Dispersion theory in hadron form factors

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Jefferson Lab

Objectives

- Review hadron interactions with electroweak fields: currents, matrix elements, form factors ↔ Seminar Vanderhaeghen
- Apply methods of amplitude analysis to form factors: Analyticity, dispersion relations, unitarity ↔ Lectures
- Learn about applications to actual experimental data
- Outline connections with other fields: QCD and partonic structure

Motivation

- Hadron reactions $\leftrightarrow$ structure
- Simple application of amplitude analysis concepts:
  Single-variable function $F(t)$
- Great practical use
- Experimental programs: Electron scattering JLab, Mainz, MIT,
  atomic physics with electronic/muonic hydrogen PSI,
  e^+e^- annihilation BES, KLOE, BABAR, etc., $p\bar{p}$ PANDA
  SM/BSM processes with hadron production,
  neutrino interactions with matter
- Connection with QCD: Generalized parton distributions,
  Lattice QCD methods

Outline

1 – Hadron interactions with external fields
Currents, matrix elements, form factors
Spacelike and timelike FFs
Elastic scattering and annihilation

2 – Analytic properties of form factors
Analytic $F(t)$, physical sheet, principal cut
Dispersion relations and convergence
Spectral function and t-channel processes
Methods for constructing spectral function

3 – Spectral functions from amplitude analysis
Unitarity relation for $\pi\pi$ cut
$\pi\pi \rightarrow NN$ partial wave amplitudes from $\pi N \rightarrow \pi N$ data
Pion form factor from $e^+e^- \rightarrow \pi^+\pi^-$ data

4 – Nucleon electromagnetic form factors
Evaluation of dispersion integral
Comparison with spacelike FF data
Open questions

5 – Extensions and connections
Nucleon scalar form factor
Nucleon axial form factors
Transverse densities and generalized parton distributions

EM interactions: Hadrons in external field

- External field — electromagnetic/weak/other

  Accelerate existing hadron
  Create/annihilate hadron-antihadron pair } Relativity

- Interaction described by Lagrangian density

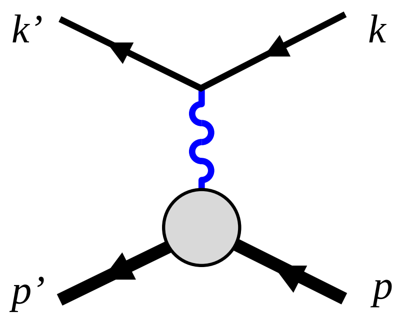
  $L_{\rm em}(x) = e\mathcal{A}_\mu(x) J^\mu(x),$ current operator

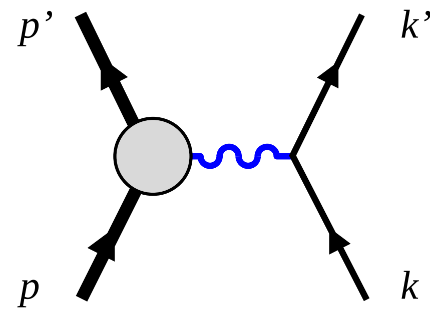
- Transition matrix elements

  $\langle h(p')|J^\mu(x)|h(p)\rangle,$ $\langle h(p')\bar{h}(p)|J^\mu(x)|0\rangle$

  Treat as abstract objects, cf. hadron scattering amplitudes

EM interactions: Lepton scattering





$$\mathcal{M}(lh \to l'h') \quad = \quad e^2 \;\underbrace{\langle h(p')|J^\mu(0)|h(p)\rangle}_{\text{Hadron current}} \;\underbrace{D_{\mu\nu}(k-k')}_{\text{Green function}} \;\underbrace{\langle l(k')|j^\nu(0)|l(k)\rangle}_{\text{Lepton current}}$$

$$\mathcal{M}(l^+l^- \to h\bar{h}) \quad = \quad e^2 \;\langle h(p')\bar{h}(p)|J^\mu(0)|0\rangle \;D_{\mu\nu}(k+k') \;\langle 0|j^\nu(0)|l(k')l(k)\rangle$$

- External field excited by lepton scattering
- Lepton current and Green function given by QED (EW theory)
- Explicit form of amplitude and cross section $\longrightarrow$ Exercise

EM interactions: Current matrix element

- Translational invariance

$$J^\mu(x) \;=\; e^{iPx} J^\mu(0) e^{-iPx}$$ finite translation operator

$$\langle h(p')|J^\mu(x)|h(p)\rangle \;=\; e^{i(p'-p)x}\, \langle h(p')|J^\mu(0)|h(p)\rangle$$ x-dep explicit

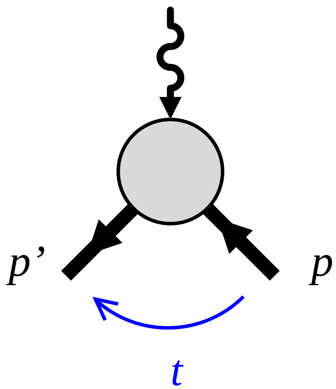
- Current conservation

$$\partial_\mu J^\mu(x) \;=\; 0 \;\;\rightarrow\;\; \partial_\mu \,\langle h(p')|J^\mu(x)|h(p)\rangle \;= 0$$

$$(p'-p)_\mu \,\langle h(p')|J^\mu(0)|h(p)\rangle \;=\; 0$$ transversality

- Similar relations for $\langle h(p')\bar{h}(p)|J^\mu(x)|0\rangle$ with $p \rightarrow -p$

EM interactions: Form factor spin-0 hadron



- Structural decomposition

$$\langle h(p')|J^\mu(0)|h(p)\rangle \quad \leftrightarrow \quad \text{structures}$$

Lorentz 4-vector: $\langle \ |J^\mu| \ \rangle \ \propto \ (p'+p)^\mu, \ (p'-p)^\mu$

Current conserved: $(p-p')_\mu \langle \ |J^\mu| \ \rangle \ = \ 0$

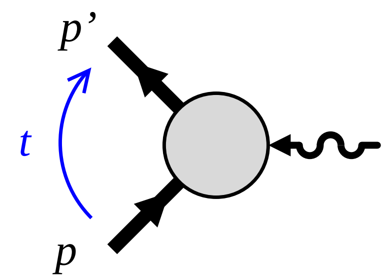
$$\langle \ |J^\mu| \ \rangle \ = \ (p'+p)^\mu \, F(t), \qquad t \ \equiv \ (p'-p)^2$$

- $F(t)$ invariant form factor, cf. invariant amplitudes

  Physical region for scattering: $t < 0$

  $F(0) =$ hadron charge (in units of e)

EM interactions: Form factor spin-0 hadron



- Similar decomposition

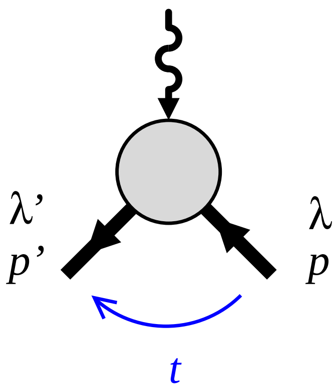
$$\langle h(p')\bar{h}(p)|J^\mu(0)|0\rangle \;=\; (p'-p)^\mu \, F(t), \qquad t = p' + p$$

Physical region for $h\bar{h}$ creation: $\quad t > 4M_h^2 > 0$

- Crossing symmetry: Scattering and annihilation matrix elements are described by a single analytic function

$$F(t), \quad \begin{cases} t < 0, & \text{scattering} \quad \text{“spacelike FF”} \\ t > 4M_h^2, & \text{annihilation} \quad \text{“timelike FF”} \end{cases}$$

EM interactions: Form factors spin-1/2 hadron



- Lorentz invariance requires

$$\langle h(p', \lambda')| J^\mu(0) | h(p, \lambda)\rangle \;\; = \bar{u}(p', \lambda')\, \Gamma^\mu \, u(p, \lambda)$$

Bilinear form in hadron bispinors $u, \bar{u}$

- Independent structures in Γ^μ ?

