Conventions for the 2017 Summer School on Reaction Theory

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1 Generalities and notations

In the production of n particle $a + b \to 1 + 2 + \ldots + n$ we will denote by m_i, E_i, λ_i and p_i with $i \in \{a, b, 1, 2, \ldots, n\}$ the mass, energy, helicities and four-vector of the particle i. The three dimensional space component of four vectors are denoted $\boldsymbol{p}_i$. The physical component are the covariant ones, $p^\mu = (E_i, p_x, p_y, p_z)$ and the metric is such that an on-shell particle satisfies

$$p_i^2 = m_i^2 = E_i - |\boldsymbol{p}_i|^2. \tag{1}$$

The Lorenz invariant relativistic measure is

$$\frac{1}{(2\pi)^3} \int \frac{d^3\boldsymbol{p}_i}{2E_i} = \int \frac{d^4 p_i}{(2\pi)^4}\, 2\pi\delta(p_i^2 - m_i^2)\theta(p_i^0) = \int \frac{d^4 p_i}{(2\pi)^4}\, 2\pi\delta_+(p_i^2 - m_i^2) \tag{2}$$

One often uses the triangle function

$$\begin{aligned} \lambda(s, m_i^2, m_j^2) &= s^2 + m_i^4 + m_j^4 - 2sm_i^2 - 2m_i^2 m_j^2 - 2m_j^2 s \\ &= \left(s - (m_i + m_j)^2\right)\left(s - (m_i - m_j)^2\right) \equiv \lambda_{ij}(s) \end{aligned} \tag{3}$$

Operators, such as the $S-$matrix $\widehat{S} = \widehat{\mathrm{I}} + i\widehat{T}$, are hatted. They act on states normalized as

$$\langle p_i \lambda_i | p_i' \lambda_i' \rangle = (2\pi)^3 \delta^3(\boldsymbol{p}_i' - \boldsymbol{p}_i) 2E_i \delta_{\lambda_i, \lambda_i'} \tag{4}$$

When contracted on the in $|\alpha\rangle = |p_a\lambda_a p_b \lambda_b\rangle$ and out $\langle\beta| = \langle p_1 \lambda_1 \ldots p_n \lambda_n|$ states, we pull out the conservation of energy and momenta with

$$S_{\beta\alpha} \equiv \langle p_1\lambda_1 \ldots p_n\lambda_n | \widehat{S} | p_a\lambda_a p_b\lambda_b\rangle = \mathrm{I}_{\beta\alpha} + (2\pi)^4 \delta^4(P_\alpha - P_\beta) i\mathcal{M}_{\beta\alpha}, \tag{5}$$

with the notation $\mathrm{I}_{\beta\alpha} = \langle\beta|\alpha\rangle = (2\pi)^3\delta^3(\boldsymbol{p}_a - \boldsymbol{p}_1)2E_a\delta_{\lambda_a,\lambda_1}(2\pi)^3\delta^3(\boldsymbol{p}_b - \boldsymbol{p}_2)2E_b\delta_{\lambda_b,\lambda_2}$. In the case of identical particles, there would be the cross term $\delta^3(\boldsymbol{p}_a - \boldsymbol{p}_2)\delta^3(\boldsymbol{p}_b - \boldsymbol{p}_1)$. We also used $P_\alpha = p_a + p_b$ and $P_\beta = \sum_i p_i$.

The n particle differential cross section is

$$d\sigma(a + b \to 1 + \ldots + n) = \frac{1}{F_I}(2\pi)^4\delta^4(P_\alpha - P_\beta)\overline{\sum}_{\lambda_i} \prod_{i=1}^{n} \frac{1}{(2\pi)^3}\frac{d^3\boldsymbol{p}_i}{2E_i}|\mathcal{M}_{\beta\alpha}|^2 \tag{6}$$

The symbol $\overline{\sum}_{\lambda_i} = (2s_a+1)^{-1}(2s_b+1)^{-1}\sum_{\lambda_i}$ stands for the sum over final helicities and average over the initial ones. The previous formula is valid for different particles. In the presence of identical particles, there is an extra factor $1/(n_l!)$ for each group of n_l identical particle (*i.e.* $\sum_l n_l = n$). The flux factor is $F_I = m_b p_{\text{lab}}$ with p_{lab} the momentum of the beam (particle a) in the laboratory frame (the target rest frame).

