the *strong* coupling limit a bound on $\alpha(0)$ which grows with the *square root* of g^2. We now wish to argue that this behavior for large g^2 is in fact plausible for the true $\alpha(0)$ and that our bound is therefore a good one, apart from constant factors, in the limit of large g^2.

We consider the analogy to the Yukawa potential Here the effective potential, including the centrifugal barrier term, is

$$V_{\text{eff}}=-(\lambda/r)e^{-\mu r}+[\alpha(\alpha+1)/r^2].$$

For fixed binding energy the ratio $\alpha(\alpha+1)/\lambda$ clearly cannot increase indefinitely as $\lambda\to\infty$ because V_{eff} would then eventually become repulsive for all values of r and could not maintain a fixed bound state. Similarly $\alpha(\alpha+1)/\lambda$ cannot decrease indefinitely towards zero as $\lambda\to\infty$ because V_{eff} would then grow more and more attractive over an increasingly large range of r. In fact it is easy to conclude that, in order for a fixed bound state to be maintained, it is necessary that

$$\lim_{\lambda\to\infty}\frac{\alpha(\alpha+1)}{\lambda}=\frac{1}{\mu e},\quad e=2.7183\cdots.$$

This corresponds to the situation where the two zeros of V_{eff} approach each other as $\lambda\to\infty$, while the depth of the potential between them grows indefinitely. This property is quite general[14]: for any attractive potential that is less singular than r^{-2} at the origin and that falls off more rapidly than r^{-2} at infinity, $\alpha(\lambda)$ must satisfy

$$\lim_{\lambda\to\infty}\frac{\alpha(\alpha+1)}{\lambda}=\text{const}>0,$$

the constant depending on the shape of the potential.

[14] This has been noted independently by R. Blankenbecler (private communication).

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Approximation Techniques in Three-Body Scattering Theory*

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With the aid of some operator algebra the Lippmann-Schwinger integral equations for three-body transition amplitudes are recast in a form which involves two-body transition operators rather than two-body potentials. These equations, which are uncoupled and apply to all channels, are ideally suited to be the basis for approximation schemes, of the impulse approximation type, which have the distinctive feature of preserving unitarity. Two such approximations are described. With either of these as the leading term, a method of successive approximations is developed which yields an expansion for the exact amplitude whose convergence properties are expected to be considerably improved over the usual Born and multiple-scattering expansions. At high energies and low momentum transfers we obtain a unitary version of the strip approximation. Here the integral equation is quite tractable and represents the nondispersion-theoretic analog of multiparticle N/D techniques which have been applied recently to N-N and π-N reactions.

1. INTRODUCTION

In a previous paper[1] we have formulated a scheme for calculating three-body scattering amplitudes which generalizes the well-known impulse approximation by taking into account the constraints imposed by unitarity; effectively, one has summed an infinite set of diagrams of the impulse approximation type. A generalized N/D procedure was employed, in a model in which the incident particle interacts with only one of the target particles. An alternative to the N/D procedure which is in fact much more convenient and direct, particularly when none of the two-body potentials are ignored, will be described here. We again obtain amplitudes which satisfy a generalized unitarity relation which, however, can be derived without reliance on the multiple scattering expansions employed in Ref. 1. In fact, in Sec. 2, we derive the exact integral equations whose iterations give rise to the multiple scattering expansions. These integral equations are essentially the Lippmann-Schwinger equations recast, with the aid of some operator algebra, into a form which involves the two-body T operator, rather than the two-body potential. Such a reformulation is particularly desirable in the light of the observation[2] that the ordinary Born expansion of the three-body amplitudes in powers of the two-body potentials is essentially useless as a calculational tool. Similar T-operator integral equations were obtained earlier by Faddeev.[3] In the form given here they lend

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[1] L. Rosenberg, Phys. Rev. **131**, 874 (1963).

[2] R. Aaron, R. D. Amado, and B. W. Lee, Phys. Rev. **121**, 319 (1961).

[3] L. D. Faddeev, Zh. Eksperim. i Teor. Fiz. **39**, 1459 (1960) [English transl.: Soviet Phys.—JETP **12**, 1014 (1961)]. These equations are highly coupled; they take the form of matrix inetgral equations. A more compact form, applicable to many-particle scattering problems, has been developed by S. Weinberg, Phys. Rev. **133**, B232 (1964), although the two-body potential still appears in Weinberg's formulation. Our equations, restricted here to the three-body case, combine the advantages of being uncoupled and potential-independent.

themselves rather more directly to the unitary approximation techniques which we wish to develop.

The generalization, to the case where none of the potentials vanish, of the unitary impulse approximation of Ref. 1 is formulated in Sec. 3A. The idea is simply to assume that the pair of particles which form the target bound system interact in intermediate and final states only when they are bound. It will be recalled that in the ordinary impulse approximation *all* interactions between the pair in intermediate states are ignored. Formally, our result is achieved by approximating, and thereby considerably simplifying, the propagator which appears in the kernel of the integral equation. It is shown directly, without recourse to N/D procedures, that unitarity is preserved (even when the ordinary impulse approximation violates unitarity). In another closely related method (described in Sec. 3B) the propagators are replaced by the forms appropriate for a particular type of separable potential. The unitarity of the resultant amplitudes is automatic since the separable potentials are Hermitian. The integral equations are particularly simple when the three particles are identical.

These methods share the attractive feature that the approximate integral equations are of the *two-body* Lippmann-Schwinger type (with a complex, energy-dependent potential). Furthermore, since the exact integral equations are known, it is possible to formulate corrections in a systematic way. In particular, we present a method of successive approximations in which the amplitude in the $(n-1)$th stage serves as an optical potential, in a two-body Lippmann-Schwinger equation, for the amplitude in the nth stage. This iterative technique, whose convergence properties have not been studied, is based on a method developed by Feinberg and Pais[4] for use in a different problem.

Our approximate integral equations admit of further simplifications, giving rise to a unitary version of the strip approximation,[5] in the domain of high energies and low momentum transfers. In Sec. 3C we point out the close relation between this result and the theory developed by Baker and Blankenbecler[6] to take into account inelastic effects in peripheral collision models.