Heuristic approach: Use transversality, Dirac eqn, gamma matrix identities

Systematic approach: Count helicity amplitudes → Exercise

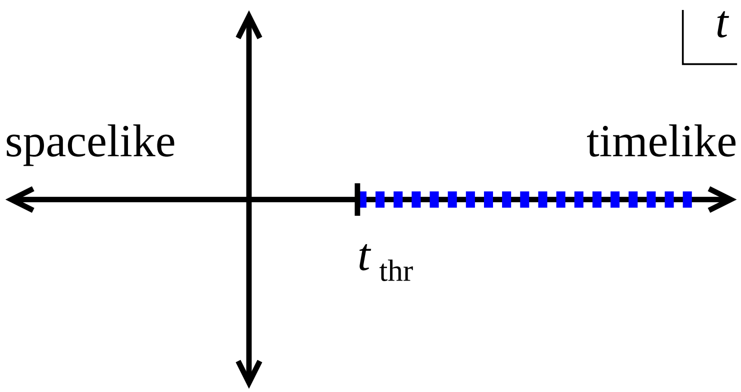
$$\Gamma^\mu = \gamma^\mu \, F_1(t) \; + \; \frac{i\sigma^{\mu\nu}(p' - p)^\nu}{2M_h} F_2(t)$$

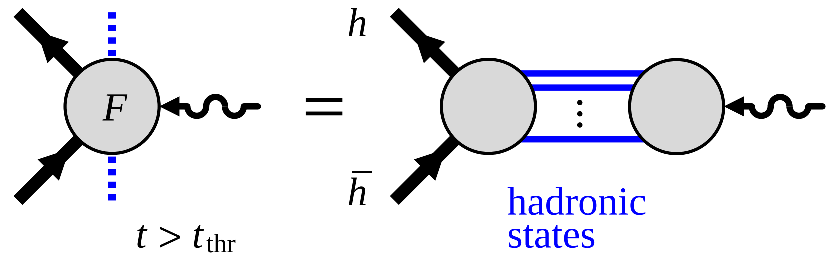
- $F_{1,2}(t)$ invariant FFs

$F_1(0) =$ charge, $F_2(0) =$ anomalous magnetic moment

Dirac and Pauli FFs; also other choices G_E, G_M

Analytic properties: Form factor





- FF analytic function of t

  Physical sheet, approached from $t < 0$

  No singularities at $t < 0$

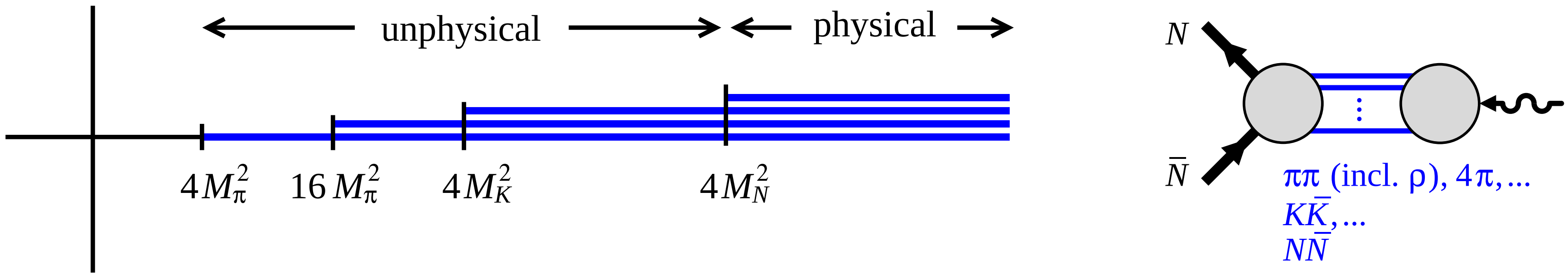
- Singularities at $t > 0$ — poles, cuts

  Result from "processes": current $\rightarrow$ hadronic state $\rightarrow h\bar{h}$

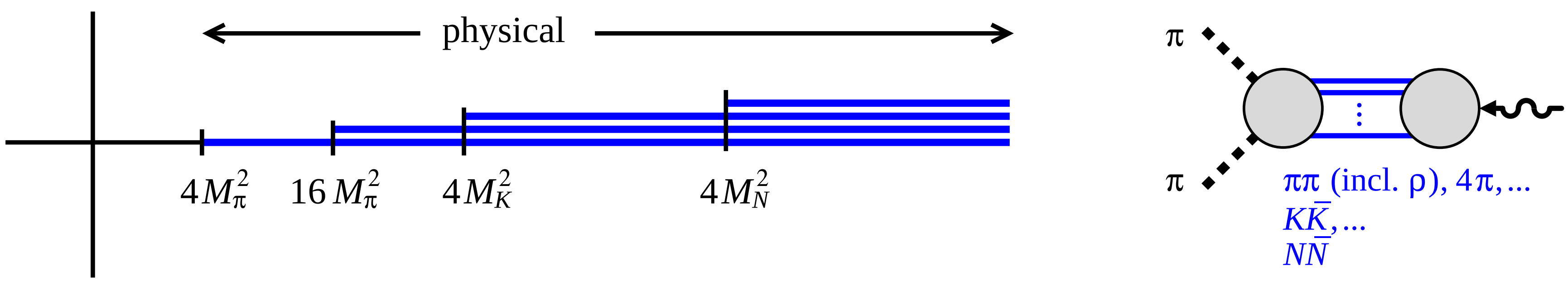
  Can occur below the physical $h\bar{h}$ threshold

Analytic properties: Examples

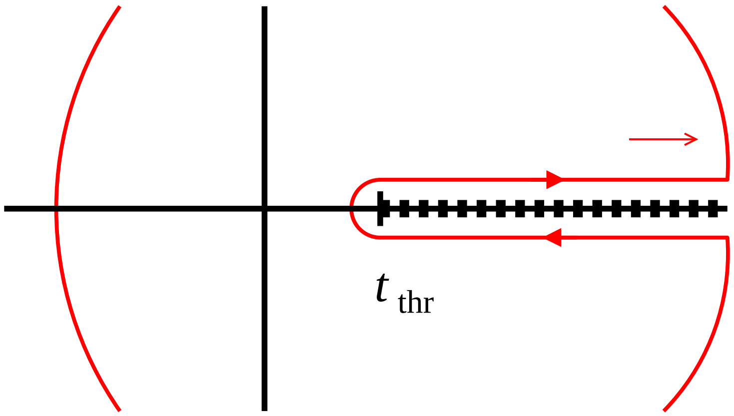
- Nucleon electromagnetic FFs, isovector



- Pion electromagnetic FF



Analytic properties: Dispersion relation



$$F(t) \;=\; \int_{t_{\text{thr}}}^{\infty} \frac{dt'}{\pi} \; \frac{\text{Im}\, F(t + i0)}{t' - t}$$

Dispersion relation

- Convergence at $|t| \to \infty$: $F(t) \sim t^{-1}$ (pion), $\sim t^{-2,3}$ (nucleon) from QCD

- DR represents FF at all t on physical sheet

  Implements correct analytic properties. Useful tool

- Spectral function $\;\text{Im}\, F(t + i0) \;=\; [F(t + i0) - F(t - i0)]/(2i)$

  Need to know it in order to evaluate integral. Unphysical region!

