Day 1

- POINCARÉ GROUP

$\mathbb{R}^4 \rtimes SO(3,1)$

$$ds^2 = c^2 dt^2 - d\vec{r}^{\,2} = g_{\mu\nu} dx^\mu dx^\nu$$

$$g_{\mu\nu} = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix} \qquad x^\mu = (t, \vec{r})$$

$$\Lambda^\mu{}_\nu \, g^{\nu\alpha} \, \Lambda^\beta{}_\alpha = g^{\mu\beta}$$

$$\det(\Lambda) = 1 \qquad \Lambda^0{}_0 > 0$$

GENERATOR OF TRASLATIONS P^μ $\quad [P^\mu, P^\nu] = 0$

GENERATOR OF LORENTZ TR. $J^{\mu\nu} = -J^{\nu\mu}$

$$J^{ij} = \varepsilon^{ijk} J^k \qquad J^{0i} = K^i$$

$$[J^{\mu\nu}, P^\rho] = i\left(g^{\mu\rho} P^\nu - g^{\nu\rho} P^\mu\right)$$

$$[J^{\mu\nu}, J^{\alpha\beta}] = i\left(J^{\mu\beta} g^{\nu\alpha} - J^{\mu\alpha} g^{\nu\beta} - J^{\nu\beta} g^{\mu\alpha} + J^{\nu\alpha} g^{\mu\beta}\right)$$

$[P^i, P^j] = 0 \qquad [P^i, E] = 0 \qquad E = P^0$

$[J^i, P^j] = i\varepsilon^{ijk} P^k \qquad [J^i, E] = 0 \qquad \mu = 0, \overbrace{\underset{x}{1}, \underset{y}{2}, \underset{z}{3}}^{i}$

$[K^i, P^j] = -i\delta^{ij} E \qquad [K^i, E] = -iP^i \qquad \varepsilon^{ijk} \quad \varepsilon^{123} = 1$

$[J^i, J^j] = i\varepsilon^{ijk} J^k \qquad [J^i, K^j] = i\varepsilon^{ijk} K^k$

$[K^i, K^j] = -i\varepsilon^{ijk} J^k$

CASIMIR OPERATORS: QUADRATIC COMBINATIONS OF THE GEN. WHICH COMMUTE WITH ALL THE GENERATORS

$[P^2, \cdot\,] = 0$

$$P^2 |m\rangle = m^2 |m\rangle$$

STATES ARE:

- EIGENSTATES OF P^2
- $m^2 \geq 0$
- $P^0 |\ \rangle = E |\ \rangle \quad E > 0$

GROUP DESCRIBES A SYMMETRY

CONTINUOUS SYMMETRIES

LIE GROUPS , DIFFERENTIABLE GROUP

INFINITESIMAL TRASFORMATIONS CLOSE TO IDENTITY

$$\sim \mathbb{1} + i\varepsilon\, \underset{\text{GENERATOR OF THE GROUP}}{\mathcal{J}}$$

ALL GENERATORS VECTOR SPACE

ALGEBRA OF THE GROUP

$[\,\cdot\,,\,\cdot\,]$ COMMUTATOR

$[\,\cdot^a, \cdot^b\,] = i f^{abc} \cdot^c$ $\qquad f^{abc}$ = STRUCTURE CONSTANT OF THE ALGEBRA

