SSRT Lecture 1 : Partial Wave Expansions

Group 2

Wednesday June 14, 2017

8:30 – 9:30

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- Particles are irreducible representations of Poincaré group

$$\mathcal{P} = \mathbb{R}^{1,3} \rtimes SO(1,3)$$

$$|\text{Particle}\rangle = |\vec{p}, \sigma\rangle$$

$\vec{p}$: 3-momentum; σ: spin projection

$$\langle \vec{p}', \sigma' | \vec{p}, \sigma\rangle = (2\pi)^3 2E(\vec{p})\, \delta^3(\vec{p}' - \vec{p})\, \delta_{\sigma'\sigma}$$

$$E(\vec{p}) = \sqrt{\vec{p}^2 + m^2}$$

$$P^2 |\vec{p}, \sigma\rangle = m^2 |\vec{p}, \sigma\rangle$$

$$W^2 |\vec{p}, \sigma\rangle = -m^2 s(s+1) |\vec{p}, \sigma\rangle$$

$$P^\mu |\vec{p}, \sigma\rangle = p^\mu |\vec{p}, \sigma\rangle$$

$$W^3 |\vec{p} = 0, \sigma\rangle = -m\sigma |\vec{p}, \sigma\rangle$$

- If system has other symmetries (e.g., isospin), symmetry group is $\underset{\text{internal symmetry group}}{G} \times \underset{\text{Poincaré group}}{\mathcal{P}}$

Two particle state (Assume distinguishable particles)

$$|\vec{p}_1, \sigma_1 ; \vec{p}_2, \sigma_2\rangle = |\vec{p}_1, \sigma_1\rangle |\vec{p}_2, \sigma_2\rangle$$

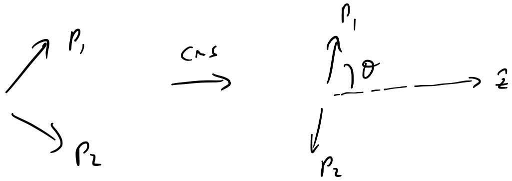
Can go to CMS frame

$$|\vec{p}_1, \sigma_1 ; \vec{p}_2, \sigma_2\rangle \xrightarrow[\text{CMS}]{} |\vec{P}, \sqrt{s}, \hat{p}, \sigma_1, \sigma_2\rangle$$

$\sqrt{s}$: invariant mass of 2 particle system

$\vec{P}$: total momentum, $\vec{P} = \vec{p}_1 + \vec{p}_2 = \vec{0}$ in CMS

$\hat{p}$: angle of particle 1 in CMS frame, $\hat{p} = (\theta, \varphi)$



$$s = (p_1 + p_2)^2$$

Two particle states obey same Relativistic symmetries

→ can recouple to state of definite spin (like LS in NRQM)

In LS basis (spin quantized in some fixed $\hat{z}$ direction)

$$|\sigma_1,\sigma_2\rangle = \sum_{s,m_s} |s m_s\rangle\langle s m_s| s_1\sigma_1, s_2\sigma_2\rangle \equiv \sum_{sm_s} |sm_s\rangle C^{sm_s}_{s_1\sigma_1 s_2\sigma_2}$$

$$|\hat{p}\rangle = \sum_{lm_l} |l m_l\rangle\langle l m_l|\hat{p}\rangle \quad , \quad Y_{lm_l}(\hat{p}) = \langle \hat{p}| l m_l\rangle$$

So,

$$|\vec{P},\sqrt{s},\hat{p},\sigma_1,\sigma_2\rangle = \sum_{lm_l}\sum_{sm_s} |\vec{P},\sqrt{s}\rangle |lm_l\rangle |sm_s\rangle C^{sm_s}_{s_1\sigma_1 s_2\sigma_2} Y^*_{lm_l}(\hat{p})$$

Now, recouple to total angular momentum J

$$|lm_l, sm_s\rangle = \sum_{JM} |JMls\rangle\langle JM| lm_l sm_s\rangle$$

$$\equiv \sum_{JM} |JMls\rangle C^{JM}_{lm_l sm_s}$$

$$\Rightarrow |\vec{P},\sqrt{s},\hat{p},\sigma_1,\sigma_2\rangle = \sum_{JM}\sum_{\substack{lm_l\\ sm_s}} |\vec{P},\sqrt{s}; JMls\rangle Y^*_{lm_l}(\hat{p}) C^{JM}_{lm_l sm_s} C^{sm_s}_{s_1\sigma_1 s_2\sigma_2}$$

↑ state of definite momentum & orientation

↑ state of definite angular momentum

- Parity & Time reversal on $|JM\ell s\rangle$ state

$$\mathbb{P}|\vec{P},\sqrt{s},\hat{p},\sigma_1,\sigma_2\rangle = \eta_1\eta_2\,|\vec{P},\sqrt{s},\hat{p}',\sigma_1,\sigma_2\rangle$$

intrinsic parities of particles 1 & 2

w/ $\hat{p} = (\theta,\varphi)$, $\hat{p}' = (\pi-\theta,\varphi+\pi)$

$$\Rightarrow \quad \mathbb{P}|\vec{P},\sqrt{s};JM\ell s\rangle = \eta_1\eta_2(-)^{\ell}\,|\vec{P},\sqrt{s};JM\ell s\rangle$$

$$\mathbb{T}\,|\vec{P},\sqrt{s},\hat{p},\sigma_1,\sigma_2\rangle = (-)^{s_1-\sigma_1}(-)^{s_2-\sigma_2}\,|\vec{P},\sqrt{s};\hat{p}',-\sigma_1,-\sigma_2\rangle$$

Note: No intrinsic "T"-parity as $\mathbb{T}$ is antiunitary!

$$\Rightarrow \quad \mathbb{T}\,|\vec{P},\sqrt{s};JM\ell s\rangle = (-)^{J-M}\,|J-M\ell s\rangle$$

Example: Constructing the $q\bar{q}$ (Meson) Quantum #'s

Quark: $s = \frac{1}{2}$, $\sigma = \pm\frac{1}{2}$

So,

$$|q\bar{q}\rangle = |s_1 = \tfrac{1}{2}, \sigma_1 = \pm\tfrac{1}{2} \,;\, s_2 = \tfrac{1}{2}, \sigma_2 = \pm\tfrac{1}{2}\rangle$$

$$= \sum_{S m_s} |S m_s\rangle\langle S m_s | s_1 \sigma_1\, s_2 \sigma_2\rangle$$