Analytic properties: Spectral functions

- Amplitude analysis techniques + hadronic scattering data: Unitarity, analytic continuation

  Frazer, Fulco, Phys. Rev. **117**, 1603 (1960); Phys. Rev. **117**, 1609 (1960)
  Hohler, Pietarinen, Nucl. Phys. B **95**, 210 (1975)
  Hoferichter, Kubis, Ruiz de Elvira, Hammer, Meissner, Eur. Phys. J. A **52**, 331 (2016)

- Fits to spacelike FF data

  Hohler et al., Nucl. Phys. B **114** (1976) 505
  Belushkin, Hammer, Meissner, Phys. Rev. C **75**, 035202 (2007)
  Lorenz, Hammer, Meissner, Eur. Phys. J. A **48**, 151 (2012)

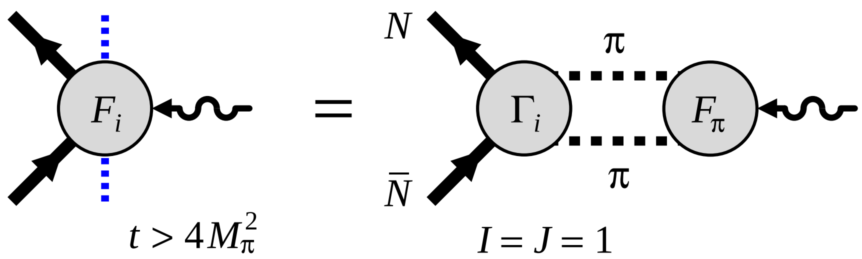
- Dynamical calculations: Chiral effective field theory

  Gasser, Sainio, Svarc, Nucl. Phys. B **307**, 779 (1988)
  Bernard, Kaiser, Meissner, Nucl. Phys. A **611**, 429 (1996)
  Becher, Leutwyler, Eur. Phys. J. C **9**, 643 (1999)
  Kubis, Meissner, Nucl. Phys. A **679**, 698 (2001)
  Kaiser, Phys. Rev. C **68**, 025202 (2003)

- Combination of above methods

  χEFT + unitarity: Granados, Leupold, Perotti, Eur. Phys. J. A **53**, 117 (2017);
  Alarcon, Hiller Blin, Vicente Vacas, Weiss, Nucl. Phys. A **964**, 18 (2017)

Analysis: Unitarity relation



$$\mathrm{Im}F_i(t) = \frac{k_{\mathrm{cm}}^3}{\sqrt{t}}\,\Gamma_i(t)\,F_\pi^*(t)$$

- At $4M_\pi^2 < t < 16M_\pi^2$ only $\pi\pi$ channel open — two-pion cut

- $\mathrm{Im}F_i(t)$ from elastic unitarity relation

  Here: Nucleon isovector EM FFs

  CM frame of timelike process, $k_{\mathrm{cm}} = \sqrt{t/4 - M_\pi^2}$ pion CM momentum

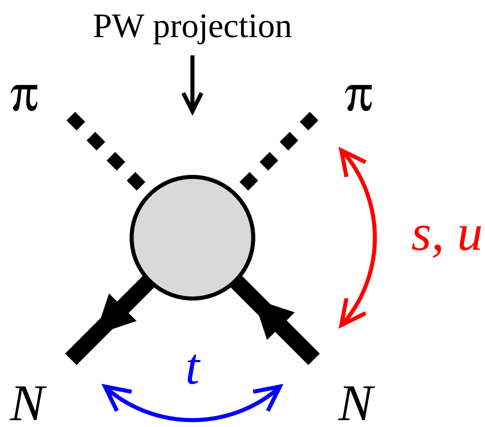
  $F_\pi(t)$ current $\rightarrow \pi\pi$ partial-wave amplitude $=$ pion timelike FF

  $\Gamma_i(t)$ $\pi\pi \rightarrow N\bar{N}$ partial-wave amplitude

  Amplitudes $F_\pi(t)$ and $\Gamma_i(t)$ have same phase — Watson theorem

  Amplitudes "contain" ρ as $\pi\pi$ resonance

Analysis: $\pi\pi \to N\bar{N}$ partial-wave amplitudes



- Idea: Construct $\pi\pi \to N\bar{N}$ PWAs $\Gamma_i(t)$ from $\pi N \to \pi N$ scattering data using amplitude analysis techniques

Two major challenges:

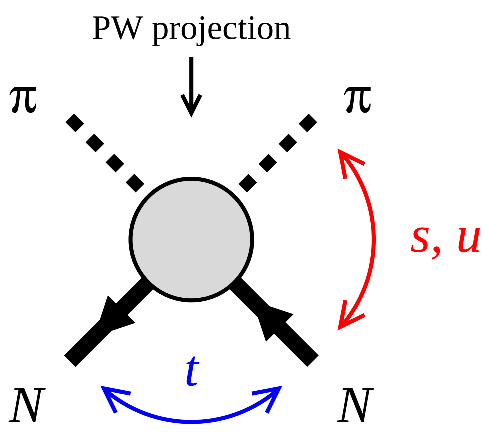
- t–channel partial wave projection requires knowledge of amplitude in unphysical region of s, u–channel processes $s, u \; < \; (M_\pi + M_N)^2$

  $\to$ analytic continuation in s, u at fixed t

- For spectral function we need t-channel PWAs at $t > 4\ M_\pi^2 > 0$, while $\pi N \to \pi N$ data are at $t < 0$

  $\to$ analytic continuation in t

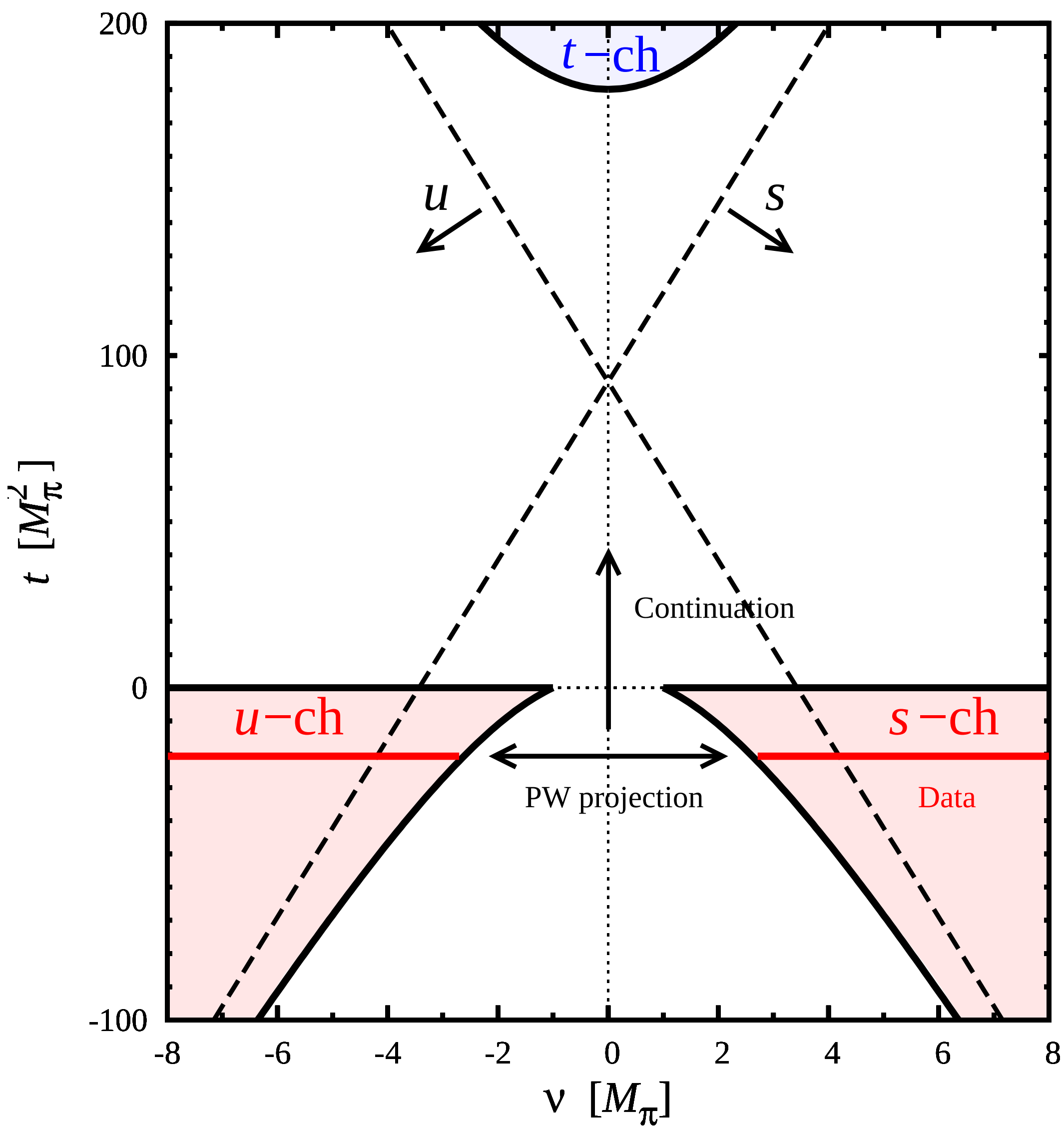
Analysis: $\pi\pi \to N\bar{N}$ partial-wave amplitudes



