Joined Physics Analysis Center

Summer Workshop on the Reaction Theory
Exercise sheet 8

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Classwork

1. Derive all the quantum numbers $I^G J^{PC}$ in the $t-$channel of the following reactions
   (a) $\pi\pi \to \pi\pi$ and $K\bar{K} \to K\bar{K}$
   (b) $\pi N \to \pi N$, $\pi N \to \eta N$ and $KN \to KN$
   (c) $\gamma N \to \eta N$ and $\gamma N \to \pi N$
   (d) $\pi\rho \to \rho\pi$

   Notation: $\pi = (\pi^+, \pi^-, \pi^0)$; $\rho = (\rho^+, \rho^-, \rho^0)$; $K = (K^+, K^0)$; $N = (p, n)$.

2. Assume that the Regge exchange form a $SU(3)$ octet and a $SU(3)$ singlet with the coupling for the octet and the singlet being different. Consider a vector and a tensor nonet (octet plus singlet). From the duality hypothesis and the absence of double charge meson, find the combination of octet-singlet tensor that decouples from $\pi\pi$. Use the SU(3) Clebsch-Gordan coefficients from Rev.Mod.Phys. 36 (1964) 1005. What are the quark content and the $K\bar{K}$ couplings of these states?

3. Assuming ideal mixing for the vector and tensor, derive the exchange degeneracy relations coming duality and the absence of resonance in the following reactions
   (a) $\pi\pi \to \pi\pi$
   (b) $K\bar{K} \to K\bar{K}$
   (c) $KN \to KN$
   (d) $\pi\rho \to \rho\pi$ (and $\pi\pi \to \rho\rho$)

4. Derive a Lorentz-covariant basis, the isospin decomposition and the crossing properties for the following reactions
   (a) $\pi N \to \pi N$ and $KN \to KN$
   (b) $NN \to NN$
   (c) $\omega \to \pi\pi\pi$ and $B \to J/\psi K\pi$
   (d) $\pi\rho \to \pi\rho$
   (e) $\gamma N \to \pi N$ and $\gamma^* N \to \pi N$ (use $F^{\mu\nu} = \epsilon^\mu k^\nu - \epsilon^\nu k^\mu$)

Solution

1. The list of exchanges having only $I = 0, 1$ is presented on Table 1. Notation: signature $\tau = (-1)^J$ and naturality $\eta = P(-1)^J$. In the quark model, $P = (-1)^{\ell+1}$ and $C = (-1)^{\ell+S}$, hence 0^{--}, $(1, 3, 5, \ldots)^{-+}$ and $(0, 2, 4, \ldots)^{+-}$ are forbidden in the quark model. Let's refer to these quantum numbers as "exotic".

Table 1: Regge Trajectories

$I^{G\tau\eta}$		J^{PC}	$I^{G\tau\eta}$		J^{PC}
0^{+++}	f_+	$(0,2,4,\ldots)^{++}$	0^{+--}	f_-	$(1,3,5,\ldots)^{++}$
0^{--+}	ω_-	$(1,3,5,\ldots)^{--}$	0^{-+-}	ω_+	$(0,2,4,\ldots)^{--}$
1^{-++}	a_+	$(0,2,4,\ldots)^{++}$	1^{---}	a_-	$(1,3,5,\ldots)^{++}$
1^{+-+}	ρ_-	$(1,3,5,\ldots)^{--}$	1^{++-}	ρ_+	$(0,2,4,\ldots)^{--}$
0^{++-}	η_+	$(0,2,4,\ldots)^{-+}$	0^{+-+}	η_-	$(1,3,5,\ldots)^{-+}$
0^{---}	h_-	$(1,3,5,\ldots)^{+-}$	0^{-++}	h_+	$(0,2,4,\ldots)^{+-}$
1^{-+-}	π_+	$(0,2,4,\ldots)^{-+}$	1^{--+}	π_-	$(1,3,5,\ldots)^{-+}$
1^{+--}	b_-	$(1,3,5,\ldots)^{+-}$	1^{+++}	b_+	$(0,2,4,\ldots)^{+-}$

(a) for $\pi\pi$: $G = +$, $\eta = +$ and $\eta(-1)^I = +$ (Bose symmetry) $\Rightarrow f_+$ and ρ_-.
for $K\bar{K}$: $\eta = +$ and $\eta(-1)^I = + \Rightarrow f_+$, a_+, ω_- and ρ_-.

(b) for $\pi\eta$: $G = -$, $\eta = +$, $I = 1$; for NN: $I = 0, 1$ and no exotic $\Rightarrow a_+$.
for $K\bar{K}$: $\eta = +$; for NN: $I = 0, 1$ and no exotic $\Rightarrow f_+$, a_+, ω_- and ρ_-.