Allowed Q#'s: $S = s_1 \otimes s_2 = \frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1$

$$m_s = \sigma_1 + \sigma_2$$

&

$$|q\bar{q}\rangle = \sum_{\ell m_\ell} |\ell m_\ell\rangle\langle \ell m_\ell | \hat{p}_q\rangle$$

$$\ell = 0, 1, 2, \ldots$$
$$m_\ell = -\ell, -\ell+1, \ldots, \ell-1, \ell$$

Total angular momentum

$$|q\bar{q}\rangle = \sum_{JM} \sum_{\substack{\ell m_\ell \\ S m_s}} |JM\ell S\rangle\, C^{JM}_{\ell m_\ell S m_s}\, C^{S m_s}_{s_1\sigma_1 s_2\sigma_2}\, Y^*_{\ell m_\ell}(\hat{p}_q)$$

Allowed Q#'s: $J = \ell \otimes S$

$J = l \otimes s$

$l=0 \rightarrow$ S-wave
$l=1 \rightarrow$ P-wave
$l=2 \rightarrow$ D-wave
$l=3 \rightarrow$ F-wave

s	l	J
0	0	0
0	1	1
0	2	2
0	3	3
1	0	1
1	1	0, 1, 2
1	2	1, 2, 3
1	3	2, 3, 4

Spectroscopic Notation

$^{2S+1}l_J$

e.g., $S=0,\ l=1,\ J=1 \rightarrow {}^1P_1$

e.g., $S=1,\ l=0,\ J=1 \rightarrow {}^3S_1$

e.g., $S=1,\ l=2,\ J=2 \rightarrow {}^3D_2$

Allowed parity of Mesons: $\eta_q = +1,\ \eta_{\bar{q}} = -1$

$\Rightarrow\ P = -(-)^l = (-)^{l+1} \Rightarrow$

$l=0 \rightarrow P = -$
$l=1 \rightarrow P = +$
$l=2 \rightarrow P = -$
$l=3 \rightarrow P = +$

$\boxed{q\bar{q}}$

S	l	J	P	J^P	$^{2S+1}l_J$
0	0	0	−	0^-	$^1S_0$
0	1	1	+	1^+	$^1P_1$
0	2	2	−	2^-	$^1D_2$
0	3	3	+	3^+	$^1F_3$
1	0	1	−	1^-	$^3S_1$
1	1	0, 1, 2	+	$0^+, 1^+, 2^+$	$^3P_0,\ ^3P_1,\ ^3P_2$
1	2	1, 2, 3	−	$1^-, 2^-, 3^-$	$^3D_1,\ ^3D_2,\ ^3D_3$
1	3	2, 3, 4	+	$2^+, 3^+, 4^+$	$^3F_2,\ ^3F_3,\ ^3F_4$

Example: $\pi\pi$ System

intrinsic parity of π

π : $J^P = 0^-$ $\Rightarrow$ Pion is pseudoscalar

spin of π

$$|\pi(p_1)\,\pi(p_2)\rangle = |\vec{P}, \sqrt{s}, \hat{p}\rangle$$

$\leftarrow$ orientation of π_1 in $\pi_1-\pi_2$ rest frame

$\vec{P}$: Total momentum of 2π system

$\sqrt{s}$: invariant mass of 2π system

Note: No spin indices!

Recouple to Total angular momentum

$$|\hat{p}\rangle = \sum_{\ell m_\ell} |\ell m_\ell\rangle\langle \ell m_\ell|\hat{p}\rangle = \sum_{\ell m_\ell} |\ell m_\ell\rangle\, Y^*_{\ell m_\ell}(\hat{p})$$

&

$$|\pi(p_1)\,\pi(p_2)\rangle = \sum_{JM}\sum_{\ell m_\ell} |\vec{P},\sqrt{s}\rangle\, |JM\rangle \underbrace{\langle JM|\ell m_\ell, 00\rangle}_{\langle JM|\ell m_\ell 00\rangle = \delta_{J\ell}\,\delta_{M m_\ell}} Y^*_{\ell m_\ell}(\hat{p})$$

$$= \sum_{JM} |\vec{P},\sqrt{s}; JM\rangle\, Y^*_{JM}(\hat{p})$$

Allowed Q#'s : $J = \ell \otimes s = \ell$ $\Rightarrow$ $J = 0, 1, 2, \ldots$

Parity : $P = \eta_\pi \eta_\pi (-)^\ell = (-)^2 (-)^\ell = (-)^\ell$

So,

ℓ	J	P	J^P	
0	0	+	0^+	$\rightarrow$ σ, $f_0(980)$
1	1	−	1^-	$\rightarrow$ $\rho(770)$
2	2	+	2^+	$\rightarrow$ $f_2(1270)$

Exercise: πN System

Find the allowed J^P quantum #'s by angular momentum addition.
Write the $|\pi N\rangle$ state as an expansion in partial waves.

Solution

$$\pi : \quad J^P = 0^-$$
$$N : \quad J^P = \frac{1}{2}^+$$

$$|\pi(p_1) N(p_2, \sigma)\rangle = |\vec{p}, \sqrt{s}, \hat{p}, \sigma\rangle$$

$$|\hat{p}\rangle = \sum_{l m_l} |l m_l\rangle\langle l m_l|\hat{p}\rangle = \sum_{l m_l} |l m_l\rangle Y_{l m_l}(\hat{p}) \qquad , \; l = 0, 1, 2, \ldots$$

$$|s_1, \sigma_1, s_2, \sigma_2\rangle = \sum_{s m_s} |s m_s\rangle\langle s m_s| 0\, 0\; s = \tfrac{1}{2}, \sigma = \pm\tfrac{1}{2}\rangle$$
$$= |s = \tfrac{1}{2}, m_s = \pm\tfrac{1}{2}\rangle$$

$$\Rightarrow \quad |\pi(p_1) N(p_2, \sigma)\rangle = \sum_{JM} |\vec{p}, \sqrt{s}; J M l s\rangle\langle J M | l m_l\, s = \tfrac{1}{2}, \sigma = \pm\tfrac{1}{2}\rangle Y_{l m_l}(\hat{p})$$

s	l	J	P	J^P	$^{2s+1}l_J$
$\frac{1}{2}$	0	$\frac{1}{2}$	$-$	$\frac{1}{2}^-$	$^2S_{\frac{1}{2}}$
$\frac{1}{2}$	1	$\frac{1}{2}, \frac{3}{2}$	$+$	$\frac{1}{2}^+, \frac{3}{2}^+$	$^2P_{\frac{1}{2}}, {}^2P_{\frac{3}{2}}$
$\frac{1}{2}$	2	$\frac{3}{2}, \frac{5}{2}$	$-$	$\frac{3}{2}^-, \frac{5}{2}^-$	$^2D_{\frac{3}{2}}, {}^2D_{\frac{5}{2}}$
$\frac{1}{2}$	3	$\frac{5}{2}, \frac{7}{2}$	$+$	$\frac{5}{2}^+, \frac{7}{2}^+$	$^2F_{\frac{5}{2}}, {}^2F_{\frac{7}{2}}$

