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Volume II. Contributions by:

W. Koch
O. Skjeggestad

G E N E V A

PREFACE

This report contains the second set of contributions for the 1964 CERN Easter School which will be held at Herceg Novi, Yugoslavia in May. This school is primarily intended for young experimental physicists engaged in the analysis of bubble chamber pictures and of events in nuclear emulsions. These notes and those in Volumes I and III deal with topics that are relevant for their work. This Volume contains two papers dealing with Phase space considerations and with the Determination of spin and decay parameters.

We wish to express our gratitude to the authors for their collaboration in the preparation of these papers. We also wish to thank Mrs. V. Cooper who has kindly assisted in this work, the Documents Typing and Reproduction Services whose efforts with the help of the Scientific Information Service made it possible to produce three volumes of the Proceedings within a short time, and to Miss C. Mason for her careful typing of the text.

Finally we wish to thank the U.S. Atomic Energy Commission for their kindness in allowing us to reproduce Tables 1, 2 and 3 (pages 20 and 21) from the Tables of the Clebsch Gordan Coefficients (NAA-SR-2123).

Editorial Board

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Geneva.

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NOTES ON PHASE SPACE

O. Skjeggestad
CERN and Institute of Physics, University of Oslo*

I INTRODUCTION

In the study of strong interactions of elementary particles a great variety of different final state interactions have been detected by the observation that relative yields, momentum and effective mass distributions of the particles deviate from the expectations from phase space. We can mention such discoveries as the hyperon isobars, the pion resonances etc. The calculation of phase space predictions for the particular interaction under investigation is, therefore, often useful and even necessary in order to extract information about the interaction between the particles in the final state.

The purpose of these notes will be to show the derivation of the general phase space formula for a system of n particles in the final state and demonstrate the use of a recursion relation in the calculation of the Lorentz invariant phase space. We will discuss the momentum spectrum of a single random particle and the angular distribution between any two particles in the centre of mass of the n particles. The effective mass distribution of any number of particles can also be calculated with the help of a recursion relation. For some special cases we will also discuss the effect of a resonance between two particles on the effective mass distribution of any two of the particles in the final state. We will also write down some of the special properties of the 3-body phase space first used by Dalitz in his special representation (Dalitz plot). Finally we will make some comparison between the predictions from phase space and the experimental data on hyperon resonances.

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The concept of "phase space" is closely connected to the calculation of transition rates, and to introduce and define the concept we will therefore consider the example of multipion production in a nucleon nucleon collision

$$N + N \rightarrow N + N + n\pi$$

which was first discussed by Fermi (Progr. Theoret. Phys. (Japan) 5, 570 (1950)) in his theory for pion production.

The probability per unit time that the above reaction will take place (the transition rate W) can be expressed by Fermi's well-known golden rule No. 2

$$W = \frac{2\pi}{\hbar} \left| \langle \psi_f | H' | \psi_i \rangle \right|^2 F \qquad (1)$$

where ψ_i and ψ_f are initial and final state wave functions resepctively, and $\langle \psi_f | H' | \psi_i \rangle$ is the matrix element (M) for the transition from i to f caused by a perturbation of the Hamiltonian H'. The multiplication factor F is what we call the phase space or density of state factor. F is a function of the total energy of the system (E) and of the masses of the individual particles in the final state.

We can express Eq. (1) in two formally different ways as:

$$W = \frac{2\pi}{\hbar} |M'|^2 \rho(E) \qquad (1a)$$

and

$$W = \frac{2\pi}{\hbar} |M''|^2 R(E) \qquad (1b)$$

where in Eq. (1a) the matrix element is expressed in a form which is not invariant under Lorentz transformations, (as is the case for matrix elements calculated in non relativistic quantum theory). In Eq. (1b) M'' is Lorentz invariant, (as is the case for the so-called Feynman amplitudes, i.e. the matrix elements calculated in relativistic quantum field theory by application of the Feynman rules.) Since W should be the same in both cases, it follows that ρ(E) and R(E) are different; ρ(E) is not invariant under Lorentz transformations whereas R(E) is an invariant.

