SSRT Lecture 2 : Resonances

Group 2

Wednesday June 14, 2017

9:30 – 10:30

A. Jachura

Resonances

- What are resonances?
- How do we describe resonances?
- How do we relate resonances to observables?

Resonances are unstable hadrons

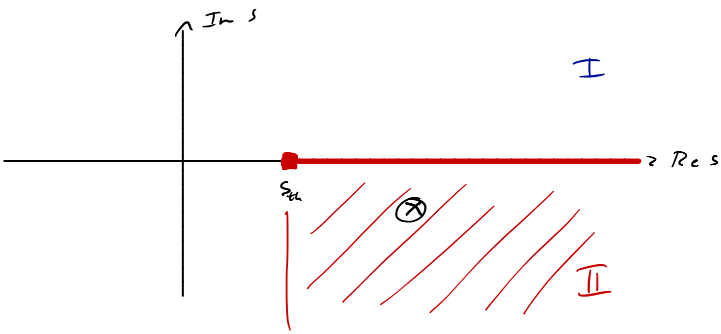
e.g., ρ, ω, N^*, Δ

We associate resonances a poles of the scattering amplitude in some partial wave amplitude.

Note: poles must appear on unphysical sheets from Causality.

Consider Scalar $2 \to 2$ scattering in examples

$$\text{Im}\, \hat{a}_\ell(s) = \rho_\ell(s)\, |\hat{a}_\ell(s)|^2 \qquad (\text{Elastic})$$



$$\hat{a}_\ell^{\,II}(s - i\epsilon) = \hat{a}_\ell^{\,I}(s + i\epsilon)$$

On the real axis,

$$\mathrm{Im}\,\hat{a}_\ell(s+i\epsilon) = \frac{\Delta_s}{2i}\hat{a}_\ell(s) = \frac{1}{2i}\left(\hat{a}_\ell(s+i\epsilon) - \hat{a}_\ell(s-i\epsilon)\right)$$

$$\hat{a}_\ell(s) \equiv \hat{a}_\ell^{I}(s)$$

unitarity $\Rightarrow$ $\Delta_s \hat{a}_\ell(s) = 2i\rho_\ell(s)\,|\hat{a}_\ell(s)|^2$

$$\hat{a}_\ell(s+i\epsilon) - \hat{a}_\ell(s-i\epsilon) = 2i\rho_\ell(s+i\epsilon)\,\hat{a}_\ell(s+i\epsilon)\,\hat{a}_\ell(s-i\epsilon)$$

Since, from Schwartz reflection principle

$$f(z^*) = f^*(z)$$

Now, $\hat{a}_\ell(s+i\epsilon) = \hat{a}_\ell^{II}(s-i\epsilon)$

$$\Rightarrow \hat{a}_\ell^{II}(s-i\epsilon) - \hat{a}_\ell(s-i\epsilon) = 2i\rho_\ell(s+i\epsilon)\,\hat{a}_\ell^{II}(s-i\epsilon)\,\hat{a}_\ell(s-i\epsilon)$$

Now, $\rho_\ell(s+i\epsilon) - \rho_\ell(s-i\epsilon) = 2\rho_\ell(s+i\epsilon)$

$$\Rightarrow \rho_\ell(s+i\epsilon) = -\rho_\ell(s-i\epsilon)$$

$$\Rightarrow \boxed{\hat{a}_\ell^{II}(s-i\epsilon) = \frac{\hat{a}_\ell(s-i\epsilon)}{1+2i\rho_\ell(s-i\epsilon)\,\hat{a}_\ell(s-i\epsilon)}}$$

Partial wave amplitude on 2nd sheet for $\mathrm{Im}\, s < 0$

Poles of amplitude are zeroes of

$$1 + 2i\rho_\ell(s_R)\,\hat{a}_\ell(s_R) = 0$$

- Define mass & width

$$m_R = \mathrm{Re}\,\sqrt{s_R}$$
$$\Gamma_R = -2\,\mathrm{Im}\,\sqrt{s_R}$$

- Spin of resonance is ℓ

- Can extend these ideas for particles w/ spin / multichannel problems.
  - $\rightarrow$ extend to matrix equations

e.g., 2 channel spinless scattering $(\pi\pi \to \pi\pi,\ \pi\pi \to K\bar{K},\ K\bar{K} \to K\bar{K})$

