Determination of the Pion-Nucleon Scattering Amplitude from Dispersion Relations and Unitarity. General Theory

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A method is proposed for using relativistic dispersion relations, together with unitarity, to determine the pion-nucleon scattering amplitude. The usual dispersion relations by themselves are not sufficient, and we have to assume a representation which exhibits the analytic properties of the scattering amplitude as a function of the energy and the momentum transfer. Unitarity conditions for the two reactions $\pi+N\rightarrow\pi+N$ and $N+\bar{N}\rightarrow 2\pi$ will be required, and they will be approximated by neglecting states with more than two particles. The method makes use of an iteration procedure analogous to that used by Chew and Low for the corresponding problem in the static theory. One has to introduce two coupling constants; the pion-pion coupling constant can be found by fitting the sum of the threshold scattering lengths with experiment. It is hoped that this method avoids some of the formal difficulties of the Tamm-Dancoff and Bethe-Salpeter methods and, in particular, the existence of ghost states. The assumptions introduced are justified in perturbation theory.

As an incidental result, we find the precise limits of the region for which the absorptive part of the scattering amplitude is an analytic function of the momentum transfer, and hence the boundaries of the region in which the partial-wave expansion is valid.

1. INTRODUCTION

In recent years dispersion relations have been used to an increasing extent in pion physics for phenomenological and semiphenomenological analyses of experimental data,[1] and even for the calculation of certain quantities in terms of the pion-nucleon scattering amplitude.[2] It is therefore tempting to ask the question whether or not the dispersion relations can actually replace the more usual equations of field theory and be used to calculate all observable quantities in terms of a finite number of coupling constants—a suggestion first made by Gell-Mann.[3] At first sight, this would appear to be unreasonable, since, although it is necessary to use all the general principles of quantum field theory to derive the dispersion relations, one does not make any assumption about the form of the Hamiltonian other than that it be local and Lorentz-invariant. However, in a perturbation expansion these requirements are sufficient to specify the Hamiltonian to within a small number of coupling constants if one demands that the theory be renormalizable and therefore self-consistent. It is thus very possible that, even without a perturbation expansion, these requirements are sufficient to determine the theory. In fact, if the "absorptive part" of the scattering amplitude, which appears under the integral sign of the dispersion relations, is expressed in terms of the scattering amplitude by means of the unitarity condition, one obtains equations which are very similar to the Chew-Low[4] equations in static theory. These equations have been used by Salzman and Salzman[5] to obtain the pion-nucleon scattering phase shifts.

It is the object of this paper to find a relativistic analog of the Chew-Low-Salzman method, which could be used to calculate the pion-nucleon scattering amplitude in terms of two coupling constants only. As in the static theory, the unitarity equation will involve the transition amplitude for the production of an arbitrary number of mesons, and, in this case, of nucleon pairs as well. In order to make the equations manageable, it is necessary to neglect all but a finite number of processes; as a first approximation, the "one-meson" approximation, we shall neglect all processes except elastic scattering.

The equations obtained from the dispersion relations and the one-meson approximation differ from the static Chew-Low equations in two important respects. Whereas, in the static theory, there was only P-wave scattering, we now have an infinite number of angular momentum states, and the crossing relation, if expressed in terms of angular momentum states, would not converge. Further, in the relativistic theory, the dispersion relations involve the scattering amplitude in the "unphysical" region, i.e., through angles whose cosine is less than -1. For these reasons, the method of procedure will be more involved than in the static theory. We shall require, not only the analytic properties of the scattering amplitude as a function of energy for fixed momentum transfer, which are expressed by the dispersion relations, but its analytic properties as a function of both variables. The required analytic properties have not yet been proved to be consequences of microscopic causality. In order to carry out the proof,

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[1] Chew, Goldberger, Low, and Nambu, Phys. Rev. **106**, 1345 (1957). This paper contains further references.

[2] Chew, Karplus, Gasiorowicz, and Zachariasen, Phys. Rev. **110**, 265 (1958).

[3] M. Gell-Mann, *Proceedings of the Sixth Annual Rochester Conference High-Energy Physics, 1956* (Interscience Publishers, Inc., New York, 1956), Sec. III, p. 30.

[4] G. F. Chew and F. E. Low, Phys. Rev. **101**, 1570 (1956).

[5] G. Salzman and F. Salzman, Phys. Rev. **108**, 1619 (1957).

one would almost certainly have to consider simultaneously several Green's functions together with the equations connecting them which follow from unitarity. It is unlikely that such a program will be carried through in the immediate future. However, if the solution obtained by the use of these analytic properties were to be expanded in a perturbation series, we would obtain precisely those terms of the usual perturbation series included in the one-meson approximation. The assumed analytic properties are, therefore, probably correct, at any rate in the one-meson approximation.

As we have to resort to perturbation theory in order to justify our assumptions, we do not yet have a theory in which the general principles of quantum theory are supplemented only by the assumption of microscopic causality. Nevertheless, the approximation scheme used has several advantages over the approximations previously applied to this problem, such as the Tamm-Dancoff or Bethe-Salpeter approximations. It refers throughout only to renormalized masses and coupling constants. The Tamm-Dancoff equations, by contrast, are unrenormalizable in higher approximations and the Bethe-Salpeter equations, while they are covariant and therefore renormalizable in all approximations, present difficulties of principle when one attempts to solve them. Further, we may hope that the one-meson approximation is more accurate than the Tamm-Dancoff approximations. The latter assumes that those components of the state vector containing more than a certain number of bare mesons are negligibly small—an approximation that is known to be completely false for the experimental value of the coupling constant. The one-meson approximation, on the other hand, assumes that the cross section for the production of one or more *real* mesons is small except at high energies. While this approximation is certainly not quantitatively correct, it is nevertheless probably a good deal more accurate than the Tamm-Dancoff approximation. Finally, the one-meson approximation, unlike the Tamm-Dancoff or Bethe-Salpeter approximations, possesses crossing symmetry. Now it is very probable that the "ghost states" which have been plaguing previous solutions of the field equations are due to the neglect of crossing symmetry. As evidence of this, we may cite the case of charged scalar theory without recoil, for which the one-meson approximation has been solved completely.[6,7] The solution obtained with neglect of the crossing term possesses the usual ghost state if the source radius is sufficiently small. The Lee model,[8] which has no crossing symmetry, shows a similar behavior. If the crossing term in the charged scalar model is included, however, there is no ghost state.

It has been pointed out by Castillejo, Dalitz, and Dyson[7] that the dispersion relations, at any rate in the charged scalar model, do not possess a unique solution. This might have been expected, since it is possible to alter the Hamiltonian without changing the dispersion relations. One simply has to introduce into the theory a baryon whose mass is greater than the sum of the masses of the meson and nucleon. Such a baryon would be unstable, and would therefore not appear as a separate particle or contribute a term to the dispersion relations. In perturbation theory, the simplest of the solutions found by Castillejo, Dalitz, and Dyson, i.e., the solution without any zero in the scattering amplitude, agrees with the solution obtained from a Hamiltonian in which there are no unstable particles, and the more complicated solutions correspond to the existence of unstable baryons. We shall assume that this is so independently of perturbation theory, and shall concern ourselves with the simplest solution. There is no physical reason why one of the other solutions may not be the correct one, but it seems worthwhile to try to compare with experiment the consequences of a theory without unstable particles. It should in any case be emphasized that the ambiguity is not a specific feature of this method of solution, but is inherent in the theory itself. The difference is that, in other methods, it occurs in writing down the equations, whereas in this method it occurs in solving them.

In Sec. 2 we shall discuss the analytic properties of the scattering amplitude, and, in Sec. 3, we shall show how these properties can be used together with the unitarity condition to solve the problem. We shall in this section ignore the "subtraction terms" in the dispersion relations. As in the corresponding static problem, we have to use an iteration procedure in which the crossing term is taken from the result of the previous iteration. The details of this solution will be entirely different from the static problem, the reason being that the part of the amplitude corresponding to the lowest angular momentum states, which is a polynomial in the momentum transfer, actually appears as a subtraction term in the dispersion relation with respect to this variable and has thus not yet been taken into account. In this and the next section we shall also be able to specify details of the analytic representation that were left undetermined in Sec. 2, in particular, we shall be able to give precise limits to the values of the momentum transfer within which the partial-wave expansion converges. In Sec. 4 we shall investigate the subtraction terms in the dispersion relations. We shall find that, in order to determine them, we shall require the unitarity condition for the lowest angular momentum states, not only in pion-nucleon scattering, but also in the pair-annihilation reaction $N+\bar{N}\rightarrow 2\pi$, which is represented by the same Green's function. The coupling constant for meson-meson scattering is thus introduced into the theory; as its value is not known experimentally it will have to be determined by fitting one of the results of the calculation, such as the sum of the S-wave scattering lengths at threshold, with experiment. The calculations

[6] T. D. Lee and R. Serber (unpublished).
[7] Castillejo, Dalitz, and Dyson, Phys. Rev. **101**, 453 (1956).
[8] K. W. Ford, Phys. Rev. **105**, 320 (1957).

of these low angular momentum states would be done in the same spirit as the Chew-Low calculations, and the details will not be given in this paper. We thus have a procedure in which the first few angular momentum states are calculated by methods similar to those used in the static theory, while the remaining part of the scattering amplitude, which will be called the "residual part," is calculated by a different procedure which does not make use of a partial-wave expansion. Needless to say, the two parts of the calculation become intermingled by the iteration procedure.

It is only in the calculation of the subtraction terms that use has to be made of the unitarity condition for the pair-annihilation reaction. For the residual part, it is only necessary to use the unitarity condition for pion-nucleon scattering. Had it been possible to use the unitarity condition exactly instead of in the one-meson approximation, the result would also satisfy the unitarity condition for the annihilation reaction in a consistent theory. As it is, we find that the residual part consists of a number of terms which correspond to various intermediate states in the annihilation reaction. In Sec. 5 it is pointed out that the calculation is greatly simplified if we keep only those terms of the residual part corresponding to pair annihilation through states with fewer than a certain number of particles. Such an approximation has already been made in calculating the subtraction terms. The unitarity condition for pion-nucleon scattering is no longer satisfied except for the low angular momentum states. However, the terms neglected are of the order of magnitude of, and probably less than, terms already neglected. The two reactions of pion-nucleon scattering and pair annihilation are now treated on an equivalent footing.

It will be found that the unitarity condition, in the one-meson approximation, cannot be satisfied at all energies if crossing symmetry and the analytic properties are to be maintained. The reason is that the unitarity condition for the scattering reaction is not completely independent of the unitarity condition for the "crossed" reaction with the two pions interchanged, and they contradict one another if an approximation is made. There is, of course, no difficulty in the region where the one-meson approximation is exact. For sufficiently small values of the coupling constant, we shall still be able to obtain a unique procedure. For values of the coupling constant actually encountered, one part of the crossing term may have to be cut off at the threshold for pair production in pion-nucleon scattering. It is unlikely that the result will be sensitive to the form and the precise value of the cutoff.