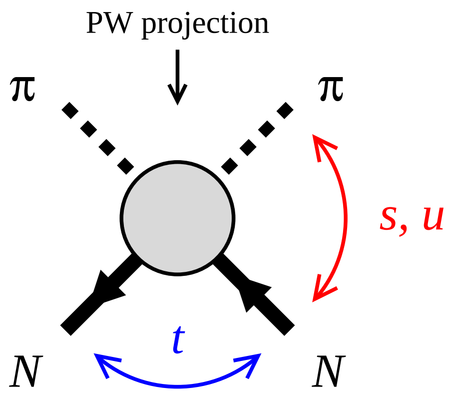
- Mandelstam diagram

$$\nu = \frac{s-u}{4M_N}$$

crossing-symmetric variable



Analysis: $\pi\pi \to N\bar{N}$ partial-wave amplitudes



I) Construct $\pi N \to \pi N$ invariant amplitudes $A_\pm$ with proper analyticity in s, u at fixed $t < 0$

Data in physical region $\quad s, u > (M_\pi + N_N)^2$

Continue to unphysical $\quad s, u < (M_\pi + N_N)^2$

II) Calculate t-channel partial-wave projections

$$f_\pm(t) \;=\; (\text{factor}) \times \int_{-1}^{1} dz \left\{ \begin{array}{c} P_1(z) \\ P_0(z) - P_2(z) \end{array} \right\} A_\pm(\nu, t) \qquad \leftrightarrow \Gamma_{1,2}(t)$$

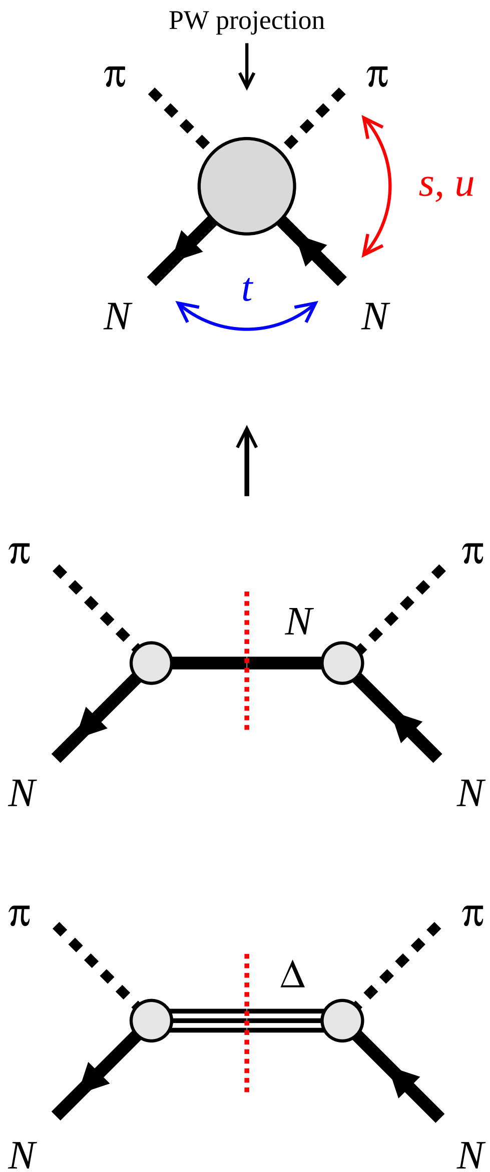
$$\nu \;=\; \frac{s-u}{4M_N}, \qquad z \;=\; -\cos\theta_t \;=\; \frac{M_N \nu}{\sqrt{M_N^2 - t/4}\sqrt{M_\pi^2 - t/4}}$$

III) Analytically continue PWAs to region $\quad t > 4M_\pi^2$

Use DR for PWA (left-hand cut), N/D method, estimate uncertainties

Hohler et al 74

Analysis: $\pi\pi \to N\bar{N}$ partial-wave amplitudes



- What hadronic processes contribute to the $\pi N \to \pi N$ amplitude?

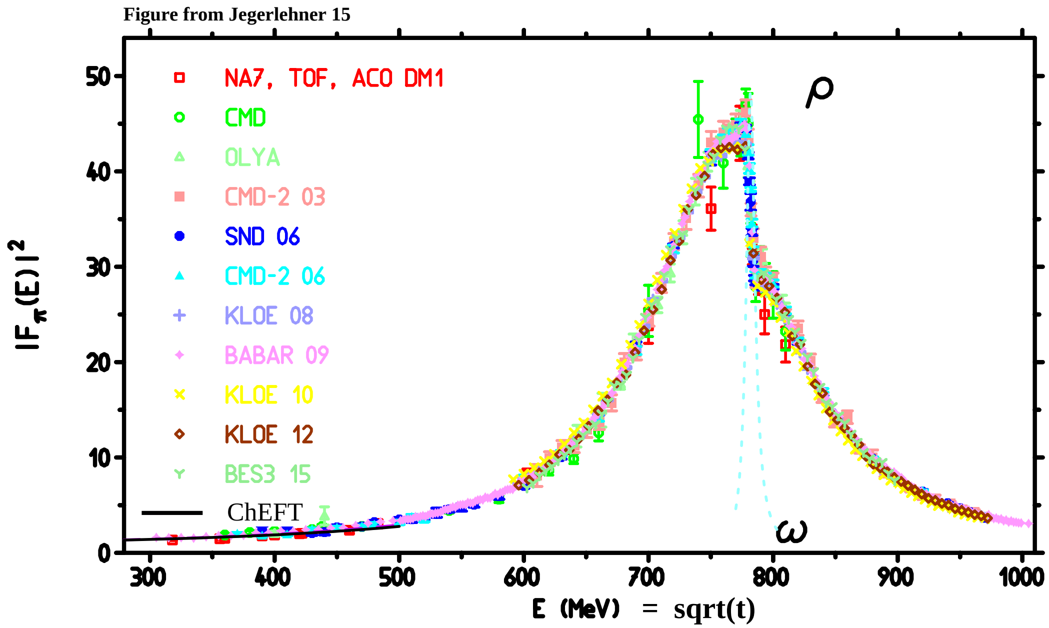
- Born term amplitudes with intermediate N, Δ

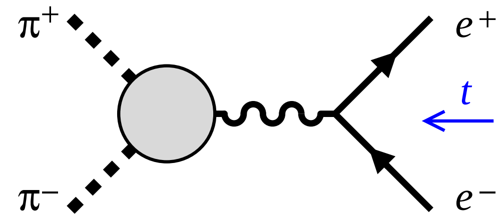
  Singularities at $s, u = M_N^2, M_\Delta^2$

  Contribute to left-hand cut of PWA

- Also other intermediate states πN etc.

