfrac{}{}{0pt}{}{\text{even}}{\text{odd}}$

if $D_u A(-z,t) \equiv 0 \to f^+ \equiv f^-$. Consequences?







No resonances in $\pi^+\pi^+ \to \Im A(u,t) \simeq 0$ in the u-channel.

$$D_u A(-z,t) \simeq 0$$

Resonances in the other channels?

$$I = 0, 1, (2) \qquad \left\{ \begin{array}{lll} I = 0 & \& \; C = + & \text{"f"} \\ I = 1 & \& \; C = - & \text{"}\rho\text{"} \end{array} \right.$$

$$G = C(-)^I = +$$

Conjugation charge of 2π is $C(\pi^+\pi^-) = (-)^\ell \pi^+\pi^-$

So f are only even $(-)^\ell = + \to f_0, f_2, f_4, \ldots$ $(0,2,4)^{++} = J^{PC}$

ρ odd $(-)^\ell = - \to \rho_1, \rho_3, \rho_5, \ldots$ $(1,3,5)^{--} = J^{PC}$

Degeneracy between f & ρ trajectories & couplings!

$$f^- = \frac{\beta^\rho(t)}{\ell-\alpha_\rho(t)} = f^+ = \frac{\beta^f(t)}{\ell-\alpha_f(t)} \qquad \forall \ell$$

$$\Rightarrow \alpha_\rho = \alpha_f \quad \text{and} \quad \beta_\rho = \beta_f$$

Another EXD derivation

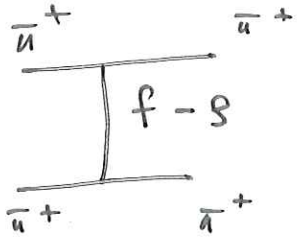
Consider "exotic" channel, e.g. $\pi^+\pi^+ \to \pi^+\pi^+$

$\mathrm{Im}\,(\text{Resonance}) \sim 0 \Rightarrow \mathrm{Im}\,(\text{Regge}) \sim 0$

low E $\longleftrightarrow$ hig E

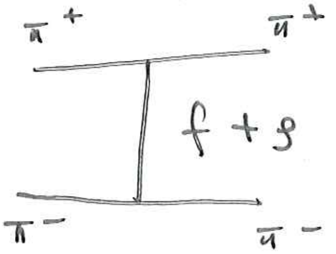
Regge form at high energy: $A(s,t) = -\beta(t)\,\dfrac{\pm 1 - e^{-i\pi\alpha(t)}}{\sin \pi\alpha(t)}\, s^{\alpha(t)}$

$$\mathrm{Im}\,A(s,t) = \beta(t)\, s^{\alpha(t)}$$



$\mathrm{Im}\,A = 0 \Rightarrow \beta_f s^{\alpha_f} - \beta_\rho s^{\alpha_\rho} = 0$

$$\boxed{\beta_\rho = \beta_f \quad \text{and} \quad \alpha_\rho = \alpha_f}$$



because of Conjugation charge

Consider $\pi\pi \to K\bar{K}$ and $K\bar{K} \to K\bar{K}$. What are the resonances in $K\bar{K}$?

7.21

$I = 0, 1$

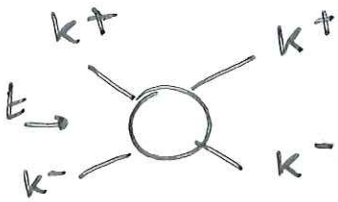
$G = \pm$ $\Longrightarrow$

I \ G	+	−
0	f	ω
1	ρ	a

Charge conjugation yields as well: $C(K^+K^-) = (-)^\ell K^+K^-$

$\Rightarrow$ ρ and ω are odd signature $(1, 3, 5, \dots)^{--}$

f and a " even " $(0, 2, 4, \dots)^{++}$



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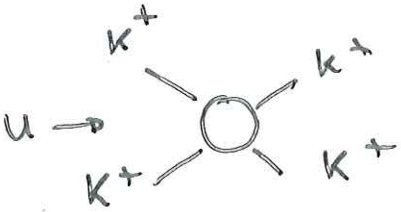
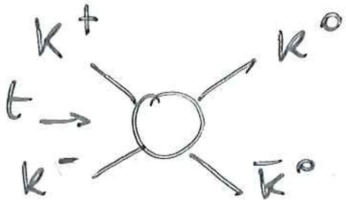