2. DISPERSION RELATIONS AND ANALYTICITY PROPERTIES OF THE TRANSITION AMPLITUDE

The kinematical notation to be used in writing down the dispersion relations will be similar to that of Chew *et al.*[1] The momenta of the incoming and outgoing pions will be denoted by q_1 and q_2, those of the incoming and outgoing nucleons by p_1 and p_2. We can then define two invariant scalars

$$\nu=-(p_1+p_2)(q_1+q_2)/4M, \tag{2.1}$$

$$t=-(q_1-q_2)^2. \tag{2.2}$$

The latter is minus the square of the invariant momentum transfer. The laboratory energy will be given by the equation

$$\omega=\nu-(t/4M). \tag{2.3a}$$

It is more convenient to use, instead of the laboratory energy, the square of the center-of-mass energy (including both rest-masses), which is linearly related to it by the equation

$$s=M^2+\mu^2+2M\omega. \tag{2.3b}$$

The Green's function relevant to the process under consideration,

$$\pi_1+N_1\rightarrow\pi_2+N_2, \tag{I}$$

also gives the processes

$$\pi_2+N_1\rightarrow\pi_1+N_2 \tag{II}$$

and

$$N_1+\bar{N}_2\rightarrow\pi_1+\pi_2. \tag{III}$$

The matrix elements for the process II can be obtained from those for the process I by crossing symmetry; the laboratory energy and the square of the center-of-mass energy will now be

$$\omega_c=-\nu-(t/4M)=-\omega-(t/2M), \tag{2.4a}$$

$$s_c=M^2+\mu^2+2M\omega_c=2M^2+2\mu^2-s-t. \tag{2.4b}$$

The square of the momentum transfer will be $-t$ as before. For the process III, the square of the center-of-mass energy will be t. The square of the momentum transfer between the nucleon N_1 and the pion π_2 will be s_c and that between the nucleon N_1 and the pion π_1 will be s.

The kinematics for the three reactions are represented diagrammatically in Fig. 1 in which t has been plotted against ν. AB represents the line $s=(M+\mu)^2$, or $\omega=\mu$, and lines for which s is constant will be parallel to it. The region for which the process I is energetically possible is therefore that to the right of AB. However, only the shaded part of this area is the "physical region"; in the unshaded part, though the energy of the meson is greater than its rest-mass, the cosine of the scattering angle is not between -1 and $+1$. The physical region is bounded above by the line $t=0$, i.e., the line of forward scattering, and below by the line of backward scattering. Similarly CD is the line $s_c=(M+\mu)^2$; the region for which the process II is energetically possible is that to the left of CD, and the shaded area represents the physical region for this reaction. Lines of constant energy for the reaction III are horizontal lines. The reaction will be energetically possible above the line EF, at which $t=4M^2$, and again the shaded area represents the physical region.

We now examine the analytic properties of the scattering amplitude. To simplify the writing, we shall first neglect spin and isotopic spin; the transition amplitude will then be a scalar function $A(\nu,t)$ of the two invariants ν and t. Its analytic properties as a function of ν, with t constant, are exhibited by the usual dispersion relations

$$A(\nu,t)=\frac{g^2}{2M}\left(\frac{1}{\nu_B-\nu}+\frac{1}{\nu_B+\nu}\right)+\frac{1}{\pi}\int_{\mu+(t/4M)}^{\infty}d\nu'\,\frac{A_1(\nu',t)}{\nu'-\nu}-\frac{1}{\pi}\int_{-\infty}^{-\mu-(t/4M)}d\nu'\,\frac{A_2(\nu',t)}{\nu'-\nu},\quad(2.5)$$

where $\nu_B=-(\mu^2/2M)+(t/4M)$. In this and all subsequent such equations, the energy denominators are taken to have a small imaginary part. A_1 and A_2 are the "absorptive parts" associated with the reactions I and II, respectively, and are given by the equations

$$(2\pi)^4A_1(\nu,t)\delta(p_1+q_1-p_2-q_2)=(2\pi)^6\left(\frac{4p_{01}p_{02}q_{01}q_{02}}{M^2}\right)^{\frac{1}{2}}\times\sum_n\langle N(p_1)\pi(q_1)|n\rangle\langle n|N(p_2)\pi(q_2)\rangle,\quad(2.6)$$

$$(2\pi)^4A_2(\nu,t)\delta(p_1+q_1-p_2-q_2)=(2\pi)^6\left(\frac{4p_{01}p_{02}q_{01}q_{02}}{M^2}\right)^{\frac{1}{2}}\times\sum_n\langle N(p_1)\pi(-q_2)|n\rangle\langle n|N(p_2)\pi(-q_1)\rangle.\quad(2.7)$$

The symbol $\langle N(p_1)\pi(q_1)|$ denotes a state with an ingoing nucleon of momentum p_1 and an ingoing pion of momentum q_1. The sum $\sum_n$ is to be taken over all intermediate states. A_1 and A_2 are nonzero to the right of AB, and to the left of CD, respectively.

Equation (2.5) indicates that A is an analytic function of ν in the complex plane, with poles at $\pm\nu_B$, and cuts along the real axis from $\mu+(t/4M)$ to ∞ and from $-\infty$ to $-\mu-(t/4M)$.

On Fig. 1, (2.5) will be represented by an integration along a horizontal line below the ν axis. The poles will occur where this line crosses the dashed lines; apart from them, the integrand will be zero between AB and CD. Except for forward scattering, the region where the integrand is nonzero will lie partly in the unphysical region, where the energy is above threshold but the angle imaginary.

Equation (2.5) is only true as it stands if the functions A, A_1, and A_2 tend to zero sufficiently rapidly as ν tends to infinity; otherwise it will be necessary to perform one or more subtractions in the usual way. Whenever such a dispersion relation is written down, the possibility of having to perform subtractions is implied.

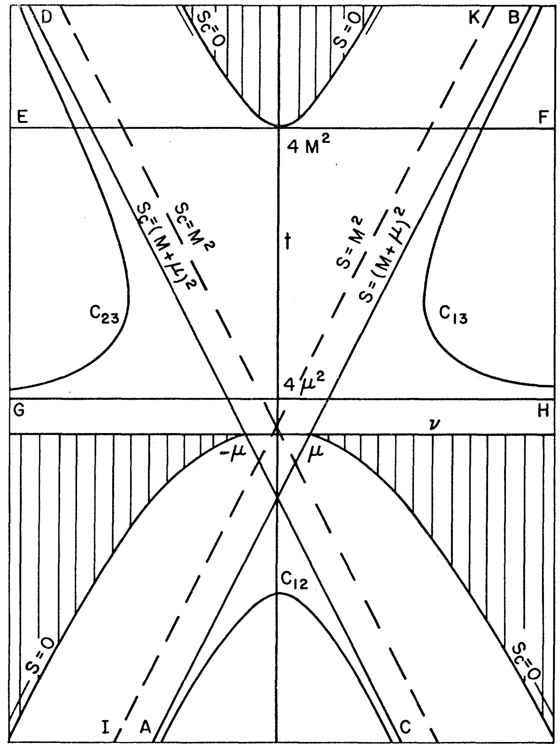

FIG. 1. Kinematics of the reactions I, II, and III.

We next wish to obtain analytic properties of A as a function of t. In order to do this we shall write the scattering amplitude, not as the expectation value of the time-ordered product of the two meson current operators between two one-nucleon states, as is done in the proof of the usual dispersion relations,[9,10] but as the expectation value of the product of a meson current operator and a nucleon current operator between a nucleon state and a meson state. Thus

$$(2\pi)^4A\delta(p_1+p_2-q_1-q_2)=(2\pi)^3\left(\frac{2p_{01}q_{02}}{M}\right)^{\frac{1}{2}}i\int dxdx'\times e^{-iq_1x+ip_2x'}\langle N(p_1)|T\{j(x)\bar{a}(x')\}|\pi(q_2)\rangle,\quad(2.8)$$

where $a(x')$ is a nucleon current operator. From this expression, we can obtain dispersion relations in which the momentum transfer between the incoming nucleon and the outgoing pion, rather than between the two nucleons, is kept constant—the proof is exactly the same as the usual heuristic proof of the ordinary dispersion relations.[9,10] As this momentum transfer is just s_c, we obtain dispersion relations in which s_c is kept constant; if A is written as a function of s_c and t, they

[9] M. L. Goldberger, Phys. Rev. **99**, 979 (1955).
[10] R. H. Capps and G. Takeda, Phys. Rev. **106**, 1337 (1956).

take the form

$$A(s_c,t)=\frac{g^2}{s_c+t-M^2-2\mu^2}-\frac{1}{\pi}\int_{\infty}^{(M-\mu)^2-s_c}dt'\frac{A_1(s_c,t')}{t'-t}+\frac{1}{\pi}\int_{4\mu^2}^{\infty}dt'\frac{A_3(s_c,t')}{t'-t}. \tag{2.9}$$

The absorptive parts in the integrand are as usual obtained by replacing the time-ordered product in (2.8) by half the commutator. The first term, in which the operators are in the order $j(x)\bar{a}(x')$, is exactly A_1, and will therefore be nonzero to the right of AB and have a δ function along IK. The second term, however, in which the operators are in the order $\bar{a}(x')j(x)$, will now be related to the process III. It will be given by the equation

$$(2\pi)^4A_3(s_c,t)\delta(p_1+q_1-p_2-q_2)=(2\pi)^6\left(\frac{4p_{01}p_{02}q_{01}q_{02}}{M^2}\right)^{\frac{1}{2}}\times\sum_n\langle N(p_1)\bar{N}(-p_2)|n\rangle\langle n|\pi(-q_1)\pi(q_2)\rangle. \tag{2.10}$$

The state n of lowest energy will now be the two-meson state. A_3 will therefore be nonzero above the line $t=4\mu^2$, represented by GH in Fig. 1 (since t is square of the center-of-mass energy of the process III). The dispersion relation (2.10) is represented by an integration along a line parallel to CD and to the right of the line $s_c=0$. It implies that A is an analytic function of t for fixed s_c, with a pole at $t=M^2+2\mu^2-s_c$, and cuts along the real axis from $-\infty$ to $(M-\mu)^2-s_c$ and from $4\mu^2$ to ∞.

As in the usual dispersion relation, part of the range of integration in Eq. (2.9) will lie in the unphysical region. This region now includes, besides imaginary angles at permissible energies, the entire area between the lines $t=4\mu^2$ and $t=4M^2$, where there are contributions to A_3 from intermediate states with two or more pions. The rigorous proof of (2.9) is therefore much more difficult than that of (2.5), and probably cannot be carried out without introducing the unitarity equations.

By interchanging the two pions in the expression (2.8), we can obtain a third dispersion relation in which s is kept constant:

$$A(s,t)=\frac{g^2}{s+t-M^2-2\mu^2}-\frac{1}{\pi}\int_{-\infty}^{(M-\mu)^2-s}dt'\frac{A_2(s,t')}{t'-t}+\frac{1}{\pi}\int_{4\mu^2}^{\infty}dt'\frac{A_3(s,t')}{t'-t}. \tag{2.11}$$

On Fig. 1, this would be represented by an integration along a line parallel to AB, and to the left of the line $s=0$.