Analysis: Pion timelike form factor





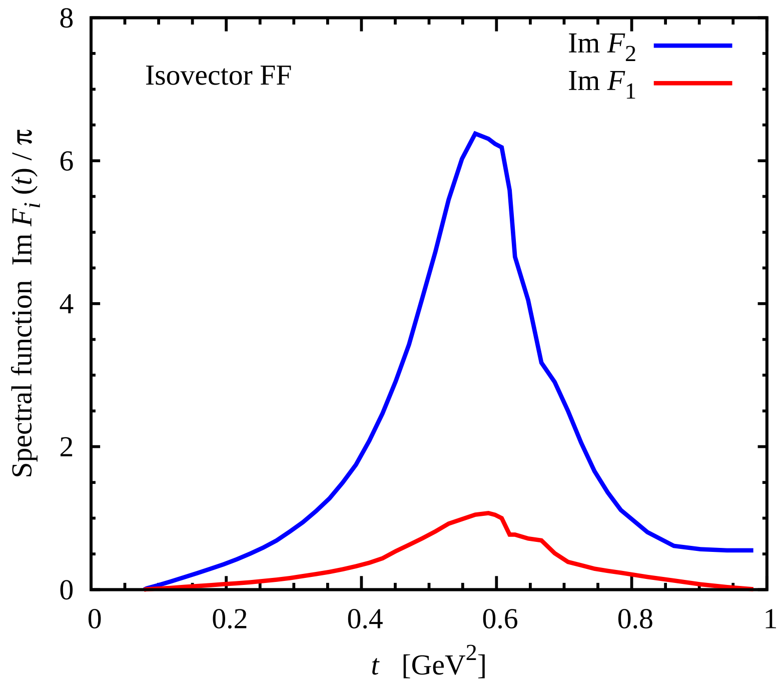
- $|F_\pi(t)|^2$ measured in $e^+e^- \to \pi^+\pi^-$ exclusive annihilation

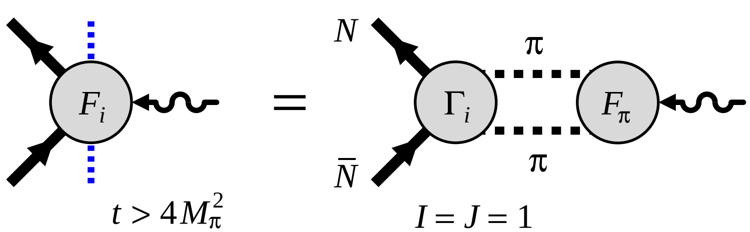
  ρ as $\pi\pi$ resonance

- Phase (Im/Re) determined by fit to resonant amplitude parametrization

  Gounaris-Sakurai: Effective range expansion of $\pi\pi$ phase shift, good description of line shape Gounaris, Sakurai, PRL 21 (1968) 244

Analysis: Spectral function





- Spectral function on $\pi\pi$ cut calculated

  Elastic unitarity condition
  $+$ data $\pi N \to \pi N$, $e^+e^- \to \pi^+\pi^-$
  $+$ analytic continuation

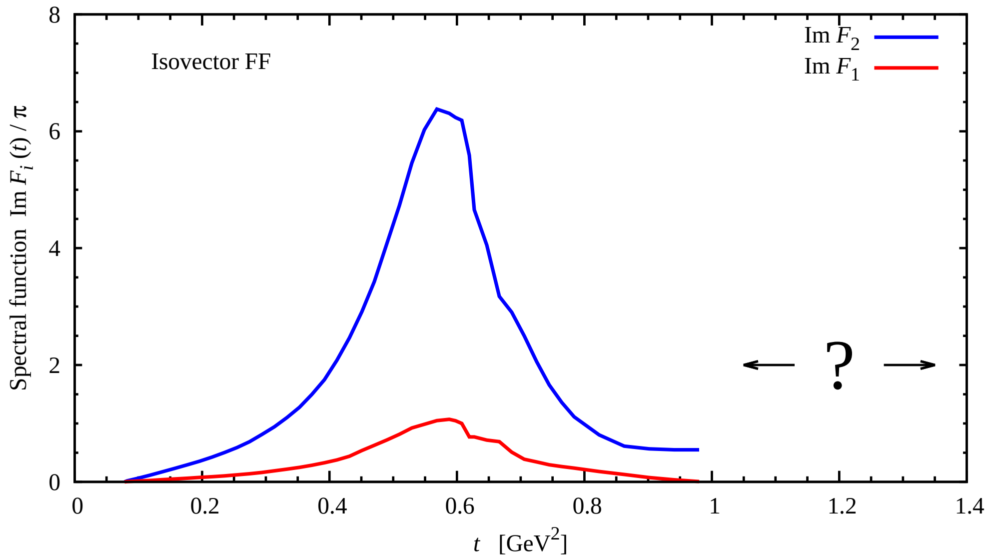
  Valid up to $t \sim 1$ GeV^2

  Contains ρ resonance

- Alt. approach: Dynamical calculations

  χEFT $+$ N/D method
  Alarcon, Hiller Blin, Vicente Vacas, Weiss, NPA **964**, 18 (2017)

Nucleon FF: Dispersion integral



- Evaulate dispersion integral

$$F_i(t) \;=\; \int_{4M_\pi^2}^{\infty} \frac{dt'}{\pi} \, \frac{\text{Im}\, F_i(t+i0)}{t'-t}$$

- What about higher t' ?

- Constraints on spectral function

$$F_1(t) \sim t^{-2} \quad \rightarrow \quad \int_{4M_\pi^2}^{\infty} \frac{dt'}{\pi} \; \text{Im}\, F_i(t') = 0$$

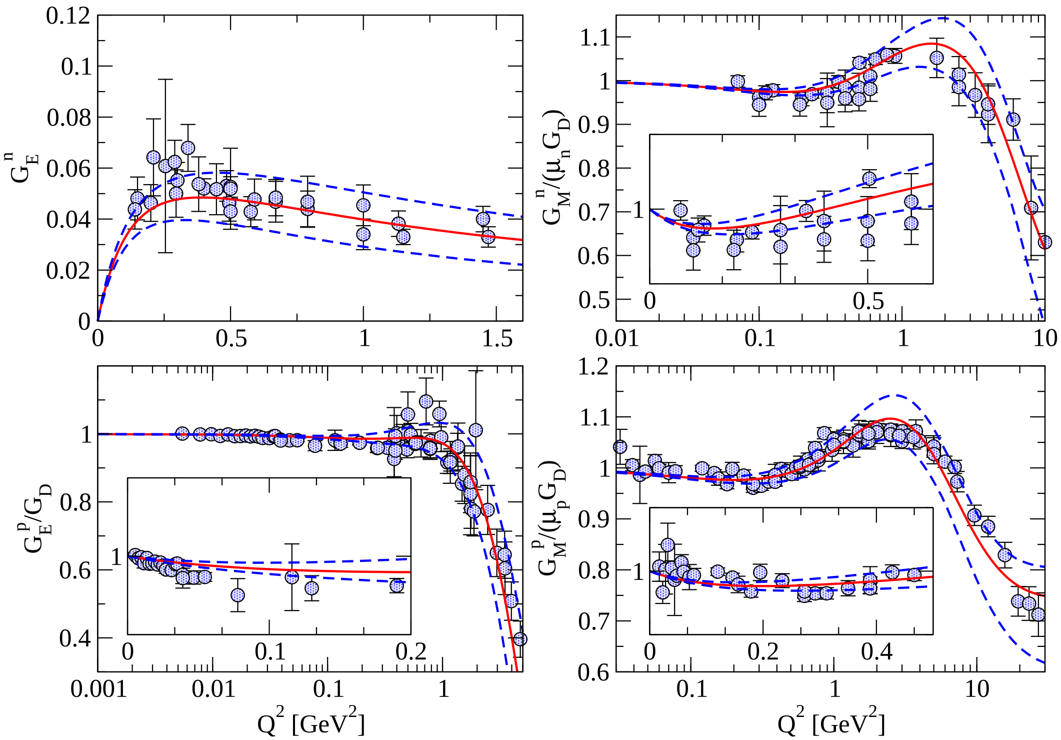
asymptotic behavior

$$F_1(0) \;=\; \text{charge} \;=\; \int_{4M_\pi^2}^{\infty} \frac{dt'}{\pi} \, \frac{\text{Im}\, F_i(t')}{t'}$$

charge

- Parametrize isovector spectral function at high t': Simple pole, multiple poles
Isoscalar: ω, ϕ + high t' poles Hohler et al NPB **114** 505 (1976); Belushkin et al PRC **75** 035202 (2007).

Nucleon FF: Fits to spacelike data



- High–t' spectral functions determined by fit to spacelike FF data
  Belushkin et al PRC **75** 035202 (2007); Lorenz et al EPJA **48**, 151 (2012)

- Good description of FF data

  Analytic parametrization of FF. Permits $t \to 0$ extrapolation, derivatives

Nucleon FF: Derivatives

$$F(t) \;=\; \int_{4M_\pi^2}^{\infty} \frac{dt'}{\pi} \; \frac{\mathrm{Im}\, F_i(t+i0)}{t'-t} \quad \rightarrow \quad F'(t=0) \;=\; \int_{4M_\pi^2}^{\infty} \frac{dt'}{\pi} \; \frac{\mathrm{Im}\, F_i(t+i0)}{(t')^2}$$

- Derivatives of FFs at $t = 0$ given by well-convergent integrals

  Dominated by $t' < 1$ GeV^2. Predictions of unitarity!