(c) for $\gamma\eta$ and $\gamma\pi^0$: $C = -$; for NN: $I = 0, 1$ and no exotic. $\Rightarrow \omega_\pm$, $\rho_\pm$, b_- and h_-.
for $\gamma\pi^+$: $I = 1$; for NN: $I = 0, 1$ and no exotic. $\Rightarrow a_\pm$, $\rho_\pm$, b_- and π_-.

(d) for $\pi\rho$: $G = - \Rightarrow a_\pm$, $\pi_\pm$, $\omega_\pm$ and $h_\pm$

Table 2: Exchanges

(a)	$\pi^+\pi^\mp \to \pi^+\pi^\mp$	$f_+ \pm \rho_-$
	$\pi^0\pi^0 \to \pi^0\pi^0$	f_+
	$K^+K^\mp \to K^+K^\mp$	$f_+ \pm \omega_- + a_+ \pm \rho_-$
	$K^+K^0 \to K^0K^+$	$a_+ - \rho_-$
(b)	$\pi^-p \to \eta n$	a_+
	$\pi^-p \to \pi^0 n$	$\sqrt{2}\rho_+$
	$\pi^\mp p \to \pi^\mp p$	$f_+ \pm \rho_+$
	$\pi^\mp n \to \pi^\mp n$	$f_+ \mp \rho_+$
	$K^-p \to \bar{K}^0 p$	$\sqrt{2}\,(\rho_- + a_+)$
	$K^+n \to K^0 p$	$\sqrt{2}\,(\rho_- - a_+)$
	$K^\mp p \to K^\mp p$	$f_+ \pm \rho_- + a_+ \pm \omega_-$
	$K^\mp n \to K^\mp n$	$f_+ \mp \rho_- - a_+ \pm \omega_-$
(c)	$\gamma p \to \eta p$	$(\omega_- + \rho_-) + (h_- + b_- + \omega_+ + \rho_+)$
	$\gamma p \to \pi^0 p$	$(\omega_- + \rho_-) + (h_- + b_- + \omega_+ + \rho_+)$
	$\gamma p \to \pi^+ n$	$(\rho_- + a_+) + (b_- + \pi_+ + \rho_+ + a_-)$
	$\gamma n \to \pi^- p$	$(\rho_- - a_+) + b_- - \pi_+ + \rho_+ - a_-)$
(d)	$\pi^+\rho^0 \to \rho^0\pi^+$	$(a_+ + \pi_-) + (a_- + \pi_+)$
	$\pi^+\rho^+ \to \rho^+\pi^+$	$(\omega_- - a_+ + h_+ - \pi_-) + (\omega_+ - a_- + h_- - \pi_+)$
	$\pi^+\rho^+ \to \pi^+\rho^+$	$f_+ - \rho_-$

2. For a general treatment of exchange degeneracy using group theory, see Ref. [1].

We assume that the residues obey a $SU(3)$ symmetry:

$$\beta^R_{ac}(t) \propto \langle \mathbf{8}\, Y_a\, I_a\, I_{a3}; \mathbf{8}\, Y^*_c\, I_c\, I^*_{c3} | \mathbf{8}\, Y_R\, I_R\, I_{R3} \rangle\,, \tag{1}$$

where $Y^* = -Y$ and $I^*_3 = -I_3$. The hypercharge Y is the strangeness and I_3 is the isospin projection. The Clebsch-Gordan coefficients for $SU(3)$ are listed in Ref. [2]. Note the extra minus sign for the π^+ and K^-.

The four couplings are $\beta_{\mathbf{8}V}, \beta_{\mathbf{8}T}, \beta_{\mathbf{1}V}$ and $\beta_{\mathbf{1}T}$ for the octet/ singlet for the tensor and vector trajectories. The absence of isospin 2 meson in $\pi^+K^+ \to K^+\pi^+$ and in $\pi^+\pi^+ \to \pi^+\pi^+$ lead to

$$\pi^+K^+ \to K^+\pi^+ : \qquad \frac{3}{10}\beta^2_{\mathbf{8}T} s^{\alpha_{\mathbf{8}T}} - \frac{1}{6}\beta^2_{\mathbf{8}V} s^{\alpha_{\mathbf{8}V}} = 0 \tag{2a}$$

$$\pi^+\pi^+ \to \pi^+\pi^+ : \qquad \frac{1}{8}\beta^2_{\mathbf{1}T} s^{\alpha_{\mathbf{1}T}} + \frac{1}{5}\beta^2_{\mathbf{8}T} s^{\alpha_{\mathbf{8}T}} - \frac{1}{3}\beta^2_{\mathbf{8}V} s^{\alpha_{\mathbf{8}T}} = 0 \tag{2b}$$

