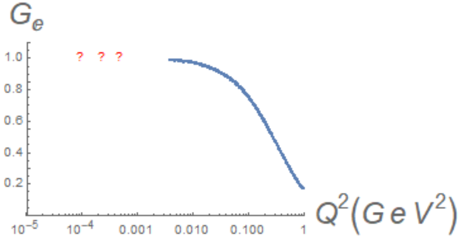
A new approach to limit the extrapolation errors of the proton size problem

Proton charge radius of e-scattering has been estimated by extrapolation of Sachs electric form factor $G_E(Q^2) = F_1(Q^2) - (Q^2/4M^2)F_2(Q^2)$ to $Q^2 = 0$.



Current estimates CONTROVERSIAL: fits allow $r_p \in [0.84, 89]$ *fm* (Horbatsch 2016).
The plan: limit extrapolation errors by making use of *analytic structure* of G_E.

Two relevant theorems: theorem #1

Theorem #1, on the convergence of Chebyshev polynomials

Chebyshev polynomial fits to a function f on $[-1, 1]$ converge geometrically as $n \to \infty$ iff f is analytic [Bernstein 1911, 1912].

Cheby polynomials: set of orthogonal polynomial functions ...
Geometric convergence: convergence at a rate $O(C^{-n})$ for $C > 1$
Errors look like straight lines on a semilog scale.

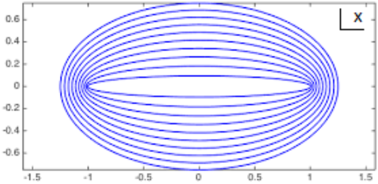
Intro to Cheby polynomials: **Chebfun (L.N. Trefethen)**

Two relevant theorems: theorem #2

Theorem #2, on Chebyshev coefficients of analytic functions

Let a function f analytic in $[-1,1]$ be analytically continuable to the open Bernstein ellipse E_ρ, where it satisfies $|f(x)| < M$ for some M.
Then its Chebyshev coefficients satisfy $|c_0| < M$ and $|c_k| < 2M\rho^{-k}$, $k > 1$.

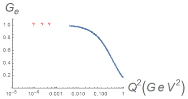
Bernstein Ellipses E_ρ



How the theorems fit together: the big picture

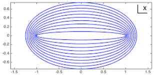
Thm #1:

We do Chebyshev fit to G_E data. Having G_E data is NOT same as having an analytic function for G_E. Thm #1 doesn't automatically apply; exponential convergence as $n \to \infty$ is not guaranteed.



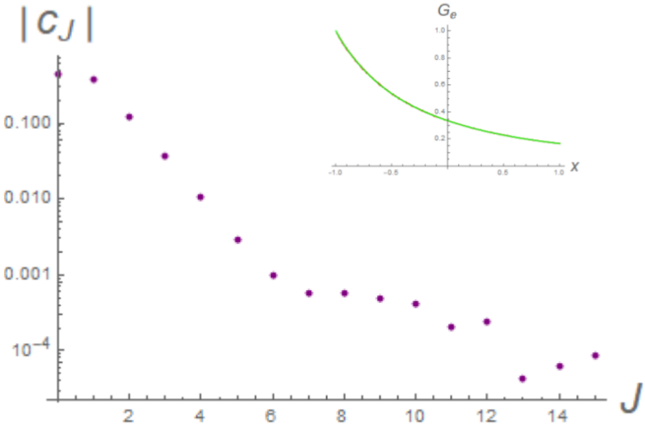
Thm #2:

FF ansatz (dipole + branch cut) + thm #2 provide a test of the convergence of the Cheby fit above– i.e. they constrain $\{|c_k|\}$.