Let us now try to obtain the analytic properties of A considered as a function of two complex variables. The simplest assumption we could make is that it is analytic in the entire space of the two variables except for cuts along certain hyperplanes. We can then determine the location of the cuts from the requirement that A must satisfy the dispersion relations (2.5), (2.9), and (2.11); there will be a cut when s is real and greater than $(M+\mu)^2$, a cut when s_c is real and greater than $(M+\mu)$, and a cut when t is real and greater than $4\mu^2$. The discontinuities across these cuts will be, respectively, $2A_1$, $2A_2$, and $2A_3$. In addition, A will have poles when $s=M^2$ and when $s_c=M^2$. By a double application of Cauchy's theorem, it can be shown that a function with cuts and poles in these positions can be represented in the form

$$A=\frac{g^2}{M^2-s}+\frac{g^2}{M^2-s_c}+\frac{1}{\pi^2}\int_{(M+\mu)^2}^{\infty}ds'\int_{4\mu^2}^{\infty}dt'\frac{A_{13}(s',t')}{(s'-s)(t'-t)}+\frac{1}{\pi^2}\int_{(M+\mu)^2}^{\infty}ds_c'\int_{4\mu^2}^{\infty}dt'\frac{A_{23}(s_c',t')}{(s_c'-s_c)(t'-t)}+\frac{1}{\pi^2}\int_{(M+\mu)^2}^{\infty}ds'\int_{(M+\mu)^2}^{\infty}ds_c'\frac{A_{12}(s',s_c')}{(s'-s)(s_c'-s_c)}. \tag{2.12}$$

This is a generalization of a representation first suggested by Nambu.[11] While we have for convenience used the three variables s, s_c, and t, which are the energies of the three processes, they are connected by the relation

$$s+s_c+t=2(M^2+\mu^2), \tag{2.13}$$

so that A is really a function of two variables only. A_{13}, A_{23} and A_{12}, which will be referred to as the "spectral functions," are nonzero in the regions indicated at the top right, top left and bottom of Fig. 1. The precise boundaries C_{13}, C_{23}, and C_{12} of the regions will be determined by unitarity in the following sections; from the reasoning given up till now, all that can be said is that the regions must lie within the respective triangles as indicated, and that the boundary must approach the sides of the triangles asymptotically (or it could touch them at some finite point). The spectral functions are always zero in the physical region.

As in the case of ordinary dispersion relations, the representation (2.12) will not be true as it stands, but will require subtractions. The subtractions will modify one or both of the energy denominators in the usual way and, in addition, they will require the addition of extra terms. These terms will not now be constants, but functions of one of the variables, e.g., if there is a subtraction in the s integration of the first term, the extra term will be a function of t. These functions must then have the necessary analytic properties in their

[11] Y. Nambu, Phys. Rev. **100**, 394 (1955).

variables, so that they will have the form

$$\frac{1}{\pi}\int_{(M+\mu)^2}^{\infty} ds' \frac{f_1(s')}{s'-s} + \frac{1}{\pi}\int_{(M+\mu)^2}^{\infty} ds_c' \frac{f_2(s_c')}{s_c'-s_c} + \frac{1}{\pi}\int_{4\mu^2}^{\infty} dt' \frac{f_3(t')}{t'-t}. \quad (2.14)$$

If more than one subtraction is involved, we may have similar terms multiplied by polynomials. Even if the spectral functions in (2.12) tend to zero as one of the variables tends to infinity, so that no subtraction in that variable is necessary, it is still not precluded that the corresponding term in (2.14) does not appear, as the function still has the required analytic properties. For pion-nucleon scattering, however, there is no undetermined over-all term, independent of both variables, to be added, as the requirement that the scattering amplitude for each angular momentum wave have the form $e^{i\delta}\sin\delta/k$, with $\mathrm{Im}\delta<0$, forces A to tend to zero in the physical region when both s and t become infinite.

The Nambu representations for the complete Green's functions are known to be invalid, even in the lowest nontrivial order of perturbation theory. The representation quoted here, however, restricts itself to the mass shells of the particles, and has not been shown to be invalid. In fact, in the case of Compton scattering, the fourth-order terms, which have been worked out by Brown and Feynman,[12] are found to have this representation, and, as we have stated in the introduction, all the perturbation terms included in the one-meson approximation can be similarly represented.

The dispersion relations are an immediate consequence of the representation (2.12). To obtain the usual dispersion relation (2.5), the third integral in (2.12) must be written as[13]

$$-\frac{1}{\pi^2}\int_{(M+\mu)^2}^{\infty} ds' \int_{-\infty}^{t_2(s)} dt' \frac{A_{12}(s',t')}{(s'-s)(t'-t)} - \frac{1}{\pi^2}\int_{(M+\mu)^2}^{\infty} ds' \int_{-\infty}^{t_2(s_c)} dt' \frac{A_{12}(s_c',t')}{(s_c'-s_c)(t'-t)}.$$

It then follows that

$$A=\frac{g^2}{M^2-s}+\frac{g^2}{M^2-s_c}+\frac{1}{\pi^2}\int_{(M+\mu)^2}^{\infty} ds' \frac{A_1(s',t)}{s'-s} + \frac{1}{\pi^2}\int_{(M+\mu)^2}^{\infty} ds_c' \frac{A_2(s_c',t)}{s_c'-s_c}, \quad (2.15)$$

where

$$A_1(s,t)=\frac{1}{\pi}\int_{t_1(s)}^{\infty} dt' \frac{A_{13}(s,t')}{t'-t} - \frac{1}{\pi}\int_{-\infty}^{t_2(s)} dt' \frac{A_{12}(s,t')}{t'-t}, \quad (2.16)$$

$$A_2(s_c,t)=\frac{1}{\pi}\int_{t_1(s_c)}^{\infty} dt' \frac{A_{23}(s_c,t')}{t'-t} - \frac{1}{\pi}\int_{-\infty}^{t_2(s_c)} dt' \frac{A_{12}(s_c,t')}{t'-t}. \quad (2.17)$$

Equation (2.15) is, however, just the dispersion relation (2.5), since s, s_c, and ν are connected by the relations (2.4) and t is being kept constant. We also see that the absorptive parts A_1 and A_2 themselves satisfy dispersion relations in t, with s (or s_c) constant; the imaginary parts which appear in the integrand are now simply the spectral functions. Equation (2.16) will be represented in Fig. 1, by an integration along a line parallel to AB and to the right of it. The limits t_1 and t_2 are the points at which this line crosses the curves C_{13} and C_{12}. They satisfy the inequalities

$$t_1>4\mu^2, \quad (2.18a)$$

$$t_2<(M-\mu)^2-s. \quad (2.18b)$$

A_1 will be nonzero for $s>(M+\mu)^2$, as it should, as long as the curves C_{13} and C_{12} approach the line AB at some point and do not cross it.

The dispersion relations (2.9) and (2.11) can be proved from (2.12) in a similar way; the absorptive part A_3 will then satisfy a dispersion relation in ν with s constant:

$$A_3=\frac{1}{\pi}\int_{\nu_3(t)}^{\infty} d\nu' \frac{A_{13}(\nu',t)}{\nu'-\nu} - \frac{1}{\pi}\int_{-\infty}^{-\nu_3(t)} d\nu' \frac{A_{23}(\nu',t)}{\nu'-\nu}. \quad (2.19)$$

This dispersion relation will be represented by an integration along a horizontal line above GH. ν_3 and $-\nu_3$ will be the points at which the line of integration crosses C_{13} and C_{23}.

Finally, then, the scattering amplitude A satisfies dispersion relations in which any of the quantities t, s_c, and s are kept constant. Further, it follows from (2.12), by the reasoning just given, that the values of the quantity which is being kept constant need no longer be restricted in sign. Thus, for example, we now know the analytic properties of A, as a function of momentum transfer, for fixed energy greater than (as well as less than) $(M+\mu)^2$. They are given by the dispersion relation (2.11), so that A is an analytic function of the square of the momentum transfer, with a pole at $t=M^2+2\mu^2-s$, and cuts along the real axis from $t=4\mu^2$ to ∞ and from $t=-\infty$ to $(M-\mu)^2-s$. For $s>(M+\mu)^2$, these cuts and poles are entirely in the nonphysical region. It has already been shown rigorously

[12] L. M. Brown and R. P. Feynman, Phys. Rev. 85, 231 (1952).

[13] When we make a change of variables, we imply of course that the spectral functions still have the same value at the same point, and not that we must take the same function of the new variables.

by Lehmann[14] that A is analytic in t in an area including the physical region. The absorptive parts A_1, A_2 and A_3 will themselves satisfy dispersion relations, provided that the correct variable be kept constant (s, s_c, and t for A_1, A_2, and A_3, respectively). The weight functions for these dispersion relations are entirely in the nonphysical region, and the boundaries of the areas in which they are nonzero are yet to be determined. In particular, we see that the absorptive part A_1 has the same analytic properties as a function of the momentum transfer [for s constant and greater than $(M+\mu)^2$] as the scattering amplitude, except that there is now no pole, and the cuts only extend from t_1 to ∞ and from $-\infty$ to t_2. According to the inequalities (2.14), these cuts do not reach as far inward as the cuts of A considered as a function of the momentum transfer. This agrees with another result of Lehmann[14] who showed that the region of analyticity of A_1 as a function of t was larger than the region of analyticity of A as a function of t.

The modifications introduced into the theory by spin and isotopic spin are trivial. The transition amplitude will now be given by the expression

$$-A+\tfrac{1}{2}i\gamma(q_1+q_2)B, \tag{2.20}$$

and both A and B will have representations of the form (2.12). There will, further, be two amplitudes corresponding to isotopic spins of $\frac{1}{2}$ and $\frac{3}{2}$. It is sometimes more convenient to use the combinations

$$A^{(+)}=\tfrac{1}{3}(A^{(\frac{1}{2})}+2A^{(\frac{3}{2})}), \tag{2.21a}$$

$$A^{(-)}=\tfrac{1}{3}(A^{(\frac{1}{2})}-A^{(\frac{3}{2})}), \tag{2.21b}$$

and similar combinations $B^{(+)}$ and $B^{(-)}$. We then have the simple crossing relations

$$A^{(\pm)}(\nu,t)=\pm A^{(\pm)}(-\nu,t), \tag{2.22a}$$

$$B^{(\pm)}(\nu,t)=\mp B^{(\pm)}(-\nu,t), \tag{2.22b}$$

or, in terms of the spectral functions,

$$A_{13}{}^{(\pm)}(s,t)=\pm A_{23}{}^{(\pm)}(s_c,t), \tag{2.23a}$$

$$A_{12}{}^{(\pm)}(s,s_c)=\pm A_{12}{}^{(\pm)}(s_c,s), \tag{2.23b}$$

$$B_{13}{}^{(\pm)}(s,t)=\mp B_{23}{}^{(\pm)}(s_c,t), \tag{2.23c}$$

$$B_{12}{}^{(\pm)}(s,s_c)=\mp B_{12}{}^{(\pm)}(s_c,s). \tag{2.23d}$$

The poles in (2.12) and in the dispersion relations will only occur in the representation for $B^{(\pm)}$ (in pseudoscalar theory), and the second term will have a minus or plus sign in the equations for $B^{(+)}$ and $B^{(-)}$, respectively.

