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Complex angular momentum, Riemann sheets, Regge poles and factorization

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FOREWORD

These notes are complementary to the MATHEMATICA notebook that can be downloaded from the Workshop webpage: http://www.indiana.edu/ jpac/lectures2017.html. You are free to modify and distribute this document and the MATHEMATICA notebook as long as you acknowledge the original authorship.

1 SINGLE CHANNEL

$m_1 + m_2 \to m_1 + m_2$

$$S = I + 2iC_\ell(s)\varphi_\ell(s) = I + 2i\tau(s)\,t_\ell(s) = I + 2i\tau(s)B_\ell(s)\varphi_\ell(s) \tag{1.1}$$

where $t_\ell(s)$ is the amplitude and $\varphi_\ell(s)$ is the reduced amplitude, which is analytical in both the complex s and complex angular momentum planes. We define

$$s_{th} = (m_1+m_2)^2\,, \qquad \text{[threshold]} \tag{1.2a}$$

$$q^2(s) = \frac{m_1 m_2}{s_{th}}\,[s - s_{th}]\,, \qquad \text{["momentum" squared]} \tag{1.2b}$$

$$q(s) = \sqrt{q^2(s)}, \qquad \text{["momentum"]} \tag{1.2c}$$

$$\tau(s) = \frac{q(s)}{q_0}, \qquad \text{[phase space]} \tag{1.2d}$$

$$B_\ell(s) = \left(\frac{q^2(s)}{q_0^2 + q^2(s)}\right)^\ell, \qquad \text{[angular momentum barrier]} \tag{1.2e}$$

$$C_\ell(s) = \tau(s)B_\ell(s), \qquad \text{[phase space with angular momentum barrier]} \tag{1.2f}$$

where, for convenience, $q_0 = 1$ GeV and m_1^2, m_2^2 and s are in units of GeV^2. Notice that $B_\ell(s)$ is a model for the angular momentum barrier and relates to the lefthand cut. Our definition of "momentum" simplifies the model and allows us to compute everything analytically The unitarity relation reads

$$\operatorname{Disc}\varphi_\ell(s) \equiv \varphi_\ell\,(s+i0) - \varphi_\ell\,(s-i0) = 2i\,\tau(s+i0)\,B_\ell(s+i0)\,\varphi_\ell\,(s+i0)\;\varphi_\ell\,(s-i0)\,. \tag{1.3}$$

Think about why $\tau(s)$ and $B_\ell(s)$ are defined at $s+i0$.[1]

1.1 PHASE SPACE: ANALYTIC CONTINUATION

The the analytically continued phase space $\rho_\ell(s)$ is defined through the once substracted dispersion relation

$$\rho_\ell(s) = b(s_{th}) + \frac{s-s_{th}}{\pi}\int_{s_{th}}^\infty ds'\,\frac{C_\ell(s')}{(s'-s)\,(s'-s_{th})} \tag{1.4a}$$

$$= b(s_{th}) + \frac{s-s_{th}}{\pi}\int_{s_{th}}^\infty ds'\,\frac{\tau(s')B_\ell(s')}{(s'-s)\,(s'-s_{th})} \tag{1.4b}$$

$$= b(s_{th}) + \frac{s-s_{th}}{\pi}PV\int_{s_{th}}^\infty ds'\,\frac{\tau(s')B_\ell(s')}{(s'-s)\,(s'-s_{th})} + i\,\tau(s)B_\ell(s). \tag{1.4c}$$

[1] Solution: By convention we define the physical amplitude at $s+i0$, hence the kinematical factors $\tau(s)$ and $B_\ell(s)$ have to be computed consistently with that convention. You can choose a different convention, but be sure to be consistent!

Where $b(s_{th})$ is the substraction constant and PV stands for principal value. Because the substraction is performed at s_{th}, $b(s_{th}) = 0$. This integral can be computed analytically

$$\begin{aligned} \rho_\ell(s) = & -\frac{a_k^{\ell+1/2}(s-s_k)^\ell\sqrt{s_k-s}}{[1+a_k\,(s-s_k)]^\ell} + \frac{\Gamma(\ell+\frac{1}{2})}{\sqrt{\pi}\Gamma(\ell+1)} \\ & \times\left([1+a_k(s-s_k)]\,{}_2F_1\left[1,\ell+\frac{1}{2},-\frac{1}{2},\frac{1}{a_k(s_k-s)}\right]\right. \\ & \left.-[3+2\ell+a_k(s-s_k)]\,{}_2F_1\left[1,\ell+\frac{1}{2},\frac{1}{2},\frac{1}{a_k(s_k-s)}\right]\right), \end{aligned} \tag{1.5}$$