The unitary relation $\widehat{S}^\dagger \widehat{S} = \widehat{\mathrm{I}}$ with γ_k, a $k-$particle intermediate state, reads:

$$i\left(M^\dagger_{\beta\alpha} - M_{\beta\alpha}\right) = (2\pi)^4 \sum_{k=1}^{\infty} \prod_{i=1}^{k} \int \frac{1}{(2\pi)^3} \frac{d^3\boldsymbol{k}_i}{2E_i} M^\dagger_{\beta\gamma_k} M_{\gamma_k\alpha}\, \delta^4(P_{\gamma_k} - P_\alpha), \tag{7}$$

with $P_{\gamma_k} = \sum_{i=1}^k k_i$. The left-hand-side can also be written $i\left(M^\dagger_{\beta\alpha} - M_{\beta\alpha}\right) = 2\,\mathrm{Im}\, M_{\beta\alpha}$. The unitarity relation for elastic scattering $\alpha = \beta$ allows one to obtain the total cross section in a simple form

$$\sigma(\alpha \to X) = \frac{1}{2F_I} \overline{\sum}_{\lambda_i} \mathrm{Im}\,\mathcal{M}_{\alpha\alpha}. \tag{8}$$

Note that since the final and initial four-momenta are equal, the matrix element $\mathcal{M}_{\alpha\alpha}$ corresponds to the elastic scattering in the forward direction.

In the above formulas we implicitly use natural units $1 \equiv \hbar c = 0.19732$ fm.GeV. We can convert the cross section in physical units by reinstalling the factor

$$10^4(\hbar c)^2 = 389.35\ \mu\text{b.GeV}^2. \tag{9}$$

2 2-to-2 scattering

In this section we particularize the formula for the process $a + b \to 1 + 2$. The Mandelstam variables are

$$s = (p_a + p_b)^2 \quad t = (p_a - p_1)^2 \quad u = (p_a - p_2)^2 \quad s + t + u = m_a^2 + m_b^2 + m_1^2 + m_2^2 \equiv \Sigma_2 \tag{10}$$

The matrix element depends on the four helicities μ_i and two Mandelstam variables $\mathcal{M} \equiv \mathcal{M}_{\mu_i}(s,t)$. In the center-of-mass of the reaction, also called the $s-$channel, the initial and final break-up momenta are

$$q_{ab}(s) = \frac{1}{2\sqrt{s}} \lambda_{ab}^{1/2}(s) \qquad\qquad q_{12}(s) = \frac{1}{2\sqrt{s}} \lambda_{12}^{1/2}(s) \tag{11}$$

The scattering takes place in the xz plane such that the beam is aligned with the z direction $\boldsymbol{p}_a = q_{ab}(s)(0,0,1)$ and $\boldsymbol{p}_1 = q_{12}(s)(\sin\theta_s, 0, \cos\theta_s)$. The scattering angle θ_s in this frame is

$$\cos\theta_s = 1 + \frac{t - t_0}{2q_{ab}(s)q_{12}(s)} = \frac{s(t-u) + (m_a^2 - m_b^2)(m_1^2 - m_2^2)}{4sq_{ab}(s)q_{12}(s)} \tag{12}$$

It is traditional to denote $t' = t - t_0$ with t_0 and t_1 being the limits of the physical region of the scattering $t_1 \leq t \leq t_0 \leq 0$. We have, with $\Delta = (m_b^2 - m_2^2) + (m_a^2 - m_1^2)$:

$$t_{1,0} = \frac{\Delta^2}{4s} + (q_{ab}(s) \pm q_{12}(s))^2 \qquad\qquad t_0 - t_1 = -4q_{ab}(s)q_{12}(s) \tag{13}$$