The matrix elements M′ and M″ are in general completely unknown, especially for transitions caused by strong interactions. The purpose of the statistical theory introduced by Fermi for calculation of transition rates, is therefore to make certain assumptions about the behaviour of the matrix element. The simplest assumption one can make, is that the matrix element is a constant, independent of the individual momenta of the particles in the final state (but not necessarily independent of E). In this case, for a constant matrix element, the transition rate as well as individual particle momenta in the final state, are determined by the phase space factor alone.

It may be worth while here to stress that the approximation of a constant matrix element may well be a crude one, and it is not a priori given which of the approximations:

M′ = constant (phase space <u>not</u> invariant)

or

M″ = constant (phase space invariant)

is the best one. In his original treatment Fermi used a non invariant phase space. Lately it has become the fashion to use a Lorentz invariant phase space. The best, and perhaps only, justification for this is that the invariant phase space is the easiest to calculate. We will in these notes mainly discuss the Lorentz invariant phase space.

II NON INVARIANT PHASE SPACE

a) Definition and general formula

For one particle a definite state of motion (i.e. specified position (x,y,z) and momentum (p_x,p_y,p_z)) can be represented as a point in a 6 dimensional phase space. Conversely each point in phase space corresponds to a definite state of motion of the particle.

Classical mechanics places no limitations on the density of the representation points. A given value of x can be combined with any value of p_x, etc. It is in principle possible to make simultaneous measurements of x and p_x with infinite accuracy and then localize a point in phase space. Thus classically there will be an infinite number of points available in phase space for a particle confined to a certain region in space and with a certain energy.

Quantum mechanics on the other hand requires the representation points to be separated by finite distances. The uncertainty principle states that it is impossible to measure position and momentum simultaneously with infinite precision. For each pair of canonical variables

$$\Delta x_j \, \Delta p_j \geq 2\pi\hbar \quad \text{etc.}$$

A state of motion can only be given with this indefiniteness and corresponds in phase space to a finite volume, or an elementary cell, of size $(2\pi\hbar)^3$.

The number of final states N_1 available to one particle will therefore be finite and equal to the total volume of the phase space, divided by the size of the elementary cell,

$$N_1 = \frac{1}{(2\pi\hbar)^3} \int dx\, dy\, dz\, dp_x\, dp_y\, dp_z \equiv \frac{1}{(2\pi\hbar)^3} \int d^3x\, d^3p \ .$$

If the particle is confined to a geometrical volume V we can write

$$N_1 = \frac{V}{(2\pi\hbar)^3} \int d^3 p \ .$$

For a particle of total energy less than or equal to E and mass m, N_1 will be the number of cells in a volume enclosed in momentum space by the sphere

$$p_x^2 + p_y^2 + p_z^2 = E^2 - m^2 .$$

Given the total number of states we now <u>define</u> the density in phase space as the number of states per unit energy interval

$$\rho_1(E) = \frac{dN_1}{dE} \ . \qquad (2)$$

This is for short, called "phase space" for one particle.

The extension to a system of n particles with energy $\leq$ E is quite simple. The number of final states N_n will be the product of the number of final states for each particle, thus

$$N_n = \left[\frac{1}{(2\pi\hbar)^3}\right]^n \int\limits_{x_i p_i} \prod_{i=1}^{n} d^3x_i \; d^3p_i \; .$$

Since in practice all the particles are confined to the same geometrical volume $V = V_i = \int d^3x_i$ we can write

$$N_n = \left[\frac{V}{(2\pi\hbar)^3}\right]^n \int\limits_{p_i} \prod_{i=1}^{n} d^3p_i \; . \qquad (3)$$

For instance in our example of multipion production, the interaction is assumed to take place in a sphere with as radius the Compton wavelength of the pion.

This formula gives the number of available cells in the final state for a system of n <u>spinless</u> particles. If the particle i has spin S_i the above expression should be multiplied by

$$\prod_{i=1}^{n} (2S_i + 1) \; .$$

Now, since the geometrical volume factor, spin factor etc. can be included in the final normalization of the phase space integral, we will neglect these factors here and simply put:

$$\rho_n(E) = \frac{dN_n}{dE} = \frac{d}{dE} \int \prod_{i=1}^{n} d^3p_i \; .$$

As already said this integral should be extended over all possible values of p_i. Now, in order to conserve total momentum ($\vec{P}$), the n particle momenta are not independent but constrained by the equation

$$\sum_{i=1}^{n} \vec{p}_i - \vec{P} = 0 \; . \qquad (4)$$

It is usual to indicate this restriction by putting

$$\rho_n(E) = \frac{d}{dE} \int \prod_{i=1}^{n-1} d^3 p_i$$

where one does not include particle n since the momentum of this particle is already given by Eq. (4).