3. COMBINATION OF THE DISPERSION RELATIONS WITH THE UNITARITY CONDITION

The dispersion relations given in the previous section must now be combined with the unitarity equations in order to determine the scattering amplitude. We shall again begin by neglecting spin and isotopic spin; the unitarity condition (2.7) then becomes, in the one-meson approximation,

$$A_1(s,\cos\theta_1)=\frac{1}{32\pi^2}\frac{q}{W}\int\sin\theta_2 d\theta_2 d\phi_2\, A^*(s,\cos\theta_2)\times A(s,\cos(\boldsymbol{\theta}_1,\boldsymbol{\theta}_2)),$$

or

$$A_1(s,z_1)=\frac{1}{32\pi^2}\frac{q}{W}\int_{-1}^{1}dz_2\int_0^{2\pi}d\phi\, A^*(s,z_2)\times A\{s,\, z_1z_2+(1-z_1^2)^{\frac{1}{2}}(1-z_2^2)^{\frac{1}{2}}\cos\phi\}, \tag{3.1}$$

where $z=\cos\theta$ and $\boldsymbol{\theta}_i(i=1,2)$ is a unit vector in the (θ_i,ϕ_i) direction. W is the center-of-mass energy (equal to $\sqrt{s}$), and q is the momentum in the center-of-mass system, given by the equation

$$q^2=\{s-(M+\mu)^2\}\{s-(M-\mu)^2\}/4s. \tag{3.2}$$

z is related to the momentum transfer by the simple relation

$$z=1+(t/2q^2). \tag{3.3}$$

The unitarity requirements only prove that Eq. (3.2) is true in the physical region. A_1 must then be obtained in the unphysical region by analytic continuation. In order to do this, A can be expressed as an analytic function of t or, equivalently, of z, by means of Eq. (2.11), in which the energy is kept fixed. Equation (3.3) shows that we can simply replace t by z in (2.12), so that we may write

$$A^*(s,z_2)=\frac{1}{\pi}\int dz_2'\frac{A_2^*(s,z_2')+A_3^*(s,z_2')}{z_2'-z_2}, \tag{3.4a}$$

$$A\{s,\, z_1z_2+(1-z_1^2)^{\frac{1}{2}}(1-z_2^2)^{\frac{1}{2}}\cos\phi\}=\frac{1}{\pi}\int dz_3'\frac{A_2(s,z_3')+A_3(s,z_3')}{z_3'-z_1z_2-(1-z_1^2)^{\frac{1}{2}}(1-z_2^2)^{\frac{1}{2}}\cos\phi}. \tag{3.4b}$$

For simplicity we have included the absorptive parts A_2 and A_3 under the same integral sign, but they will of course contribute in different regions of the variable of integration. $A_2(s,z)$ will be nonzero only if $z<1-\{s-(M-\mu)^2\}/2q^2$, apart from a δ function at $z=1-(s-M^2-2\mu^2)/2q^2$, and $A_3(s,z)$ will be nonzero only if $z>1+2\mu^2/q^2$. The dispersion relations have been written down on the (incorrect) assumption that there are no subtractions necessary; we shall see in the following section how the theory must be modified to take them into account.

On substituting (3.4) into (3.2) and performing the integrations over z_2 and ϕ, we are left with the equation

$$A_1(s,z_1)=\frac{1}{16\pi^3}\frac{q}{W}\int dz_2'\int dz_3'\frac{1}{\sqrt{k}}\ln\frac{z_1-z_2'z_3'+\sqrt{k}}{z_1-z_2'z_3'-\sqrt{k}}\times\{A_2^*(s,z_2')+A_3^*(s,z_2')\}\{A_2(s,z_3')+A_3(s,z_3')\}, \tag{3.5}$$

[14] H. Lehmann (to be published).

where

$$k=z_1^2+z_2'^2+z_3'^2-1-2z_1z_2'z_3'. \tag{3.6}$$

We must take that branch of the logarithm which is real in the physical region $-1<z_1<1$. Equation (3.5) then gives the value of A_1 in the entire complex z_1 plane.

According to Eq. (2.16), $A(s,z_1)$ must be an analytic function of t, and therefore of z, with discontinuities of magnitude $2A_{13}$ and $2A_{12}$ as z_1 crosses the positive and negative real axes. It is easily seen that the expression for A_1 in (3.5) has this property, and, on identifying the discontinuities along the real axis with A_{13} and A_{12}, we arrive at the equations

$$A_{13}(s,z_1)=\frac{1}{8\pi^2}\frac{q}{W}\int dz_2\int dz_3\, K_1(z_1,z_2,z_3)$$
$$\times\{A_3^*(s,z_2)A_3(s,z_3)+A_2^*(s,z_2)A_2(s,z_3)\}, \tag{3.7a}$$

$$A_{12}(s,z_1)=\frac{1}{8\pi^2}\frac{q}{W}\int dz_2\int dz_3\, K_2(z_1,z_2,z_3)$$
$$\times\{A_2^*(s,z_2)A_3(s,z_3)+A_3^*(s,z_2)A_2(s,z_3)\}. \tag{3.7b}$$

The primes on z_2 and z_3 have been suppressed. K_1 and K_2 are defined by the equations

$$\begin{aligned} &K_1(z_1,z_2,z_3)\\ &=-1/[k(z_1,z_2,z_3)]^{\frac12},\ z_1>z_2z_3+(z_2^2-1)^{\frac12}(z_3^2-1)^{\frac12}\\ &=0 \qquad\qquad\qquad\quad z_1<z_2z_3+(z_2^2-1)^{\frac12}(z_3^2-1)^{\frac12} \end{aligned} \tag{3.8a}$$

$$\begin{aligned} &K_2(z_1,z_2,z_3)\\ &=1/[k(z_1,z_2,z_3)]^{\frac12},\ z_1<z_2z_3-(z_2^2-1)^{\frac12}(z_3^2-1)^{\frac12}\\ &=0, \qquad\qquad\qquad z_1>z_2z_3-(z_2^2-1)^{\frac12}(z_3^2-1)^{\frac12}. \end{aligned} \tag{3.8b}$$

The points $z_1=z_2z_3\pm(z_2^2-1)^{\frac12}(z_3^2-1)^{\frac12}$ are the points at which k changes sign.

Let us now transform back from z to our original variables. As we shall use the dispersion relations (2.17) and (2.19), it is convenient to express A_2 and A_{12} as functions of s and s_c and A_3 and A_{13} as functions of s and t. Equations (3.7) then become

$$A_{13}(s,t)=\frac{1}{32\pi^2q^3W}\bigg[\int dt_2\int dt_3$$
$$\times K_1(s;t_1,t_2,t_3)A_3^*(s,t_2)A_3(s,t_3) \tag{3.9a}$$
$$+\int ds_{c2}\int ds_{c3}\,K_1(s;t_1,s_{c2},s_{c3})A_2^*(s,s_{c2})A_2(s,s_{c3})\bigg],$$

$$A_{12}(s,s_{c1})=\frac{1}{32\pi^2q^3W}\int dt_2\int ds_{c3}\,K_2(s;s_{c1},t_2,s_{c3})$$
$$\times[A_3^*(s,t_2)A_2(s,s_{c3})+A_2^*(s,s_{c3})A_3(s,t_2)]. \tag{3.9b}$$

Note that s is fixed in these equations, while s_c and t vary. K must be re-expressed as a function of the new variables by (3.3) and (2.13).

The use of Eq. (3.9), together with the dispersion relations, in order to determine the spectral functions is greatly facilitated by the fact that K is zero unless the variables satisfy certain inequalities; for all s,

$$K_1(s;t_1,t_2,t_3)\;=0 \quad\text{unless}\quad t_1^{\frac12}>t_2^{\frac12}+t_3^{\frac12}, \tag{3.10a}$$

$$K_1(s;t_1,s_{c2},s_{c3})=0 \quad\text{unless}\quad t_1^{\frac12}>s_{c2}^{\frac12}+s_{c3}^{\frac12}, \tag{3.10b}$$

$$K_2(s;s_{c1},t_2,s_{c3})=0 \quad\text{unless}\quad s_{c1}^{\frac12}>t_2^{\frac12}+s_{c3}^{\frac12}. \tag{3.10c}$$

(For any particular s, the restrictions on the variables could be strengthened.) Equations (3.10) are true as long as s_{c2}, s_{c3}, t_2, and t_3 are in the regions $s_c>M^2$, $t>4\mu^2$, outside which A_2 and A_3 vanish. It follows from (3.9) that, *for any given value of t (or s_c), $A_{13}(s,t)$ [or $A_{12}(s,s_c)$] can be calculated in terms of $A_3(s,t')$ and $A_2(s,s_c')$, where the values of t' and s_c' involved are all less than t (or s_c).* On the other hand, by writing the dispersion relations (2.17) and (2.19) in the form

$$A_2(s,s_t)=\frac{1}{\pi}\int_{s_2(s_c)}^{\infty}ds'\,\frac{A_{12}(s',s_c)}{s'-s}$$
$$+\frac{1}{\pi}\int_{t_1(s_c)}^{\infty}dt'\,\frac{A_{23}(s_c,t')}{t'-t}, \tag{3.11a}$$

$$A_3(s,t)=\frac{1}{\pi}\int_{s_3(t)}^{\infty}ds'\,\frac{A_{13}(s',t)}{s'-s}$$
$$+\frac{1}{\pi}\int_{s_3(t)}^{\infty}ds'\,\frac{A_{23}(s',t)}{s_c'-s_c}, \tag{3.11b}$$

it is evident that $A_3(s,t)$ and $A_2(s,s_c)$ can be found in terms of $A_{12}(s',s_c)$ and $A_{13}(s',t)$, if for the moment we neglect the second term in these equations. We can therefore calculate A_{13}, A_{12}, A_3, and A_2 for all values of s and successively larger values of s_c and t. The lowest value of s_c or t for which either A_2 or A_3 is nonzero is $s_c=M^2$, at which there is a contribution of $g^2\delta(s_c-M^2)$ to A_2 from the one-nucleon state. From (3.9) and (3.10) it follows that A_{13} and A_{12} are zero if t and s_c are less than $4M^2$; for a range of values of t above this, A_{13} is nonzero and can be calculated by inserting the δ-function contribution to A_2 into (3.9a). The rest of A_2 and A_3 will still not contribute owing to (3.10). Once we have the procedure thus started, we can proceed to larger and larger values of t and s_c by alternate application of (3.9) and (3.11).[15]

Before discussing how to take the second terms of (3.11) into account, let us study in more detail the form of the functions A_{13} and A_{12} calculated thus far. In order to do this, we require the precise values of t and s_c, at a given value of s, for which the kernels K vanish; we find that

[15] It will be noticed that, though we have brought the pole in the crossing term from the one-nucleon intermediate state into our calculations, we have not yet introduced the pole in the direct term. This pole is actually a subtraction term of Eq. (2.11) and will be treated in the following section.

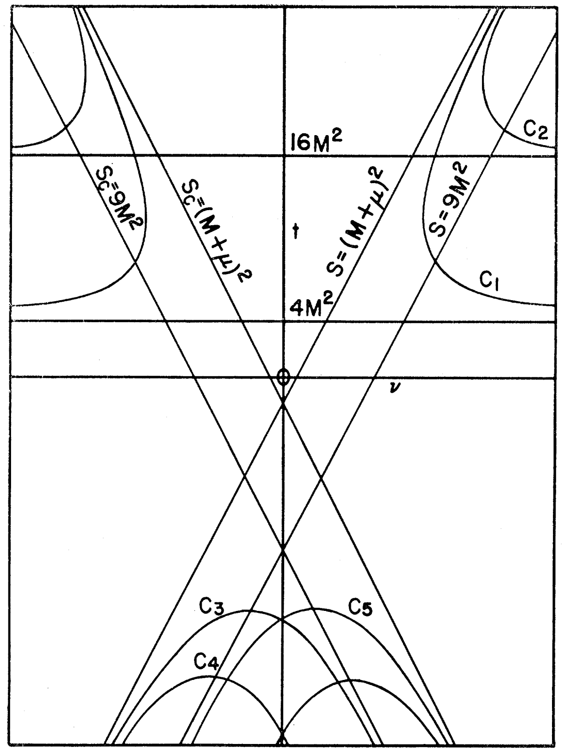

FIG. 2. Properties of the spectral functions.