Unitarity

$$\hat{a}_{jk}(s) = \sum_n \rho_n(s)\,\hat{a}_{jn}(s)\,\hat{a}_{nk}(s) \qquad \hat{a}_{jk} = \begin{pmatrix} \pi\pi\to\pi\pi & \pi\pi\to K\bar{K} \\ K\bar{K}\to\pi\pi & K\bar{K}\to K\bar{K} \end{pmatrix}$$

$\uparrow$ note: T-reversal for strong interactions

$\Rightarrow \hat{a}_{jk} = \hat{a}_{kj}$

$\Rightarrow$ Poles appear when

$$\det\left(\delta_{jk} + 2i\rho_k(s_R)\,\hat{a}_{jk}(s_R)\right) = 0$$

- Will not discuss this further

Elastic scattering – Phase shifts

$$\operatorname{Im} \hat{a}_\ell(s) = \frac{1}{2}\frac{1}{8\pi}\frac{2|\vec{k}|}{\sqrt{s}}^{2\ell+1} |\hat{a}_\ell(s)|^2$$

Define: $\rho_\ell(s) = \frac{1}{2}\frac{1}{8\pi}\frac{2|\vec{k}|}{\sqrt{s}} b_\ell^2(s)$ where $b_\ell(s) = \left(\frac{|\vec{k}|^2}{|\vec{k}|^2+\Lambda^2}\right)^{\ell/2}$

is regularized barrier factor

So,

$$\operatorname{Im} \hat{a}_\ell(s) = \rho_\ell(s)\, |\hat{a}_\ell(s)|^2 \quad \Rightarrow \quad \operatorname{Im} \hat{a}_\ell^{-1}(s) = -\rho_\ell(s)$$

$$\hat{a}_\ell(s) = |\hat{a}_\ell(s)|\, e^{i\delta_\ell(s)}$$

$$\Rightarrow \quad |\hat{a}_\ell(s)| \sin\delta_\ell(s) = \rho_\ell(s)\, |\hat{a}_\ell(s)|^2 \quad \Rightarrow \quad |\hat{a}_\ell(s)| = \frac{\sin\delta_\ell(s)}{\rho_\ell(s)}$$

$$\Rightarrow \quad \boxed{\hat{a}_\ell(s) = e^{i\delta_\ell(s)}\frac{\sin\delta_\ell(s)}{\rho_\ell(s)}}$$

$\delta_\ell(s)$ = "phase shift"

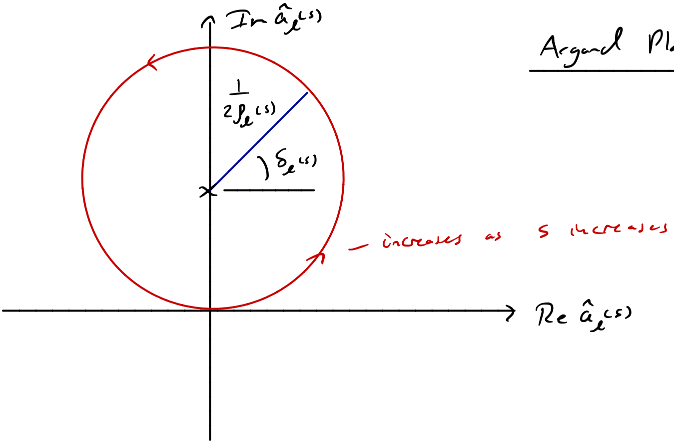
Note: from unitarity

$$\operatorname{Im} \hat{a}_\ell(s) = \rho_\ell(s)\,[\operatorname{Re} \hat{a}_\ell(s)]^2 + \rho_\ell(s)\,[\operatorname{Im} \hat{a}_\ell(s)]^2$$