We will, however, include the restrictions Eq. (4) by the introduction of the Dirac δ function. We use the fact that from the definition of the δ function

$$\int d^3 p_n \, \delta^3 (\vec{p}_n - (\vec{P} - \sum_{i=1}^{n-1} \vec{p}_i)) = 1$$

for all integrations including

$$\vec{p}_n = \vec{P} - \sum_{i=1}^{n-1} \vec{p}_i$$

(that is for momentum balance).

We also want to introduce the requirement of energy conservation

$$\sum_{i=1}^{n} E_i - E = 0 .$$

Since

$$\int \delta \left(\sum_{i=1}^{n} E_i - E \right) dE = 1$$

and

$$\frac{d}{dE} \int \delta\left(\sum_{i=1}^{n} E_i - E \right) dE = \delta\left(\sum_{i=1}^{n} E_i - E \right)$$

we can replace d/dE by

$$\delta\left(\sum_{i=1}^{n} E_i - E \right).$$

We then get the general formula for the non invariant phase space.

$$\rho_n(E) = \int \prod_{i=1}^{n} d^3p_i \; \delta^3(\sum_{i=1}^{n} \vec{p}_i - \vec{P}) \; \delta(\sum_{i=1}^{n} E_i - E) . \qquad (5)$$

This formula is symmetrical in all the n particles, and the requirements of energy and momentum conservation are explicitly given.

To illustrate the explicit calculation of $\rho(E)$ for one particular reaction we will consider one simple example, namely that of two particles in the final state. The formulae for 3 and 4-body non invariant phase space are given by M. Block in Phys. Rev. 101, 796 (1956).

b) Two-body non invariant phase space

Let the masses of the two particles in the final state be m_1 and m_2 and their momenta $\vec{p}_1$ and $\vec{p}_2$ in the centre of mass. From Eq. (5) we have

$$\rho_2(E) = \int d^3p_1 \; d^3p_2 \; \delta^3(\vec{p}_1 + \vec{p}_2) \; \delta(E_1 + E_2 - E)$$

$$= \int d^3p_1 \; d^3p_2 \; \delta^3(\vec{p}_1 + \vec{p}_2) \; \delta(\sqrt{m_1^2 + p_1^2} + \sqrt{m_2^2 + p_2^2} - E) .$$

In the following we will make use of some general rules for integration of δ functions given in Appendix A.

Integration over p_2 gives

$$\rho_2(E) = \int d^3p_1 \; \delta(\sqrt{m_1^2 + p_1^2} + \sqrt{m_2^2 + p_1^2} - E) .$$

In polar coordinates we can write

$$d^3p_1 = p_1^2 \; dp_1 \; d\Omega_1 = p_1^2 \; dp_1 \; d(\cos\Theta_1) \; d\Phi_1 \quad .$$

The integrations over $\cos\Theta$ and Φ give a factor 4π, thus

$$\rho_2(E) = \int 4\pi \; p_1^2 \; dp_1 \; \delta\underbrace{(\sqrt{m_1^2 + p_1^2} + \sqrt{m_2^2 + p_1^2} - E)}_{A}$$

Since

$$\frac{dA}{dp_1} = \frac{p_1}{E_1} + \frac{p_1}{E_2} = \frac{p_1}{E_1}\frac{E}{E_2}$$

and A = 0 for

$$p_1 = \frac{\{[E^2 - (m_2 + m_1)^2]\ [E^2 - (m_2 - m_1)^2]\}^{1/2}}{2E}$$

we get by integration over p_1

$$\rho_2(E) = \frac{4\pi\ p_1 E_1 E_2}{E} \qquad (6)$$

$$= \frac{4\pi}{E}\ \frac{\{[E^2 - (m_2 - m_1)^2]\ [E^2 - (m_2 + m_1)^2]\}^{1/2}}{2E}\ \frac{E^4 - (m_2^2 - m_1^2)^2}{4E^2}$$

This is the expression for non invariant 2-body phase space.