$K_1(s;t_1,t_2,t_3)=0$ unless

$$t_1^{\frac{1}{2}}>t_2^{\frac{1}{2}}(1+t_3/4q^2)^{\frac{1}{2}}+t_3^{\frac{1}{2}}(1+t_2/4q^2)^{\frac{1}{2}}, \tag{3.12a}$$

$K_1(s;t_1,s_{c2},s_{c3})=0$ unless

$$t_1^{\frac{1}{2}}>(s_{c2}-u)^{\frac{1}{2}}\{1+(s_{c3}-u)/4q^2\}^{\frac{1}{2}} +(s_{c3}-u)^{\frac{1}{2}}\{1+(s_{c2}-u)/4q^2\}^{\frac{1}{2}}, \tag{3.12b}$$

$K_2(s;s_{c1},t_2,s_{c3})=0$ unless

$$(s_1-u)^{\frac{1}{2}}>t_2^{\frac{1}{2}}\{1+(s_{c3}-u)/4q^2\}^{\frac{1}{2}} +(s_{c3}-u)^{\frac{1}{2}}(1+t_2/4q^2)^{\frac{1}{2}}, \tag{3.12c}$$

where

$$u=(M^2-\mu^2)^2/s. \tag{3.13}$$

As the smallest value of s_c or t which contributes to the integrand in Eq. (3.9a) is $s_c=M^2$, where A_2 has a δ-function singularity, it follows from (3.12b) that the smallest value of t for which $A_{13}(s,t)$ is nonzero (for any given value of s) is given by

$$t^{\frac{1}{2}}=2(M^2-u)^{\frac{1}{2}}\{1+(M^2-u)/4q^2\}^{\frac{1}{2}}. \tag{3.14}$$

For very large s, this value of t approaches $4M^2$, but, as s decreases, t becomes larger and larger until, at $s=(M+\mu)^2$, it becomes infinite. Equation (3.14) has been plotted as C_1 in Fig. 2. A_{13} will be nonzero above C_1, and, near it, it will behave like $(t-t_0)^{-\frac{1}{2}}$, where t_0 is the value of t given by (3.14). It follows from (3.11b) that $A_3(s,t)$ is nonzero if $t>4M^2$, and behaves like $(t-4M^2)^{\frac{1}{2}}$ just above this limit. The value $t=4M^2$ is precisely the threshold for the process III, and we would have obtained the same results from our general reasoning in the previous section if we had neglected intermediate states containing two or more mesons but no nucleon pairs. This indicates that our assumptions are probably correct, as we have not considered the process III explicitly in this section. When we treat the subtraction terms in the dispersion relations, we shall see that A_{13} is also nonzero between $t=4\mu^2$ and $t=4M^2$, and that the region in which A_{13} is nonzero must be enlarged. The curve C_1 is therefore not yet the curve C_{13} of Fig. 1.

For a range of values of t above the curve C_1, the entire contribution to the integrand in (3.9a) comes from the δ function in A_2. At a certain point, however, the other terms in A_2 and A_3 begin to contribute. If for the moment we neglect the second term in (3.9a), the new contribution begins at the value of t obtained by putting $t_2=t_3=4M^2$ in (3.12a), since this is (at the present stage of the calculation) the lowest value of t for which A_3 is nonzero. The result has been plotted against s in Fig. 2 to give the curve C_2. As this curve approaches the line $t=16M^2$ asymptotically, there will be a corresponding new contribution to A_3 above this value, and, near it, the new contribution will behave like $(t-16M^2)^{\frac{1}{2}}$. The value $t=16M^2$ is just the threshold for the production of an additional nucleon pair in the process III, and A_3 would be expected to show such a behavior at this threshold.

We find similar discontinuities in the higher derivatives of A_{13} at series of curves (there will now be more than one for each threshold) approaching asymptotically the lines $t=4n^2M^2$, so that A_3 will have the expected behavior at the thresholds for producing n nucleon pairs.

The functions A_{12} and A_2 will exhibit the same sort of characteristics. In Eq. (3.9b), the lowest values of t_2 and s_{c3} which contribute to the integrand are $t_2=4M^2$, $s_{c3}=M^2$, so that the boundary of the region in which A_{12} is nonzero is obtained by inserting these values into (3.12c). The result is represented by the curve C_3 in Fig. 2; it approaches the line $s_c=9M^2$ as s tends to infinity. As with A_{13}, the region in which A_{12} is nonzero will be widened in the following section. From (3.19a), it follows that A_2 will (at present) be nonzero for $s_c>9M^2$, which is the threshold for pair production in the reaction II. A_{12} will also have discontinuities in the higher derivatives at series of curves such as C_4 which approach asymptotically the lines $s_c=(2n+1)^2M^2$. Finally, it can be seen that the second term of (3.9a) will give rise to further curves at which the higher derivatives of A_{13} are discontinuous, but these curves will all approach asymptotically the lines $t=4n^2M^2$.

We must now return to the second term in the Eq. (3.11), which we have so far neglected in the calculation. It can be taken into account by introducing the requirement of crossing symmetry, which has not yet been used. As in the static theory, one now has to use an iteration procedure. The function A_{23}, which only affects the crossing term in the dispersion relation (2.5), is first neglected, and the calculation done as described. A_{23} is then found from the calculated value of A_{13} and the crossing-symmetry relations (2.23), and inserted into Eq. (3.11) for the next iteration. However, the scattering amplitude calculated by this procedure would still not satisfy the equations of crossing symmetry since, while A_{13} and A_{23} are connected by (2.23a), A_{12} does not satisfy (2.23b). We have seen that the dispersion relations together with the equation of unitarity determine A_{12} uniquely, and the result is not a symmetric function of s and s_c; even the region in which it is nonzero is not symmetric. It therefore appears that we cannot satisfy simultaneously the requirements of analyticity, unitarity (in the one-meson approximation), and crossing symmetry.

The reason why this is so is easily seen in perturbation theory. Among the graphs included in the first iteration of the one-meson approximation is Fig. 3(a). The topologically similar graph Fig. 3(b) will also be included, since Fig. 3(a) by itself would have square roots in the energy denominators and would not have the necessary analytic properties. If, therefore, crossing symmetry is to be maintained, Fig. 3(c) must also be included. In this graph, however, there is an intermediate state of a nucleon and a pair, so that the unitarity condition in the one-meson approximation is not satisfied.

This example also indicates how we should modify our iteration procedure. In addition to inserting a term A_{23}, obtained by crossing symmetry from the previous iteration, into (3.11), we must insert a term $A_{12}'(s,s_c)$ equal to $A_{12}(s_c,s)$ as calculated in the previous iteration. The contribution from this term is to be added to the contribution from $A_{12}(s,s_c)$ calculated in the normal way. A_{12}' will be nonzero above the curve C_5 in Fig. 2, and, in particular, it will be zero for all values of s_c if s is less than $9M^2$. Complete crossing symmetry is now maintained, but the addition of A_{12} violates the unitarity condition (in the one-meson approximation) for values of s greater than $9M^2$, and a perturbation expansion would include graphs such as Fig. 3(c). As these graphs will appear in higher approximations, the fact that we are forced to include them here should not be considered a disadvantage of our method. In any case, the unitarity condition is only violated where the one-meson approximation is far from correct.

The iteration procedure is found to give rise to further curves, like C_2 and C_4 (Fig. 2), at which the higher derivatives of the spectral functions are discontinuous. These new discontinuities correspond to the production of mesons together with nucleon pairs. We still do not have discontinuities at all possible thresholds.

The inclusion of the spin does not change any of the essential features of the theory, though the details are rather more complicated. Following Chew *et al.*,[1] we write the pion-nucleon T matrix in the form

$$T=-\frac{2\pi W}{Ew}(a+\boldsymbol{\sigma}\cdot\mathbf{q}_2\boldsymbol{\sigma}\cdot\mathbf{q}_1 b), \tag{3.15}$$

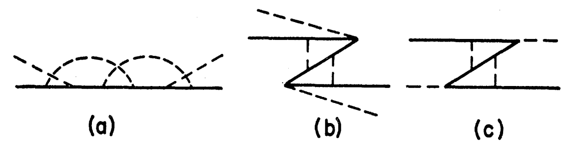

FIG. 3. Graphs which bring in intermediate states with pairs.

where E is the center-of-mass energy of the nucleon and w that of the pion. a and b are related to the quantities A and B in the expression (2.20) by the formulas

$$a=\frac{E+M}{2W}\left(\frac{A+(W-M)B}{4\pi}\right), \tag{3.16a}$$

$$b=\frac{E-M}{2W}\left(\frac{-A+(W+M)B}{4\pi}\right). \tag{3.16b}$$

The unitarity condition corresponding to (3.7) can now be worked out in terms of a and b; the equation obtained is

$$a_{13(12)}(s,z_1)=\sum_\alpha \frac{q}{\pi}\int dz_2\int dz_3\, K_{1(2)}(z_1,z_2,z_3)$$
$$\times\left\{a_\alpha^*(s,z_2)a_\alpha(s,z_3)+\frac{z_2-z_3z_1}{1-z_1^2}b_\alpha^*(s,z_2)a_\alpha(s,z_3)\right.$$
$$\left.+\frac{z_3-z_2z_1}{1-z_1^2}a_\alpha^*(s,z_2)b_\alpha(s,z_3)\right\}, \tag{3.17a}$$

$$b_{13(12)}(s,z_1)=\sum_\alpha \frac{q}{\pi}\int dz_2\int dz_3\, K_{1(2)}(z_1,z_2,z_3)$$
$$\times\left\{\frac{z_3-z_2z_1}{1-z_1^2}b_\alpha^*(s,z_2)a_\alpha(s,z_3)+\frac{z_2-z_3z_1}{1-z_1^2}\right.$$
$$\left.\times a_\alpha^*(s,z_2)b_\alpha(s,z_3)+b_\alpha^*(s,z_2)b_\alpha(s,z_3)\right\}, \tag{3.17b}$$

where $\sum_\alpha$ indicates that terms of the form $a_\alpha^*a_\alpha$ are to be replaced by $a_2^*a_2+a_3^*a_3$ in the calculation of a_{13} and b_{13} and by $a_2^*a_3+a_3^*a_2$ in the calculation of a_{12} and b_{12}, exactly as in (3.7). a_2 and b_2, a_3 and b_3, a_{12} and b_{12}, and a_{13} and b_{13} are related respectively to A_2 and B_2, A_3 and B_3, A_{12} and B_{12}, and A_{13} and B_{13} by Eqs. (3.16). The unitarity condition (3.17) can be rewritten in terms of A and B; it then becomes

$$A_{13(12)}(s,z_1)=\sum_\alpha \frac{q}{4\pi^2 W}\int dz_2\int dz_3\, K_{1(2)}(z_1,z_2,z_3)$$
$$\times\Bigg\{\left(1-\frac{w}{2W}\frac{1-z_2-z_3+z_1}{1+z_1}\right)A_\alpha^*(s,z_2)A_\alpha(s,z_3)$$
$$+\left(\frac{\omega}{2}\frac{1-z_2+z_3-z_1}{1-z_1}+\frac{Mw}{2W}\frac{1-z_2-z_3+z_1}{1+z_1}\right)$$
$$\times A_\alpha^*(s,z_2)B_\alpha(s,z_3)+\left(\frac{\omega}{2}\frac{1+z_2-z_3-z_1}{1-z_1}\right.$$
$$\left.+\frac{Mw}{2W}\frac{1-z_2-z_3+z_1}{1+z_1}\right)B_\alpha^*(s,z_2)A_\alpha(s,z_3)$$
$$+\frac{W^2-M^2}{2W}\frac{1-z_2-z_3+z_1}{1+z_1}B_\alpha^*(s,z_2)B_\alpha(s,z_3)\Bigg\},\quad (3.18\text{a})$$

$$B_{13(12)}(s,z_1)=\sum_\alpha \frac{q}{4\pi^2 W}\int dz_2\int dz_3\, K_{1(2)}(z_1,z_2,z_3)$$
$$\times\Bigg\{\frac{E}{2MW}\frac{1-z_2-z_3+z_1}{1+z_1}A_\alpha^*(s,z_2)A_\alpha(s,z_3)$$
$$+\left(\frac{1+z_2-z_3-z_1}{2(1-z_1)}-\frac{E}{2W}\frac{1-z_2-z_3+z_1}{1+z_1}\right)$$
$$\times A_\alpha^*(s,z_2)B_\alpha(s,z_3)+\left(\frac{1-z_2+z_3-z_1}{1-z_1}\right.$$
$$\left.-\frac{E}{2W}\frac{1-z_2-z_3+z_1}{1+z_1}\right)B_\alpha^*(s,z_2)A_\alpha(s,z_3)$$
$$+\left(\omega-\frac{(w^2-M^2)E}{2MW}\frac{1-z_2-z_3+z_1}{1+z_1}\right)$$
$$\times B_\alpha^*(s,z_2)B_\alpha(s,z_3)\Bigg\}.\quad (3.18\text{b})$$

Equations (3.17) and (3.18) will hold separately for the amplitudes corresponding to isotopic spin $\frac{1}{2}$ and $\frac{3}{2}$.