$$\Rightarrow \quad [\operatorname{Re} \hat{a}_\ell(s)]^2 + [\operatorname{Im} \hat{a}_\ell(s)]^2 - \frac{1}{\rho_\ell(s)} \operatorname{Im} \hat{a}_\ell(s) = 0$$ ← complete square

$$[\operatorname{Re} \hat{a}_\ell(s)]^2 + \left[\operatorname{Im} \hat{a}_\ell(s) - \frac{1}{2\rho_\ell(s)}\right]^2 = \frac{1}{4\rho_\ell^2(s)}$$ ← equation of circle of radius $1/2\rho_\ell(s)$ centered at $\left(0, \frac{1}{2\rho_\ell(s)}\right)$

$$\left[\mathrm{Re}\,\hat{a}_\ell(s)\right]^2 + \left[\mathrm{Im}\,\hat{a}_\ell(s) - \frac{1}{2\rho_\ell(s)}\right]^2 = \frac{1}{4\rho_\ell^2(s)}$$



Argand Plot

Check parameterization:

$$\hat{a}_\ell(s) = \frac{e^{i\delta_\ell(s)}\sin\delta_\ell(s)}{\rho_\ell(s)} \Rightarrow \quad \mathrm{Re}\,\hat{a}_\ell(s) = \frac{\cos\delta_\ell(s)\sin\delta_\ell(s)}{\rho_\ell(s)}, \quad \mathrm{Im}\,\hat{a}_\ell(s) = \frac{\sin^2\delta_\ell(s)}{\rho_\ell(s)}$$

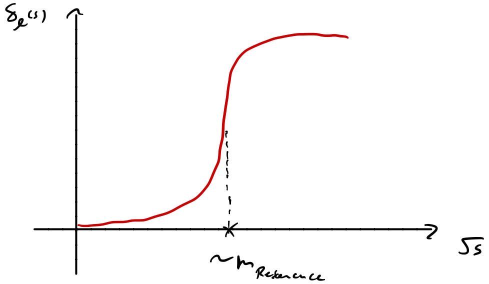
$$\left[\mathrm{Re}\,\hat{a}_\ell(s)\right]^2 + \left[\mathrm{Im}\,\hat{a}_\ell(s) - \frac{1}{2\rho_\ell(s)}\right]^2 = \frac{1}{4\rho_\ell^2(s)}$$

$$\Rightarrow \frac{1}{\rho_\ell^2(s)}\cos^2\delta_\ell(s)\sin^2\delta_\ell(s) + \frac{\sin^4\delta_\ell(s)}{\rho_\ell^2(s)} - \frac{1}{\rho_\ell(s)}\frac{\sin^2\delta_\ell(s)}{\rho_\ell(s)} = 0$$

$$= \frac{1}{\rho_\ell^2(s)}\sin^2\delta_\ell(s) - \frac{\sin^2\delta_\ell(s)}{\rho_\ell^2(s)} = 0 \quad \checkmark$$

Narrow Resonances – Breit-Wigner

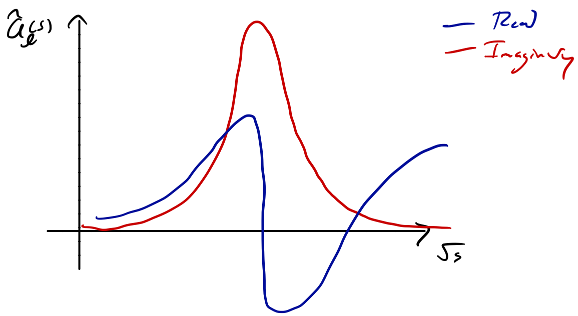
Resonances are associated w/ a rapid phase change through $\pi/2$



Simple model: $\tan\delta_\ell(s) = \dfrac{m\Gamma\rho_\ell(s)/\rho_\ell(m^2)}{m^2 - s}$

$$\Rightarrow \quad \boxed{\hat{a}_\ell(s) = \frac{m\Gamma/\rho_\ell(m^2)}{m^2 - s - im\Gamma\rho_\ell(s)/\rho_\ell(m^2)}}$$

(Breit-Wigner amplitude)



Note: Often one associates m & Γ as the mass & width of the resonance, but truly, the pole position is what characterizes a resonance.