REPRESENTATION OF A GROUP

PAULI-LUBANSKI VECTOR

$$W_\mu = -\frac{1}{2}\varepsilon_{\mu\nu\rho\sigma} P^\nu J^{\rho\sigma}$$

$$\varepsilon_{0123} = 1$$

$$\varepsilon^{0123} = -1$$

$$W^2 = W_\mu W^\mu \qquad [W^2, \cdot\,] = 0$$

$$J^{\mu\nu} = -J^{\nu\mu}$$

$$W_\mu = -\frac{1}{2}\varepsilon_{\mu 0\rho\sigma} P^0 J^{\rho\sigma} =$$

$$= +\frac{1}{2}\varepsilon_{0\mu\rho\sigma} P^0 J^{\rho\sigma} =$$

$$= +\frac{1}{2}\varepsilon_{0ijk} P^0 J^{jk} =$$

$$= -\frac{1}{2}\varepsilon^{0ijk} P^0 J^{jk} = - \quad J^i m$$

$$J^{ij} = \varepsilon^{ijk} J^k$$

$$J^{0i} = K^i$$

$$\boxed{(m, 0, 0, 0)}$$

$$W_\mu = (0, -m\vec{J}\,) \qquad W^2 = m^2 \vec{J}^{\,2}$$

$$W^\mu = (0, m\vec{J}\,)$$

- FINITE-DIMENSIONAL IRREDUCIBLE REPRESENTATION

$\Downarrow$

SPIN

$$|\vec{P}, s, m, n\rangle$$

$$\langle \vec{p}', s', m', n' | \vec{p}, s, m, n \rangle = \delta^3(\vec{p}-\vec{p}')\, \delta_{ss'}\, \delta_{mm'}\, \delta_{nn'}$$

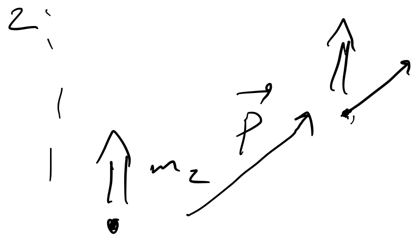
$(E, 0, 0, E)$

$(2\pi)^3 (2E_p)$

$|p, s, m, n\rangle =$ PARTICLE OF MOMENTUM $\vec{p}$
AND ENERGY E
AND W^2 EIGENVALUE $-m^2 s(s+1)$
AND PROJECTION OF $J_z = m$
AND (WHATEVER) n

$$J^2 |\vec{0}, m_z\rangle = s(s+1) |\vec{0}, m_z\rangle \qquad J_z |\vec{0}, m_z\rangle = m_z |\vec{0}, m_z\rangle$$

$$\vec{P} |\vec{0}, m_z\rangle = \vec{0}$$



$$|\vec{p}, m_z\rangle = L(\vec{p}) |\vec{0}, m_z\rangle$$

CANONICAL STATE

$$|\vec{p}, \mu\rangle = R(-R) L_z(|\vec{p}|) |\vec{0}, m_z\rangle$$

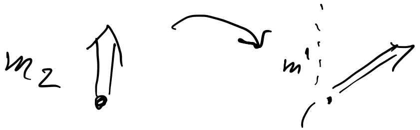
$$R(\Omega) = e^{-iJ_z\alpha}\, e^{-iJ_y\beta}\, e^{-iJ_z\gamma}$$

$$R(\Omega)\,|s, m_z\rangle \doteq D^s_{m_z m'}(\alpha, \beta, \gamma)\,|s, m'\rangle$$

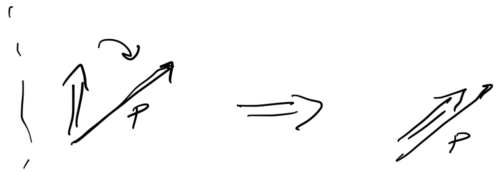
$$D^s_{m_z m'}(\varphi, \theta, \gamma)\,|s, m'\rangle =$$

$$\langle s, m'|\, D^s_{m_z m'}(\varphi, \theta, 0)\, e^{-im_z\gamma}\, |s, m'\rangle = e^{-i\varphi m'}\, d^s_{m_z m'}(\theta)\, e^{-im_z\gamma}$$

CONV. $\begin{cases} \gamma = 0 & \text{I DON'T CARE} \\ \gamma = -\varphi & \Longrightarrow \quad \theta = 0 \quad e^{-im'\varphi}\, \delta_{m_z m'}\, e^{-im_z(-\varphi)} = 1 \end{cases}$



$$|\vec{p}, \mu\rangle = D^{s}_{\mu m_z}(\varphi, \theta, -\varphi)\, |\vec{p}, m_z\rangle$$

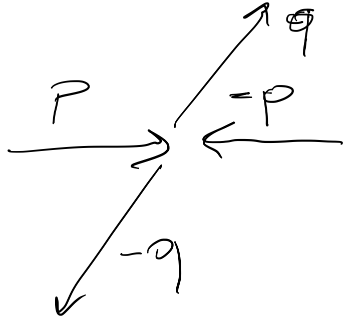


$\frac{\vec{J}\cdot\vec{p}}{|\vec{p}|}$ COMMUTES (SMALL) BOOSTS ALONG THE DIRECTION OF MOTION