The boundary of the physical region is given by the Kibble function

$$\phi(s,t) \equiv s(t - t_0)(t - t_1) = 4sq_{ab}^2(s)q_{12}^2(s)\sin^2\theta_s \tag{14}$$

The differential cross section is

$$\frac{d\sigma}{dt} = \frac{1}{64\pi F_I^2} \overline{\sum} |\mathcal{M}|^2 \qquad\qquad \frac{d\sigma}{dt} = \frac{\pi}{q_{ab(s)}q_{12}(s)} \frac{d\sigma}{d\Omega} \tag{15}$$

Allowing only one two-body intermediate state γ, the unitarity takes the simple form

$$2\,\mathrm{Im}\,M_{\beta\alpha}(s,\theta_{\alpha\beta}) = \rho_2(s)\int \frac{d\Omega_{\alpha\gamma}}{4\pi} M^\dagger_{\beta\gamma}(s,\theta_{\gamma\beta}) M_{\gamma\alpha}(s,\theta_{\gamma\beta}). \tag{16}$$

The angle $\theta_{\alpha\beta}$ is the angle between the state α and β. The two-body phase space is, with m_x and m_y being the masses of the two intermediate state γ,

$$\rho_2(s) = \frac{1}{8\pi s}\lambda^{1/2}(s, m_x^2, m_y^b). \tag{17}$$

By convention we have left the factor of two in the left-hand-side of Eq. (16) outside the two-body phase-space factor $\rho_2(s)$.

The partial wave expansion for scalar particles reads

$$\mathcal{M}(s,\cos\theta_s) = \sum_{\ell=0}^{\infty}(2\ell+1)a_\ell(s)P_\ell(\cos\theta_s). \tag{18}$$

The partial wave expansion diagonalize the unitarity equation (7). In the elastic approximation we obtain

$$2\,\mathrm{Im}\,a_\ell(s) = \rho_2(s)|a_\ell(s)|^2 \tag{19}$$

Another convenient and equivalent way to write the elastic unitarity equation for the partial wave is

$$2\,\mathrm{Im}\,a_\ell^{-1}(s) = -\rho_2(s). \tag{20}$$

The partial decomposition can also defined for particle with spin

$$\mathcal{M}_{\mu_i}(s,\theta_s) = \sum_{J=0}^{\infty}(2J+1)a^J_{\mu\mu'}(s)d^J_{\mu\mu'}(\theta_s) \qquad \mu = \mu_1-\mu_2 \qquad \mu' = \mu_3-\mu_4 \tag{21}$$

3 Spinors

3.1 Generalities

See Perl, ArXiv:0703214 and ArXiv:9405376 for the conventions. The physical components are the contra-variant ones $x^\mu = (t, x, y, z)$ etc. The gamma matrices are

$$\gamma^0 = \begin{pmatrix} \mathrm{I}_2 & 0 \\ 0 & -\mathrm{I}_2 \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \qquad \gamma^5 = \begin{pmatrix} 0 & \mathrm{I}_2 \\ \mathrm{I}_2 & 0 \end{pmatrix} \tag{22a}$$

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{22b}$$

They satisfy $\gamma_\mu^\dagger = \gamma_0\gamma_\mu\gamma_0$ and $C^{-1}\gamma_\mu C = -\gamma_\mu^T$ with $C = i\gamma^0\gamma^2$. The commutation relations are

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}\ \mathrm{I}_4 \qquad \{\sigma^i, \sigma^j\} = 2\epsilon^{ijk}\sigma_k. \tag{23}$$