where $s_k = s_{th}$ for channel k and $a_k = m_1 m_2/s_{th}$ (this equation is prepared for coupled channels case). Notice that our $\rho_\ell(s)$ is analytical in both the s complex plane *and* the ℓ complex plane. Equation (1.5) is valid for

$$[s_k \geq \Re\{s\} \cup \Im\{s\} \neq 0] \cap \Re\{\ell\} > -\frac{1}{2}.$$

1.2 Single pole amplitude

Let us build a very simple model for the amplitude. We assume that the amplitude is dominated everywhere by a single pole. Then the amplitude reads

$$t_\ell(s) = B_\ell(s)\,\varphi_\ell(s) = B_\ell(s)\,\frac{\beta}{\ell-\alpha(s)}, \tag{1.6}$$

$$\varphi_\ell(s) = \frac{\beta}{\ell-\alpha(s)}, \tag{1.7}$$

where β is a real number and $\alpha(s)$ is the Regge trajectory. If we use the single channel unitarity relation in Eq. (1.3) we obtain (do the derivation)

$$\frac{1}{2i}\left[\alpha(s+i0)-\alpha(s-i0)\right] = \Im\{\alpha(s)\} = C_\ell(s)\,\beta, \tag{1.8}$$

So, if we assume a *linear* Regge trajectory we obtain

$$\alpha(s) = \alpha_0 + \alpha' s + i\Im\{\alpha(s)\} = \alpha_0 + \alpha' s - \beta\,\rho_\ell(s), \tag{1.9}$$

where α_0 and α' are two real numbers and we have analytically continued $C_\ell(S) \to \rho_\ell(s)$. Please notice that a $\alpha_0 + \alpha' s$ is a linear relation, but unitarity through the phase space $\rho_\ell(s)$ breaks the linearity. Actually, Regge trajectories are never linear. Hence the reduced amplitude reads

$$\varphi_\ell(s) = \frac{\beta}{\ell-\alpha_0-\alpha' s-\beta\,\rho_\ell(s)}. \tag{1.10}$$

Please notice how this amplitude looks a lot like a Breit-Wigner.

Now we need to analytically continue the reduced amplitude $\varphi_\ell(s)$ to the 2nd Riemann sheet $[\varphi_\ell^{II}(s)]$. That is straightforward using the unitarity relation in Eq. (1.3) if we remember that $\varphi_\ell^{II}(s-i0)=\varphi_\ell(s+i0)$:

$$\varphi_\ell\,(s+i0)-\varphi_\ell\,(s-i0) = 2i\,C_\ell(s)\,\varphi_\ell\,(s+i0)\;\varphi_\ell\,(s-i0)\,, \tag{1.11}$$
$$\varphi_\ell^{II}\,(s-i0)-\varphi_\ell\,(s-i0) = 2i\,C_\ell(s)\,\varphi_\ell^{II}\,(s-i0)\;\varphi_\ell\,(s-i0)\,, \tag{1.12}$$

and because all the energy dependence is written as $s-i0$ we substitute $s-i0\to s$ and we isolate $\varphi_\ell^{II}\,(s)$ obtaining

$$\varphi_\ell^{II}(s) = \frac{\varphi_\ell(s)}{1-2i\,C_\ell(s)\,\varphi_\ell(s)}, \tag{1.13}$$

and we have the 2nd Riemann sheet reduced amplitude $\varphi_\ell^{II}(s)$ written in terms of the 1st Riemann sheet reduced amplitude $\varphi_\ell(s)$. We substitute $\varphi_\ell(s)$ and we obtain

$$\varphi_\ell^{II}(s) = \frac{\beta}{\ell-\alpha(s)-2i\,\beta\,C_\ell(s)} \tag{1.14}$$

Notice that $C_\ell(S)$ has the square-root cut to the left, so $C_\ell(s+i0)=C_\ell(s-i0)$.