It remains to justify the claim that the result calculated by our procedure, if expanded in a perturbation series, would give a subset of the usual perturbation series. The proof is somewhat awkward because we were unable to satisfy the unitarity condition in the one-meson approximation at all values of the energy. Let us first ignore this. The nth term in the perturbation series $A^{(n)}$ is then determined uniquely in the physical region by the following two requirements:

(i) For sufficiently small values of the momentum transfer $\{$less than $2\mu[\frac{2}{3}(2M+\mu)/(2M-\mu)]^{\frac{1}{2}}\}$, $A^{(n)}$ must satisfy the dispersion relation (2.5), a result which has been proved rigorously.[14] The absorptive part A_1 (and hence, by crossing symmetry, A_2) is known, since it is determined by unitarity in terms of lower order perturbation terms in the physical region, and by analytic continuation (with s constant) outside it.[14]

(ii) For a fixed value of s, $A^{(n)}$ is an analytic function of the momentum transfer throughout the physical region.[14]

As the functions calculated by our method certainly fulfil these requirements, they must generate the correct perturbation series.

However, our result does not satisfy the unitarity condition in the one-meson approximation at all energies, and we must examine more closely how A_1 is to be determined. Let us assume that our method gives the correct perturbation series up to the $(n-1)$th order. The reasoning developed in this section then shows that the nth-order contribution to A_1 will be of the form

$$A_1^{(n)}=\frac{1}{\pi}\int dt'\frac{A_{13}^{(n)}(s,t')}{t'-t}-\frac{1}{\pi}\int dt'\frac{A_{12}^{(n)}(s,t')}{t'-t},\quad (3.19)$$

where $A_{13}^{(n)}$ and $A_{12}^{(n)}$ are certainly zero below C_1 and above C_3, respectively, in Fig. 2. Inserting this expression into (2.5), we find that

$$A_d^{(n)}=\frac{1}{\pi^2}\int ds'\int dt'\frac{A_{13}^{(n)}(s',t')}{(s'-s)(t'-t)}$$
$$-\frac{1}{\pi^2}\int ds'\int dt'\frac{A_{12}^{(n)}(s',t')}{(s'-s)(t'-t)}.\quad (3.20)$$

The suffix d indicates that we are considering the direct and not the crossing term. The second term of (3.20) will not be an analytic function of t in the physical region, but it will have a branch point at the largest value of t for which A_{12} is nonzero. We can make it analytic by adding to A_2 the expression

$$-\frac{1}{\pi}\int dt'\frac{A_{12}^{(n)}(s_c,t')}{t'-t},\quad (3.21)$$

which we would expect from (2.17), if our representation is correct. By inserting this into (2.5) and adding the result to the second term of (3.20), we obtain

$$\frac{1}{\pi^2}\int ds'\int ds_c'\frac{A_{12}^{(n)}(s',s_c')}{(s'-s)(s_c'-s_c)},\quad (3.22)$$

which is analytic in the physical region. The contribution (3.21) to $A_2^{(n)}$ is uniquely determined from the requirement that $A^{(n)}$ be an analytic function of the momentum transfer in the physical region, and is nonzero only for $s_c>9M^2$. It corresponds to adding a graph such as Fig. 3(b) to Fig. 3(a); as A_1 for Fig. 3(c) is nonzero for $s>9M^2$, A_2 for Fig. 3(b) will be nonzero for $s_c>9M^2$.

Finally, then, the nth-order perturbation term can be determined from the lower order perturbation terms without using any unproved properties of the scattering amplitude as follows:

(i) Calculate A_1 by unitarity, and extend it into the nonphysical region for momentum transfers less than $2\mu[\frac{2}{3}(2M+\mu)/(2M-\mu)]^{\frac{1}{2}}$ by analytic continuation.

(ii) Calculate a contribution $A_{2d}{}^{(n)}$ to $A_2{}^{(n)}$, for $s_c>9M^2$, from the requirement that if it, together with A_1, be inserted into (2.5), the resulting function $A_d{}^{(n)}$ must be an analytic function of the momentum transfer in the physical region. By doing this we partially include intermediate states with nucleon pairs, which is necessary if we are to maintain the required analytic properties and crossing symmetry.

(iii) Now calculate $A_2{}^{(n)}$ and the extra contribution to $A_1{}^{(n)}$ by crossing symmetry from $A_1{}^{(n)}$ and the extra contribution to $A_2{}^{(n)}$.

(iv) Find $A^{(n)}$ from (2.5) for values of the momentum transfer less than $2\mu[\frac{2}{3}(2M+\mu)/(2M-\mu)]^{\frac{1}{2}}$, and calculate it in the rest of the physical region by analytic continuation in t.

This procedure defines a one-meson approximation in perturbation theory. From what has been said, it is clear that our solution will give precisely this perturbation expansion, so that our assumptions are justified in perturbation theory.

4. SUBTRACTION TERMS IN THE DISPERSION RELATIONS

We have thus far assumed that the dispersion relations are true without any subtractions. As we have pointed out in the first section, by doing this we neglect what is physically the most important part of the scattering amplitude. In this section we shall investigate how many subtractions are necessary for each dispersion relation and shall outline how they can be calculated, leaving the details for a further paper.

Let us first consider Eqs. (2.11) and (2.16), which were used in obtaining the unitarity condition (3.9) [or (3.18) for nucleons with spin]. Even if these dispersion relations are written with subtraction terms, it is found that (3.9) is unchanged, so that the subtraction terms are only needed in the final evaluation of A from A_2 and A_3 by means of (2.11), or of A_1 from A_{12} and A_{13} by means of (2.16). The number of subtractions will depend on the behavior of A_{12}, A_{13}, A_2, and A_3, as calculated by our procedure, as s_c and t tend to infinity—we shall have to perform at least enough subtractions for (2.11) and (2.16) to converge.

It is difficult to make an estimate of the behavior of these functions at infinite values of s_c and t from the equations determining them, and we shall use indirect arguments which, though not rigorous, are very plausible. We shall find that, if the coupling constant is small enough, the functions tend to zero at infinity, so that one can write the dispersion relations without any subtractions. For larger values of the coupling constant, more and more subtractions will be needed. The reader who is prepared to accept this may omit the following two paragraphs.

We consider only the first iteration, since subsequent iterations proceed in a similar way and the results are unlikely to be qualitatively different. The result can then be expanded in a perturbation series. If the solutions obtained for this problem by other methods, such as the Tamm-Dancoff or Bethe-Salpeter methods, are expanded in a perturbation series, it is found that the series for each angular momentum state converges as long as the coupling constant is within a certain radius of convergence, and that this radius of a convergence tends to infinity with the angular momentum.[16] Our perturbation series would be different from the perturbation series obtained by these methods, partly because the intermediate states with pairs which we include are not the same as those included by either of them, and partly because, in calculating the subtraction terms (other than those at present under discussion), we shall not take into account terms corresponding to all graphs included by these approximations. Such differences would not be expected to affect qualitatively the convergence properties of the angular momentum states, and we shall assume that the results quoted above are true for our perturbation series too.

The transition amplitude for the state of total angular momentum j and orbital angular momentum $j\pm\frac{1}{2}$ can be shown to be

$$f_{j\pm}=\int_{-1}^{1}dz\, a(s,z)P_{j\pm\frac{1}{2}}(z)+\int_{1}^{1}dz\, b(s,z)P_{j\mp\frac{1}{2}}(z), \quad (4.1)$$

where a and b are the functions defined in (3.15) and (3.16). Now it is easily seen that each term in the perturbation series for $a_2(s,z)$, $a_3(s,z)$, $b_2(s,z)$, and $b_3(s,z)$ tends to zero like $1/z$ as z tends to infinity, so that the dispersion relation (2.11) for each term can be written down without any subtractions. Hence

$$f_{j\pm}{}^{(n)}=\int_{-1}^{1}dz\int dz'\left\{\frac{a_2{}^{(n)}(s,z')+a_3{}^{(n)}(s,z')}{z'-z}P_{j\pm\frac{1}{2}}(z)+\frac{b_2{}^{(n)}(s,z')+b_3{}^{(n)}(s,z')}{z'-z}P_{j\mp\frac{1}{2}}(z)\right\} \quad (4.2)$$

$$=\int dz'\{[a_2{}^{(n)}(s,z')+a_3{}^{(n)}(s,z')]\phi_{j\pm\frac{1}{2}}(z')+[b_2{}^{(n)}(s,z')+b_3{}^{(n)}(s,z')]\phi_{j\mp\frac{1}{2}}(z')\}, \quad (4.3)$$

[16] Note that the "potential" in the Tamm-Dancoff or Bethe-Salpeter equation involved includes only the crossing term and not the direct term, which has still to be brought into the calculation.

where

$$\phi_n(z')=\int_{-1}^{1}dz\frac{P_n(z)}{z'-z}\approx 1/z'^{n+1}\quad\text{as}\quad z'\to\infty. \tag{4.4}$$

Let us suppose that the value of the coupling constant is such that the perturbation series for states of angular momentum j_1 converges. If each term in the perturbation series for this angular momentum state is expressed by (4.3), and if we assume that we can interchange the order of summation and integration, we arrive at the equation

$$f_{j_1\pm}=\int dz'\{\textstyle\sum_n[a_2^{(n)}(s,z')+a_3^{(n)}(s,z')]\phi_{j_1\pm\frac{1}{2}}(z')+\sum_n[b_2^{(n)}(s,z')+b_3^{(n)}(s,z')]\phi_{j_1\mp\frac{1}{2}}(z')\}. \tag{4.5}$$

In order for the integrand to exist, we see from (4.4) that a and b must be smaller than $z^{j_1-\frac{1}{2}}$ at infinite z. The dispersion relations can therefore be written down with not more than $j-\frac{1}{2}$ subtractions. In particular, if the coupling constant is small enough the dispersion relations can be written down without any subtractions.[17]

If the coupling constant is such that n subtractions are required, the unitarity condition for the states of angular momentum $\frac{1}{2}$ to $n-\frac{1}{2}$ will have to be applied separately. The wave functions for these states are polynomials of degree not greater than $n-1$ in the variable z (or s_c and t), and are not determined from the absorptive parts in the dispersion relations (2.11) and (2.16).

The calculation must be done after each iteration, as the result will be needed for the next iteration. The details of the calculation will not be discussed here, but they will in principle be similar to those of Chew and Low[4] and Dalitz, Castillejo, and Dyson,[5] and will involve considering the reciprocal of the scattering amplitude. The analytic properties of the individual angular momentum states are not as simple as in the static theory, but they can be determined from the assumed analytic properties of the transition amplitude, and, as in the static theory, the singularities not on the positive real axis can be found from the previous iteration.