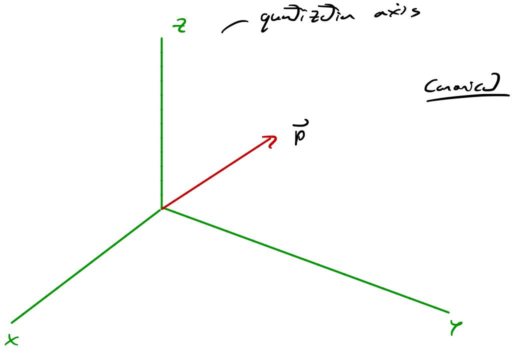
$$M = m_l \pm \tfrac{1}{2}$$
$$J = l \otimes s$$
$$= l \pm \tfrac{1}{2}$$
$$P = \eta_\pi \eta_N (-1)^l$$
$$= (-)^{l+1}$$

Helicity states

So far, we have discussed canonical (LS) spin states

$\rightarrow$ states quantized wrt some fixed axis in space

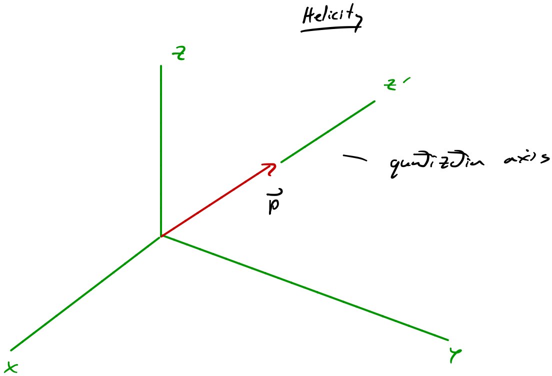
$$|\vec{p}, \sigma\rangle$$



Can quantize spin along direction of momentum – Denote this as helicity

$$\sigma \rightarrow \lambda$$

$$|\vec{p}, \lambda\rangle$$



- Canonical states are defined

$$|\vec{p}, \sigma\rangle = U(R)\,U(B)\,U^{-1}(R)\,|\vec{0}, \sigma\rangle$$

- Helicity states are defined

$$|\vec{p}, \lambda\rangle = U(R)\,U(B)\,|\vec{0}, \lambda\rangle$$

Two particle helicity states

$$|\vec{p}_1 \lambda_1, \vec{p}_2 \lambda_2\rangle = |\vec{p}_1 \lambda_1\rangle |\vec{p}_2 \lambda_2\rangle$$

$$\xrightarrow[\text{cms}]{} |\vec{P}, \sqrt{s}, \hat{p}, \lambda_1 \lambda_2\rangle$$

Recouple to total angular momentum

$$|\hat{p}, \lambda_1 \lambda_2 \rangle = \sum_{\substack{JM \\ \lambda_1' \lambda_2'}} |JM \lambda_1' \lambda_2' \rangle \langle JM \lambda_1' \lambda_2' | \hat{p} \lambda_1 \lambda_2 \rangle$$

$$\langle \hat{p} \lambda_1 \lambda_2 | JM \lambda_1' \lambda_2' \rangle = \sqrt{\frac{2J+1}{4\pi}} \, \mathcal{D}^{J*}_{M\lambda}(\hat{p}) \, \delta_{\lambda_1 \lambda_1'} \delta_{\lambda_2 \lambda_2'}$$

$\lambda = \lambda_1 - \lambda_2$

Convenient normalization

$$\Rightarrow \quad |\hat{p}, \lambda_1 \lambda_2 \rangle = \sum_{JM} |JM \lambda_1 \lambda_2 \rangle \sqrt{\frac{2J+1}{4\pi}} \, \mathcal{D}^{J*}_{M\lambda}(\hat{p})$$

Two-particle helicity state

Parity: $\mathbb{P} |JM \lambda_1 \lambda_2 \rangle = \eta_1 \eta_2 (-)^{J - s_1 - s_2} |JM -\lambda_1 -\lambda_2 \rangle$

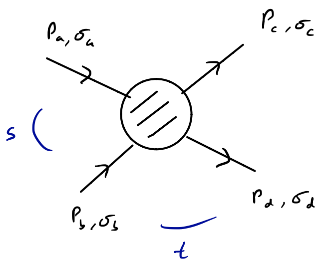
T-reversal: $\mathbb{T} |JM \lambda_1 \lambda_2 \rangle = (-)^{J-M} |J -M \lambda_1 \lambda_2 \rangle$

Partial Wave Expansions

→ Expand reaction amplitudes into definite angular momentum

Consider $2 \to 2$ scattering

- kinematic variables



$$s = (p_a + p_b)^2$$
$$t = (p_a - p_c)^2$$
$$u = (p_a - p_d)^2$$

$$s + t + u = \sum_i m_i^2$$

In cms:
$$\vec{p} \equiv \vec{p}_a = -\vec{p}_b \qquad \vec{p} = |\vec{p}|\,(\sin\theta\cos\varphi,\ \sin\theta\sin\varphi,\ \cos\theta)$$
$$\vec{p}\,' \equiv \vec{p}_c = -\vec{p}_d \qquad \vec{p}\,' = |\vec{p}\,'|\,(\sin\theta'\cos\varphi',\ \sin\theta'\sin\varphi',\ \cos\theta')$$