1.3 Regge Trajectories

1.3.1 Moving poles in the s complex plane

We can track the Regge trajectory in the complex s plane changing the value of ℓ with the restrictions $\ell\in\mathbb{R}$ and $\ell>-\frac{1}{2}$ and tracking the s values that are solution to

$$\ell-\alpha(s)-2i\,\beta\,C_\ell(s)=0. \tag{1.15}$$

This gives us the Regge trajectory. Keep in mind that the only values that we can *measure* are the physical $\ell=0,1,2,\ldots$

1.3.2 Moving poles in the complex angular momentum plane

In the same way, we can track down the movement of the poles (*a.k.a.* Regge trajectory) in the complex angular momentum plane through the solution of

$$\bar{\ell}-\alpha(s)=0, \tag{1.16}$$

where $\bar{\ell}\equiv\ell-2i\,\beta\,C_\ell(s)$.

2 Coupled channels

Every time a new channel opens, *e.g.* we have $\pi\pi\to\pi\pi$ and we increase the energy of the incoming π until the channel $\pi\pi\to K\bar{K}$ opens, 2 new Riemann sheets open, because a new unitarity cut joins the party.

The considered processes are

$$1+\bar{1} \to 1+\bar{1} \tag{2.1a}$$
$$1+\bar{1} \to 2+\bar{2} \tag{2.1b}$$
$$2+\bar{2} \to 2+\bar{2} \tag{2.1c}$$

2.1 Unitarity relations

For the coupled channel case, the unitarity relations read

$$\begin{aligned}\varphi_\ell^{11}(s+i0)-\varphi_\ell^{11}(s-i0) &= 2i\,\tau_1(s)\,B_\ell^{11}(s)\,\varphi_\ell^{11}(s-i0)\,\varphi_\ell^{11}(s+i0)\\ &\quad +2i\,\tau_2(s)\,B_\ell^{22}(s)\,\varphi_\ell^{12}(s-i0)\,\varphi_\ell^{12}(s+i0)\end{aligned} \tag{2.2a}$$
$$\begin{aligned}\varphi_\ell^{12}(s+i0)-\varphi_\ell^{12}(s-i0) &= 2i\,\tau_1(s)\,B_\ell^{11}(s)\,\varphi_\ell^{11}(s-i0)\,\varphi_\ell^{12}(s+i0)\\ &\quad +2i\,\tau_2(s)\,B_\ell^{22}(s)\,\varphi_\ell^{12}(s-i0)\,\varphi_\ell^{22}(s+i0)\end{aligned} \tag{2.2b}$$
$$\begin{aligned}\varphi_\ell^{22}(s+i0)-\varphi_\ell^{22}(s-i0) &= 2i\,\tau_1(s)\,B_\ell^{11}(s)\,\varphi_\ell^{12}(s-i0)\,\varphi_\ell^{12}(s+i0)\\ &\quad +2i\,\tau_2(s)\,B_\ell^{22}(s)\,\varphi_\ell^{22}(s-i0)\,\varphi_\ell^{22}(s+i0)\end{aligned} \tag{2.2c}$$

Below the second threshold $s_2=(m_2+m_2)^2$, $\tau_2(s)=0$ so the unitarity relations reduce to

$$\varphi_\ell^{11}(s+i0)-\varphi_\ell^{11}(s-i0) = 2i\,\tau_1(s)\,B_\ell^{11}(s)\,\varphi_\ell^{11}(s-i0)\,\varphi_\ell^{11}(s+i0) \tag{2.3a}$$
$$\varphi_\ell^{12}(s+i0)-\varphi_\ell^{12}(s-i0) = 2i\,\tau_1(s)\,B_\ell^{11}(s)\,\varphi_\ell^{11}(s-i0)\,\varphi_\ell^{12}(s+i0) \tag{2.3b}$$
$$\varphi_\ell^{22}(s+i0)-\varphi_\ell^{22}(s-i0) = 2i\,\tau_1(s)\,B_\ell^{11}(s)\,\varphi_\ell^{12}(s-i0)\,\varphi_\ell^{12}(s+i0) \tag{2.3c}$$

where $\tau_j(s)=\frac{q_j(s)}{q_0}=\frac{m_j^2}{s_j}\left(s-s_j\right)$, $s_j=4m_j^2$, $B_\ell^{jj}(s)=\left(\frac{q_j^2(s)}{q_0+q_j^2(s)}\right)^\ell$ and $C_\ell^{jj}=\tau_j(s)B_\ell^{jj}(s)$.