2. DERIVATION OF THE INTEGRAL EQUATIONS

We consider a model in which three distinguishable, spinless particles interact by means of two-body local, central potentials. The T-matrix elements of interest are given by the familiar expressions

$$T_{\alpha\beta}^{(\pm)}=(\Phi_\alpha,V_\alpha\Psi_\beta^{(\pm)})=(\Psi_\alpha^{(\mp)},V_\beta\Phi_\beta)\,,\quad \alpha,\beta=a,b,c,o \tag{2.1}$$

where

$$(H-E)\Psi_\alpha^{(\pm)}=(K+V_{12}+V_{13}+V_{23}-E)\Psi_\alpha^{(\pm)}=0\,, \tag{2.2}$$

$$(H-E-V_\alpha)\Phi_\alpha=0\,. \tag{2.3}$$

K is the center-of-mass kinetic-energy operator and we have introduced the notation

$$\begin{aligned} V_a&=V_{12}+V_{13}\,,\quad V_b=V_{13}+V_{23}\,,\\ V_c&=V_{12}+V_{23}\,,\quad V_o=V_{12}+V_{13}+V_{23}\,. \end{aligned} \tag{2.4}$$

Channels a, b, and c are "two-body" channels; Φ_a corresponds to a state in which particles 2 and 3 are bound, while the relative motion of the center of mass of the bound system and particle 1 is described by a plane wave. Similar definitions hold for Φ_b and Φ_c, with appropriate permutations of particle labels according to Eqs. (2.4). All three particles are unbound and noninteracting in channel o. We also define a channel d, with $V_d=V_a$, in which the three particles are unbound although particles 2 and 3 interact through V_{23}. The (somewhat unconventional) retention of this potential in the definition of the three-body channel wave function Φ_d will be convenient in the following. It has the consequence that T_{dd} differs from the full scattering amplitude T_{oo}, for three particles free in initial and final states, by a "disconnected" part as shown explicitly below.[7]

The two-body transition operators T_{ij} are defined by

$$\begin{aligned} T_{ij}^{(\pm)}(E)&=V_{ij}+V_{ij}G_o^{(\pm)}(E)T_{ij}^{(\pm)}(E)\\ &=V_{ij}+V_{ij}G_{ij}^{(\pm)}(E)V_{ij}\,, \end{aligned} \tag{2.5}$$

where

$$G_o^{(\pm)}(E)=(E\pm i\eta-K)^{-1}\,, \tag{2.6}$$

$$G_{ij}^{(\pm)}(E)=(E\pm i\eta-K-V_{ij})^{-1}\,. \tag{2.7}$$

The relation

$$G_{ij}=G_o+G_oT_{ij}G_o \tag{2.8}$$

will be useful in the following. It will be convenient to introduce, in addition, the operators G_α, defined, e.g., by means of eigenfunction expansions which we indicate schematically as

$$G_\alpha(E)=\mathbf{S}_\alpha\frac{|\Phi_{\alpha(n)}(E_n)\rangle\langle\Phi_{\alpha(n)}(E_n)|}{E-E_n}\,. \tag{2.9}$$

Here the sum over states, symbolized by $\mathbf{S}_\alpha$, is an appropriately weighted integration over momentum vari-

[4] G. Feinberg and A. Pais, Phys. Rev. **131**, 2724 (1963).

[5] G. F. Chew and S. C. Frautschi, Phys. Rev. **123**, 1478 (1961).

[6] M. Baker and R. Blankenbecler, Phys. Rev. **128**, 415 (1962). The similarity between the Baker-Blankenbecler theory and our unitary impulse approximation is not surprising since both are suggested by the multiparticle N/D relations. It is our purpose here to point out that the use of linear integral equations can serve as a useful alternative to the N/D relations as a starting point for generating approximation techniques. This point of view has also been emphasized recently by R. D. Amado (see Ref. 12).

[7] We should, for completeness, display the boundary conditions satisfied by the wave functions $\Psi_\beta^{(\pm)}$. In the interest of brevity we do not do so here (see, however, Ref. 13). Information concerning the momentum variables needed to complete the definition of the channel wave functions is assumed to be absorbed in the channel indices. We shall at times write $\Phi_{\alpha(n)}$ to make this more explicit.

ables for the state Φ_α. We have, accordingly, the relation

$$G_{23}=G_a+G_d. \tag{2.10}$$

It will be helpful in the following to have at our disposal an operator identity which has previously appeared (in perhaps slightly altered form) in the literature.[8] A brief derivation is included here for the reader's convenience. Consider a scattering system whose state vector $\Psi^{(\pm)}$ satisfies the Lippmann-Schwinger equation

$$\Psi^{(\pm)}=\Phi+G^{(\pm)}(E)V\Psi^{(\pm)}, \tag{2.11}$$

with

$$G^{(\pm)}(E)=(E\pm i\eta-H+V)^{-1}, \tag{2.12}$$

and

$$(H-V-E)\Phi=0 \tag{2.13}$$

so that

$$(H-E)\Psi^{(\pm)}=0. \tag{2.14}$$

A transition operator T and a wave operator Ω may be defined such that

$$T\Phi=V\Psi \tag{2.15}$$

and

$$\Psi=\Omega\Phi. \tag{2.16}$$

With the aid of Eq. (2.11), the relations

$$T=V+VGT, \tag{2.17}$$

$$T=V\Omega, \tag{2.18}$$

$$\Omega=1+GT, \tag{2.19}$$

are easily verified. Now consider two different systems, distinguished by the subscripts A and B. The identity

$$\begin{aligned}T_A^{(+)}=T_A^{(+)}&-T_B^{\dagger(-)}[\Omega_A-G_A^{(+)}T_A^{(+)}-1]\\&+[\Omega_B^{\dagger(-)}-T_B^{\dagger(-)}G_B^{\dagger(-)}-1]T_A^{(+)}\end{aligned} \tag{2.20}$$

obviously holds since both bracketed terms vanish. The identity we seek is obtained by rewriting Eq. (2.20) in the form

$$\begin{aligned}T_A^{(+)}=T_B^{\dagger(-)}&+\Omega_B^{\dagger(-)}(V_A-V_B^\dagger)\Omega_A^{(+)}\\&+T_B^{\dagger(-)}[G_A^{(+)}-G_B^{\dagger(-)}]T_A^{(+)},\end{aligned} \tag{2.21}$$

where use has been made of Eq. (2.18). In the following we assume that

$$V^\dagger=V, \quad G^{\dagger(-)}=G^{(+)}.$$

As a first application of Eq. (2.21) we return to our model problem and make the choices

$$\begin{aligned}&V_A=V_{12}+V_{13}+V_{23}, \quad T_A=T\\&V_B=V_{23}, \qquad\qquad\quad T_B=T_{23}\\&G_A=G_B=G_o.\end{aligned} \tag{2.22}$$

We find immediately that

$$T^{(+)}=T_{23}^{(+)}+\Omega_{23}^{\dagger(-)}(V_{12}+V_{13})\Omega^{(+)}, \tag{2.23}$$

so that

$$\begin{aligned}T_{o(f)o(i)}&=(\Phi_{o(f)},T\Phi_{o(i)})\\&=(\Phi_{o(f)},T_{23}\Phi_{o(i)})+T_{d(f)d(i)}.\end{aligned} \tag{2.24}$$

[8] See, e.g., H. A. Bethe, B. H. Brandow, and A. G. Petschek, Phys. Rev. **129**, 225 (1963).