  "ρ meson dominance"

- Nucleon charge/magnetic radii

$$\langle r^2 \rangle_1 \;=\; 6\, F_1'(0) \qquad\qquad \langle r^2 \rangle_2 \;=\; 6\, \frac{F_2'(0)}{F_2(0)}$$

  Interpretation in context of non-relativistic systems:
  Radius of 3D charge/magnetization density in Breit frame $q^\mu = (0, \boldsymbol{q})$

  Relativistic systems: Use 2D transverse densities!

Extensions: Scalar nucleon FF

- Amplitude analysis methods described here can be applied to variety of meson and baryon form factors

- Scalar nucleon FF

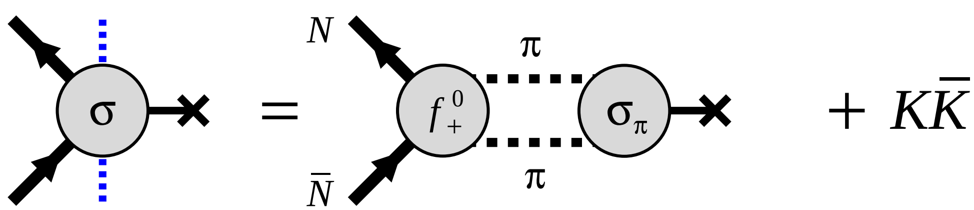
Gasser, Leutwyler, Sainio, PLB **253**, 260 (1991)

$$O_\sigma(x) = \sum_{u,d} m\bar{\psi}(x)\psi(x), \qquad \langle N(p')|O_\sigma(0)|N(p)\rangle = \bar{u}'u\,\sigma(t)$$

Fundamental interest: Mass term of QCD Hamiltonian, trace of QCD energy-momentum tensor, "σ term"

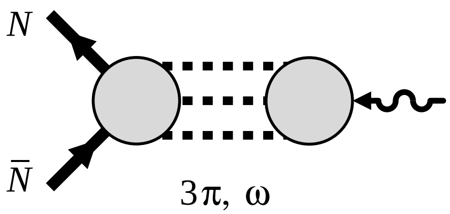
Two-pion cut, unitarity condition. Coupling to $K\bar{K}$ at $t > 4M_K^2$

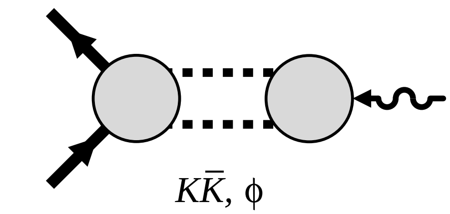
Pion scalar FF from dispersion theory with χEFT input



Extensions: Vector and axial FFs

- Isoscalar vector FF



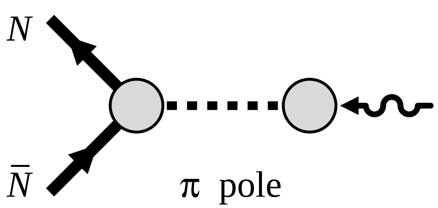


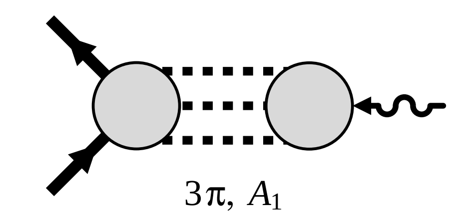
Intermediate states $3\pi, K\bar{K}$ + higher

3-body unitarity: Techniques being developed
JPAC: Pilloni, Szczepaniak

Coupled channel problem

- Isovector pseudoscalar and axial FF



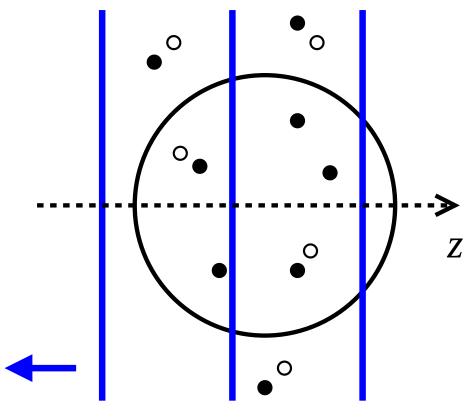


$$J^{5\mu}(x) = \bar{\psi}(x)\gamma^{\mu}\gamma^{5}\psi(x) \;\; [u-d]$$

Pion pole in induced pseudoscalar FF

3π + higher intermediate states in axial FF

Extensions: Transverse densities



- Structure of relativistic system is naturally described at fixed light-front time $x^+ = x^0 + x^3$

  Boost-invariant definition of "time"

  Frame-independent wave functions, densities

  Unambiguous "particle content" of system

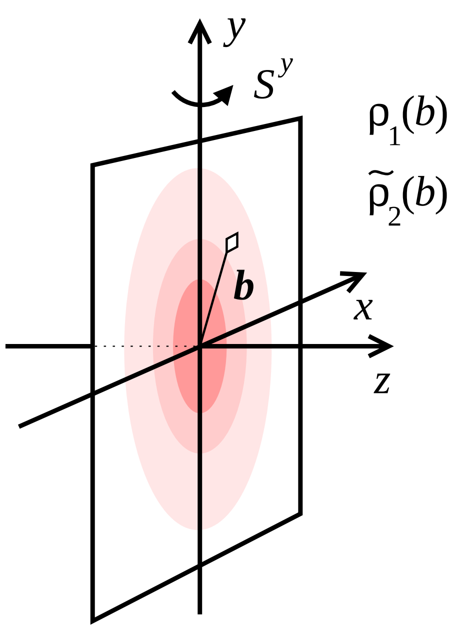
- FFs expressed through transverse densities
  Soper 76, Burkardt 00, Miller 07

$$F_{1,2}(t = -\boldsymbol{\Delta}_T^2) \; = \; \int d^2b \; e^{i\boldsymbol{\Delta}_T \boldsymbol{b}} \; \rho_{1,2}(b)$$

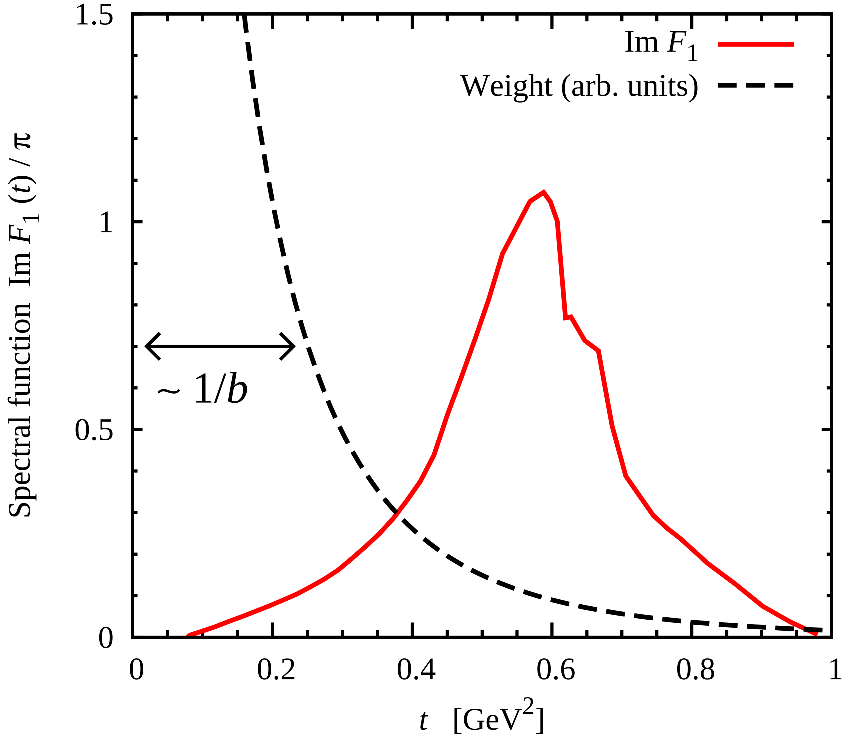
  Cumulative charge/magnetization density at transverse position b