III LORENTZ INVARIANT PHASE SPACE

a) General formula

The phase space formula Eq. (5) is not symmetrical in E and p and therefore clearly not invariant under Lorentz transformations. The simplest way to find an invariant expression for the phase space formula is to replace d^3p_i in formula (5) by $d^3p_i/2E_i$. This corresponds to the relation between non relativistic matrix elements M′, Eq. (1a) and Feynman amplitude M″, Eq. (1b). We then get

$$R_n(E) = \int \prod_{i=1}^{n} \frac{d^3p_i}{2E_i}\ \delta^3(\sum_{i=1}^{n} \vec{p}_i - \vec{P})\ \delta(\sum_{i=1}^{n} E_i - E) \qquad (7a)$$

which is invariant under Lorentz transformations. The factor $2E_i$ enters from a normalization of the wave function in field theory. We may qualitatively understand the meaning of this normalization factor as follows. In non relativistic quantum mechanics we express the probability density of, say one particle, as $|\psi|^2$, where ψ is the wave function describing the particle and where $\int |\psi|^2 \, dx\, dy\, dz = 1$ when integrated over total space. This expression for the probability density is not a relativistic invariant. When applied to a relativistic case, the density (probability per cm^3) observed from a moving system appears greater by a factor $\gamma = (1 - v^2/c^2)^{-1/2}$ because of Lorentz contraction of the volume element. It is important to note, however, that the total energy of the particle has also changed by the same factor γ. If we therefore use as a probability density the expression $|\sqrt{2E}\,\psi|^2$, the density will be relativistically invariant. (The factor 2 is a convention). Introducing this normalization into formula Eq. (1a) we can write

$$W = \frac{2\pi}{\hbar} |M'|^2 \rho(E) = \frac{2\pi}{\hbar} |M'|^2 \left(\prod_{i=1}^{n} 2E_i\right) \left(\prod_{i=1}^{n} \frac{1}{2E_i}\right) \rho(E)$$

$$= \text{constant } |M''|^2 R(E)$$

where now

$$R(E) = \rho(E) \prod_{i=1}^{n} \frac{1}{2E_i} .$$

This formula which is also expressed in Eq. (7a) can be written in a more symmetrical form by introducing the four vector

$$q_i \equiv (\vec{p}_i, E_i) = (p_{ix}, p_{iy}, p_{iz}, E_i) \text{ of length } q_i^2 = E_i^2 - p_i^2 .$$

Using the rules for integration over a δ function given in the Appendix we find that

$$\int d^4q \; \delta(q^2 - m^2) \equiv \int d^3p \; dE \; \delta[E^2 - (p^2 + m^2)] \quad (\text{for } E > 0)$$

$$= \int \frac{d^3p}{2E} \quad \text{for } E^2 = p^2 + m^2 .$$

We must, however, eliminate the negative root of $p^2 + m^2$, and do this by making the convention that all integrations over E, E_i are limited to positive values.

Similarly we introduce $Q = (\vec{P},E)$ and write

$$\delta^4(\sum_{i=1}^{n} q_i - Q) \equiv \delta^3(\sum_{i=1}^{n} \vec{p}_i - \vec{P})\ \delta(\sum_{i=1}^{n} E_i - E) \ .$$

Introducing these expressions in Eq. (7a) gives

$$R_n(E) = \int \prod_{i=1}^{n} [d^4 q_i\ \delta(q_i^2 - m_i^2)]\ \delta^4(\sum_{i=1}^{n} q_i - Q) \tag{7b}$$

Expression (7b) is in most literature presented as the definition of Lorentz invariant phase space.

We can qualitatively understand the meaning of the term $\delta(q_i^2 - m_i^2)$, since $q_i^2 - m_i^2 = E_i^2 - p_i^2 - m_i^2$ will be zero for q_i^2 evaluated on the mass shell of the particle. This term therefore essentially represents the constraint that for all i's we must have

$$E_i^2 = m_i^2 + p_i^2 \ .$$

(By convention we have limited E_i to positive values).