The precise number of subtractions required cannot be determined without calculating the result, but it is almost certainly not less than two. It is difficult to see how the observed resonant behavior of the $P_{\frac{3}{2}}$ state could be reproduced by means of the calculations described in the last section, whereas it follows quite naturally from a Chew-Low-type calculation. If the coupling constant were large enough to bind the (3,3) resonance state, and for a certain range of values of the coupling constant below this, we would definitely have to perform two subtractions. The precise range involved is difficult to determine, but it would be expected to include those values of the coupling constant for which the (3,3) state still has the appearance of an unstable isobar. Until we state otherwise, however, we shall suppose that the coupling constant is sufficiently small for the functions $A(s,z)$ and $B(s,z)$ to tend to zero at infinite z, as the situation with regard to the other subtractions is much simpler in this case. Even then, we would have to perform one subtraction for each of A and B, since the calculations of the previous section did not include the pole of the scattering amplitude from the one-nucleon intermediate state; only the pole in the crossing term was included. The pole affects the states with $j=\frac{1}{2}$ alone, so that, if we apply the unitarity condition for these states separately by the Chew-Low method, we can include it correctly. We thereby change A and B by a quantity independent of z.

When we calculate the scattering amplitudes for the states with $j=\frac{1}{2}$, we find a ghost state in the first iteration, just as in all other models. In subsequent iterations, however, where the crossing terms contribute, it does not follow from the form of the equations that we shall necessarily find a ghost state, and, judging from the charged scalar model, we may hope that the ghost state does not in fact occur.

We now turn to consider the subtraction terms in the other dispersion relations used in the calculations, Eq. (3.11). By putting the δ-function contribution to A_2 into (3.18), it can be seen that the lowest order term in $A_{13}(s,t)$ tends to a constant as s tends to infinity, whereas the lowest order term in $B_{13}(s,t)$ behaves like $1/s$. For a certain range of values of t, only the lowest order term contributes to A_{13} and B_{13}, so that there will certainly be one subtraction in Eq. (3.11b) for A_3, while the equation for B_3 could be written down without any subtractions. We find similarly that both $A_{12}(s,s_c)$ and $B_{12}(s,s_c)$ tend to zero like $1/s$ as s tends to infinity. It would therefore appear that the dispersion relations (3.11a) did not require any subtractions. However, we have seen that $A_1(s,s_c)$ and $B_1(s,s_c)$ behave like a constant for large s_c with s constant, even for small values of the coupling constant, so that, by crossing symmetry, $A_2(s,s_c)$ and $B_2(s,s_c)$ will behave like a constant for large s. There will therefore be one subtraction term in Eqs. (3.11a) for both A_2 and B_2.

The determination of the subtraction terms in Eq. (3.11a) is not difficult, since the contributions to A_2 and B_2 from the states with $j=\frac{1}{2}$ (with the energy s_c of the reaction II kept constant) can be found by crossing symmetry from the corresponding contributions to A_1 and B_1 in the previous iteration. However, for the subtraction terms in Eq. (3.11b), we require

[17] We should emphasize that it is only in the first iteration that we relate the number of subtractions needed to the convergence of the angular momentum states. We say nothing at all about the convergence of the perturbation series in subsequent iterations, but assume simply that the behavior of the spectral functions at infinite values of z is not likely to be qualitatively different from their behavior in the first iteration.

the unitarity condition for A_3, which involves the reaction III. As there is one subtraction, only the S waves will be involved. Again we have to limit the intermediate states considered; in this first approximation we would consider the two-meson states ("two-meson approximation") and perhaps the nucleon-antinucleon intermediate states ("two-meson plus pair approximation") as well. We shall then require the meson-meson scattering amplitude (and the nucleon-antinucleon scattering amplitude if nucleon-antinucleon intermediate states are being considered). The determination of these scattering amplitudes would be as extensive a calculation as the determination of the pion-nucleon scattering amplitude, but neglect of the crossing term would probably not give rise to too great an error in our final result, in which case the S-wave amplitudes could be written down immediately in the two-meson or two-meson plus pair approximations. The meson-meson coupling constant is thereby introduced into the calculation, as has been mentioned in the introduction. Once the meson-meson and nucleon-antinucleon scattering amplitudes are known, the transition amplitude for the reaction III can be calculated. Since the integral equation is now linear, the details will be different from those of the Chew-Low calculations, but, as in their case, the solution could be written down exactly if there were no other singularities of the transition amplitude, and we can use an iteration procedure for the actual problem. The iterations will again be interspersed between the iterations of the main calculation. The S-wave portion of A_3, as calculated by this procedure, will be nonzero for $t>4\mu^2$, so that the scattering amplitude now has the expected spectral properties. The boundaries of the regions in which the spectral functions are nonzero will thereby also be changed; this will be discussed in more detail at the end of the section.

We have seen that, as long as the coupling constant is sufficiently small, we require one subtraction for each of the dispersion relations except the dispersion relation (3.11b) for B_3, for which we do not require any subtractions. It is also easily seen that this behavior is consistent—the functions as calculated in the last section, with the calculations modified by the subtraction terms, will not at any stage become too large at infinity. If, however, one were to make any additional subtractions, one would find that, on performing the calculations, one would need more and more subtractions as the work progressed, and one could not obtain any final result. The number of subtractions to be performed is therefore determined uniquely. There is one exception to this statement: we could perform one subtraction in Eq. (3.11b) for B_3. Such a subtraction is, however, excluded by the requirement that the theory remain consistent when the interaction with the electromagnetic field is introduced. If one were to make this subtraction, the scattering amplitude would behave like $f(t)\gamma(q_1+q_2)$ for large values of s. It then follows from gauge invariance that the matrix element for the processes

$$\pi^{\pm}+n\rightarrow\pi^{\pm}+n+\nu \quad \text{or} \quad \pi^0+p\rightarrow\pi^0+p+\nu$$

will contain a term which behaves like $f(t)\gamma$ for large s, where t is now minus the square of the momentum transfer of the neutral particle.[18] The contribution to B_1 and B_{13} from the $\pi-N-\gamma$ intermediate state therefore tends to infinity at least as fast as s for infinite s, so that one would require two subtractions for the dispersion relation in question and the theory would not be consistent.

Since the unitarity conditions for the two $j=\frac{1}{2}$ states of the pion-nucleon system, and for the S state of the pion-pion system, have to be applied separately by the Chew-Low method, there will be Castillejo-Dalitz-Dyson ambiguities associated with these states. The ambiguities will of course affect all states in subsequent iterations. They correspond to the existence of unstable baryons of spin $\frac{1}{2}$ and either parity, or of heavy unstable mesons of spin zero. There are no ambiguities associated with states of higher angular momentum; this is in agreement with perturbation theory, according to which it is impossible to renormalize systems containing particles of spin 1 or more. Had there been no interaction with the electromagnetic field, we could have introduced a further subtraction term which would have necessitated a separate application of the unitarity condition for the P state of the pion-pion system. The resulting Castillejo-Dalitz-Dyson ambiguity would have been associated with a heavy unstable meson of spin 1. This corresponds to the Bethe-Beard mixture of vector and scalar mesons, which can be renormalized in perturbation theory as long as there is no interaction with the electromagnetic field.

Now let us consider the situation that occurs in practice, when the coupling constant is sufficiently large for the scattering amplitude and its absorptive parts to tend to infinity with z (or s_c and t) when s remains constant. The function A_{12}' which, according to our procedure, must be added to A_{12} in iterations other than the first, will now tend to infinity with s, so that A_2, as calculated from (3.11a), would show a similar behavior. In practice, when the unitarity condition for states with $j=\frac{3}{2}$ as well as with $j=\frac{1}{2}$ must be applied separately, $A_{12}'(s,s_c)$ and $A_{23}(s_c,t)$ will tend to infinity faster than s or t, and the dispersion relation (3.11a) will require two subtractions. The subtraction terms can be determined by crossing symmetry as before. However, we have seen that, if A_2 tends to infinity with s, we cannot consistently perform the calculation, so that we shall have to introduce some further modifications.

The reason for the difficulty is probably the in-

[18] This can be shown by using a generalization of the Ward identity due to H. S. Green, Proc. Phys. Soc. (London) **66**, 873 (1953), and T. D. Lee, Phys. Rev. **95**, 1329 (1954), and proved by Y. Takahashi, Nuovo cimento **6**, 372 (1957).

adequacy of the one-meson approximation. The breakdown occurs just at the value of the coupling constant for which the contribution to the scattering amplitude from A_{12}' is comparable to the remainder of the scattering amplitude when s is large. Since that part of A_1 calculated from A_{12}' represents a partial effect of states with one or more pairs, the contribution of these intermediate states is now important at high energies and it seems reasonable that, if one could take them into account properly, one could still perform the calculations for large values of the coupling constant. In the one-meson approximation, one would have to make some sort of a cutoff to the contribution to A_2 from the crossing term above $s=9M^2$. As this entails modifying the unitarity condition in the region where it is in any case inaccurate, it is consistent with our approximations, and it may be hoped that the theory is not very sensitive to the precise location and form of the cutoff. If one were to go to further approximations in which intermediate states with pairs were included, the cutoff would always be applied only at or above the threshold for processes which were neglected.

Once we are prepared to introduce cutoffs into our approximations, we might legitimately ask whether or not we should perform more than one subtraction in Eq. (3.11b). This could only be determined by examining the behavior of the scattering amplitude and its absorptive parts at large values of s when we go beyond the one-meson approximation. However, if A and B have the behavior assumed thus far (A remains constant and B behaves like $1/s$), the cross section would tend to zero like $1/s$ at large s, whereas the experimental results indicate that the cross section remains constant. It therefore may be necessary to perform an additional subtraction and to introduce the unitarity condition of the reaction III in P states.

At first sight it would seem as though there were Castillejo-Dalitz-Dyson ambiguities associated with all states for which the unitarity condition has to be applied separately, not only with the $j=\frac{1}{2}$ states. However, it is also possible that only the solution without any of the extra terms in the higher angular momentum waves would converge as we introduced more and more states into the unitarity equations. This solution would be an analytic continuation of the solution obtained for small values of the coupling constant, whereas the other solutions could not be continued below a certain value of the coupling constant and would have no perturbation expansion. While we can by no means exclude such a behavior, it nevertheless gives us grounds to suppose that the ambiguity exists only for meson-nucleon states with $j=\frac{1}{2}$ and for S-wave meson-meson states, even when the coupling constant is large.

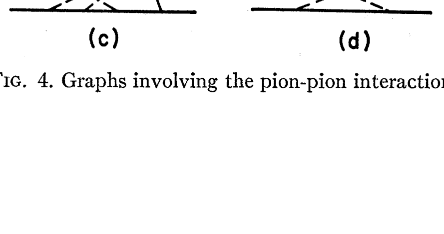

FIG. 4. Graphs involving the pion-pion interaction.