u exotic $\to f^+(\ell, t) = f^-(\ell, t)$

$$f^+ = \frac{\beta_f}{\ell - \alpha_f} + \frac{\beta_a}{\ell - \alpha_a} = f^- = \frac{\beta_\rho}{\ell - \alpha_\rho} + \frac{\beta_\omega}{\ell - \alpha_\omega}$$



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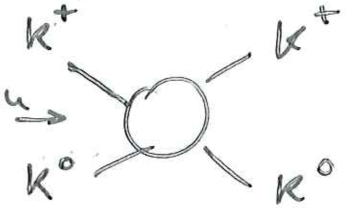


$K^+ : u\bar{s}$ $K^0 : d\bar{s}$

$K^- : \bar{u}s$ $\bar{K}^0 : \bar{d}s$

K^+K^0 is exotic

$$\tilde{f}^+ = \frac{\beta_a}{\ell - \alpha_a} - \frac{\beta_f}{\ell - \alpha_f} = \tilde{f}^- = \frac{\beta_\rho}{\ell - \alpha_\rho} - \frac{\beta_\omega}{\ell - \alpha_\omega} \Rightarrow \begin{array}{l} a \equiv \rho \\ f \equiv \omega \end{array}$$

because the SU(3) Clebsch-Gordan:

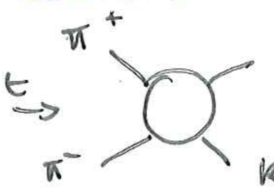
$$\langle \underset{K^+}{\overset{Y\ y_3\ I\ i_3}{1\ 1\ \tfrac{1}{2}\ \tfrac{1}{2}}} ; \underset{K^-}{1\ {-1}\ \tfrac{1}{2}\ {-\tfrac{1}{2}}} | \overset{\omega/f}{0\ 0\ 0\ 0} \rangle = - \langle \overset{K^0}{1\ 1\ \tfrac{1}{2}\ {-\tfrac{1}{2}}} ; \overset{\bar{K}^0}{1\ {-1}\ \tfrac{1}{2}\ {-\tfrac{1}{2}}} | 0\ 0\ 0\ 0 \rangle$$

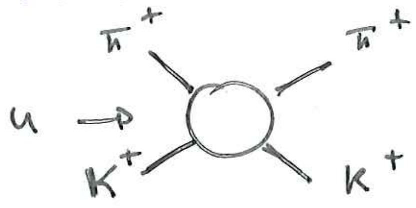
$$| \underset{\rho/a}{0\ 0\ 1\ 0} \rangle = + \langle \qquad | 0\ 0\ 1\ 0 \rangle$$

For $K\bar{K}$, we find that

$$\begin{matrix} \rho \equiv a \\ \omega \equiv f \end{matrix} \quad \text{EXD (Exchange degeneracy)}$$

Consider $\pi\pi \to K\bar{K}$:

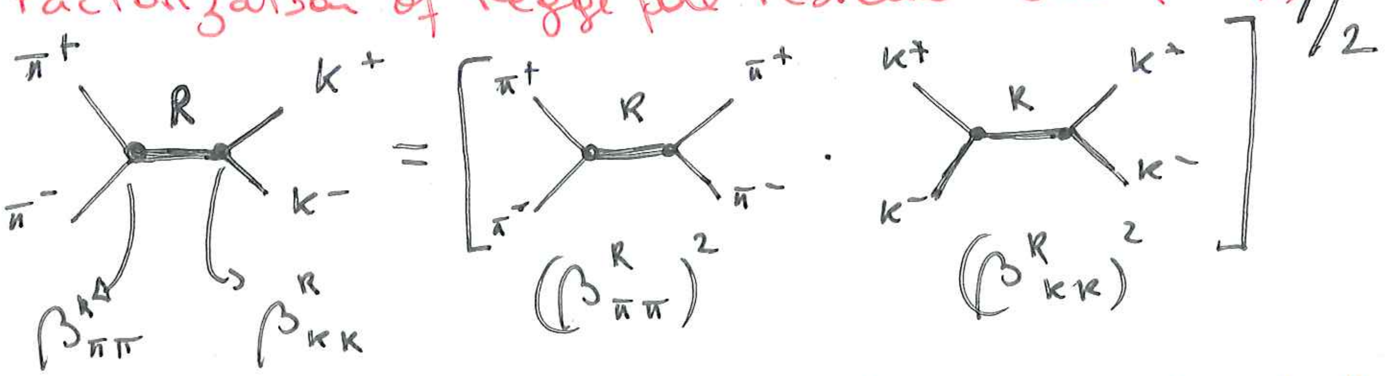




exotic

$$f^+ = f^- \to \frac{\beta_\rho}{\ell - \alpha_\rho} = \frac{\beta_f}{\ell - \alpha_f}$$

Factorization of Regge pole residue (no proof)



Exotic channel $\pi^+\pi^+$, K^+K^+, K^+K^0 lead to

$$\boxed{\begin{gathered} \alpha_\rho = \alpha_f = \alpha_a = \alpha_\omega \\ \\ \beta^\rho_{\pi\pi} = \beta^f_{\pi\pi} \\ \beta^\rho_{KK} = \beta^f_{KK} = \beta^\omega_{KK} = \beta^a_{KK} \end{gathered}}$$

Total cross section:

Optical theorem leads to:

$$\sigma_{tot}(ab \to X) = \frac{(\hbar c)^2}{2 m_b p_{Lab}} \, \mathcal{I}m \; A(ab \to ab)\Big|_{t=0}$$

Regge formula at high energy:

$$A(ab \to ab)\Big|_{t=0} = - \sum_R \beta^R_{aa} \beta^R_{bb} \frac{\pm 1 - e^{-i\pi \alpha_R(0)}}{\sin \pi \alpha_R(0)} \left(s/s_0\right)^{\alpha_R(0)}$$

we obtain:

$$\sigma_{tot}(ab \to X) = \sum_R \beta^R_{aa} \beta^R_{bb} \left(s/s_0\right)^{\alpha_R(0)}$$

Relative contribution:

$$\pi^{\mp} p : \; \mathbb{P} + f \pm \rho$$

$$\left.\begin{array}{l} K^{\mp} p : \; \mathbb{P} + f \pm \rho \pm \omega + a \\ K^{\mp} n : \; \mathbb{P} + f \mp \rho \pm \omega - a \end{array}\right\} \; f = \omega \quad \rho = a$$

$$\left.\begin{array}{l} \bar{p} p : \; \mathbb{P} + f \mp \rho \mp \omega + a \\ \bar{p} n : \; \mathbb{P} + f \pm \rho \mp \omega - a \end{array}\right\} \; f = \omega \quad \rho = a$$