Scattering angle: $\cos\Theta = \hat{p}\cdot\hat{p}' = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\varphi - \varphi')$

$$s = (E_a + E_b)^2$$
$$t = m_a^2 + m_c^2 - 2E_aE_c + 2|\vec{p}||\vec{p}\,'|\cos\Theta$$

$$E_a = \frac{s + m_a^2 - m_b^2}{2\sqrt{s}} \qquad E_c = \frac{s + m_c^2 - m_d^2}{2\sqrt{s}} \qquad |\vec{p}| = \frac{\lambda^{\frac{1}{2}}(s, m_a^2, m_b^2)}{2\sqrt{s}}$$
$$E_b = \frac{s + m_b^2 - m_a^2}{2\sqrt{s}} \qquad E_d = \frac{s + m_d^2 - m_c^2}{2\sqrt{s}} \qquad |\vec{p}\,'| = \frac{\lambda^{\frac{1}{2}}(s, m_c^2, m_d^2)}{2\sqrt{s}}$$

$$\cos\Theta = 1 - \frac{t'}{2|\vec{p}||\vec{p}\,'|}, \quad t' = t - t_{min}$$

- Scattering Amplitude

$$S = \mathbb{1} + iT$$

$\Rightarrow \quad \langle \vec{p}_c\sigma_c, \vec{p}_d\sigma_d | T | \vec{p}_a\sigma_a, \vec{p}_b\sigma_b \rangle$ Reaction matrix element

Now, $|\vec{p}_a\sigma_a, \vec{p}_b\sigma_b\rangle = |\vec{P}\sqrt{s}\,\hat{p}, \sigma_a\sigma_b\rangle$

$|\vec{p}_c\sigma_c, \vec{p}_d\sigma_d\rangle = |\vec{P}'\sqrt{s'}\,\hat{p}', \sigma_c\sigma_d\rangle$

$$\Rightarrow \langle \vec{p}_c\sigma_c, \vec{p}_d\sigma_d | T | \vec{p}_a\sigma_a, \vec{p}_b\sigma_b \rangle = \langle \vec{P}'\sqrt{s'}\,\hat{p}', \sigma_c\sigma_d | T | \vec{P}\sqrt{s}\,\hat{p}, \sigma_a\sigma_b \rangle$$

$$= (2\pi)^4 \delta^4(P'-P) \langle \hat{p}', \sigma_c\sigma_d | T | \hat{p}, \sigma_a\sigma_b \rangle$$

$$\equiv (2\pi)^4 \delta^4(P'-P)\, A_{\substack{\sigma_a\sigma_b\\ \sigma_c\sigma_d}}(s,t)$$

depends on s since $s'=s$

- Canonical (LS) basis

We can expand the state $|\hat{p}, \sigma_a\sigma_b\rangle = \sum_{JM} \sum_{\substack{\ell m_\ell \\ s m_s}} |JM\ell s\rangle\, C^{JM}_{\ell m_\ell s m_s}\, C^{s m_s}_{s_a\sigma_a s_b\sigma_b}\, Y_{\ell m_\ell}(\hat{p})$

$$\Rightarrow A_{\substack{\sigma_a\sigma_b\\ \sigma_c\sigma_d}}(s,t) = \langle \hat{p}', \sigma_c\sigma_d | T | \hat{p}, \sigma_a\sigma_b \rangle$$

$$= \sum_{JM} \sum_{\substack{\ell m_\ell \\ s m_s}} \sum_{J'M'} \sum_{\substack{\ell' m_\ell' \\ s' m_s'}} \langle J'M'\ell's' | T | JM\ell s\rangle \times$$

$$\times\, C^{J'M'^*}_{\ell' m_\ell' s' m_s'}\, C^{JM}_{\ell m_\ell s m_s}\, C^{s' m_s'^*}_{s_c\sigma_c s_d\sigma_d}\, C^{s m_s}_{s_a\sigma_a s_b\sigma_b}\, Y^*_{\ell' m_\ell'}(\hat{p}')\, Y_{\ell m_\ell}(\hat{p})$$

Total angular momentum is conserved

$$\langle J'M'\ell's'|T|JM\ell s\rangle = \delta_{JJ'}\,\delta_{MM'}\,\langle \ell's'|T^J|\ell s\rangle$$

$$= \delta_{J'J}\,\delta_{M'M}\; a^J_{\ell s,\,\ell's'}(s)$$

independent of M
(Wigner-Eckart)

So,

$$\boxed{A_{\sigma_a\sigma_b,\,\sigma_c\sigma_d}(s,t) = \sum_{JM}\sum_{\ell m_\ell\, s m_s}\sum_{\ell' m_\ell'\, s' m_s'} a^J_{\ell s,\,\ell' s'}(s)\; C^{JM\,*}_{\ell' m_\ell'\, s' m_s'}\; C^{JM}_{\ell m_\ell\, s m_s}\; C^{s' m_s'\,*}_{s_c\sigma_c\, s_d\sigma_d}\; C^{s m_s}_{s_a\sigma_a\, s_b\sigma_b}\; Y^*_{\ell' m_\ell'}(\hat{p}')\; Y_{\ell m_\ell}(\hat{p})}$$

Partial wave expansion in LS basis

- Helicity Basis

$$\langle \vec{p}_c\lambda_c, \vec{p}_d\lambda_d|T|\vec{p}_a\lambda_a, \vec{p}_b\lambda_b\rangle = (2\pi)^4\,\delta^4(p'-p)\; A_{\lambda_a\lambda_b,\,\lambda_c\lambda_d}(s,t)$$

Now, expand

$$|\hat{p}, \lambda_a\lambda_b\rangle = \sum_{JM}\sqrt{\frac{2J+1}{4\pi}}\; |JM\lambda_a\lambda_b\rangle\; D^J_{M\lambda}(\hat{p})$$

$\lambda = \lambda_a - \lambda_b$

$$\Rightarrow\quad A_{\lambda_a\lambda_b,\,\lambda_c\lambda_d}(s,t) = \sum_{JM}\sum_{J'M'}\sqrt{\frac{2J+1}{4\pi}}\sqrt{\frac{2J'+1}{4\pi}}\;\langle J'M'\lambda_c\lambda_d|T|JM\lambda_a\lambda_b\rangle\; D^{J'*}_{M'\lambda'}(\hat{p}')\; D^J_{M\lambda}(\hat{p})$$

$\lambda' = \lambda_c - \lambda_d$

Now, $\langle J'M'\lambda_c\lambda_d|T|JM\lambda_a\lambda_b\rangle = \delta_{J'J}\,\delta_{M'M}\; a^J_{\lambda_a\lambda_b,\,\lambda_c\lambda_d}(s)$

$$A_{\substack{\lambda_a\lambda_b\\ \lambda_c\lambda_d}}(s,t) = \sum_{JM}\left(\frac{2J+1}{4\pi}\right) a^J_{\substack{\lambda_a\lambda_b\\ \lambda_c\lambda_d}}(s)\, D^{J*}_{M\lambda'}(\hat{p}')\, D^J_{M\lambda}(\hat{p})$$