Before leaving this section, let us state the boundaries of the region in which the spectral functions A_{13}, A_{23}, and A_{12} are nonzero, i.e., the position of the curves C_{13}, C_{23}, and C_{12} in Fig. 1. Since A_3 is now nonzero for $t^2>4\mu^2$, C_{13} in the one-meson approximation is obtained by putting $t_2=t_3=4\mu^2$ in (3.12a), so that

$$t_{1a}^{\frac{1}{2}}=4\mu(1+\mu^2/q^2)^{\frac{1}{2}},$$

or

$$t_{1a}=\frac{16\mu^2(s-M^2+\mu^2)^2}{[s-(M+\mu)^2][s-(M-\mu)^2]}. \tag{4.6}$$

For any given value of s, A_{13} will be nonzero if $t>t_{1a}$. We notice that, as s tends to infinity, t_{1a} approaches the value $16\mu^2$. This is not the expected result—we have shown in Sec. 2 that it should approach the value $4\mu^2$. The reason for the discrepancy is that, in our approximation, the reaction III takes place purely through S waves for $4\mu^2<t<16\mu^2$, and A_3 will be a function only of t in this region. Had it been possible for the reaction III to go through an intermediate state of one pion, A_3 would have had a δ function at $t=\mu^2$, and, on putting this value into (3.12a), we would have obtained the expected result. As it is, however, we shall have to go beyond the one-meson approximation to get the correct boundary of A_{13}.

The reaction $N+\bar{N}\rightarrow 3\pi$ can go through a one-pion intermediate state by means of the process represented in Fig. 4(a). If, therefore, we treat the outgoing pions in the reaction $N+\pi\rightarrow N+2\pi$ as one particle with fixed energy and angular momentum, and represent the transition amplitude in the same way as we have represented the transition amplitude for pion-nucleon scattering, the absorptive part corresponding to A_3 will have a δ function at $t=\mu^2$. We can work out the resulting contribution to A_{13} (of the pion-nucleon scattering amplitude) by unitarity in the same way as we worked out the contributions from the one-meson approximation. z_2 and z_3 in Eqs. (3.4)–(3.8) will now refer to the center-of-mass deflection of the nucleon in the production reaction, and will be connected with the momentum transfer by the relation

$$z=\{q^2+q_1^2+t-[(M^2+q^2)^{\frac{1}{2}}-(M^2+q_1^2)^{\frac{1}{2}}]^2\}/2qq_1,$$

where q_1, is the center-of-mass momentum of the outgoing nucleon. The value of q_1 will depend on the relative energy of the two pions; we shall require the maximum value of q_1 (for a fixed s), which occurs when the pions are at rest with respect to one another and is given by

$$q_{1m}^2=\{s-(M+2\mu)^2\}\{s-(M-2\mu)^2\}/4s. \quad (4.7)$$

We then find that the boundary of this contribution to A_{13} has the equation

$$t_{1b}=\frac{4\mu^2(s-M^2-2\mu)^2}{[s-(M+2\mu)^2][s-(M-2\mu)^2]}. \quad (4.8)$$

The curve represented by (4.8) approaches asymptotically the lines $t=4\mu^2$ and $s=(M+2\mu)^2$. Thus, as would be expected, this contribution to A_{13} only occurs above the threshold for pion production.

A_{13} is therefore nonzero for $t>t_1$, where

$$\begin{aligned} t_1&=t_{1a}, && (M+\mu)^2<s<(M+2\mu)^2; \\ t_1&=\min(t_{1a},t_{1b}), && (M+2\mu^2)<s<\infty; \end{aligned} \quad (4.9)$$

and $t=t_1$ is the curve C_{13} of Fig. 1. We cannot be sure that contributions from other intermediate states will not extend beyond this curve, but this is unlikely owing to the greater mass of these states.

The curve C_{23} is obtained from C_{13} simply by changing s to s_c. C_{12} can be calculated in a similar way; we find that

$$\begin{aligned} s_{c2}&=s_{c2a}, && (M+\mu)^2<s<(M+2\mu)^2 \\ &=\min(s_{c2a},s_{c2b}), && (M+2\mu)^2<s<\infty, \end{aligned} \quad (4.10)$$

where

$$(s_{c2a}-u)^{\frac{1}{2}}=2\mu\left\{\frac{s^2-s(3M^2+2\mu^2)+2(M^2-\mu^2)^2}{[s-(M+\mu)^2][s-(M-u)^2]}\right\}^{\frac{1}{2}} +\left\{\frac{[M^2s-(M^2-\mu^2)^2][s^2-2s(M^2+3\mu^2)+(M^2-\mu^2)^2]}{s[s-(M+\mu)^2][s-(M-\mu)^2]}\right\}^{\frac{1}{2}}, \quad (4.11)$$

$$s_{c2b}(s)=s(s_{c2a}). \quad (4.12)$$

The equation $s_c=s_{c2b}$ represents in fact the boundary of the region in which A_{12}' is nonzero. We observe that, once the pion-pion interaction has been included, this region approaches asymptotically the line $s=(M+2\mu)^2$ rather than the line $s=9M^2$. The reason is that processes represented by graphs such as Fig. 4(b) are now included in our approximation, so that the crossing term will include the contribution from Fig. 4(c), the intermediate state of which involves a nucleon and two pions.

For a given real value of s, the absorptive part A of the scattering amplitude will be an analytic function of the momentum transfer as long as

$$t_2<t<t_1, \quad (4.13a)$$

where t_1 is given by (4.9), and t_2 by (4.10) and (2.13). The expansion in partial waves will converge if

$$-t_1-4q^2<t<t_1, \quad (4.13b)$$

as $-t_1-4q^2$ is always greater than t_2.

We may note finally one interesting point concerning the spectral properties of the scattering amplitude. The unitarity condition should, strictly, be used in the physical region only, and the results extended to the unphysical region by analytic continuation. This has actually been done for the reaction I, as well as for the reaction III with $t>4M^2$. For the reaction III in the region $4\mu^2<t<4M^2$, we should apply the unitarity condition with the nucleon masses taken, not on the mass shell, but at some smaller value where all the momenta would be real. The result should then be continued analytically onto the mass shell. In our case this is found to make no difference, but if, in addition to the nucleon, we had a baryon whose mass M_B satisfied the inequality

$$M_B^2<M^2-\mu^2, \quad (4.14)$$

it would be necessary to do the calculation in this way. On making the continuation to the mass shell, it would be found that the absorptive part A_3 extended below the limit $t^2=4\mu^2$. It has been shown by several workers[19] that, if an inequality such as (4.14) is satisfied, the vertex function would show similar spectral properties. The simplest graph to exhibit them in our case would be Fig. 4(d), which will obviously have properties similar to those of a vertex graph. It is thus seen that these spectral abnormalities would not limit the applicability of our method, but, on the contrary, follow from it.

5. APPROXIMATION SCHEME FOR OBTAINING THE SCATTERING AMPLITUDE

In the methods developed in the previous sections, the unitarity condition for the reaction I is satisfied for all angular-momentum states in the one-meson approximation. The unitarity condition for the reaction III is satisfied only for S states in the two-meson or two-meson plus pair approximations. The unitarity condition for higher angular momentum states of the reaction III is not satisfied, but the scattering amplitude shows the expected behavior at the threshold for competing real processes.

These properties suggest immediately a further approximation which would be consistent with our other approximations. The major portion of the work, and certainly the major part of the computing time, would be employed in calculating the spectral functions, as this involves finding double integrals which are themselves functions of two variables. The calculations would therefore be simplified if we neglected those contributions to the spectral functions which begin at the threshold for processes involving more than two particles. The only contributions to A_{13} and A_{23} left would be those beginning at $t=4M^2$, and they could be obtained by inserting the δ-function contribution to B_2

[19] Karplus, Sommerfield, and Wichman, Phys. Rev. **111**, 1187 (1958); Y. Nambu, Nuovo cimento **9**, 610 (1958); R. Oehme, Phys. Rev. **111**, 1430 (1958).

into (3.18). The spectral function A_{12} would be zero in this approximation.

The unitarity condition for the higher angular momentum states of the reaction I is no longer satisfied. However, the terms neglected appear by their form to arise from intermediate states of the reaction III with more than two particles, so that the approximation is in the spirit of the approximations already made. We have in fact made precisely this approximation in the unitarity condition for the S waves of the reaction III. The unitarity condition for the low angular momentum states of the reaction I, and in particular for the states with $j=\frac{1}{2}$ or $\frac{3}{2}$, is still satisfied, as it has been introduced separately. The present approximation treats the reactions I, II, and III on the same footing.

To summarize, then, our method of procedure will be the following: The first few angular momentum states of A_1 and A_3 are found on the assumption that each angular momentum state is an analytic function of the square of the center-of-mass energy except for the perturbation singularities and the cuts on the positive real axis. This calculation can be done exactly if the discontinuity across the cut along the positive real axis is determined by unitarity (complications arise, as the relations connecting a and b with A and B involve square roots of kinematical factors, but the methods can be modified accordingly). A_{13} and A_{12} are also found as just described. The analytic properties of the low angular momentum states are now determined from the analytic properties of the scattering amplitude given by (2.12). The singularities can be calculated in terms of A_1, A_2, A_3. These absorptive parts can in turn be found from A_{13} and A_{23} by means of the dispersion relations (2.16), (2.17), (2.19), with subtraction terms which can be obtained from the low-angular-momentum states. In the next iteration, all the singularities of the low angular momentum states except that along the positive real axis are found from the quantities calculated in the first iteration, and the singularity along the positive real axis is redetermined from the unitarity condition. The iteration procedure is repeated until it converges. As in the calculations of Sec. 4, it is found necessary to cut off the absorptive parts A_1, A_2 and A_3 at high energies, before calculating the singularities of the low angular momentum states in the next iteration. However, the cutoff is only applied above the threshold for processes neglected in the unitarity condition, and in particular, above the threshold for pair production in the reaction I.

This approximation could be regarded as the first of a series of approximations in which more and more of the contributions to the spectral functions are included, until we ultimately reach a solution in which the unitarity condition in the one-meson approximation is satisfied for every angular momentum state. In the higher approximations the spectral functions are no longer determined by perturbation theory, but, once the contribution from the crossing term enters, they will have to be recalculated after each iteration. However, it would be more worthwhile to go beyond the one-meson approximation at the same time as we took the higher contributions to the spectral functions into account. In other words, we continue to put the reactions I, II, and III on the same footing, bringing in the higher intermediate states of all three together. If the approximation scheme converged, the exact unitarity condition of the three reactions would finally be satisfied for all angular momentum states. Needless to say, one would not in practice be able to go beyond the first one or two approximations.

The number of angular momentum states for which the unitarity condition is applied separately will, as has been explained in the last section, depend on the behavior of A and B as t (or s_c) tends to infinity with s constant. However, in our first approximation, it should be sufficient to treat separately only states with $j=\frac{1}{2}$ and $j=\frac{3}{2}$, as the other angular momentum states will not be important below the threshold for pion production. If we went beyond the one-meson approximation we would probably have to treat some higher angular momentum states separately in any case, since, for instance, two pions both in a (3,3) resonance state with a nucleon could form a $D_{\frac{5}{2}}$ state. For reaction III, one would have to treat separately S states and possibly P states as well.

If one neglected the nucleon-antinucleon intermediate state in the reaction III and only took the two-pion intermediate state into account, all three spectral functions A_{13}, A_{23}, and A_{12} would be zero, since they all begin above the threshold for processes which are being neglected. The entire scattering amplitude would then consist of "subtraction terms" for one or other of the dispersion relations. This may be the best first approximation from the point of view of the amount of work required and the accuracy of the result, as the nucleon-antinucleon intermediate state is a good deal heavier than multipion states which are being neglected. Though the spectral functions are not now brought in at all, it will of course be realized that the only justification for the approximation is that it is the first of a series of approximations which do involve the spectral functions. In this approximation, if the crossing term is neglected in the calculation of the pion-pion scattering amplitude, only intermediate S states occur in reaction III, so that the unitarity condition for the P states will not enter.

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