WHAT HAPPENS IF I BOOST IN A RANDOM DIRECTION ?

$$[K^i, K^j] = -i\varepsilon^{ijk} J^k$$

WIGNER ROTATION



$$t+u = 2m_1^2 + m_3^2 + m_4^2 - 2E_1 \overbrace{(E_3+E_4)}^{\sqrt{s}}$$

$$= 2m_1^2 + m_3^2 + m_4^2 - 2\,\frac{s+m_1^2-m_2^2}{\cancel{2\sqrt{s}}}\,\cancel{\sqrt{s}}$$

$$= m_1^2 + m_2^2 + m_3^2 + m_4^2 - s$$

$$\boxed{s+t+u = \sum m_i^2}$$

$$t = (p_1-p_3)^2 = m_1^2 + m_3^2 - 2E_1E_3 + 2pq\cos\theta$$

$$4pq\,s = \lambda^{\frac12}(\ )\,\lambda^{\frac12}(\ )$$

$$u = (p_1-p_4)^2 = m_1^2 + m_4^2 - 2E_1E_4 - 2pq\cos\theta$$

$$t-u = 4pq\cos\theta + \cancel{m_1^2} + m_3^2 - \cancel{m_1^2} - m_4^2 - 2E_1(E_3-E_4)$$

$$-2E_1\,\frac{(E_3^2-E_4^2)}{E_3+E_4} = -\frac{2E_1(m_3^2-m_4^2)}{\sqrt{s}}$$

$$4pq\cos\theta + \cancel{m_1^2} - \cancel{m_4^2} + \frac{(m_3^2-m_4^2)}{\sqrt{s}}\Big(\overbrace{\sqrt{s} - \cancel{2E_1}}^{E_2-E_1}\Big) \qquad \frac{m_2^2-m_1^2}{\sqrt{s}}$$

$$4pq\cos\theta + \frac{(m_3^2-m_4^2)(m_2^2-m_1^2)}{s} \qquad \cos\theta = \frac{(t-u)s + (m_1^2-m_2^2)(m_3^2-m_4^2)}{4pq\,s}$$

$$\frac{d^3p_1}{\cancel{(2\pi)^3}\,2E_1}\,\frac{d^3p_2}{\cancel{(2\pi)^3}\,2E_2}\,(2\pi)^{\cancel{4}\,2}\,\delta^4(P-p_1-p_2)$$

$$\frac{d^3p_2}{2E_2} = d^4p_2\;\delta(p_2^2-m_2^2)\,\theta(p_2^0)$$

$$\vec{P}=\vec{0}$$

$$\frac{d^3p_1}{2E_1}\,\cancel{d^4p_2}\;\delta(p_2^2-m_2^2)\;\theta(p_2^0)\;\cancel{\delta^4(P-p_1-p_2)}$$

$$\frac{d^3p_1}{2E_1}\;\delta\left((P-p_1)^2-m_2^2\right)\;\theta(P^0-p_1^0)$$

$$\frac{\delta\left(E_1-\frac{s+m_1^2-m_2^2}{2\sqrt{s}}\right)}{2\sqrt{s}}$$

$$\frac{d^3p_1}{2E_1}\;\underbrace{\delta\left(s-2\sqrt{s}E_1+m_1^2-m_2^2\right)\theta(P^0-p_1^0)}_{}$$

$$p_1\,dp_1 = E_1\,dE_1$$

$$\frac{d\Omega\,p_1^2\,dp_1}{2E_1}\qquad\frac{d\Omega\,p_1\,dE_1}{2} = \frac{d\Omega}{2}\,\frac{p_1}{2\sqrt{s}}\,4\pi^2 = \frac{d\Omega}{4\pi}\underbrace{\left(\frac{p_1}{4\pi\sqrt{s}}\right)}_{\rho(s)}$$

$$\rho(s) = \frac{1}{4\pi} \frac{\sqrt{\left(s - (m_1 + m_2)^2\right)\left(s - (m_1 - m_2)^2\right)}}{2s}$$

THRESHOLD

PSEUDO THRESHOLD