Since a^J is independent of M

$$\Rightarrow \quad \sum_M D^{J*}_{M\lambda'}(\hat{p}')\, D^J_{M\lambda}(\hat{p}) = D^{J*}_{\lambda\lambda'}(\hat{p}\cdot\hat{p}')$$

$$\Rightarrow \quad \boxed{A_{\substack{\lambda_a\lambda_b\\ \lambda_c\lambda_d}}(s,t) = \sum_J \left(\frac{2J+1}{4\pi}\right) a^J_{\substack{\lambda_a\lambda_b\\ \lambda_c\lambda_d}}(s)\, D^{J*}_{\lambda\lambda'}(\hat{p}\cdot\hat{p}')}$$

Partial wave expansion in helicity basis

Special case: Scalar particles

e.g., $\pi^+\pi^- \to \pi^+\pi^-$

$$s_a = s_b = s_c = s_d = 0$$

in LS: $\quad S=0, m_s=0 \quad \Rightarrow \quad J = \ell = \ell' \quad \Rightarrow \quad C^{JM}_{\ell' m_\ell' 00} = \delta_{J\ell'}\,\delta_{Mm_\ell'}$

$$C^{JM}_{\ell m_\ell 00} = \delta_{J\ell}\,\delta_{Mm_\ell}$$

$$A(s,t) = \sum_{JM}\sum_{\ell m_\ell}\sum_{\ell' m_\ell'} a^J_{\ell\ell'}(s)\,\delta_{J\ell}\,\delta_{Mm_\ell}\,\delta_{J\ell'}\,\delta_{Mm_\ell'}\, Y^*_{\ell' m_\ell'}(\hat{p}')\, Y_{\ell m_\ell}(\hat{p})$$

$$= \sum_{JM} a^J(s)\, Y^*_{JM}(\hat{p}')\, Y_{JM}(\hat{p})$$

$$A(s,t) = \sum_{JM} a^J(s)\, Y^*_{JM}(\hat{p}')\, Y_{JM}(\hat{p})$$

Spherical Harmonic addition $\sum_{M} Y^*_{JM}(\hat{p}')\, Y_{JM}(\hat{p}) = \frac{2J+1}{4\pi} P_J(\hat{p}'\cdot\hat{p})$

$$\Rightarrow \boxed{A(s,t) = \sum_J \left(\frac{2J+1}{4\pi}\right) a^J(s)\, P_J(\hat{p}'\cdot\hat{p})}$$

Scalar partial wave expansion

Projection: $a^J(s) = \int_0^{2\pi} d\varphi \int_{-1}^{1} d\cos\Theta \left(\frac{2J+1}{4\pi}\right) P_J(\hat{p}'\cdot\hat{p})\, A(s, t(\Theta))$

Since $\int_0^{2\pi} d\varphi \int_{-1}^{1} d\cos\Theta\, P_{J'}(\cos\Theta)\, P_J(\cos\Theta) = \frac{4\pi}{2J+1}\delta_{J'J}$

Exercise: Reduce the helicity basis partial wave expansion for the case of scalar particles

Solution: $A(s,t) = \sum_J \left(\frac{2J+1}{4\pi}\right) a^J(s)\, D^J_{00}(\hat{p}'\cdot\hat{p})$

but, $D^J_{00}(\hat{p}'\cdot\hat{p}) = P_J(\hat{p}'\cdot\hat{p})$

$\Rightarrow A(s,t) = \sum_{JM} \left(\frac{2J+1}{4\pi}\right) a^J(s)\, P_J(\hat{p}'\cdot\hat{p})$ ← Same as before!

Unitarity

$$S = \mathbb{1} + iT$$

$$S^\dagger S = \mathbb{1} \Rightarrow (\mathbb{1}+iT)^\dagger(\mathbb{1}+iT) = \mathbb{1} \Rightarrow T - T^\dagger = iT^\dagger T$$

Consider initial state $|i\rangle = |\vec{p}_1\sigma_1, \vec{p}_2\sigma_2, \dots, \vec{p}_n\sigma_n\rangle$

final state $|f\rangle = |\vec{p}_1{}'\sigma_1{}', \vec{p}_2{}'\sigma_2{}', \dots, \vec{p}_{n'}{}'\sigma_{n'}{}'\rangle$

for $n \to n'$ scattering

$$d\tilde{k} = \frac{d^3k}{(2\pi)^3 2E(\vec{k})}$$

$$\Rightarrow \quad \langle f|T - T^\dagger|i\rangle = i\langle f|T^\dagger T|i\rangle$$

Insert complete set of intermediate states (m-particles)

$$\mathbb{1} = \sum\!\!\!\!\!\!\int_n |n\rangle\langle n| = \sum_{\sigma_1}\int d\tilde{k}_1\, |\vec{k}_1\sigma_1''\rangle\langle \vec{k}_1\sigma_1''| + \sum_{\sigma_1\sigma_2}\int d\tilde{k}_1\, d\tilde{k}_2\, |\vec{k}_1\sigma_1'', \vec{k}_2\sigma_2''\rangle\langle\vec{k}_1\sigma_1'', \vec{k}_2\sigma_2''|$$

$$+ \sum_{\sigma_1\sigma_2\sigma_3}\int d\tilde{k}_1\, d\tilde{k}_2\, d\tilde{k}_3\, |\vec{k}_1\sigma_1'', \vec{k}_2\sigma_2'', \vec{k}_3\sigma_3''\rangle\langle\vec{k}_1\sigma_1'', \vec{k}_2\sigma_2'', \vec{k}_3\sigma_3''| + \dots$$

$$+ \dots + \sum_{\{\sigma_m\}}\int d\tilde{k}_1 \cdots d\tilde{k}_m\, |\vec{k}_1\sigma_1'', \dots, \vec{k}_m\sigma_m''\rangle\langle\vec{k}_1\sigma_1'', \dots, \vec{k}_m\sigma_m''|$$

$$\Rightarrow \quad \langle f|T|i\rangle - \langle f|T^\dagger|i\rangle = i\sum\!\!\!\!\!\!\int_n \langle f|T^\dagger|n\rangle\langle n|T|i\rangle$$

If T-reversal is a good symmetry $\Rightarrow$ $\langle f|T|i\rangle = \langle i|T|f\rangle$