Here

$$T_{d(f)d(i)}=(\Phi_{d(f)}^{(-)},V_d\Psi_{d(i)}^{(+)}) \tag{2.25}$$

is evidently the difference between the true amplitude and a disconnected part which describes the process in which particles 2 and 3 interact and particle 1 is free. No such disconnected part can exist, for energy conserving reactions, if particles 2 and 3 are bound in initial or final states so that[9]

$$T_{oa}=(\Phi_d^{(-)},V_d\Psi_a^{(+)}), \tag{2.26}$$

$$T_{ao}=(\Psi_a^{(-)},V_d\Phi_d^{(+)}). \tag{2.27}$$

We proceed by defining operators $\mathcal{T}_{aa}^{(\pm)}$ and $\tau_{aa}^{(\pm)}$ according to

$$\mathcal{T}_{aa}^{(\pm)}=V_a+V_aG_{23}^{(\pm)}\mathcal{T}_{aa}^{(\pm)} \tag{2.28}$$

and

$$\tau_{aa}^{(\pm)}=V_a+V_aG_o^{(\pm)}\tau_{aa}^{(\pm)}. \tag{2.29}$$

If, in Eq. (2.21), we choose

$$\begin{aligned}&V_A=V_{12}+V_{13}, \quad T_A=\mathcal{T}_{aa}, \quad G_A=G_{23},\\&V_B=V_{12}+V_{13}, \quad T_B=\tau_{aa}, \quad G_B=G_o,\end{aligned} \tag{2.30}$$

we obtain

$$\begin{aligned}\mathcal{T}_{aa}&=\tau_{aa}+\tau_{aa}(G_{23}-G_o)\mathcal{T}_{aa}\\&=\tau_{aa}+\tau_{aa}G_oT_{23}G_o\mathcal{T}_{aa}.\end{aligned} \tag{2.31}$$

The significance of the operator $\mathcal{T}_{aa}$ lies in the relation

$$\mathcal{T}_{aa}\Phi_a=V_a\Psi_a, \tag{2.32}$$

which follows from Eq. (2.28) and the Lippmann-Schwinger equation for Ψ_a. Therefore, the elastic amplitude can be written as

$$T_{aa}=(\Phi_a,\mathcal{T}_{aa}\Phi_a). \tag{2.33}$$

It is seen from Eqs. (2.25)–(2.27) and Eq. (2.32) that the inelastic amplitudes can also be represented in terms of $\mathcal{T}_{aa}$ as

$$T_{oa}=(\Phi_d^{(-)},\mathcal{T}_{aa}\Phi_a), \tag{2.34}$$

$$T_{ao}=(\Phi_a,\mathcal{T}_{aa}\Phi_d^{(+)}), \tag{2.35}$$

$$T_{dd}=(\Phi_d^{(-)},\mathcal{T}_{aa}\Phi_d^{(+)}). \tag{2.36}$$

In obtaining Eq. (2.35) we have made use of the reciprocity relation $\mathcal{T}_{aa}^{\dagger(-)}=\mathcal{T}_{aa}^{(+)}$ which follows directly from Eq. (2.21) with $A=B$. Similar reciprocity relations hold for the operators τ_{aa} and T_{ij}. Consequently, Eq. (2.31) may be written as

$$\mathcal{T}_{aa}=\tau_{aa}+\mathcal{T}_{aa}G_oT_{23}G_o\tau_{aa}. \tag{2.37}$$

Our task now is to replace Eq. (2.29) by a set of equations which determine τ_{aa} but which do not involve the potentials. Toward this end we introduce the decomposition

$$\tau_{aa}=\tau_2+\tau_3, \tag{2.38}$$

[9] Equation (2.26) (along with its time-reversed counterpart) was derived by K. M. Watson, Phys. Rev. **88**, 1163 (1952), as the basis for his final-state interaction theory.

with

$$\tau_2 = V_{13} + V_{13}G_o\tau_{aa}, \tag{2.39}$$

$$\tau_3 = V_{12} + V_{12}G_o\tau_{aa}. \tag{2.40}$$

It is easy to see that Eqs. (2.39) and (2.40) may be replaced by

$$\tau_2 = T_{13} + T_{13}G_o\tau_3, \tag{2.41}$$

$$\tau_3 = T_{12} + T_{12}G_o\tau_2. \tag{2.42}$$

Thus, Eq. (2.41) may be written as

$$\begin{aligned}\tau_2 &= V_{13} + V_{13}G_oT_{13} + V_{13}G_o\tau_3 + V_{13}G_oT_{13}G_o\tau_3 \\ &= V_{13} + V_{13}G_o(T_{13} + T_{13}G_o\tau_3 + \tau_3) \\ &= V_{13} + V_{13}G_o(\tau_2 + \tau_3).\end{aligned} \tag{2.43}$$

In a similar manner the expressions for τ_3 in Eqs. (2.40) and (2.42) can be shown to be equivalent. The equations can be simplified by a transformation to uncoupled form. By combining Eqs. (2.41) and (2.42) we find that

$$\tau_2 = \hat{\tau}_2(1 + G_oT_{12}), \tag{2.44}$$

$$\tau_3 = \hat{\tau}_3(1 + G_oT_{13}), \tag{2.45}$$

where $\hat{\tau}_2$ and $\hat{\tau}_3$ satisfy

$$\hat{\tau}_2 = T_{13} + T_{13}G_oT_{12}G_o\hat{\tau}_2, \tag{2.46}$$

$$\hat{\tau}_3 = T_{12} + T_{12}G_oT_{13}G_o\hat{\tau}_3. \tag{2.47}$$

In summary, Eqs. (2.46) and (2.47) are to be solved for $\hat{\tau}_2$ and $\hat{\tau}_3$, from which τ_{aa} can be constructed; Eq. (2.31) is then to be solved for $\mathcal{T}_{aa}$. A knowledge of $\mathcal{T}_{aa}$ leads not only to the elastic and break-up amplitudes [see Eqs. (2.33)–(2.36)] but to the rearrangement amplitudes as well, as is shown below. It should be noted that the kernels in Eqs. (2.31), (2.46), and (2.47) contain no disconnected parts so that Fredholm techniques may be applied. This point is discussed in detail by Weinberg[3], who devised a different method for separating off the disconnected parts.

The expression of the scattering operator $\mathcal{T}_{aa}$ as the sum of two parts, as in Eq. (2.31) has a simple interpretation. The operator τ_{aa} represents the sum of all distinct multiple scattering events, each described by either T_{12} or T_{13}, in which the incident particle strikes one target particle, then the other, and then the first again, etc. The second part represents the additional terms (sometimes called "potential" corrections in discussions of the impulse approximation) which take into account virtual collisions between the target particles themselves. This representation appears to be an ideal starting point in a problem in which the impulse approximation (τ_{aa} replaced by $T_{12}+T_{13}$) represents a fairly good first approximation and one is looking for corrections to it.