The Lorentz invariant expression given by Eq. (7) gives the total volume in momentum space available for n particles of given masses and total energy E. Clearly this volume R_n for a given energy E is just a number. The knowledge of this number is necessary for estimations of cross-sections and relative yields.

Before performing the integrations over all the momenta, p_i, we may, however, (at least in practice) regard R_n as a function of all p_i, thus $R_n(E) = R_n(E, p_1 \dots p_n)$. This expression can now provide us with the differential momentum spectrum of any of the particles, k, simply by evaluation of dR_n/dp_k. Clearly, dR_n/dp_k is just the expression one obtains from formula Eq. (7) by omitting the integration over the k'th particle.

b) Lorentz invariant two-body phase space

For two particles of masses m_1 and m_2 with momenta $\vec{p}_1$ and $\vec{p}_2$ in the centre of mass we get from Eq. (7b)

$$R_2 = \int d^4 q_1 \; d^4 q_2 \; \delta(q_1^2 - m_1^2) \; \delta(q_2^2 - m_2^2) \; \delta^4 (q_1 + q_2 - Q)$$

$$= \int d^3 p_1 \; d^3 p_2 \; dE_1 \; dE_2 \; \delta[E_1^2 - (m_1^2 + p_1^2)] \; \delta[E_2^2 - (m_2^2 + p_2^2)]$$

$$\times \; \delta^3 (\vec{p}_1 + \vec{p}_2) \; \delta(E_1 + E_2 - E) \; .$$

By using the integration rules for a δ function we obtain

$$R_2 = \int \frac{d^3 p_1}{2E_1} \frac{d^3 p_2}{2E_2} \; \delta^3 (\vec{p}_1 + \vec{p}_2) \; \delta(E_1 + E_2 - E)$$

which we could have written down directly from Eq. (7a). We further get by integration over p_2 ,

$$R_2 = \int \frac{1}{4E_1 \; E_2(p_1)} \; d^3 p_1 \; \delta(E_1 + E_2(p_1) - E)$$

$$= \int \frac{1}{4E_1 \; E_2(p_1)} \; d\Omega_1 \; p_1^2 \; dp_1 \; \delta(E_1 + E_2(p_1) - E) \; .$$

Since the two particles are emitted isotropically in space the integration over $d\Omega_1$ gives a factor 4π. Thus integration over p_1 gives

$$R_2 = \frac{\pi \; p_1^2}{E_1 E_2} \; \frac{1}{\frac{p_1}{E_1} + \frac{p_1}{E_2}}$$

and finally

$$R_2 = \frac{\pi \; p_1}{E} = \frac{\pi}{E} \; \frac{\{[E^2 - (m_2 - m_1)^2] \; [E^2 - (m_2 + m_1)^2]\}^{1/2}}{2E} \; . \qquad (8)$$

We note that this Lorentz invariant expression for 2-body phase space R_2 is different from the non invariant expression ρ_2 evaluated earlier. For this special case (2-body) we also see that

$$R_2 = \rho_2 \cdot \frac{1}{2E_1} \cdot \frac{1}{2E_2} \; .$$

We can proceed similarly to calculate R_3 using the general formula Eq. (7). Then we will soon find however, that the expressions cannot be integrated straightforwardly. Instead of attempting to derive the 3-body phase space directly from Eq. (7) we will first evaluate a useful recursion relation for phase space, first given by Srivastava and Sudarshan (Phys. Rev. 110, 765 (1958)).

c) Recursion relation

We re-write formula (7) for the Lorentz invariant phase space for n particles with initial state four vector $Q = (\vec{P},E)$ as

$$R_n(\vec{P},E) = \int \prod_{i=1}^{n} [d^4 q_i \; \delta(q_i^2 - m_i^2)] \; \delta^4 \; (\sum_{i=1}^{n} q_i - Q) \; . \qquad (9)$$

In the centre of mass of the n particles we can write (compare our preceding example for 2-body phase space)

$$R_n(0,E) = \int \prod_{i=1}^{n} \frac{d^3 p_i}{2E_i} \delta^3 \; (\sum_{i=1}^{n} \vec{p}_i) \; \delta(\sum_{i=1}^{n} E_i - E)$$

$$= \int \frac{d^3 p_n}{2E_n} \int \prod_{i=1}^{n-1} \frac{d^3 p_i}{2E_i} \delta^3[\sum_{i=1}^{n-1} \vec{p}_i - (-\vec{p}_n)] \; \delta[\sum_{i=1}^{n-1} E_i - (E - E_n)] \, .$$

We see that the last integral is the phase space for n - 1 particles with total momentum $(-\vec{p}_n)$ and total energy $(E - E_n)$.