& $\langle f|T^\dagger|i\rangle = \langle i|T|f\rangle^*$

So, have

$$\langle f|T|i\rangle - \langle f|T|i\rangle^* = i \sum\!\!\!\!\!\!\int_n \langle n|T|f\rangle^* \langle n|T|i\rangle$$

$$\langle f|T|i\rangle - \langle f|T|i\rangle^* = 2i\,\mathrm{Im}\,\langle f|T|i\rangle$$

Now, $\langle f|T|i\rangle = (2\pi)^4 \delta^4(P_i - P_f)\,\mathcal{A}(i\to f)$

$$\langle f|T|n\rangle = (2\pi)^4 \delta^4(P_f - P_n)\,\mathcal{A}(n\to f)$$

$$\langle n|T|i\rangle = (2\pi)^4 \delta^4(P_n - P_i)\,\mathcal{A}(i\to n)$$

w/ $\mathcal{A}(i\to f) = \mathcal{A}_{\{\sigma_n,\sigma'_{n'}\}}(\{\vec{p}_n, \vec{p}'_{n'}\})$

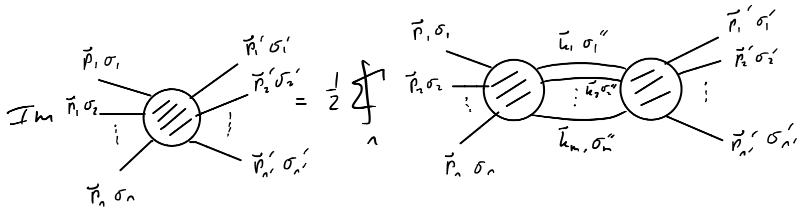
$$\mathcal{A}(n\to f) = \mathcal{A}_{\{\sigma''_m,\sigma'_{n'}\}}(\{\vec{k}_m, \vec{p}'_{n'}\})$$

$$\mathcal{A}(i\to n) = \mathcal{A}_{\{\sigma_n,\sigma''_m\}}(\{\vec{p}_n, \vec{k}_m\})$$

So,

$$(2\pi)^4\delta^4(P_f - P_i)\,2i\,\mathrm{Im}\,\mathcal{A}(i\to f) = i \sum\!\!\!\!\!\!\int_n (2\pi)^4 \delta^4(P_f - P_n)\,\mathcal{A}^*(n\to f)\,(2\pi)^4\delta^4(P_n - P_i)\,\mathcal{A}(i\to n)$$

$$\Rightarrow \quad \mathrm{Im}\,\mathcal{A}(i\to f) = \frac{1}{2}\sum\!\!\!\!\!\!\int_n (2\pi)^4\delta^4(P_n - P_i)\,\mathcal{A}^*(n\to f)\,\mathcal{A}(i\to n)$$



Consider $2 \to 2$ scattering

$$a\, b \to c\, d$$

1-body

$$\Rightarrow \operatorname{Im} \mathcal{A}_{\substack{\sigma_a \sigma_b \\ \sigma_c \sigma_d}}(s, \hat{p}\cdot\hat{p}') = \frac{1}{2}\sum_{\sigma_1} \int d\tilde{k}_1 \, (2\pi)^4 \delta^4(p_a + p_b - k_1)\, \mathcal{A}^*_{\substack{\sigma_c\sigma_d \\ \sigma_1}}(s, \hat{p}'\cdot\hat{k}_1)\, \mathcal{A}_{\substack{\sigma_a\sigma_b \\ \sigma_1}}(s, \hat{p}\cdot\hat{k}_1)$$

$$+ \frac{1}{2}\sum_{\sigma_1\sigma_2} \int d\tilde{k}_1 d\tilde{k}_2 \, (2\pi)^4 \delta(p_a + p_b - k_1 - k_2)\, \mathcal{A}^*_{\substack{\sigma_c\sigma_d \\ \sigma_1\sigma_2}}(s, \hat{p}'\cdot\hat{k})\, \mathcal{A}_{\substack{\sigma_a\sigma_b \\ \sigma_1\ \sigma_2}}(s, \hat{p}\cdot\hat{k})$$

2-body

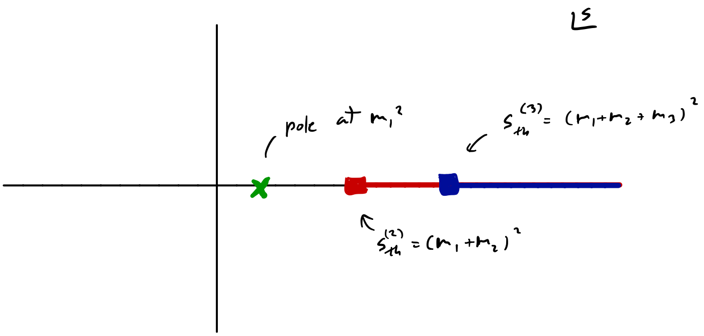
$$+ \cdots$$

3 body & higher $\equiv \mathcal{R}_{\substack{\sigma_a\sigma_b \\ \sigma_c\sigma_d}}(s, \hat{p}'\cdot\hat{p})$

1-body : Gives pole contribution in complex s-plane at $s = m_1^2$ ← mass of intermediate state

2-body : Gives branch cut at $s = (m_1 + m_2)^2$ (square root)

3-body : " " " " $s = (m_1 + m_2 + m_3)^2$ (Logarithmic)



Consider only 2-body intermediate states

$$\operatorname{Im} A_{\substack{\sigma_a\sigma_b\\ \sigma_c\sigma_d}}(s,\hat{p}\cdot\hat{p}') = \frac{1}{2}\sum_{\sigma_1\sigma_2}\int d\tilde{k}_1 d\tilde{k}_2 (2\pi)^4 \delta^4(p_a+p_b-k_1-k_2) A^*_{\substack{\sigma_c\sigma_d\\ \sigma_1\sigma_2}}(s,\hat{p}'\cdot\hat{k}) A_{\substack{\sigma_a\sigma_b\\ \sigma_1\ \sigma_2}}(s,\hat{p}\cdot\hat{k})$$

Now, $\displaystyle\int d\tilde{k}_1 d\tilde{k}_2 (2\pi)^4 \delta^4(p_a+p_b-k_1-k_2) = \frac{1}{8\pi}\frac{2|\vec{k}|}{\sqrt{s}}\int \frac{d\hat{k}}{4\pi}\Theta(s-s_{th})$ in CMS of ab

$$\text{w/}\quad |\vec{k}| = \frac{\lambda^{\frac{1}{2}}(s,m_1^2,m_2^2)}{2\sqrt{s}}$$