$\hookrightarrow \beta^f_{pp} = \beta^\omega_{pp}$

$\beta^\rho_{pp} = \beta^a_{pp}$

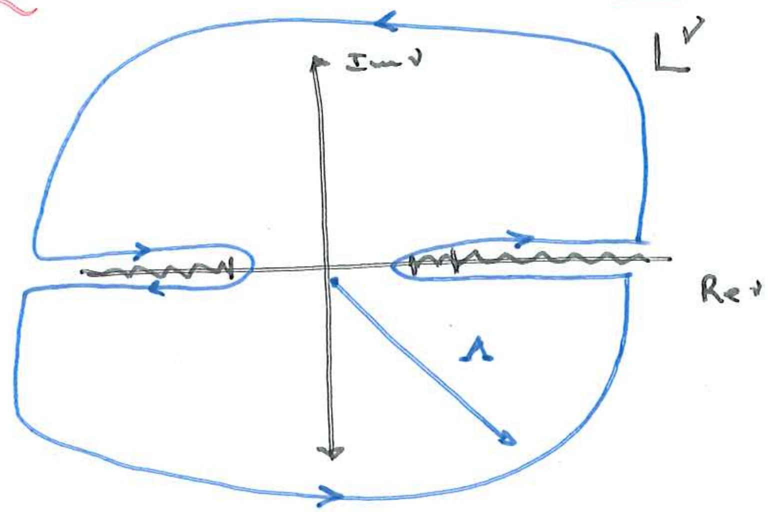
$p \leftrightarrow n$: isoscalar changes sign

$K^+ \leftrightarrow K^-$
$p \leftrightarrow \bar{p}$: $C = -$ change sign
$\pi^+ \leftrightarrow \pi^-$

Exotic channels: $K^+ p$, pp, $K^+ n$, pn

Finite Energy Sum Rules

7.24



Use $\nu = \frac{s-u}{2}$

$$\oint_C \nu^k A^{\boxed{\pm} \rightarrow \tau}(\nu,t)\,\frac{d\nu}{2i} = 0 \qquad k \in \mathbb{N}$$

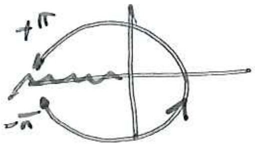
$$\int_{\nu_0}^{\Lambda} \left[D_s A(\nu,t) \mp (-)^k D_u A(\nu,t) \right] \nu^k d\nu = \frac{-1}{2i} \oint_{C_\Lambda} \nu^k A^{\pm}(\nu,t)\, d\nu$$

$$\int_{\nu_0}^{\Lambda} D_s^{\eta} A(\nu,t)\, \nu^k d\nu = \int_{\nu_0}^{\Lambda} \mathrm{Im}\, A^{\eta}(\nu,t)\, \nu^k d\nu \qquad \begin{aligned} \eta &= \mp (-)^k \\ &= -\tau (-)^k \end{aligned}$$

For large Λ: $A^{\tau}(\nu,t) \sim -\beta \dfrac{\tau \nu^{\alpha} + (-\nu)^{\alpha}}{\sin \pi\alpha}$

Split the 2 term:

$$-\frac{1}{2i} \cdot (-\beta)\, \tau \int_{-\pi}^{\pi} \frac{\Lambda e^{i\phi(\alpha+k+1)}}{\sin \pi\alpha} i\, d\phi\, \Lambda^{k+\alpha} \qquad \nu = \Lambda e^{i\phi}$$

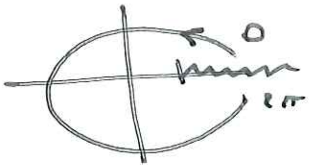


$$= \frac{\beta\tau}{2} \frac{\Lambda^{\alpha+k+1}}{\sin \pi\alpha} \cdot \frac{e^{i\pi(\alpha+k+1)} - e^{-i\pi(\alpha+k+1)}}{(\alpha+k+1)\, i}$$

$$= -\tau\beta(-)^k \frac{\Lambda^{k+\alpha+1}}{\alpha+k+1}$$

the other term has a different cut:

$$+\frac{1}{2i}\beta\int_0^{2\pi}\Lambda^k e^{i\phi k}\cdot e^{-i\pi\alpha}\Lambda^\alpha e^{i\phi\alpha}\frac{i\Lambda e^{i\phi}d\phi}{\sin\pi\alpha}$$