A complete base of 4×4 is given by the 16 matrices, collectively denoted by γ^A

$$\gamma^A = \{\mathrm{I}_4, \gamma_5, \gamma^\mu, \gamma_5\gamma^\mu, \sigma^{\mu\nu}\}, \tag{24}$$

with the notation $\sigma^{\mu\nu} = (i/2)\,[\gamma^\mu, \gamma^\nu] = (i/2)\,(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)$. γ_5 satisfy the following properties

$$\gamma_5 = \frac{1}{4!}\epsilon_{\alpha\beta\mu\nu}\gamma^\alpha\gamma^\beta\gamma^\mu\gamma^\nu \qquad \gamma_5^2 = \mathrm{I}_4 \qquad \mathrm{Tr}\,(\gamma_5) = 0 \tag{25a}$$

$$= i\gamma^0\gamma^1\gamma^2\gamma^3 \qquad [\gamma_\mu, \gamma_5] = 0 \qquad [\sigma_{\mu\nu}, \gamma_5] = 0 \tag{25b}$$

Let $\hat{\boldsymbol{p}} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ being a unit vector $(\hat{\boldsymbol{p}} \cdot \hat{\boldsymbol{p}} = 1)$ with direction (θ, ϕ). The variables domains are $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi[$. The two component spinors along $\hat{\boldsymbol{p}}$ are

$$\chi_+(\hat{\boldsymbol{p}}) = \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \end{pmatrix} \qquad \chi_-(\hat{\boldsymbol{p}}) = \begin{pmatrix} -e^{-i\phi}\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \end{pmatrix} \qquad \frac{1}{2}\boldsymbol{\sigma}\cdot\hat{\boldsymbol{p}}\ \chi_\lambda = \lambda\chi_\lambda \tag{26}$$

They obey the normalization and the useful relations:

$$\chi_\pm^\dagger(\hat{\boldsymbol{p}})\chi_\pm(\hat{\boldsymbol{p}}) = 1 \qquad \chi_\pm^\dagger(\hat{\boldsymbol{p}})\,\boldsymbol{\sigma}\,\chi_\pm(\hat{\boldsymbol{p}}) = \pm\hat{\boldsymbol{p}} \qquad \chi_\pm(\hat{\boldsymbol{p}})\chi_\pm^\dagger(\hat{\boldsymbol{p}}) = \frac{1}{2}\left(\mathrm{I}_2 \pm \hat{\boldsymbol{p}}\cdot\boldsymbol{\sigma}\right) \tag{27}$$

$$\chi_\pm^\dagger(\hat{\boldsymbol{p}})\chi_\mp(\hat{\boldsymbol{p}}) = 0 \qquad \chi_\mp^\dagger(\hat{\boldsymbol{p}})\,\boldsymbol{\sigma}\,\chi_\pm(\hat{\boldsymbol{p}}) = \sqrt{2}\hat{\boldsymbol{\epsilon}}^{(*)} \qquad \chi_\pm(\hat{\boldsymbol{p}})\chi_\mp^\dagger(\hat{\boldsymbol{p}}) = \frac{1}{\sqrt{2}}\hat{\boldsymbol{\epsilon}}^{(*)}\cdot\boldsymbol{\sigma} \tag{28}$$

We introduced the unit vector $\hat{\boldsymbol{\epsilon}} = (1/\sqrt{2})e^{i\phi}(\cos\theta\cos\phi - i\sin\phi, \cos\theta\sin\phi + i\cos\phi, -\sin\theta)$ orthogonal to $\hat{\boldsymbol{p}}$ (($\hat{\boldsymbol{\epsilon}}\cdot\hat{\boldsymbol{\epsilon}} = 1$) and $\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{\epsilon}}$=0). The complex conjugation in $\hat{\boldsymbol{\epsilon}}^{(*)}$ is taken for the lower helicity combination.