Exercise: Show this

Solution:

$$\varphi_2 \equiv \int d\tilde{k}_1 d\tilde{k}_2 (2\pi)^4 \delta^4(p_a+p_b-k_1-k_2) = \frac{(2\pi)^4}{(2\pi)^6}\frac{1}{4}\int \frac{d^3\vec{k}_1 d^3\vec{k}_2}{E_1(\vec{k}_1)E_2(\vec{k}_2)} \delta(\sqrt{s}-E_1-E_2)\delta^3(\vec{k}_1+\vec{k}_2)$$

$$\vec{k}\equiv \vec{k}_1 = -\vec{k}_2 \quad , \quad E_1(\vec{k}_1) = \sqrt{\vec{k}_1^{\,2}+m_1^2} \quad , \quad E_2(\vec{k}_2) = \sqrt{\vec{k}_2^{\,2}+m_2^2}$$

$$\Rightarrow \varphi_2 = \frac{1}{(2\pi)^2}\frac{1}{4}\int \frac{d^3\vec{k}}{E_1(\vec{k})E_2(\vec{k})}\delta(\sqrt{s}-E_1-E_2) = \frac{1}{(2\pi)^2}\frac{1}{4}\int \frac{\vec{k}^2 d|\vec{k}|\, d\hat{k}}{E_1(\vec{k})E_2(\vec{k})}\delta(\sqrt{s}-E_1-E_2)$$

Now, $x(\vec{k}) = E_1(\vec{k}) + E_2(\vec{k}) - \sqrt{s} = \sqrt{\vec{k}^2+m_1^2} + \sqrt{\vec{k}^2+m_2^2} - \sqrt{s}$

$$\frac{\partial x(\vec{k})}{\partial|\vec{k}|} = \frac{|\vec{k}|}{E_1(\vec{k})} + \frac{|\vec{k}|}{E_2(\vec{k})} = |\vec{k}|\frac{(E_1(\vec{k})+E_2(\vec{k}))}{E_1(\vec{k})E_2(\vec{k})} = \frac{|\vec{k}|\sqrt{s}}{E_1(\vec{k})E_2(\vec{k})}$$

So, $\displaystyle d|\vec{k}| = \frac{\partial|\vec{k}|}{\partial x(\vec{k})}dx = \frac{E_1(\vec{k})E_2(\vec{k})}{\sqrt{s}\,|\vec{k}|}$, $s_{th}\equiv(m_1+m_2)^2$

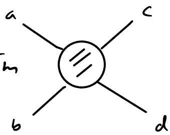
$$\Rightarrow \varphi_2 = \frac{1}{(2\pi)^2}\frac{1}{4}\frac{|\vec{k}|}{\sqrt{s}}\int d\hat{k}\,\Theta(s-s_{th}) = \frac{1}{8\pi}\frac{2|\vec{k}|}{\sqrt{s}}\int\frac{d\hat{k}}{4\pi}\Theta(s-s_{th}) \quad \blacksquare$$

So, unitarity is

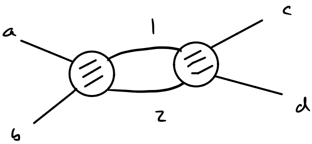
$$\mathrm{Im}\, \mathcal{A}_{\substack{\sigma_a\sigma_b\\ \sigma_c\sigma_d}}(s,\hat{p}\cdot\hat{p}') = \frac{1}{2} \sum_{\sigma_1\sigma_2} \frac{1}{8\pi}\frac{2|\vec{k}|}{\sqrt{s}}\int \frac{d\hat{k}}{4\pi}\, \mathcal{A}^*_{\substack{\sigma_c\sigma_d\\ \sigma_1\sigma_2}}(s,\hat{p}'\cdot\hat{k})\, \mathcal{A}_{\substack{\sigma_a\sigma_b\\ \sigma_1\sigma_2}}(s,\hat{p}\cdot\hat{k})\, \Theta(s-s_{th})$$

Imaginary part exists for $s > s_{th}$

Pictorially

Im 

$$= \frac{1}{2}\sum_{\sigma_1\sigma_2} \frac{1}{8\pi}\frac{2|\vec{k}|}{\sqrt{s}}\int\frac{d\hat{k}}{4\pi}$$



) s

$ab \to cd$ amplitude

Two body phase space of intermediate states

$ab \to 12$ amplitude

$12 \to cd$ amplitude

If in helicity basis, simply $\sigma \to \lambda$

Partial Wave Unitarity

Consider $2 \to 2$ scattering w/ only 2 particles in the intermediate state

$$\mathrm{Im}\, A_{\substack{\lambda_a\lambda_b \\ \lambda_c\lambda_d}}(s, \hat{p}\cdot\hat{p}') = \frac{1}{2}\sum_{\lambda_1\lambda_2} \frac{1}{8\pi}\frac{2|\vec{k}|}{\sqrt{s}}\int \frac{d\hat{k}}{4\pi} A^*_{\substack{\lambda_c\lambda_d \\ \lambda_1\lambda_2}}(s, \hat{p}'\cdot\hat{k})\, A_{\substack{\lambda_a\lambda_b \\ \lambda_1\lambda_2}}(s, \hat{p}\cdot\hat{k})\, \Theta(s - s_{th})$$

Now, in helicity basis

$$A_{\substack{\lambda_a\lambda_b \\ \lambda_c\lambda_d}}(s, \hat{p}\cdot\hat{p}') = \sum_J \left(\frac{2J+1}{4\pi}\right) a^J_{\substack{\lambda_a\lambda_b \\ \lambda_c\lambda_d}}(s)\, D^{J*}_{\lambda\lambda'}(\hat{p}\cdot\hat{p}')$$

$\lambda = \lambda_a - \lambda_b$ $\quad \lambda' = \lambda_c - \lambda_d$

& similar for other amplitudes

- J is conserved
- use $$\int \frac{d\hat{k}}{4\pi} D^{J'}_{\lambda''\lambda'}(\hat{p}'\cdot\hat{k})\, D^{J*}_{\lambda\lambda''}(\hat{p}\cdot\hat{k}) = \frac{\delta_{JJ'}}{2J+1} D^{J*}_{\lambda\lambda'}(\hat{p}\cdot\hat{p}')$$