We now turn to a discussion of rearrangement collisions and show that as a generalization of the preceding discussion the amplitude $T_{\alpha\beta}$, for any pair of entrance and exit channels, can be represented as

$$T_{\alpha\beta} = (\Phi_\alpha, \mathcal{T}_{\alpha\beta}\Phi_\beta). \tag{2.48}$$

Here the operators $\mathcal{T}_{\alpha\beta}$ can be obtained as the solutions of uncoupled linear integral equations in which the operators T_{ij}, rather than the two-body potentials, appear; the wave functions Φ_α are obtained by solving two-body problems. To see how this comes about we first set $\alpha=a$ and $\beta=b$. According to Eq. (2.48) an operator $\mathcal{T}_{ab}$ must be found such that

$$\mathcal{T}_{ab}\Phi_b = V_a\Psi_b. \tag{2.49}$$

We look for $\mathcal{T}_{ab}$ in the form

$$\mathcal{T}_{ab} = V_{12} + R = V_{12} + R_2 + R_3, \tag{2.50}$$

where R_2 and R_3 are to satisfy

$$R_2\Phi_b = V_{13}\Psi_b, \tag{2.51}$$

$$(V_{12} + R_3)\Phi_b = V_{12}\Psi_b. \tag{2.52}$$

If, in addition, we introduce the operator R_1 such that

$$R_1\Phi_b = V_{23}\Psi_b, \tag{2.53}$$

then the integral equation

$$\Psi_b = \Phi_b + G_{12}(V_{13} + V_{23})\Psi_b \tag{2.54}$$

leads to the relations

$$R_1 = V_{23} + V_{23}G_{12}(R_1 + R_2), \tag{2.55}$$

$$R_2 = V_{13} + V_{13}G_{12}(R_1 + R_2), \tag{2.56}$$

$$R_3 = V_{12}G_{12}(R_1 + R_2), \tag{2.57}$$

from which the R_i may be determined. It is easily verified that Eqs. (2.55)–(2.57) can be written in the equivalent form

$$R_1 = T_{23} + T_{23}G_o(R_2 + R_3), \tag{2.58}$$

$$R_2 = T_{13} + T_{13}G_o(R_1 + R_3), \tag{2.59}$$

$$R_3 = T_{12}G_o(R_1 + R_2). \tag{2.60}$$

If, for example, one replaces T_{23} in Eq. (2.58) by its form given in Eq. (2.5) the resultant expression for R_1 can, by reapplication of Eq. (2.58) itself, and by use of Eqs. (2.60) and (2.8), be reduced to that given in Eq. (2.55). The algebraic details are omitted here. It might be thought that our expression for $\mathcal{T}_{ab}$ is still not potential independent due to the appearance of V_{12} in Eq. (2.50). However, when the appropriate matrix element is formed, explicit dependence on the potential can be eliminated since the Born term

$$(\Phi_a, V_{12}\Phi_b)$$

can be expressed in terms of the two-body bound-state wave functions for the pairs (1,2) and (2,3).[10] By means of additional algebraic manipulations it is easy to show that Eqs. (2.58)–(2.60) imply the relations

$$R_2 = \hat{\tau}_2 + \tau_2G_oT_{23}(1 + G_oR), \tag{2.61}$$

$$R_3 = T_{12}G_o\hat{\tau}_2 + \tau_3G_oT_{23}(1 + G_oR). \tag{2.62}$$

[10] See, e.g., Eq. (4.22) of Ref. 1.

Therefore, by adding Eqs. (2.61) and (2.62) we see that R is given implicitly by

$$R=r+\tau_{aa}G_oT_{23}+\tau_{aa}G_oT_{23}G_oR, \tag{2.63}$$

with

$$r\equiv(1+T_{12}G_o)\hat{\tau}_2. \tag{2.64}$$

Equations (2.31) and (2.63) may be used to verify the alternate form

$$R=r+\mathcal{T}_{aa}G_oT_{23}G_or+\mathcal{T}_{aa}G_oT_{23}. \tag{2.65}$$

Since operation of R on Φ_b is always understood, and since Φ_b satisfies the eigenvalue equation

$$\Phi_b(E)=G_0(E)V_{12}\Phi_b(E), \tag{2.66}$$

we may write

$$\mathcal{T}_{aa}G_oT_{23}=\mathcal{T}_{aa}G_oT_{23}G_oV_{12} \tag{2.67}$$

in Eq. (2.65). Finally, we have [recall Eq. (2.50)]

$$\mathcal{T}_{ab}=\tau_{ab}+\mathcal{T}_{aa}G_oT_{23}G_o\tau_{ab}, \tag{2.68}$$

with

$$\tau_{ab}\equiv V_{12}+r. \tag{2.69}$$

We note the equivalent form

$$\mathcal{T}_{ab}=\tau_{ab}+\tau_{aa}G_oT_{23}G_o\mathcal{T}_{ab}. \tag{2.70}$$

It is clear that $\mathcal{T}_{ba}$ should be defined as

$$\mathcal{T}_{ba}=\mathcal{T}_{ab}{}^{\dagger(-)} \tag{2.71}$$

to ensure the relation

$$(\Phi_b,\mathcal{T}_{ba}\Phi_a)=T_{ba}. \tag{2.72}$$

In fact, with the aid of Eqs. (2.71) and (2.49) the left-hand side of Eq. (2.72) can be written as

$$(\mathcal{T}_{ab}{}^{(-)}\Phi_b,\Phi_a)=(\Psi_b{}^{(-)},V_a\Phi_a) \tag{2.73}$$

which, according to Eq. (2.1), is just T_{ba}.

Integral equations for the operators $\mathcal{T}_{\alpha\beta}$ appropriate for other pairs of entrance and exit channels are easily deduced from the equations given above by suitable permutations of particle indices. Note that all the amplitudes for inelastic and rearrangement collisions can be constructed (assuming knowledge of the Φ_α) once the elastic transition operators $\mathcal{T}_{\alpha\alpha}$ have been found.

3. UNITARY APPROXIMATIONS

A. Unitary Impulse Approximation

The first approximation to be discussed is a generalization of the unitary impulse approximation formulated in Ref. 1. From the present point of view the approximation arises from a simple modification (in the spirit of the ordinary impulse approximation) of the propagator which appears in the integral equation for the elastic amplitude T_{aa}. We begin by defining a modified scattering operator $\mathcal{T}_{aa}{}^{(0)}$ according to

$$\begin{aligned}\mathcal{T}_{aa}{}^{(0)}&=\tau_{aa}+\mathcal{T}_{aa}{}^{(0)}G_a\tau_{aa}\\&=\tau_{aa}+\tau_{aa}G_a\mathcal{T}_{aa}{}^{(0)},\end{aligned} \tag{3.1}$$