Inelasticities

inelastic channels

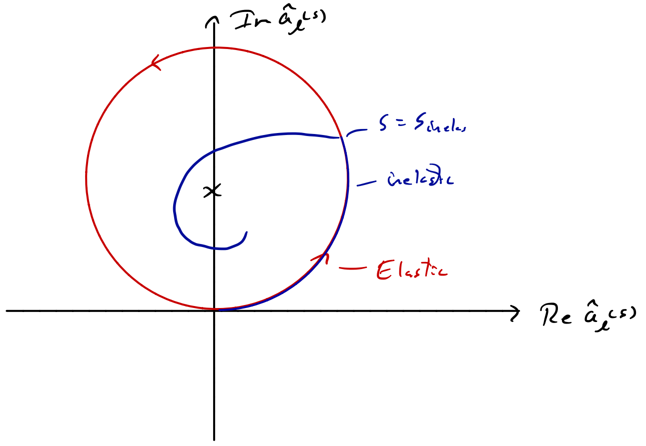
$$\mathrm{Im}\, \hat{a}_\ell(s) = \rho_\ell(s)\, |\hat{a}_\ell(s)|^2 + \mathcal{R}_\ell^2(s) \qquad (\mathcal{R}_\ell^2(s) > 0)$$

opens up at $s > s_{inel}$

$\Rightarrow$

$$[\mathrm{Re}\, \hat{a}_\ell(s)]^2 + \left[\mathrm{Im}\, \hat{a}_\ell(s) - \frac{1}{2\rho_\ell(s)}\right]^2 = \frac{1}{4\rho_\ell^2(s)} - \frac{1}{\rho_\ell(s)} \mathcal{R}_\ell^2(s)$$

Radius decreases as s increases



Can define complex phase shift $\varphi_\ell(s)$

$$\varphi_\ell(s) = \delta_\ell(s) + i\beta_\ell(s)$$

w/ $\beta_\ell(s) = -\frac{1}{4} \ln\left(1 - 4\rho_\ell(s)\, \mathcal{R}_\ell^2(s)\right)$

Then, phase shift parametrization

$$\hat{a}_\ell(s) = \frac{e^{i\varphi_\ell(s)} \sin\delta_\ell(s)}{\rho_\ell(s)} = \frac{e^{2i\varphi_\ell(s)} - 1}{2i\rho_\ell(s)}$$

$$= \frac{e^{2i\delta_\ell(s) - 2\beta_\ell(s)} - 1}{2i\rho_\ell(s)} = \frac{e^{-2\beta_\ell(s)} e^{2i\delta_\ell(s)} - 1}{2i\rho_\ell(s)}$$

Now, define inelasticity $\eta_\ell(s) = e^{-2\beta_\ell(s)}$; $\eta_\ell(s) \in \mathbb{R}$, $0 < \eta_\ell(s) \leq 1$

$$\Rightarrow \boxed{\hat{a}_\ell(s) = \frac{\eta_\ell(s)\, e^{2i\delta_\ell(s)} - 1}{2i\rho_\ell(s)}}$$

General Parameterizations

K-matrix: Assume no left-hand cuts (as in BW amplitude)

but now, <u>no</u> narrow resonance limit

unitarity can be written

$$\text{Im}\, \hat{a}_\ell^{-1}(s) = -\rho_\ell(s)$$

Dispersive phase space
$\equiv$ Chew-Mandelstam phase space

$$\Rightarrow \quad \hat{a}_\ell^{-1}(s) = K_\ell^{-1}(s) - \frac{1}{\pi}\int_{s_{th}}^{\infty} ds' \frac{\rho_\ell(s')}{s'-s}$$

Real function

To have integral converge, perform subtraction at s_0

$$\Rightarrow \quad \hat{a}_\ell^{-1}(s) = K_\ell^{-1}(s) - \frac{s-s_0}{\pi}\int_{s_{th}}^{\infty} ds' \frac{\rho_\ell(s')}{(s'-s_0)(s'-s)}$$