  Connection with GPDs in QCD

$$\rho_1(b) \; = \; \textstyle\sum_q e_q \; \displaystyle\int_0^1 dx \; [q - \bar{q}](x, b)$$



Extensions: Transverse densities



- Dispersive representation

$$\rho(b) \;=\; \int_{4M_\pi^2}^{\infty} \frac{dt}{2\pi}\, K_0(\sqrt{t}b)\, \frac{\mathrm{Im}\, F(t)}{\pi}$$

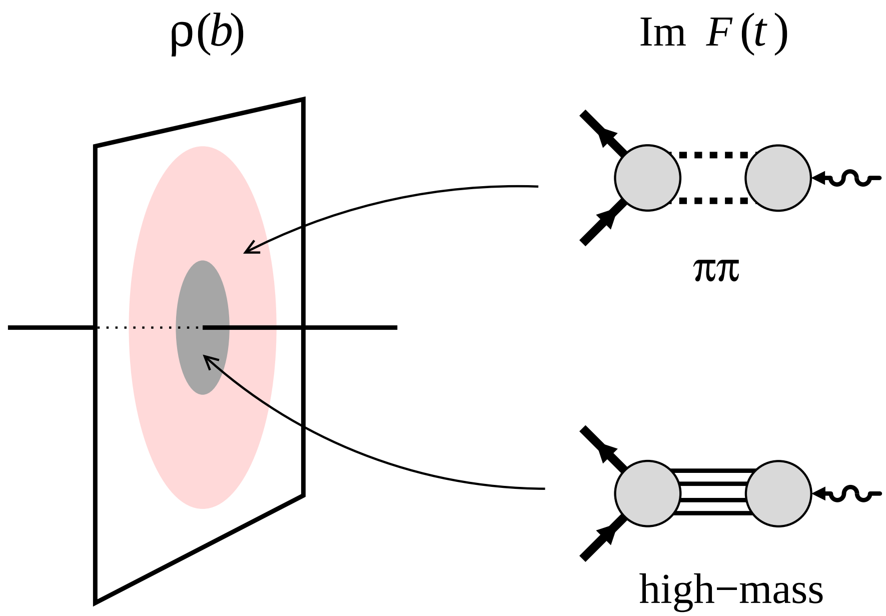
$K_0 \sim e^{-b\sqrt{t}}$ ensures exponential suppression of large t

Distance b selects $\sqrt{t} \sim 1/b$: Filter for spectral function

Peripheral densities dominated by lowest–mass intermediate states

Strikman, Weiss PRC **82** 042201 (2010);
Miller, Strikman, Weiss PRC **84** 045205 (2011)

Extensions: Transverse densities

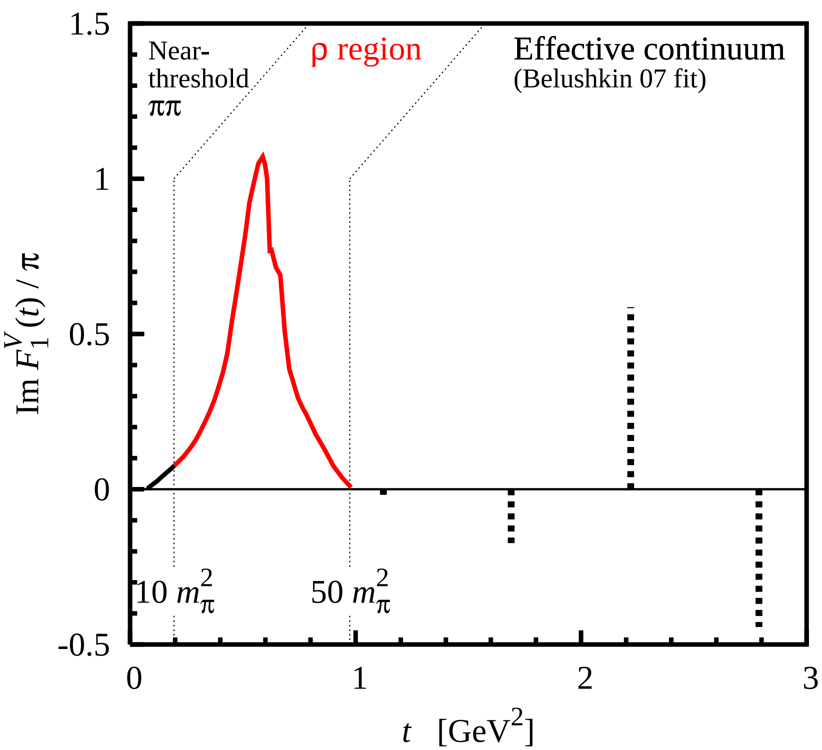


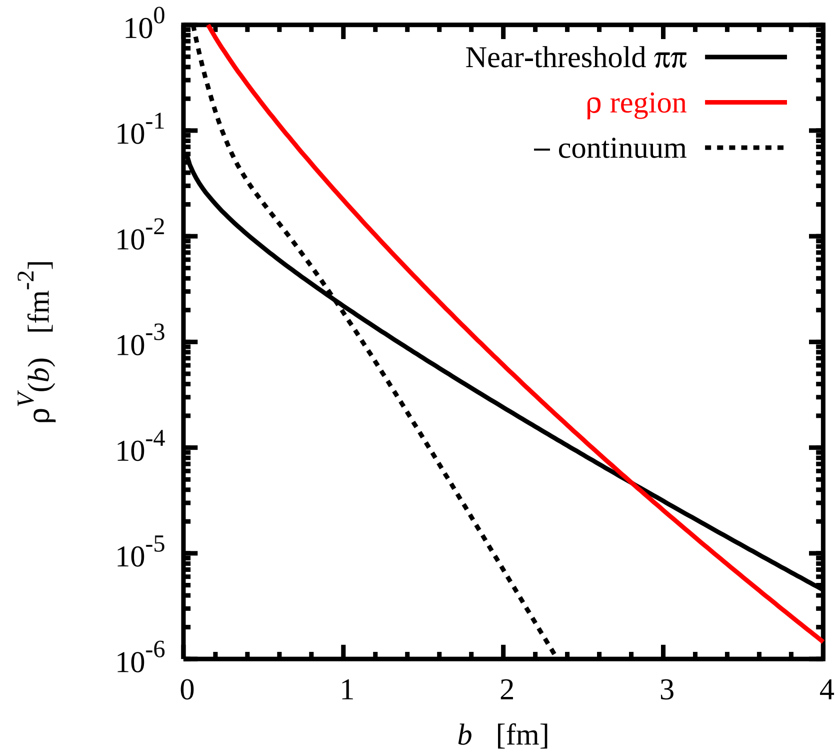
- Spatial structure of t-channel processes

Peripheral $b \gtrsim 1$ fm	$\leftrightarrow$	$\pi\pi$ intermediate state (includes ρ)
Central $b \ll 1$ fm	$\leftrightarrow$	higher-mass intermediate states

- Quantify "range" of exchange mechanisms

Extensions: Transverse densities





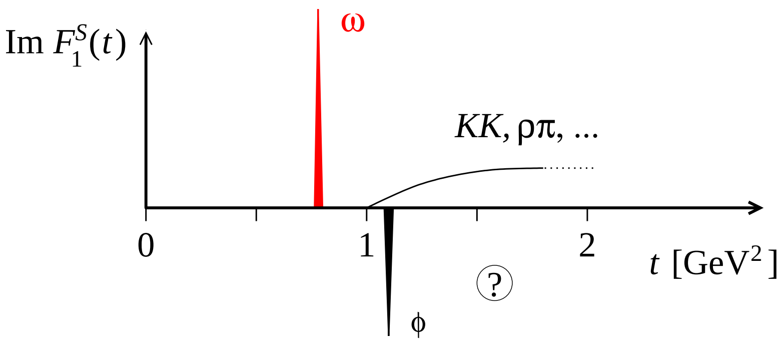
- Spectral analysis of isovector charge density

