Joined Physics Analysis Center

Summer Workshop on the Reaction Theory
Exercise sheet 7

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Classwork

1. Using $\int_0^\infty e^{-ax}dx = 1/a$, compute the series (and make the factor s^α appear)

$$F(s,\alpha) = \sum_{J=0}^{\infty} \frac{s^J}{J-\alpha}. \tag{1}$$

2. Consider $\pi\pi \to \pi\pi$ with m being the pion mass. The reduce amplitude φ_ℓ is defined by removing the barrier factor $B_\ell = (t-4m^2)^\ell$ from the (elastic) partial amplitude $t_\ell(s) = B_\ell(s)\varphi_\ell$. The phase space factor is $\rho(s) = (1/16\pi)\sqrt{1-4m^2/s}$. Use the unitarity equation $\operatorname{Im} t_\ell(s) = \rho(s)|t_\ell|^2$ to deduce the unitarity equation for the reduce amplitude.

3. Consider $\pi\pi \to \pi\pi$ and $\pi\pi \to K\bar{K}$ with m_1 being the pion mass and m_2 being the kaon mass. Let us denote by 1 (2) the $\pi\pi$ ($K\bar{K}$) channel so that $t_\ell^{ij}(s)$ is the partial wave for the scattering $i \to j$. The reduce amplitude φ_ℓ^{ij} is defined by removing the barrier factors $B_\ell^i(s) = (s-4m_i^2)^\ell$ from the (elastic) partial amplitude $t_\ell^{ij}(s) = \sqrt{B_\ell^i(s)B_\ell^j(s)}\varphi_\ell^{ij}(s)$. Note that $t_\ell^{ji}(s) = t_\ell^{ij}(s)$. The phase space factors are $\rho_i(s) = (1/16\pi)\sqrt{1-4m_i^2/s}$. Use the unitarity equation $\operatorname{Im} t_\ell^{ij}(s) = \sum_{k=1,2}\rho_k(s)t_\ell^{ik*}(s)t_\ell^{kj}(s)$ or equivalently

$$\operatorname{Im} t_\ell^{11}(s) = \rho_1(s)|t_\ell^{11}(s)|^2 + \rho_2(s)|t_\ell^{12}(s)|^2, \tag{2a}$$
$$\operatorname{Im} t_\ell^{12}(s) = \rho_1(s)t_\ell^{11*}(s)t_\ell^{12}(s) + \rho_2(s)t_\ell^{12*}(s)t_\ell^{22}(s), \tag{2b}$$
$$\operatorname{Im} t_\ell^{22}(s) = \rho_1(s)|t_\ell^{12}(s)|^2 + \rho_2(s)|t_\ell^{22}(s)|^2, \tag{2c}$$

to derive the unitarity equations for the reduce amplitudes φ_ℓ^{ij}.

4. In the single channel case $\pi\pi \to \pi\pi$, assume the following form for the reduce amplitude $\varphi_\ell(s) = \beta(s)/(\ell-\alpha(s))$ and derive the unitarity equation for the Regge trajectory $\alpha(s)$ using the unitarity equation for the reduce amplitude, assuming the residue $\beta(s)$ is real.

5. In the coupled channel case $\pi\pi \to \pi\pi, K\bar{K}$, assume the following form for the reduce amplitude $\varphi_\ell^{ij}(s) = \beta_{ij}(s)/(\ell-\alpha(s))$. Derive the unitarity equation for the Regge trajectory $\alpha(s)$ using the three unitarity equations for the reduce amplitudes $\varphi^{11}, \varphi^{12}$ and φ^{22}, assuming the residues $\beta_{ij}(s)$ are real (and $\beta_{12} = \beta_{21}$). Since these unitarity equations are equal, what are the resulting constraints on the residues $\beta_{ij}(s)$?

Solution

1. Use the trick given and the change of variable $t = se^{-x}$ to obtain

$$F(s,\alpha) = \sum_J^{\infty} \int_0^{\infty} (se^{-x})^J e^{\alpha x} dx = \int_0^{\infty} \frac{e^{\alpha x} dx}{1 - se^{-x}} = s^{\alpha} \int_0^{s} \frac{t^{-\alpha-1}}{1-t} dt \tag{3}$$

2. By remplacement we obtain

$$\operatorname{Im} \varphi_\ell(s) = \rho(s) B_\ell(s) |\varphi(s)|^2 \tag{4}$$

or $\operatorname{Im} \varphi_\ell^{-1}(s) = -\rho(s) B_\ell(s)$.