which differs from Eq. (2.31) for $\mathcal{T}_{aa}$ by the replacement $G_{23}=G_a+G_d\rightarrow G_a+G_o$. This corresponds to ignoring the interaction between particles 2 and 3 in continuum intermediate states, but leaving it unaltered when the pair is bound. This is of course just the essential feature of the impulse approximation, extended here to intermediate states in the scattering process. As we shall show directly, it is this extension which allows for the preservation of unitarity in the set of channels $\{a,o\}$. (We have assumed that we are particularly interested in elastic and break-up reactions for entrance channel a. This accounts for the asymmetric appearance of channel a in our equations.) The approximate scattering amplitudes are taken as

$$T_{aa}{}^{(0)}=(\Phi_a,\mathcal{T}_{aa}{}^{(0)}\Phi_a), \tag{3.2}$$

$$T_{ao}{}^{(0)}=(\Phi_a,\mathcal{T}_{aa}{}^{(0)}\Phi_o), \tag{3.3}$$

$$T_{oa}{}^{(0)}=(\Phi_o,\mathcal{T}_{aa}{}^{(0)}\Phi_a), \tag{3.4}$$

$$T_{oo}{}^{(0)}=(\Phi_o,\mathcal{T}_{aa}{}^{(0)}\Phi_o). \tag{3.5}$$

This is a consistent extension of our approximation since we have ignored the interaction of the (2,3) pair in continuum initial and final states as well. To complete the approximation scheme we define the amplitudes

$$T_{a\alpha}{}^{(0)}=(\Phi_a,\mathcal{T}_{a\alpha}{}^{(0)}\Phi_\alpha), \tag{3.6}$$

$$T_{\alpha a}{}^{(0)}=(\Phi_\alpha,\mathcal{T}_{\alpha a}{}^{(0)}\Phi_a),\quad \alpha=b,c, \tag{3.7}$$

with $\mathcal{T}_{a\alpha}{}^{(0)}$ and $\mathcal{T}_{\alpha a}{}^{(0)}$ given by

$$\mathcal{T}_{a\alpha}{}^{(0)}=\tau_{a\alpha}+\mathcal{T}_{aa}{}^{(0)}G_a\tau_{a\alpha}, \tag{3.8}$$

$$\mathcal{T}_{\alpha a}{}^{(0)}=\tau_{\alpha a}+\tau_{\alpha a}G_a\mathcal{T}_{aa}{}^{(0)}. \tag{3.9}$$

It will be convenient to discuss the unitarity conditions in terms of the operator relations

$$\begin{aligned}&\mathcal{T}_{aa}{}^{(+)}-\mathcal{T}_{aa}{}^{(-)}\\&\quad=\sum_{\alpha=a,b,c,d}\mathcal{T}_{a\alpha}{}^{(-)}[G_\alpha{}^{(+)}-G_\alpha{}^{(-)}]\mathcal{T}_{\alpha a}{}^{(+)},\end{aligned} \tag{3.10}$$

which can be obtained directly from the integral equations which define $\mathcal{T}_{aa}$. Alternatively, Eq. (3.10) can be inferred from the unitarity conditions for the scattering amplitudes, viz.,

$$\frac{1}{\pi}\mathrm{Im}T_{\alpha\beta}=\sum_{\gamma=a,b,c,o}\mathbf{S}_\gamma T_{\alpha\gamma(n)}{}^*T_{\gamma(n)\beta}\delta(E-E_n). \tag{3.11}$$

The symbol $\mathbf{S}_\gamma$, introduced previously in Eq. (2.9), represents an integration over intermediate-state momentum variables. Matrix elements of Eq. (3.10) with respect to the states Φ_a and Φ_d give rise to "generalized" unitarity relations. They differ from the ordinary unitarity relations of Eq. (3.11) in that branch cuts due to initial- and final-state interactions of particles 2 and 3 are ignored, this being exactly compensated for by the use of connected amplitudes. (Generalized unitarity relations of a similar type have been proposed for rela-

tivistic scattering amplitudes by Blankenbecler.[11]) Since the operators $\mathcal{T}_{\alpha\beta}^{(0)}$ differ from the $\mathcal{T}_{\alpha\beta}$ only in the replacement of G_d by G_o we obtain, as the analog of Eq. (3.10)

$$\mathcal{T}_{aa}^{(0)(+)}-\mathcal{T}_{aa}^{(0)(-)}=\sum_{\alpha=a,b,c,o}\mathcal{T}_{a\alpha}^{(0)(-)}[G_\alpha^{(+)}-G_\alpha^{(-)}]\mathcal{T}_{\alpha a}^{(0)(+)}. \quad (3.12)$$

If we now take matrix elements of this equation with respect to the states Φ_a and Φ_o we find that the amplitudes $T_{\alpha\beta}^{(0)}$ satisfy the ordinary unitarity conditions, Eq. (3.11), for the set of channels $\{a,o\}$. If we replace Eq. (3.5) by

$$T_{oo}^{(0)}=(\Phi_o,\mathcal{T}_{aa}^{(0)}\Phi_o)+(\Phi_o,T_{23}\Phi_o), \quad (3.13)$$

i.e., if we include the disconnected part, then Eq. (3.11) is no longer satisfied for the inelastic amplitudes, although it remains valid for the elastic amplitude ($\alpha=\beta=a$). However, the generalized unitarity relations, which are formulated in terms of the connected amplitude, are still satisfied for the set of entrance and exit channels $\{a,o\}$.

The simplifying feature of our approximation, Eq. (3.1), is the presence of the operator G_a which is essentially a two-body propagator; the (2,3) bound system behaves kinematically as a single particle. Thus, $\mathcal{T}_{aa}^{(0)}$ is determined by a two-body equation, of the Lippmann-Schwinger type, in which τ_{aa} appears as an "optical potential." Of course, we still must determine τ_{aa}. Short of solving the integral equations for τ_{aa} exactly, we expect that this approach will be most useful when multiple scattering effects are unimportant, so that $\tau_{aa}\approx T_{12}+T_{13}$. Equation (3.1) then leads to a summation of an infinite subclass of terms which are iterations of the basic impulse approximation amplitude. We shall have more to say about approximations for τ_{aa} in the following.

It is interesting to observe that with the aid of a method proposed by Feinberg and Pais[4] in connection with their peratization theory, the approximation described above can be exhibited as the leading term in an iteration scheme to determine the exact operator $\mathcal{T}_{aa}$. The method, stated in the context of the present problem, provides a solution of the integral equation for $\mathcal{T}_{aa}$ in the series form

$$\mathcal{T}_{aa}=\sum_{n=0}^{\infty}\mathcal{T}_{aa}^{(n)}. \quad (3.14)$$

The operators $\mathcal{T}_{aa}^{(n)}$ are obtained recursively from the series of integral equations

$$\mathcal{T}_{aa}^{(0)}=\tau_{aa}+\tau_{aa}G_a\mathcal{T}_{aa}^{(0)}, \quad (3.15)$$

$$\mathcal{T}_{aa}^{(n)}=\tau_{aa}[G_{23}-G_o-G_a]\mathcal{T}_{aa}^{(n-1)}+\tau_{aa}G_a\mathcal{T}_{aa}^{(n)}, \quad n\geqslant 1. \quad (3.16)$$

[11] R. Blankenbecler, Phys. Rev. 122, 983 (1961).