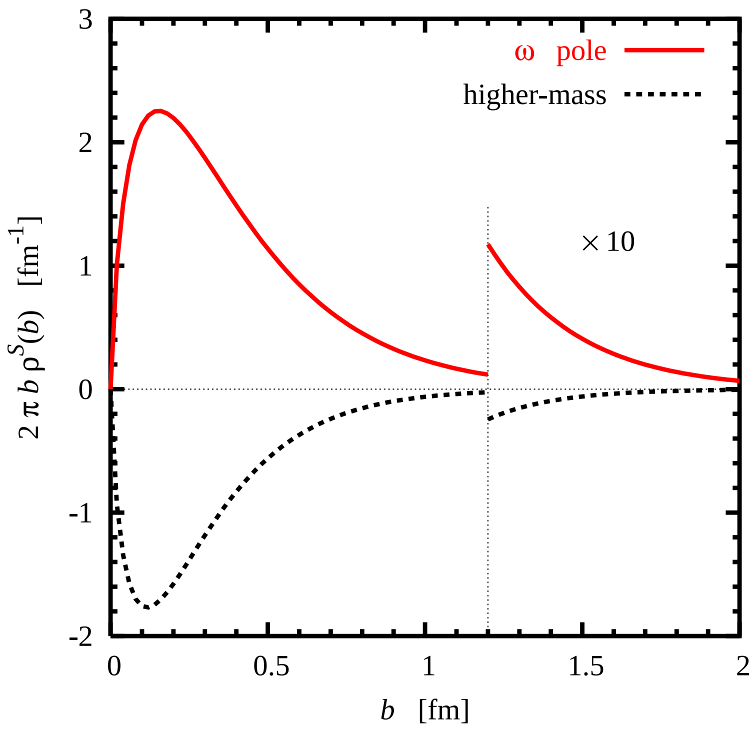
  Near–threshold $\pi\pi$ relevant only at $b > 3\,\mathrm{fm}$

  Intermediate $b = 0.5 - 1.5\,\mathrm{fm}$ dominated by ρ

  Higher–mass states relevant only at $b < 0.3$ fm

Extensions: Transverse densities





- Spectral analysis of isoscalar density — Miller, Strikman, CW 11

  ω dominates at $b > 1.5\,\mathrm{fm}$

  Large cancellations between ω and higher–mass states at $b = 0.5 - 1\,\mathrm{fm}$

Extensions: Transverse densities

- Quark-hadron duality in transverse densities

  Hadron exchanges in t–channel $\leftrightarrow$ Partonic configurations in s–channel

- Pion FF and transverse density

  $|F_\pi(t)|^2$ from $e^+e^- \to \pi^+\pi^-$ data

  Spectral function $\mathrm{Im}\, F_\pi(t)$ from fit to resonant amplitude parametrization

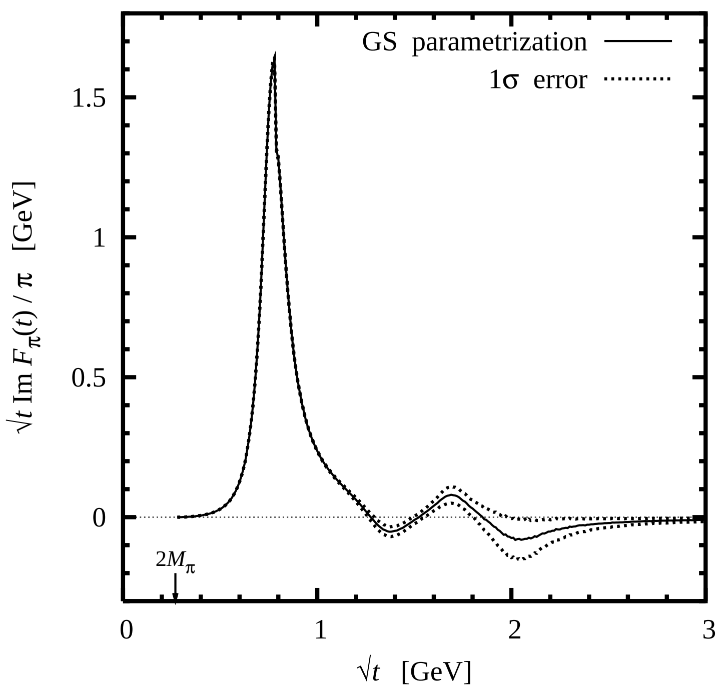
  FF and transverse density from dispersion relation

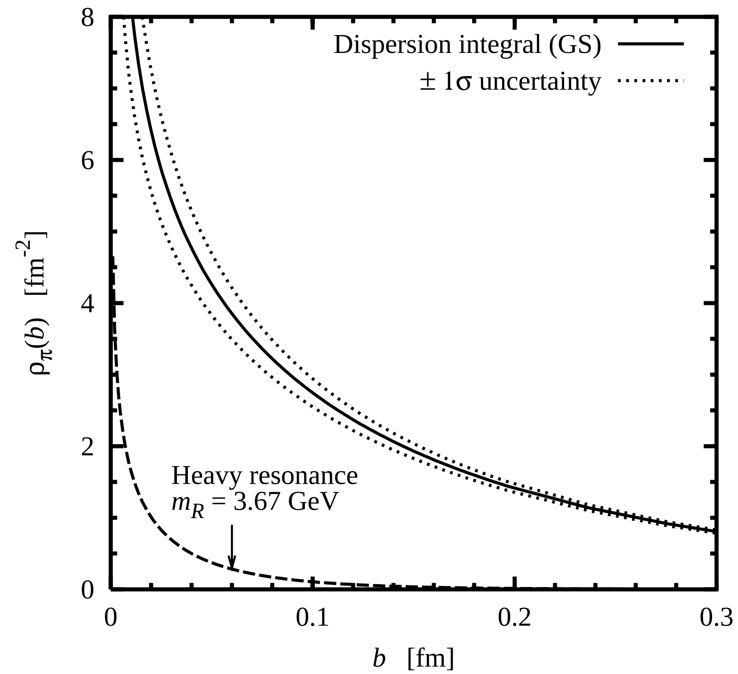
- Resonance transition FFs and densities

  $N \to N^*$ and $N^* \to N^*$ transition FFs

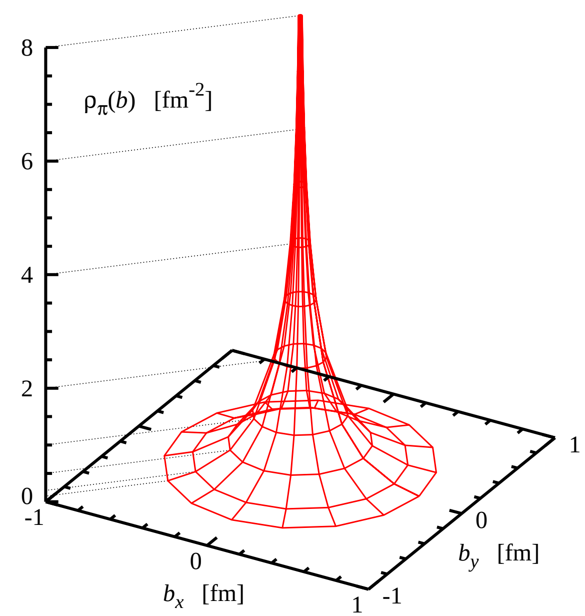
  S-matrix theory: Resonance structure defined at complex pole

Extensions: Pion FF and density





- $|F_\pi(t)|^2$ measured in $e^+e^- \rightarrow \pi^+\pi^-$ up to $t \sim 10$ GeV^2
- $\mathrm{Im}\, F_\pi(t)$ from resonant amplitude fit
- Transverse density from dispersion relation
- High density in center of pion: Pointlike $q\bar{q}$ configurations
  Miller, Strikman, Weiss, PRD **83**, 013006 (2011)



Summary

- Interaction of hadrons with external fields described by current matrix elements
- FFs have analytic properties similar to hadronic scattering amplitudes
- Spectral functions on $\pi\pi$ cut from elastic unitarity and hadronic data
- Good description of spacelike nucleon FFs and derivatives
- Transverse densities connect hadronic exchanges with partonic structure
- *Amplitude analysis is not just science, but also art!*