2.2 Coupled channel amplitudes

The amplitudes can be written

$$\varphi_\ell^{11}(s)=\frac{\beta_{11}}{\ell-\alpha(s)}, \tag{2.4a}$$
$$\varphi_\ell^{12}(s)=\frac{\beta_{12}}{\ell-\alpha(s)}, \tag{2.4b}$$
$$\varphi_\ell^{22}(s)=\frac{\beta_{22}}{\ell-\alpha(s)}. \tag{2.4c}$$

We substitute in Eq. (2.2) and we obtain three equation for $\Im\{\alpha(s)\}$

$$\Im\{\alpha(s)\}=C_\ell^{11}\beta_{11}+C_\ell^{22}\beta_{12}^2/\beta_{11}, \tag{2.5a}$$
$$\Im\{\alpha(s)\}=C_\ell^{11}\beta_{11}+C_\ell^{22}\beta_{22}, \tag{2.5b}$$
$$\Im\{\alpha(s)\}=C_\ell^{11}\beta_{12}^2/\beta_{22}+C_\ell^{22}\beta_{22}, \tag{2.5c}$$

which reduce to a single unitarity condition

$$\Im\{\alpha(s)\} = C_\ell^{11}(s)\,\beta_{11} + C_\ell^{22}(s)\,\beta_{22}, \tag{2.6}$$

and one relation among the β residues

$$\beta_{11}\,\beta_{22} = \beta_{12}^2, \tag{2.7}$$

which is pole factorization (valid even if β has an energy dependence).

2.3 Amplitude in the 2nd Riemann sheet

It is straightforward to write the amplitudes in the 2nd Riemann sheet following Eqs. (2.3). They read

$$\varphi_\ell^{11,II}(s) = \frac{\varphi_\ell^{11}(s)}{1-2iC_\ell^{11}(s)\,\varphi_\ell^{11}(s)} = \frac{\beta_{11}\,s}{\ell-\alpha(s)-2iC_\ell^{11}(s)\beta_{11}}, \tag{2.8a}$$

$$\varphi_\ell^{12,II}(s) = \frac{\varphi_\ell^{12}(s)}{1-2iC_\ell^{11}(s)\,\varphi_\ell^{11}(s)} = \frac{\beta_{12}\,s}{\ell-\alpha(s)-2iC_\ell^{11}(s)\beta_{11}}, \tag{2.8b}$$

$$\varphi_\ell^{22,II}(s) = \varphi_\ell^{22}(s) + 2iC_\ell^{11}(s)\frac{[\varphi_\ell^{12}(s)]^2}{1-2iC_\ell^{11}(s)\,\varphi_\ell^{11}(s)} = \frac{\beta_{22}\,s}{\ell-\alpha(s)-2iC_\ell^{11}(s)\beta_{11}}, \tag{2.8c}$$

and you can see that all of them have the 2nd Riemann sheet poles at the same location. Remember: poles are universal; in a given Riemann sheet, all channels have the poles at the same locations.

2.4 Regge Trajectory

Due to unitarity and coupled channels the Regge trajectory is modified –see Eq. (2.6). It reads

$$\alpha(s) = \alpha_0 + \alpha' s + \rho_\ell^{11}(s)\,\beta_{11} + \rho_\ell^{22}(s)\,\beta_{22} \tag{2.9}$$

where $\rho_\ell^{11}(s)$ and $\rho_\ell^{22}(s)$ are the analytical continuations of $C_\ell^{11}(s)$ and $C_\ell^{22}(s)$ respectively.

Then, the Regge trajectory (poles in the 2nd Riemann sheet)is given by

$$\ell - \alpha(s) - 2i\beta_{11}C_\ell^{11}(s) = 0 \tag{2.10}$$

or explicitly

$$\ell - \alpha_0 - \alpha' s - \rho_\ell^{11}(s)\,\beta_{11} - \rho_\ell^{22}(s)\,\beta_{22} - 2i\beta_{11}C_\ell(s) = 0 \tag{2.11}$$

We can play the same trick as before and define

$$\bar\ell \equiv \ell - 2i\beta_{11}C_\ell(s) \tag{2.12}$$

so we get moving poles in the complex ℓ plane

$$\bar\ell - \alpha(s) = 0 \tag{2.13}$$