$$=\frac{\beta}{2}\frac{\Lambda^{k+\alpha+1}}{\sin\pi\alpha}\int_0^{2\pi}e^{i\phi(k+\alpha+1)}e^{-i\pi\alpha}d\phi$$

$$=\frac{\beta}{2}\frac{\Lambda^{\alpha+k+1}}{\sin\pi\alpha}\cdot\frac{e^{i\pi\alpha}\overbrace{e^{2\pi i(k+1)}}^{1}-e^{-i\pi\alpha}}{i(k+\alpha+1)}=\beta\frac{\Lambda^{\alpha+k+1}}{\alpha+k+1}$$

So we arrive to:

$$\int_{\nu_0}^{\Lambda}\left[D_s A(\nu,t)-\tau(-)^k D_u A(\nu,t)\right]\frac{\nu^k d\nu}{\Lambda^{k+1}}=\left[1-\tau(-)^k\right]\beta\frac{\Lambda^\alpha}{\alpha+k+1}$$

If the u-ch = s-ch par c invariance then:

$$\boxed{\int_{\nu_0}^{\Lambda}\mathrm{Im}\,A(\nu,t)\frac{\nu^k d\nu}{\Lambda^{k+1}}=\beta\frac{\Lambda^\alpha}{\alpha+k+1}}$$

k opposite parity of the exchange $\tau=-(-)^k$

for instance in $\pi\pi\to\pi\pi$, $\pi N\to\pi N$, ...
any $ab\to ac$ of a is its own antiparticle.

Crossing Relations

Useful formulas:

$C^{-1}\gamma_\mu C = -\gamma_\mu^T$ $\qquad$ $C = i\gamma^0\gamma^2 = -C^{-1}$

$C^{-1}\gamma_5 C = +\gamma_5^T$ $\qquad$ $\gamma_5^T = \gamma_5$

$N = -C\bar{u}^T$ $\qquad$ $\bar{N} = u^T C^{-1}$

Pion-nucleon scattering.

Isospin decomposition:

$$A^{ab}_{ij} = \delta^{ab}\delta_{ij}A^{(+)} + i\epsilon^{abc}(\tau^c)_{ij}A^{(-)}$$

Lorentz decomposition:

$$A + (\not{p}_1 + \not{p}_3)B$$

Pion photoproduction

Isospin decomposition:

$$A^a_{ij} = \delta^{a3}\delta_{ij}A^{(+)} + i\epsilon^{a3c}(\tau^c)_{ij}A^{(-)} + (\tau^a)_{ij}A^{(0)}$$

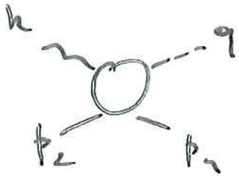
Lorentz decomposition: $\sum_{i=1}^{6} A_i M_i$

$M_1 = \frac{1}{2}\gamma_5\gamma_\mu\gamma_\nu F^{\mu\nu}$ $\qquad$ $M_5 = \gamma_5 k_\mu q_\nu F^{\mu\nu}$

$M_2 = 2\gamma_5 q_\mu p_\nu F^{\mu\nu}$ $\qquad$ $M_6 = \gamma_5 k_\mu \gamma_\nu F^{\mu\nu}$

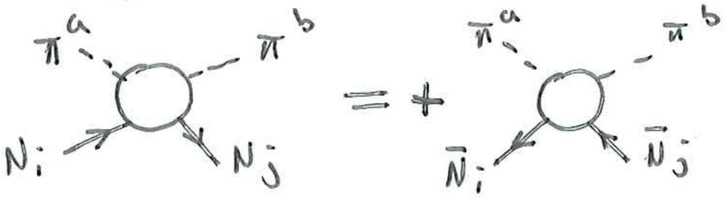
$M_3 = \gamma_5\gamma_\mu q_\nu F^{\mu\nu}$ $\qquad$ $F^{\mu\nu} = \epsilon^\mu k^\nu - k^\mu\epsilon^\nu$