We combine them to get $\alpha_{\mathbf{8}V} = \alpha_{\mathbf{8}V} = \alpha_{\mathbf{1}T}$ and $(2/5)\beta^2_{\mathbf{8}T} = (1/8)\beta^2_{\mathbf{1}T}$, I choose by convention

$$\sqrt{\frac{1}{8}}\beta_{\mathbf{1}T} = -\sqrt{\frac{2}{5}}\beta_{\mathbf{1}V}. \tag{3}$$

Let us define the octet-singlet mixing

$$\begin{pmatrix} f \\ f' \end{pmatrix} = \begin{pmatrix} \cos\theta_T & \sin\theta_T \\ -\sin\theta_T & \cos\theta_T \end{pmatrix} \begin{pmatrix} f_8 \\ f_1 \end{pmatrix} \tag{4}$$

The states are $f_8 = |\mathbf{8}; 000\rangle$ and $f_1 = |\mathbf{1}; 000\rangle$. The notation is $|\boldsymbol{R}; YII_3\rangle$. Let us impose that the f' coupling to $\pi^+\pi^+$ vanishes

$$-\sin\theta_T \left(-\sqrt{\frac{1}{5}}\beta_{\mathbf{8}T}\right) + \cos\theta_T \left(\sqrt{\frac{1}{8}}\beta_{\mathbf{1}T}\right) = 0. \tag{5}$$

With the relation between the couplings, we obtain $\sin\theta_T = \sqrt{2}\cos\theta_T$ or $\tan^2\theta_T = 1/2$. The quark content are then

$$\begin{pmatrix} \frac{u\bar{u}+d\bar{d}}{\sqrt{2}} \\ s\bar{s} \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{1}{3}} & \sqrt{\frac{2}{3}} \\ -\sqrt{\frac{2}{3}} & \sqrt{\frac{1}{3}} \end{pmatrix} \begin{pmatrix} \frac{u\bar{u}+d\bar{d}-2s\bar{s}}{\sqrt{6}} \\ \frac{u\bar{u}+d\bar{d}+s\bar{s}}{\sqrt{3}} \end{pmatrix} \tag{6}$$

The couplings are

$$\beta^f_{\pi\pi} = -\sqrt{\frac{1}{5}}\beta_{\mathbf{8}T}\cos\theta_T + \sqrt{\frac{1}{8}}\beta_{\mathbf{1}T}\sin\theta_T = -\sqrt{\frac{3}{5}}\beta_{\mathbf{8}T} \tag{7a}$$

$$\beta^f_{K^+K^+} = \sqrt{\frac{1}{20}}\beta_{\mathbf{8}T}\cos\theta_T + \sqrt{\frac{1}{8}}\beta_{\mathbf{1}T}\sin\theta_T = -\frac{1}{2}\sqrt{\frac{3}{5}}\beta_{\mathbf{8}T} \tag{7b}$$

$$\beta^{f'}_{K^+K^+} = -\sqrt{\frac{1}{20}}\beta_{\mathbf{8}T}\sin\theta_T + \sqrt{\frac{1}{8}}\beta_{\mathbf{1}T}\cos\theta_T = -\frac{1}{\sqrt{2}}\sqrt{\frac{3}{5}}\beta_{\mathbf{8}T} \tag{7c}$$

3. In this section, I only wrote the relative sign, not the relative magnitude given by $SU(2)$ and $SU(3)$ Clebsch-Gordan coefficients. In the $\pi^+\pi^+ \to \pi^+\pi^+$ case we obtain

$$0 = \beta^{f_+}(t)s^{\alpha_{f_+}(t)} - \beta^{\rho_-}(t)s^{\alpha_{\rho_-}(t)}. \tag{8}$$

Since this relation is valid in a range of s and t, we obtain $\alpha_{\rho_-}(t) = \alpha_{f_+}(t)$ and $\beta^{\rho_-}(t) = \beta^{f_+}(t)$. For particles with spin, one can repeat the argument with specific combination of helicity amplitudes having good naturality. Hence we obtain EXD relations between exchanges with the same naturality. In the case of $\pi^+\rho^+ \to \rho^+\pi^+$ case we obtain for the natural exchanges

$$0 = \beta^{\omega_-}(t)s^{\alpha_{\omega_-}(t)} - \beta^{a_+}(t)s^{\alpha_{a_+}(t)} + \beta^{h_+}(t)s^{\alpha_{h_+}(t)} - \beta^{\pi_-}(t)s^{\alpha_{\pi_-}(t)} \tag{9a}$$

$$= \left(\beta^{\omega_-}(t) - \beta^{a_+}(t)\right) s^{\alpha_N(t)} + \left(\beta^{h_+}(t) - \beta^{\pi_-}(t)\right) s^{\alpha_{EN}(t)}, \tag{9b}$$

and for the unnatural exchanges

$$0 = \beta^{\omega_+}(t)s^{\alpha_{\omega_+}(t)} - \beta^{a_-}(t)s^{\alpha_{a_-}(t)} + \beta^{h_-}(t)s^{\alpha_{h_-}(t)} - \beta^{\pi_+}(t)s^{\alpha_{\pi_+}(t)} \tag{9c}$$

$$= \left(\beta^{\omega_-}(t) - \beta^{a_+}(t) + \beta^{h_-}(t) - \beta^{\pi_+}(t)\right) s^{\alpha_U(t)}, \tag{9d}$$