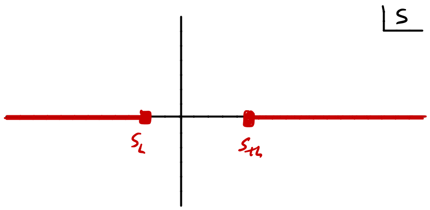
absorbed subtraction constant

Parametrize K-matrix $\quad K_\ell(s) = \sum_r \frac{\alpha_\ell^{(r)}(s)}{\beta_\ell^{(r)}(s) - s} + \sum_j \gamma_\ell^{(j)} s^{(j)}$ for example

Why do all this?
- Two nearby resonances: can't use two BW $\Rightarrow$ violates unitarity
- Wide width resonance (e.g., σ)
- Multichannel processes

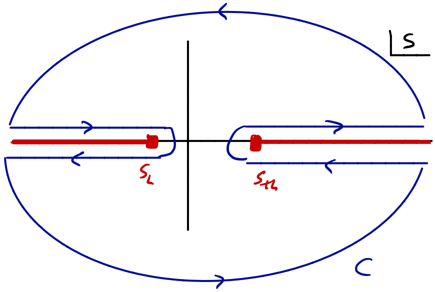
Partial Wave Integral eqns. & Dispersion relations

- Partial wave amplitudes have cuts in complex s-plane
    - RHC from unitarity
    - LHC from projections from exchanges



- Can write a dispersion relation for $a_\ell(s)$ using Cauchy's thm

$$\hat{a}_\ell(s) = \frac{1}{2\pi i}\oint_C ds' \, \frac{\hat{a}_\ell(s')}{s'-s}$$



assume $\hat{a}_\ell(s)$ converges sufficiently fast as $|s|\to\infty$ (if not, perform subtractions)

So,

$$\hat{a}_\ell(s) = \frac{1}{2\pi i}\int_{-\infty}^{s_L} ds' \, \frac{\Delta_L \hat{a}_\ell(s')}{s'-s} + \frac{1}{2\pi i}\int_{s_{th}}^{\infty} ds' \, \frac{\Delta_R \hat{a}_\ell(s')}{s'-s}$$

$$\Rightarrow \quad \hat{a}_\ell(s) = \frac{1}{\pi}\int_{-\infty}^{s_L} ds' \, \frac{\mathrm{Im}_L \hat{a}_\ell(s')}{s'-s} + \frac{1}{\pi}\int_{s_{th}}^{\infty} ds' \, \frac{\mathrm{Im}_R \hat{a}_\ell(s')}{s'-s}$$

Now, unitarity gives $\mathrm{Im}_R \hat{a}_\ell(s') = \rho_\ell(s') \, |\hat{a}_\ell(s')|^2$

s, /

$$\hat{a}_\ell(s) = \frac{1}{\pi}\int_{-\infty}^{s_L} ds' \frac{\mathrm{Im}_L\,\hat{a}_\ell(s')}{s'-s} + \frac{1}{\pi}\int_{s_{th}}^{\infty} ds' \frac{\rho_\ell(s')\,|\hat{a}_\ell(s')|^2}{s'-s}$$

- Nonlinear integral eqn. for $\hat{a}_\ell(s)$.

- Can linearize system by introducing N/D parameterization

let $\hat{a}_\ell(s) = \dfrac{N_\ell(s)}{D_\ell(s)}$ where $N_\ell(s)$ contains LHC's only, $D_\ell(s)$ contains RHC's only

$$\Rightarrow \quad \mathrm{Im}\,D_\ell(s) = -\rho_\ell(s)\,N_\ell(s) \quad \text{for } s > s_{th}$$

$$\mathrm{Im}\,N_\ell(s) = \mathrm{Im}_L\,\hat{a}_\ell(s)\,D_\ell(s) \quad \text{for } s < s_L$$

$\therefore$ write integral equations (coupled)

$$\left.\begin{aligned} N_\ell(s) &= \frac{1}{\pi}\int_{-\infty}^{s_L} ds' \frac{\mathrm{Im}_L\,\hat{a}_\ell(s')\,D_\ell(s')}{s'-s} \\ D_\ell(s) &= \frac{-1}{\pi}\int_{s_{th}}^{\infty} ds' \frac{\rho_\ell(s')\,N_\ell(s')}{s'-s} \end{aligned}\right\}$$

linear first order coupled integral eqns.