$M_4 = \frac{i}{2}\epsilon_{\alpha\beta\mu\nu}\gamma^\alpha q^\beta F^{\mu\nu}$ $\qquad$ $p = (p_2 + p_4)/2$



Conjugation charge invariance $C^{-1} T C = T$

implies



$\pi^1 = (\pi^+ + \pi^-)/\sqrt{2}$

$\pi^2 = i(\pi^- - \pi^+)/\sqrt{2}$

$\pi^3 = \pi^0$

$N_{1,2} = p, n$

$\bar{N}_{1,2} = \bar{p}, \bar{n}$

or in matrix element

$$\langle \pi^b(3)\, N_j(4) | T | \pi^a(1)\, N_i(2) \rangle = \bar{u}(4)\left[A^{ba}_{ji} + (\not{k}_1 + \not{k}_3) B^{ba}_{ji} \right] u(2)$$

$$= \langle \pi^b(3)\, \bar{N}_j(4) | T | \pi^a(1)\, \bar{N}_i(2) \rangle = - \bar{N}(-2)\left[A^{ba}_{ji} + (\not{k}_1 + \not{k}_3) B^{ba}_{ji} \right] N(-4)$$

Number → take transposition (and complex conjugation)

$$= u^T(2)\left[A^{ba}_{ij} + (\not{k}_1 + \not{k}_3)^T B^{ba}_{ij} \right] \bar{u}^T(4)$$

$$= - u^T(2)\, C^{-1}\left[A^{ba}_{ij} + C(\not{k}_1 + \not{k}_3)^T C^{-1} B^{ba}_{ij} \right] C\, \bar{u}^T(4)$$

$$= - \bar{N}(2)\left[A^{ba}_{ij} - (\not{k}_1 + \not{k}_3) B^{ba}_{ij} \right] N(4)$$

So $A^{ba}_{ij}(-\nu) = A^{ba}_{ji}(\nu)$ and $B^{ba}_{ij}(-\nu) = -B^{ba}_{ji}(\nu)$

But $i \leftrightarrow j \to A^{(\pm)} \to \pm A^{(\pm)}$ thus

$$\boxed{\begin{aligned} A^{(\pm)}(-\nu) &= \pm A^{(\pm}(\nu) \\ B^{(\pm)}(-\nu) &= \mp B^{(\pm)}(\nu) \end{aligned}}$$

Do the same for photoproduction:

$$\langle \pi^a(3)\, N_j(4) | T | \gamma(1)\, N_i(2) \rangle = \bar{u}(4) \left[\sum_k (A_k)^a_{ij}\, M_k \right] u(2)$$

$$= -\langle \pi^a\, \bar{N}_j(4) | T | \gamma(1)\, N_i(2) \rangle = +\bar{N}(-2) \left[\sum_k (A_k)^a_{ij}\, M_k \right] N(-4)$$

$$= -N(2) \left[\sum_k (A_k)^a_{ji}\; C^{-1} M_k^T C \right] N(4)$$

But we have (including the change $p \to -p$)

$$C^{-1} M_i^T C = -M_i \qquad i = 1, 2, 4$$

$$C^{-1} M_i^T C = +M_i \qquad i = 3, 5, 6$$

and $i \leftrightarrow j \Rightarrow A^{(+,0)} \to +A^{(+,0)}$ and $A^{(-)} \to -A^{(-)}$ thus

$$\boxed{\begin{array}{lll} A_i^{(+,0)}(-\nu) & = +A_i^{(+,0)} & i = 1, 2, 4 \\ A_i^{(+,0)}(-\nu) & = -A_i^{(+,0)} & i = 3, 5, 6 \\ A_i^{(-)}(-\nu) & = +A_i^{(-)} & i = 3, 5, 6 \\ A_i^{(-)}(-\nu) & = -A_i^{(-)} & i = 1, 2, 4 \end{array}}$$