For particle moving along the $-\hat{\boldsymbol{p}}$ direction, *i. e.* along the angles $(\pi - \theta, \phi \pm \pi)$ (choose the sign such that $0 \leq \phi \pm \pi < 2\pi$), we would obtain

$$\chi_\pm(-\hat{\boldsymbol{p}}) = \mp e^{\pm i\phi}\chi_\mp(\hat{\boldsymbol{p}}). \tag{29}$$

However in order to satisfy the remove the ambiguity in $\chi_\pm^\dagger(\hat{\boldsymbol{p}})\chi_\mp(-\hat{\boldsymbol{p}}) = \mp e^{\mp i\phi}$ (because in the limit where the momentum goes to zero, a particle with helicity in the direction of his momentum reduces to a particle moving in the opposite direction with helicity opposite to its momentum), we define the spinor along $-\boldsymbol{p}$ with the second particle convention of Jacob and Wick:

$$\chi_\pm(-\hat{\boldsymbol{p}}) \equiv \chi_\mp(\hat{\boldsymbol{p}}). \tag{30}$$

With $v = -C\bar{u}^T$ and $\bar{v} = u^T C^{-1}$, the four component spinors are (with $\bar{u} = u^\dagger \gamma_0$)

$$u_\pm(p) = \beta_p \begin{pmatrix} \chi_\pm(\hat{\boldsymbol{p}}) \\ \pm\alpha_p \chi_\pm(\hat{\boldsymbol{p}}) \end{pmatrix} \qquad v_\pm(p) = \beta_p \begin{pmatrix} -\alpha_p \chi_\mp(\hat{\boldsymbol{p}}) \\ \pm\, \chi_\mp(\hat{\boldsymbol{p}}) \end{pmatrix} \tag{31}$$

$$\bar{u}_\pm(p) = \beta_p \begin{pmatrix} \chi^\dagger_\pm(\hat{\boldsymbol{p}}) & \mp\alpha_p \chi^\dagger_\pm(\hat{\boldsymbol{p}}) \end{pmatrix} \qquad \bar{v}_\pm(p) = \beta_p \begin{pmatrix} -\alpha_p \chi^\dagger_\mp(\hat{\boldsymbol{p}}) & \mp\chi^\dagger_\mp(\hat{\boldsymbol{p}}) \end{pmatrix} \tag{32}$$

The satisfy the Dirac equations (with $\not{p} \equiv p_\mu \gamma^\mu$)

$$(p_\mu \gamma^\mu - m)\, u_\pm(p) = 0 \qquad (p_\mu \gamma^\mu + m)\, v_\pm(p) = 0 \tag{33}$$

$$\tag{34}$$

With the definitions and properties

$$\beta_p = \sqrt{E_p + m} \qquad \alpha_p = \frac{|\boldsymbol{p}|}{E_p + m} \qquad E_p = \sqrt{\boldsymbol{p}^2 + m^2} \tag{35}$$

$$\alpha\beta^2 = |\boldsymbol{p}| \qquad \alpha^2\beta^2 = E_p - m \qquad \bar{u} = u^\dagger \gamma^0 \tag{36}$$

$$\bar{u}_{\lambda'} u_\lambda = 2m\delta_{\lambda'\lambda} \qquad u^\dagger_{\lambda'} u_\lambda = 2E_p \delta_{\lambda'\lambda} \qquad \sum_\lambda u_\lambda \bar{u}_\lambda = \not{p} + m \tag{37}$$

$$\bar{v}_{\lambda'} v_\lambda = -2m\delta_{\lambda'\lambda} \qquad v^\dagger_{\lambda'} v_\lambda = 2E_p \delta_{\lambda'\lambda} \qquad \sum_\lambda v_\lambda \bar{v}_\lambda = \not{p} - m \tag{38}$$