- Perform subtraction on $D_\ell(s)$ to ensure convergence

$$D_\ell(s) = 1 - \frac{s-s_0}{\pi}\int_{s_{th}}^{\infty} ds' \frac{\rho_\ell(s')\,N_\ell(s')}{(s'-s_0)(s'-s)}$$

Normalization (choice)

s_0 = subtraction point

Note: Integral eqns. don't have a unique solution!

$\gamma_1 < 0$

Can always add Rational fcn $D_0(s) = \sum_r \frac{\gamma_r}{\mu_r - s}$

Provided $\mathrm{Im}_R D_0(s) = 0$ & γ_r, μ_r such that no poles on first sheet.

- $D_0(s)$ are called CDD poles = zeros of the amplitude
- Another type of CDD pole called CDD pole at ∞ can be added provided same conditions

$$D_\infty(s) = C_0 - C_1 s$$

$C_1 > 0$

- Combining all together

$$D_\ell(s) = C_0 - C_1 s + \sum_r \frac{\gamma_r}{\mu_r - s} - \frac{s-s_0}{\pi}\int_{s_{th}}^{\infty} ds' \frac{\rho_\ell(s') N_\ell(s')}{(s'-s_0)(s'-s)}$$

Can normalize $D_\ell(s_0) = 1$ if desired as before

- Can combine N_ℓ & D_ℓ eqns.

$$\Rightarrow D_\ell(s) = C_0 - C_1 s + \sum_r \frac{\gamma_r}{\mu_r - s} - \frac{s-s_0}{\pi^2}\int_{s_{th}}^{\infty} ds' \frac{\rho_\ell(s')}{(s'-s_0)(s'-s)} \int_{-\infty}^{s_L} ds'' \frac{\mathrm{Im}_L \hat{a}_\ell(s'') D_\ell(s'')}{s''-s'}$$

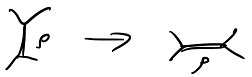
$$= C_0 - C_1 s + \sum_r \frac{\gamma_r}{\mu_r - s} - \frac{s-s_0}{\pi^2}\int_{-\infty}^{s_L} ds'' \underbrace{\int_{s_{th}}^{\infty} ds' \frac{\rho_\ell(s')}{(s'-s_0)(s'-s)(s''-s')}}_{= K_\ell(s, s'')} \mathrm{Im}_L \hat{a}_\ell(s'') D_\ell(s'')$$

$$\Rightarrow \quad D_\ell(s) = D_\infty(s) + D_0(s) + \int_{-\infty}^{s_L} ds''\, K_\ell(s,s'')\, \mathrm{Im}_L \hat{a}_\ell(s'')\, D_\ell(s'')$$

w/ kernel
$$K_\ell(s,s'') = -\frac{(s-s_0)}{\pi^2} \int_{s_{th}}^{\infty} ds' \frac{\rho_\ell(s')}{(s'-s_0)(s'-s)(s''-s')}$$

- In principle, if given input $\mathrm{Im}_L \hat{a}_\ell(s)$, can solve for $D_\ell(s)$, then $N_\ell(s)$, then we have the solution for $\hat{a}_\ell(s)$.

Issues:
- Don't know $\mathrm{Im}_L \hat{a}_\ell(s)$ exactly.
  The old bootstrap program approached this as input

  , But it failed.
- Can i. model $\mathrm{Im}_L \hat{a}_\ell(s)$ as some exchange process, & solve
- Still, don't know CDD contributions (Integral eqns. have no unique soln.)
- CDD terms contribute to resonances!

- In practice:
  - model $\mathrm{Im}_L \hat{a}_\ell(s)$ (-or- $N_\ell(s)$) w/ some LHC (e.g., exchange model)
  - fit CDD parameters to data
    - How many CDD terms? Input from elsewhere (quark models, LQCD, etc.)
  - Change LHC model to test systematics of model dependence.