In the reaction $\pi^+\pi^+ \to \rho^+\rho^+$, the exchanges pick up a sign equal to PC, we obtain

$$0 = \left(\beta^{\omega_-}(t) - \beta^{a_+}(t)\right) s^{\alpha_N(t)} + \left(\beta^{h_+}(t) - \beta^{\pi_-}(t)\right) s^{\alpha_{EN}(t)} \tag{9e}$$

$$0 = \left(\beta^{\omega_-}(t) - \beta^{a_+}(t) - \beta^{h_-}(t) + \beta^{\pi_+}(t)\right) s^{\alpha_U(t)} \tag{9f}$$

There are then EXD relation between exchanges with same naturality, same PC, same $G-$parity and opposite signature. The Regge trajectories are indicated on Fig. 1. The exchange degeneracy relations are summarized in Table 3 and in Fig. 1

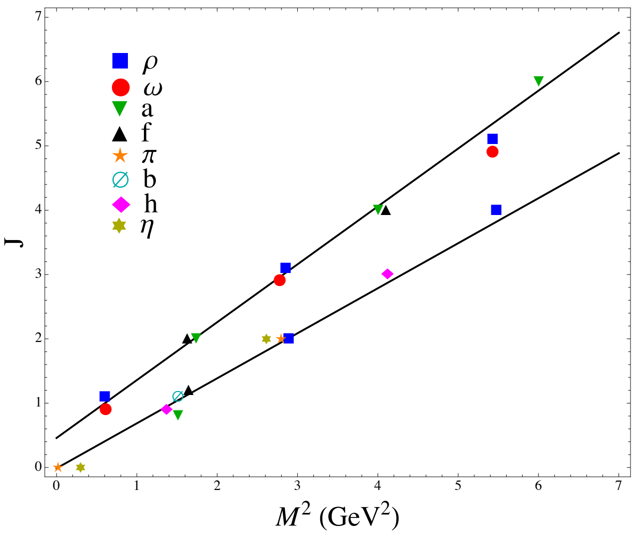

Figure 1: Regge trajectories.The solid lines are $\alpha_N(t) = 0.9(t - m_\rho^2) + 1$ and $\alpha_U(t) = 0.7(t - m_\pi^2) + 0$.

Table 3: Exchange degneracy relation

$\pi^+\pi^+ \to \pi^+\pi^+$	$\alpha_{f_+} = \alpha_{\rho_-}$	$\beta^{f_+}_{\pi^+\pi^+} = \beta^{\rho_-}_{\pi^+\pi^+}$
$K^+K^0 \to K^0K^+$	$\alpha_{a_+} = \alpha_{\rho_-}$	$\beta^{a_+}_{K^+K^0} = \beta^{\rho_-}_{K^+K^0}$
$K^+K^+ \to K^+K^+$	$\alpha_{f_+} = \alpha_{\omega_-}$	$\beta^{f_+}_{K^+K^+} = \beta^{\omega_-}_{K^+K^+}$
$K^+n \to K^+n$	$\alpha_{a_+} = \alpha_{\rho_-}$	$\beta^{a_+}_{pp} = \beta^{\rho_-}_{pp}$
$K^+p \to K^+p$	$\alpha_{f_+} = \alpha_{\omega_-}$	$\beta^{f_+}_{pp} = \beta^{\omega_-}_{pp}$
$\pi^+\rho^+ \to \rho^+\pi^+$	$\alpha_{h_+} = \alpha_{\pi_-}$	$\beta^{h_+}_{\pi^+\rho^+} = \beta^{\pi_-}_{\pi^+\rho^+}$
$\pi^+\pi^+ \to \rho^+\rho^+$	$\alpha_{h_-} = \alpha_{\pi_+}$	$\beta^{h_-}_{\pi^+\rho^+} = \beta^{\pi_+}_{\pi^+\rho^+}$
	$\alpha_{a_+} = \alpha_{\omega_-}$	$\beta^{a_+}_{\pi^+\rho^+} = \beta^{\omega_-}_{\pi^+\rho^+}$
	$\alpha_{a_-} = \alpha_{\omega_+}$	$\beta^{a_-}_{\pi^+\rho^+} = \beta^{\omega_+}_{\pi^+\rho^+}$