It is a simple matter to verify that this system of equations provides a formal solution of Eq. (2.31), although the convergence properties of the series, Eq. (3.14), has not been investigated. Note that Eq. (3.16) preserves the feature described earlier with regard to Eq. (3.1), namely, it is in the form of a two-body Lippmann-Schwinger equation; the inhomogeneous term for the nth equation can be constructed once the solution of the $(n-1)$th equation has been found.

Clearly, the same method can be used to solve Eqs. (2.46) and (2.47) to determine τ_{aa}. If no bound state exists for the (1,3) pair, say, then the Feinberg-Pais procedure applied to Eq. (2.47) reduces to the ordinary multiple scattering expansion

$$\tau_3=T_{12}+T_{12}G_oT_{13}G_oT_{12}+\cdots. \quad (3.17)$$

It is, of course, just the presence of bound states which casts doubt on the validity of this type of expansion, and which would favor the use of the integral equations we have described to sum infinite subclasses of terms.

B. Separable Potential Model

As an alternative to the above procedure we now describe an approximation scheme based on the introduction of separable potentials. Let χ_{23} represent the bound-state wave function for the (2,3) pair, with eigenenergy $-\epsilon_{23}$. (For simplicity we asume that only one bound state exists.) We consider the separable potential

$$v_{23}^{(s)}=v_{23}|\chi_{23}\rangle\langle\chi_{23}|v_{23}/\langle\chi_{23}|v_{23}|\chi_{23}\rangle, \quad (3.18)$$

where the use of a lower case v is meant to indicate that v_{23} operates in a space of two particles while V_{23} operates in a three-particle space. The transition operator $t_{23}^{(s)}$ is easily constructed. Since we are concerned with a three-body problem, we state the result in terms of the corresponding operator $T_{23}^{(s)}$. We find that

$$T_{23}^{(s)}(E)=\mathbf{S}_aV_{23}|\Phi_{a(n)}\rangle\times\frac{N(E-E_n-\epsilon_{23})}{E-E_n}\langle\Phi_{a(n)}|V_{23}, \quad (3.19)$$

where

$$[N(z)]^{-1}=\int\frac{d\mathbf{k}}{(2\pi)^3}\times\frac{g^2(\mathbf{k})}{[(\hbar^2/2\mu_{23})k^2-z][(\hbar^2/2\mu_{23})k^2+\epsilon_{23}]} \quad (3.20)$$

(μ_{23} is the reduced mass of the two-body system) and

$$g(\mathbf{k})=\langle\mathbf{k}|v_{23}|\chi_{23}\rangle. \quad (3.21)$$

We note the relation

$$N(-\epsilon_{23})=1, \quad (3.22)$$

which is easily deduced from Eqs. (3.20) and (3.21) along with the normalization condition $\langle\chi|\chi\rangle=1$.

We now replace T_{23} in Eq. (2.31) by $T_{23}^{(s)}$, giving rise to an approximate transition operator $\mathcal{T}_{aa}^{(s)}$ determined by

$$\mathcal{T}_{aa}^{(s)}=\tau_{aa}+\tau_{aa}G_a^{(s)}\mathcal{T}_{aa}^{(s)}, \tag{3.23}$$

with $G_a^{(s)}$ defined by

$$G_a^{(s)}=G_oT_{23}^{(s)}G_o. \tag{3.24}$$

Equation (3.23) shares the attractive feature of our previous approximation, Eq. (3.1), in that $G_a^{(s)}$ behaves kinematically as a two-body propagator. In fact, if we define the state vector

$$\tilde{\Phi}_{a(n)}^{(\pm)}(E)=G_o^{(\pm)}(E)V_{23}\Phi_{a(n)}(E_n), \tag{3.25}$$

which satisfies the "on-shell" relation

$$\tilde{\Phi}_{a(n)}^{(\pm)}(E_n)=\Phi_{a(n)}(E_n), \tag{3.26}$$

we obtain from Eqs. (3.19) and (3.24) the expression

$$G_a^{(s)}(E)=\mathbf{S}_a|\tilde{\Phi}_{a(n)}^{(+)}(E)\rangle \times\frac{N(E-E_n-\epsilon_{23})}{E-E_n}\langle\tilde{\Phi}_{a(n)}^{(-)}(E)|, \tag{3.27}$$

which may be compared with G_a as given by Eq. (2.9). It might be anticipated that at large scattering energies the operators $\mathcal{T}_{aa}^{(o)}$ and $\mathcal{T}_{aa}^{(s)}$ should not differ appreciably. In fact, to the extent that we need retain only those contributions to the integrals in Eqs. (2.9) and (3.27) for which $E_n\simeq E$ we see that by virtue of Eqs. (3.22) and (3.26) the difference between $G_a^{(s)}$ and G_a vanishes. It is clear that since the separable potential $V_{23}^{(s)}$ is Hermitian the unitarity relations will be preserved. We also note that the Feinberg-Pais iterative procedure can be formulated just as in Eqs. (3.14)–(3.16), with the replacement of G_a by $G_a^{(s)}$. As a final remark let us suppose that V_{12} and V_{13} can each support a bound state. Then Eqs. (2.46)–(2.47) can be replaced by

$$\hat{\tau}_2^{(s)}=T_{13}^{(s)}+T_{13}^{(s)}G_b^{(s)}\hat{\tau}_2^{(s)}, \tag{3.28}$$

$$\hat{\tau}_3^{(s)}=T_{12}^{(s)}+T_{12}^{(s)}G_c^{(s)}\hat{\tau}_3^{(s)}, \tag{3.29}$$

so that an approximation to the operator τ_{aa} can be constructed by solving two-body integral equations with relatively simple inhomogeneous terms. When all three particles are identical, the integral equations become particularly simple in form; they in fact reproduce a model introduced recently by Amado,[12] as we have shown.[13] The use of separable potentials to simplify the three-body equations has been discussed previously by Mitra[14] from a different point of view.