3. By remplacement we obtain

$$\operatorname{Im} t_\ell^{ij}(s) = \rho_1(s) t_\ell^{i1*}(s) t_\ell^{1j}(s) + \rho_2(s) t_\ell^{i2*}(s) t_\ell^{2j}(s) \tag{5a}$$

$$\begin{aligned}\operatorname{Im} \sqrt{B_\ell^i(s) B_\ell^j(s)} \varphi_\ell^{ij}(s) &= \rho_1(s) \sqrt{B_\ell^i(s) B_\ell^1(s)} \varphi_\ell^{i1*}(s) \sqrt{B_\ell^1(s) B_\ell^j(s)} \varphi_\ell^{1j}(s) \\ &\quad + \rho_2(s) \sqrt{B_\ell^i(s) B_\ell^2(s)} \varphi_\ell^{i2*}(s) \sqrt{B_\ell^2(s) B_\ell^j(s)} \varphi_\ell^{2j}(s)\end{aligned} \tag{5b}$$

$$\operatorname{Im} \varphi_\ell^{ij}(s) = \rho_1(s) B_\ell^1(s) \varphi_\ell^{i1*}(s) \varphi_\ell^{1j}(s) + \rho_2(s) B_\ell^2(s) \varphi_\ell^{i2*}(s) \varphi_\ell^{2j}(s) \tag{5c}$$

$$\operatorname{Im} \varphi_\ell^{ij}(s) = \sum_{k=1,2} \rho_k(s) B_\ell^k(s) \varphi_\ell^{ik*}(s) \varphi_\ell^{kj}(s) \tag{5d}$$

We can equivalently perform the same derivation in a matrix form. Let us define the matrices $(\boldsymbol{t}_\ell)_{ij} = t_\ell^{ij}(s)$, $(\boldsymbol{\varphi}_\ell)_{ij} = \varphi_\ell^{ij}(s)$, $(\boldsymbol{\rho})_{ij} = \rho_i(s)\delta_{ij}$ and $\left(\boldsymbol{B}_\ell^{1/2}\right)_{ij} = \sqrt{B_\ell^i(s)}\delta_{ij}$. Note that $\boldsymbol{t}_\ell^T = \boldsymbol{t}_\ell$ and $\boldsymbol{B}_\ell^{-1} = \boldsymbol{B}_\ell$. The unitarity equations read $\operatorname{Im} \boldsymbol{t}_\ell = \boldsymbol{t}_\ell^\dagger \boldsymbol{\rho} \boldsymbol{t}_\ell$ or $\operatorname{Im} \boldsymbol{t}_\ell^{-1} = -\boldsymbol{\rho}$ (by writing $\operatorname{Im} \boldsymbol{t}_\ell = (1/2i)(\boldsymbol{t}_\ell^\dagger - \boldsymbol{t}_\ell)$ and multiplying to left by $(\boldsymbol{t}_\ell^\dagger)^{-1}$ and to the right by $\boldsymbol{t}_\ell^{-1}$). Since $\boldsymbol{t}_\ell = \boldsymbol{B}_\ell^{1/2} \boldsymbol{\varphi}_\ell \boldsymbol{B}_\ell^{1/2}$, we obtain $\operatorname{Im} \boldsymbol{\varphi}_\ell = \boldsymbol{\varphi}_\ell^\dagger \boldsymbol{B}_\ell^{1/2} \boldsymbol{\rho} \boldsymbol{B}_\ell^{1/2} \boldsymbol{\varphi}_\ell$ or $\operatorname{Im} \boldsymbol{\varphi}_\ell^{-1} = -\boldsymbol{B}_\ell^{1/2} \boldsymbol{\rho} \boldsymbol{B}_\ell^{1/2} = -\boldsymbol{\rho} \boldsymbol{B}_\ell$.

4. The trajectory is $\alpha(s) = \ell - \beta(s)/\varphi_\ell(s)$. We obtain

$$\operatorname{Im} \alpha(s) = -\operatorname{Im} \frac{\beta(s)}{\varphi_\ell(s)} = \beta(s) \frac{\operatorname{Im} \varphi_\ell(s)}{|\varphi_\ell(s)|^2} = \rho(s) B_\ell(s) \beta(s), \tag{6}$$

as expected since $\operatorname{Im} \varphi^{-1} = -\operatorname{Im} \alpha/\beta = -\rho B_\ell$.

5. The trajectory is $\alpha(s) = \ell - \beta_{ij}(s)/\varphi_\ell^{ij}(s)$. We obtain

$$\operatorname{Im} \alpha(s) = \beta_{ij}(s) \frac{\operatorname{Im} \varphi_\ell^{ij}(s)}{|\varphi_\ell^{ij}(s)|^2} = \beta_{ij}^{-1}(s) \sum_{k=1,2} \rho_k(s) B_\ell^k(s) \beta_{ik}(s) \beta_{kj}(s) \tag{7}$$

More explicitly the three equations are ($ij = \{11, 12, 22\}$)

$$\begin{aligned}\operatorname{Im} \alpha(s) &= \left[\rho_1(s) B_\ell^1(s) \beta_{11}^2(s) + \rho_2(s) B_\ell^2(s) \beta_{12}^2(s)\right] / \beta_{11}(s) &\text{(8a)} \\ &= \rho_1(s) B_\ell^1(s) \beta_{11}(s) + \rho_2(s) B_\ell^2(s) \beta_{22}(s) &\text{(8b)} \\ &= \left[\rho_1(s) B_\ell^1(s) \beta_{12}^2(s) + \rho_2(s) B_\ell^2(s) \beta_{22}^2(s)\right] / \beta_{22}(s) &\text{(8c)}\end{aligned}$$

We then derive the factorization of residues

$$\beta_{12}^2(s) = \beta_{11}(s) \beta_{22}(s) \tag{9}$$