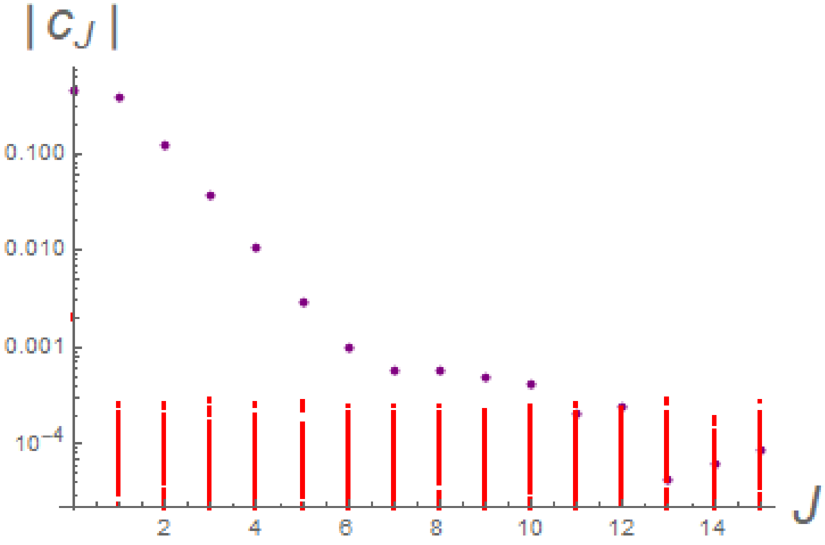


$|c_k| < 2M\rho^{-k}$, where physical singularities fix ρ

Chebyshev fit to G_E data



Chebyshev fit to G_E data

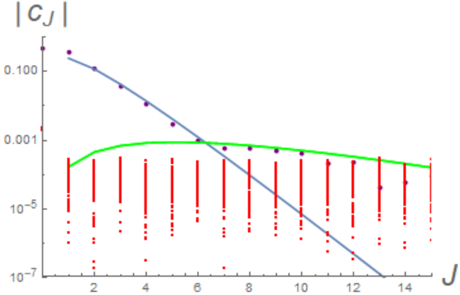


red = 2σ Gaussian noise, approximating effects of δG_E

Adding in the analyticity bounds

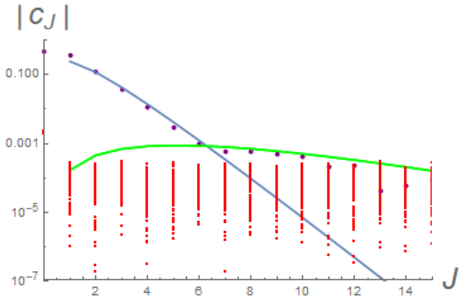
$$F_{ansatz}(Q^2) = \frac{a_1}{(Q^2 + m_{di}^2)^2} + a_2 \cdot \left(1 - \frac{\sqrt{Q^2 + 4m_\pi^2} + 0.08}{\sqrt{Q^2 + 4m_\pi^2} - 0.08}\right)^4 \xrightarrow{\text{thm 2}} bounds$$

Blue is constraint from dipole term; green from branch cut term.



Upshot: Cheb fit obeys analyticity bounds !

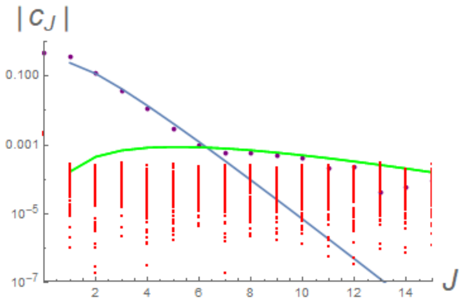
The (preliminary) final analysis



! PRELIMINARY !

We find
1) a radius value consistent with the 'small' radius values $\sim$ 0.84 *fm* and 2) a factor of 2 reduction in the relative uncertainty of r_p.

The (preliminary) final analysis



! PRELIMINARY !

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BACKUP

Model form factor is

$$F(Q^2) = \frac{a_1}{(Q^2 + m_{di}^2)^2} + a_2 \cdot (1 - \frac{\sqrt{Q^2 + 4m_\pi^2} + 0.08}{\sqrt{Q^2 + 4m_\pi^2} - 0.08})^4$$

Dipole term has a singularity at $q^2 = -m_{di}^2$ that is mathematically allowed, if a bit unphysical. It does not include the width of the resonance, and its imaginary part is quite singular.
The other term has been concocted to include a branch cut from the 2-pion continuum, beginning at $Q^2 < -4m_\pi^2$ and falling fast enough as $Q^2 \to \infty$.
The important thing is that singularities of the model occur at physically realistic locations in the time-like region