C. Unitary Strip Approximation

A study of the three-body problem in the approach described here should be of some value even when the potential picture breaks down. The reason for this lies in the formal similarity of the integral equations derived here for the potential model and those approximate integral equations which can be written down on the basis of unitarity and analyticity considerations in the relativistic theory. In the latter category we have in mind, in particular, the multiparticle N/D relations discussed by Blankenbecler[11] and others in which a proper treatment is sought for the effects of the coupling of elastic and inelastic channels; this coupling must exist by virtue of the unitarity conditions. In order to emphasize this similarity we will now show that by taking the Fourier-Bessel transform of an approximate version of the integral equations obtained above we are led to a tractable set of equations which can be compared with a set derived earlier by Baker and Blankenbecler.[6] These authors used the N/D method to study proton-proton reactions in the peripheral collision model. It will be seen that the two sets are formally identical once allowance is made for the difference in kinematics, and the correspondence between relativistic form factors and nonrelativistic bound-state wave functions is recognized.[15]

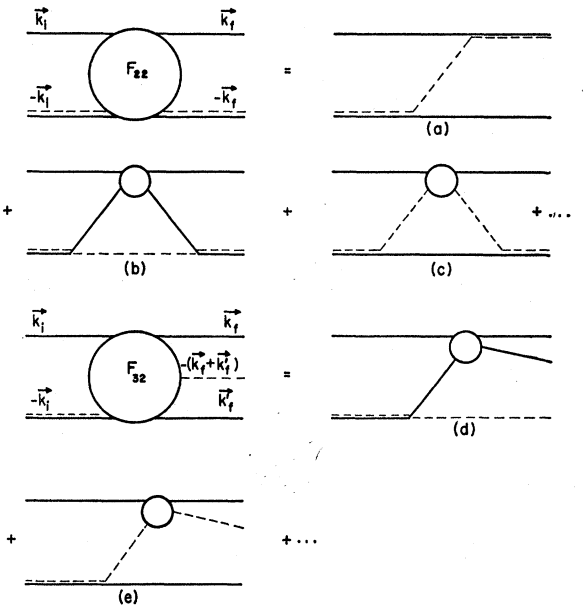

FIG. 1. Leading graphs in the diagrammatic expansions of the elastic and break-up amplitudes. Solid lines refer to the heavy particles and dashed lines to the light particle. The double line (one solid, one dashed) represents the bound two-body system.

Let us consider a case where particles 1 and 2 are identical, of mass M, and particle 3 has mass m, with $m<M$. The solution of the integral equation for the elastic amplitude [which is actually a sum of direct (T_{aa}) and exchange (T_{ba}) terms] can be interpreted as the sum of an infinite set of graphs, the first few of

[12] R. D. Amado, Phys. Rev. **132**, 485 (1963).
[13] L. Rosenberg, Phys. Rev. **134**, B937 (1964).
[14] A. N. Mitra, Nucl. Phys. **32**, 529 (1962).
[15] R. Blankenbecler, M. Goldberger, and F. Halpern, Nucl. Phys. **12**, 647 (1959); R. Blankenbecler and L. F. Cook, Phys. Rev. **119**, 1745 (1960).

which are pictured in Fig. 1. The leading graphs for the inelastic amplitude are also shown. We wish to develop an approximation scheme which will be valid in the domain of high-energy and low-momentum transfer. In this case a useful criterion for the relative importance of different graphs is the nearness to the physical region of singularities of the amplitude in the complex t plane, where t is the square of the momentum transfer. We assume that the amplitude has a branch cut on the negative t axis running from $t=-\infty$ to a threshold value t_0, and that other singularities in t are either not present or have little influence on the physical amplitude. Let the threshold values for graphs (b) and (c) in Fig. 1 be denoted by t_{0M} and t_{0m}, respectively. We assume the binding energy, ϵ, of the target system to be small enough so that these threshold values are of the "anomalous" type, depending on ϵ and not on the potential range. (This is the case for the deuteron, e.g.) Using methods described previously[16] we find [here our study is based on Eq. (3.35)] that

$$t_{0m}/t_{0M}=(m/M)^2\ll 1. \tag{3.30}$$

Consequently, we omit from our sum of graphs all those which involve heavy particle transfer, as in Fig. 1(b). Furthermore, the Born term, Fig. 1(a), is dominated by singularities in the exchange momentum transfer variable $u=\{\mathbf{k}_i+[M/(m+M)]\mathbf{k}_f\}^2$ close to the physical threshold $u_0=0$. In the domain of interest, namely, $\mathbf{k}_i\approx\mathbf{k}_f$ and $k_i^2\gg\epsilon$, u will be large. We therefore omit this graph (and iterations of it) from our sum; the general term in this diagrammatic expansion will in the form of a chain, each link of which is of the type shown in Fig. 1(c). We now make the additional approximation of replacing the propagator $G_oT_{23}G_o$ by G_a in the integral equation which sums these diagrams. This approximation can also by justified by the "nearness of singularities" argument. Because of its dependence on the bound-state wave function, the propagator G_a introduces the nearby "anomalous" threshold singularities mentioned above; the neglected part of the propagator is responsible for more distant singularities. Our use of dispersion theory is confined to the above qualitative considerations (the strip approximation).

The off-the-energy-shell elastic amplitude is denoted by $F_{22}(\mathbf{k}_f,\mathbf{k}_i;s)$. For convenience in this problem, where identical particles are present, we depart from our earlier channel notation. Here the index 2 refers to the channel in which one of the heavy particles is free and the other bound, while the index 3 will refer to the channel in which all three particles are free. The momenta $\mathbf{k}_i$ and $\mathbf{k}_f$ refer to the heavy particle in initial and final states, respectively, while s is related to the total energy E according to

$$(\hbar^2/2\mu)s-\epsilon=E, \tag{3.31}$$

with

$$\mu=(m+M)/(m+2M). \tag{3.32}$$

[16] L. Rosenberg, Phys. Rev. **129**, 968 (1963).

We have the on-shell relation

$$[F(\mathbf{k}_f,\mathbf{k}_i;s)]_{k_i^2=k_f^2=s}=f(s,t), \tag{3.33}$$

where $t=(\mathbf{k}_f-\mathbf{k}_i)^2$ and $f(s,t)$ is the physical elastic-scattering amplitude. According to our simplifying assumptions the integral equation for F_{22} takes the form

$$F_{22}(\mathbf{k}_f,\mathbf{k}_i;s)=F_{22}^{(1)}(\mathbf{k}_f,\mathbf{k}_i;s)+\int\frac{d\mathbf{k}}{(2\pi)^3}\times F_{22}^{(1)}(\mathbf{k}_f,\mathbf{k};s)\frac{1}{k^2-s-i\eta}F_{22}(\mathbf{k},\mathbf{k}_i;s). \tag{3.34}$$

Here $F_{22}^{(1)}$, the impulse approximation, corresponds to Fig. 1(c) and is given by

$$F_{22}^{(1)}(\mathbf{k}_f,\mathbf{k}_i;s)=-\frac{1}{4\pi}\left(\frac{2\mu}{\hbar^2}\right)\int\frac{d\mathbf{k}}{(2\pi)^3}\chi\left(\frac{M}{m+M}\mathbf{k}_i+\mathbf{k}\right)\chi\left(\frac{M}{m+M}\mathbf{k}_f+\mathbf{k}\right)\times t\left(\mathbf{k}_f+\frac{M}{m+M}\mathbf{k},\,\mathbf{k}_i+\frac{M}{m+M}\mathbf{k};\,E-\frac{\hbar^2}{2\mu}k^2\right), \tag{3.35}$$

where $\chi(\mathbf{k})$ is the bound-state wave function in momentum space and $t(\mathbf{k},\mathbf{k}';z)$ is related to the two-body t operator by

$$\langle\mathbf{k}|t(z)|\mathbf{k}'\rangle=t(\mathbf{k},\mathbf{k}';z). \tag{3.36}$$