$\lambda'' = \lambda_1 - \lambda_2$

$$\Rightarrow \boxed{\mathrm{Im}\, a^J_{\substack{\lambda_a\lambda_b \\ \lambda_c\lambda_d}}(s) = \frac{1}{2}\sum_{\lambda_1\lambda_2} \frac{1}{8\pi}\frac{2|\vec{k}|}{\sqrt{s}}\, a^{J*}_{\substack{\lambda_c\lambda_d \\ \lambda_1\lambda_2}}(s)\, a^J_{\substack{\lambda_a\lambda_b \\ \lambda_1\lambda_2}}(s)}$$

Helicity partial wave unitarity

Can repeat for LS-basis

find

$$\boxed{\operatorname{Im} a^{J}_{\substack{ls\\ l's'}}(s) = \frac{1}{2}\sum_{l''s''} \frac{1}{8\pi}\frac{2|\vec{k}|}{\sqrt{s}}\, a^{J*}_{\substack{l''s''\\ l's'}}(s)\; a^{J}_{\substack{ls\\ l''s''}}(s)}$$

Kinematic Singularities

In our discussions, we have ignored the effects of kinematic singularities. Unitarity constrains the dynamical singularities
$\Rightarrow$ We need to factorize the kinematic singularities from unitarity eqns.

Proceed w/ simple example: $2\to 2$ scalar scattering w/ 2 scalars in intermediate state

$$\Rightarrow \quad \langle c\, d | T | a\, b\rangle = (2\pi)^4 \delta^4(p_a+p_b-p_c-p_d)\, A(s,t)$$

- $A(s,t)$ is a Lorentz scalar

So,

$$\operatorname{Im} A(s,\hat{p}\cdot\hat{p}') = \frac{1}{2}\frac{1}{8\pi}\frac{2|\vec{k}|}{\sqrt{s}}\int\frac{d\hat{k}}{4\pi} A^*(s,\hat{p}'\cdot\hat{k})\, A(s,\hat{p}\cdot\hat{k})$$

& PW expansion

$$A(s,\hat{p}\cdot\hat{p}') = \sum_{l=0}^{\infty}\left(\frac{2l+1}{4\pi}\right) a_l(s)\, P_l(\hat{p}\cdot\hat{p}')$$

$$\operatorname{Im} \mathcal{A}(s, \hat{p}\cdot\hat{p}') = \frac{1}{2}\,\frac{1}{8\pi}\,\frac{2|\vec{k}|}{\sqrt{s}} \int \frac{d\hat{k}}{4\pi}\, \mathcal{A}^*(s, \hat{p}'\cdot\hat{k})\,\mathcal{A}(s, \hat{p}\cdot\hat{k})$$

$$\mathcal{A}(s, \hat{p}\cdot\hat{p}') = \sum_{\ell=0}^{\infty} \left(\frac{2\ell+1}{4\pi}\right) a_\ell(s)\, P_\ell(\hat{p}\cdot\hat{p}')$$

Now, $P_\ell(\hat{p}\cdot\hat{p}') = P_\ell(z_s)$ is a regular fn of z_s $\quad (z_s \equiv \cos\theta = \hat{p}\cdot\hat{p}')$

but $z_s = 1 - \dfrac{t'}{2|\vec{p}||\vec{p}'|} \sim \dfrac{1}{|\vec{p}||\vec{p}'|}$

& $P_\ell(z_s) \sim (z_s)^\ell \sim (|\vec{p}||\vec{p}'|)^{-\ell}$

So, as $s \to s_{th}^{(1)}$ or $s_{th}^{(2)}$ $\Rightarrow$ $P_\ell(z_s) \sim \left(\sqrt{s - s_{th}^{(1)}}\,\sqrt{s - s_{th}^{(2)}}\right)^{-\ell} \to \infty$

singularity

So, $a_\ell(s)$ must have this factor for $s \to s_{th}$ to have $\mathcal{A}(s,t)$ remain a Lorentz scalar

$$\Rightarrow \quad a_\ell(s) \xrightarrow[s \to s_{th}]{} (|\vec{p}||\vec{p}'|)^\ell$$

$$\Rightarrow \quad a_\ell(s) \equiv (|\vec{p}||\vec{p}'|)^\ell\, \hat{a}_\ell(s)$$

kinematic singularity free amplitude

So, Partial wave Unitarity

$$\underset{\substack{\uparrow \\ ab \to cd}}{\mathrm{Im}\, a_\ell(s)} = \frac{1}{2} \frac{1}{8\pi} \frac{2|\vec{k}|}{\sqrt{s}} \underset{\substack{\uparrow \\ 12 \to cd}}{f_\ell^*(s)} \underset{\substack{\uparrow \\ ab \to 12}}{t_\ell(s)}$$

Now,

$$a_\ell(s) \equiv (|\vec{p}||\vec{p}\,'|)^\ell \hat{a}_\ell(s) \quad \Rightarrow \quad \mathrm{Im}\, a_\ell(s) = (|\vec{p}||\vec{p}\,'|)^\ell \mathrm{Im}\, \hat{a}_\ell(s)$$

$$f_\ell(s) = (|\vec{k}||\vec{p}\,'|)^\ell \hat{f}_\ell(s)$$

$$t_\ell(s) = (|\vec{p}||\vec{k}|)^\ell \hat{t}_\ell(s)$$

So,

$$(|\vec{p}||\vec{p}\,'|)^\ell \mathrm{Im}\, \hat{a}_\ell(s) = \frac{1}{2} \frac{1}{8\pi} \frac{2|\vec{k}|}{\sqrt{s}} (|\vec{p}||\vec{k}|)^\ell (|\vec{k}||\vec{p}\,'|)^\ell \hat{f}_\ell^*(s) \hat{t}_\ell(s)$$

$$\Rightarrow \mathrm{Im}\, \hat{a}_\ell(s) = \frac{1}{2} \underset{\substack{\uparrow \\ \text{phase space w/ barrier factors}}}{\frac{1}{8\pi} \frac{2|\vec{k}|^{2\ell+1}}{\sqrt{s}}} \hat{f}_\ell^*(s) \hat{t}_\ell(s)$$

Note: In practical calculations, need to regularize the barrier factors as for $s \to \infty$, $|\vec{k}|^{2\ell+1} \to \infty$

Introduce phenomenological parameterization

Common methods: Blatt-Weiskopf factors

$$|\vec{k}|^\ell \to \left(\frac{|\vec{k}|^2}{|\vec{k}|^2 + \Lambda} \right)^{\ell/2} \begin{cases} \xrightarrow{s \to s_{th}} \sim |\vec{k}|^\ell \\ \xrightarrow{s \to \infty} \sim 1 \end{cases}$$