3.2 Spinors in $s-, t-$ and $u-$channels

In the $s-$channel $1(0) + 2(\pi) \to 3(\hat{\boldsymbol{p}}) + 4(-\hat{\boldsymbol{p}})$, the particle 2 and 4 should be defined with the 'second particle convention. In the case of nucleon-nucleon scattering we would use

$$u_\pm(p_1) = \beta_{p_1} \begin{pmatrix} \chi_\pm(\hat{\mathbf{0}}) \\ \pm\alpha_{p_1} \chi_\pm(\hat{\mathbf{0}}) \end{pmatrix} \qquad u_\pm(p_2) = \beta_{p_2} \begin{pmatrix} \chi_\mp(\hat{\mathbf{0}}) \\ \pm\alpha_{p_2} \chi_\mp(\hat{\mathbf{0}}) \end{pmatrix} \tag{39a}$$

$$\bar{u}_\pm(p_3) = \beta_{p_3} \begin{pmatrix} \chi^\dagger_\pm(\hat{\boldsymbol{p}}) & \mp\alpha_{p_3} \chi^\dagger_\pm(\hat{\boldsymbol{p}}) \end{pmatrix} \qquad \bar{u}_\pm(p_4) = \beta_{p_4} \begin{pmatrix} \chi^\dagger_\mp(\hat{\boldsymbol{p}}) & \mp\alpha_{p_4} \chi^\dagger_\mp(\hat{\boldsymbol{p}}) \end{pmatrix} \tag{39b}$$

In the $t-$channel $1(0) + \bar{3}(\pi) \to \bar{2}(\hat{\boldsymbol{p}}) + 4(-\hat{\boldsymbol{p}})$, the particle 3 and 4 should be defined with the 'second particle' convention. In the case of nucleon-nucleon scattering we would use

$$u_\pm(p_1) = \beta_{p_1} \begin{pmatrix} \chi_\pm(\hat{\mathbf{0}}) \\ \pm\alpha_{p_1} \chi_\pm(\hat{\mathbf{0}}) \end{pmatrix} \qquad v_\pm(p_2) = \beta_{p_2} \begin{pmatrix} -\alpha_{p_2} \chi_\mp(\hat{\boldsymbol{p}}) \\ \pm\, \chi_\mp(\hat{\boldsymbol{p}}) \end{pmatrix} \tag{40a}$$

$$\bar{v}_\pm(p_3) = \beta_{p_3} \begin{pmatrix} -\alpha_{p_3} \chi^\dagger_\pm(\hat{\mathbf{0}}) & \mp\chi^\dagger_\pm(\hat{\mathbf{0}}) \end{pmatrix} \qquad \bar{u}_\pm(p_4) = \beta_{p_4} \begin{pmatrix} \chi^\dagger_\mp(\hat{\boldsymbol{p}}) & \mp\alpha_{p_4} \chi^\dagger_\mp(\hat{\boldsymbol{p}}) \end{pmatrix} \tag{40b}$$

In the $u-$channel $1(0) + \bar{4}(\pi) \to 3(\hat{\boldsymbol{p}}) + \bar{2}(-\hat{\boldsymbol{p}})$, the particle 2 and 4 should be defined with the 'second particle' convention. In the case of nucleon-nucleon scattering we would use

$$u_\pm(p_1) = \beta_{p_1} \begin{pmatrix} \chi_\pm(\hat{\mathbf{0}}) \\ \pm\alpha_{p_1} \chi_\pm(\hat{\mathbf{0}}) \end{pmatrix} \qquad v_\pm(p_2) = \beta_{p_2} \begin{pmatrix} -\alpha_{p_2} \chi_\pm(\hat{\boldsymbol{p}}) \\ \pm\, \chi_\pm(\hat{\boldsymbol{p}}) \end{pmatrix} \tag{41a}$$

$$\bar{u}_\pm(p_3) = \beta_{p_3} \begin{pmatrix} \chi^\dagger_\pm(\hat{\boldsymbol{p}}) & \mp\alpha_{p_3} \chi^\dagger_\pm(\hat{\boldsymbol{p}}) \end{pmatrix} \qquad \bar{v}_\pm(p_4) = \beta_{p_4} \begin{pmatrix} -\alpha_{p_4} \chi^\dagger_\pm(\hat{\mathbf{0}}) & \mp\chi^\dagger_\pm(\hat{\mathbf{0}}) \end{pmatrix} \tag{41b}$$