Once the elastic amplitude is determined the break-up amplitude may be obtained from the formula

$$F_{32}(\mathbf{k}_f,\mathbf{k}_f';\mathbf{k}_i;s)=F_{32}^{(1)}(\mathbf{k}_f,\mathbf{k}_f';\mathbf{k}_i;s)+\int\frac{d\mathbf{k}}{(2\pi)^3}F_{32}^{(1)}(\mathbf{k}_f,\mathbf{k}_f';\mathbf{k};s)\times\frac{1}{k^2-s-i\eta}F_{22}(\mathbf{k},\mathbf{k}_i;s), \tag{3.37}$$

where $F_{32}^{(1)}$ corresponds to the impulse approximation of Fig. 1(e) and takes the form

$$F_{32}^{(1)}(\mathbf{k}_f,\mathbf{k}_f';\mathbf{k}_i;s)=-\frac{1}{4\pi}(2\mu/\hbar^2)\chi((M/m+M)\mathbf{k}_i+\mathbf{k}_f')\times t(\mathbf{k}_f+(M/m+M)\mathbf{k}_f',\,\mathbf{k}_i+(M/m+M)\mathbf{k}_f';\,E-(\hbar^2/2\mu)k_f'^2). \tag{3.38}$$

The time-reversed amplitude is given by

$$F_{23}(\mathbf{k}_f;\mathbf{k}_i,\mathbf{k}_i';s)=F_{23}^{(1)}(\mathbf{k}_f;\mathbf{k}_i,\mathbf{k}_i';s)+\int\frac{d\mathbf{k}}{(2\pi)^3}F_{22}(\mathbf{k}_f,\mathbf{k};s)\times\frac{1}{k^2-s-i\eta}F_{23}^{(1)}(\mathbf{k};\mathbf{k}_i,\mathbf{k}_i';s). \tag{3.39}$$

Finally, the fully connected part of the amplitude describing the collision in which three particles are free in initial and final states is given by

$$F_{33}^{C}(\mathbf{k}_f,\mathbf{k}_f';\mathbf{k}_i,\mathbf{k}_i';s)=\int\frac{d\mathbf{k}}{(2\pi)^3}F_{32}^{(1)}(\mathbf{k}_f,\mathbf{k}_f';\mathbf{k};s)\times\frac{1}{k^2-s-i\eta}F_{23}(\mathbf{k};\mathbf{k}_i,\mathbf{k}_i';s). \quad (3.40)$$

At this point it is possible to introduce additional simplifying approximations which bring our results in correspondence with the Baker-Blankenbecler theory. We observe that alterations of the two-body propagator which preserve its imaginary part will also preserve the unitarity property of the amplitudes. Furthermore, one might expect that at high energies it is the imaginary part which is most important. We therefore replace Eq. (3.34) by

$$F_{22}(\mathbf{k}_f,\mathbf{k}_i;s)=F_{22}^{(1)}(\mathbf{k}_f,\mathbf{k}_i;s)+I(s)\left[\int d\Omega_k F_{22}^{(1)}(\mathbf{k}_f,\mathbf{k};s)F_{22}(\mathbf{k},\mathbf{k}_i;s)\right]_{k^2=s}, \quad (3.41)$$

with

$$\mathrm{Im}I(s)=\rho_2(s)=s^{1/2}/4\pi. \quad (3.42)$$

Equations (3.37), (3.39), and (3.40) can be modified in a similar way. If we wished complete correspondence with the Baker-Blankenbecler theory we would attempt to write

$$I(s)=\int_0^\infty\frac{ds'}{\pi}\frac{\rho_2(s')}{s'-s}. \quad (3.43)$$

While this form is finite for relativistic kinematics, it diverges in our case. We therefore introduce a zero-energy subtraction and write

$$I(s)=s\int_0^\infty\frac{ds'}{\pi}\frac{\rho_2(s')}{(s'-s)s'}=i\rho_2(s), \quad (3.44)$$

which corresponds simply to the replacement of the two-body propagator G_a by its imaginary (on-shell) part. At this stage we note that use of the propagator $G_a^{(s)}$ of Sec. 3B rather than G_a would have led to the same result since, as we pointed out above, the two propagators differ only in the manner in which the amplitudes are continued off the energy shell.

The final step is the introduction of the Fourier-Bessel representation. We write[6]

$$F_{ij}=M_{ij}(s,t;v_{ij}), \quad (3.45)$$

where the v_{ij} are the remaining scalar variables needed to specify the amplitude completely, and assume that the representation

$$M_{ij}(s,t;v_{ij})=\int_0^\infty bdbJ_0(bt^{1/2})H_{ij}(s,b;v_{ij}) \quad (3.46)$$

is valid. The Fourier-Bessel transforms of the amplitudes $F_{ij}^{(1)}$ are denoted by $B_{ij}(s,b;v_{ij})$. If we employ the high-energy approximation of Blankenbecler and Goldberger,[17] our integral equations, in the new representation, take the simple form

$$H_{22}(s,b)=B_{22}(s,b)/[1-I(s)B_{22}(s,b)], \quad (3.47)$$

$$H_{32}(s,b;v_{32})=B_{32}(s,b;v_{32})[1+I(s)H_{22}(s,b)], \quad (3.48)$$

$$H_{23}(s,b;v_{23})=[1+H_{22}(s,b)I(s)]B_{23}(s,b;v_{23}), \quad (3.49)$$

$$H_{33}(s,b;v_{33})=B_{32}(s,b;v_{32})I(s)H_{23}(s,b;v_{23}), \quad (3.50)$$

which are identical in form with those proposed in Ref. 6. In our model the input amplitudes B_{ij} are given explicitly, in terms of the ordinary impulse approximation.

If particles 1 and 2 were lighter than particle 3, the arguments outlined above, in which heavy particle transfers are ignored, would again lead to a unitary strip approximation which sums all iterations of the leading graph shown in Fig. 1(b). (In this case the solid line would refer to the light particle and the dashed line to the heavy particle.) By making use of the above-mentioned correspondence between bound-state wave functions and relativistic vertex functions,[15] this scheme could be applied to peripheral interactions (or interactions in the higher partial waves) for the pion-nucleon system in which only two- and three-body intermediate states are retained. We would then obtain an alternative to the dispersion-theoretic approaches to this problem which have already been attempted.[18]

[17] See Eq. (3.9) in R. Blankenbecler and M. L. Goldberger, Phys. Rev. **126**, 766 (1962).

[18] L. F. Cook and B. W. Lee, Phys. Rev. **127**, 283 (1962); J. S. Ball, W. R. Frazer, and M. Nauenberg, *ibid.* **128**, 478 (1962).