Thus

$$R_n(0,E) = \int \frac{d^3 p_n}{2E_n} R_{n-1} [(-\vec{p}_n), \; (E - E_n)] \; .$$

Moreover, since R is Lorentz invariant, R_{n-1} must also be the same in a system with zero total momentum where the total energy is

$$\epsilon = \sqrt{(E - E_n)^2 - (-p_n)^2}$$

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so that

$$R_{n-1}\ [(-\vec{p}_n),\ (E-E_n)] = R_{n-1}\ (0,\epsilon).$$

This gives the following recursion relation between n and n- 1 body Lorentz invariant phase space.

$$R_n(0,E) = \int \frac{d^3 p_n}{2E_n} R_{n-1}\ (0,\epsilon) \qquad (10)$$

where

$$\epsilon^2 = (E-E_n)^2 - p_n^2 \ .$$

We will next use this recursion relation to derive the 3-body phase space.

d) Three-body phase space

For three particles with masses m_1, m_2, m_3 and momenta in centre of mass p_1, p_2, p_3 formula Eq. (10) gives

$$R_3(0,E) = \int \frac{d^3 p_3}{2E_3} R_2(0,\epsilon)$$

where

$$\epsilon^2 = (E-E_3)^2 - p_3^2 \ .$$

From the determination of R_2 in the preceding paragraph we get

$$R_3(0,E) = \int \frac{d^3 p_3}{2E_3} \left(\frac{\pi p'}{E'}\right)$$

where p' is the momentum of each particle in the 2-body system with energy E' in their centre of mass.

Thus

$$E' = E_1' + E_2' = \sqrt{m_1^2 + p'^2} + \sqrt{m_2^2 + p'^2} \ .$$

Since obviously $\epsilon = E'$ we get by solving for p'

$$p' = \frac{\{[\epsilon^2 - (m_1 + m_2)^2]\,[\epsilon^2 - (m_1 - m_2)^2]\}^{1/2}}{2\epsilon} .$$

Substituting this expression for p' into the above formula for $R_3(0,E)$ one gets

$$R_3(0,E) = \int \frac{4\pi p_3^2\ dp_3}{2E_3}\,\pi\,\frac{\{[E^2 + m_3^2 - 2EE_3 - (m_1 - m_2)^2]\,[E^2 + m_3^2 - 2EE_3 - (m_1 + m_2)^2]\}^{1/2}}{2(E^2 + m_3^2 - 2EE_3)} \qquad (11)$$

This integral should be taken between the minimum $p_3(\text{min})$ and maximum $p_3(\text{max})$ values of p_3. Obviously $p_3(\text{min}) = 0$, which takes place when particles 1 and 2 are emitted antiparallel with equal momenta. The maximum momentum of the third particle will be obtained when the other two are emitted parallel and with the same velocity (opposite to $\vec{p}_3$). In this case we have

$$E = \sqrt{m_3^2 + p_3^2(\text{max})} + \sqrt{(m_1 + m_2)^2 + p_3^2(\text{max})}$$

which when solved for $p_3(\text{max})$ gives

$$p_3(\text{max}) = \frac{\{[E^2 - (m_1 + m_2 - m_3)^2]\,[E^2 - (m_1 + m_2 + m_3)^2]\}^{1/2}}{2E} . \qquad (12)$$

For the general case where the three particles have different masses, Eq. (11) is an elliptical integral.

Finally, we want to point out that the differential momentum spectrum of a particle from 3-body final state is given by dR_3/dp_3 which is explicitly given in Eq. (11). We see that the momentum spectrum is a function of the total energy E of the system and the masses of the three particles m_1, m_2 and m_3 only.

IV ANGULAR DISTRIBUTION

a) General formula

The general invariant phase space integral Eq. (7) can be used to find an expression for the angular distribution between any two particles among a system of n particles. The distribution will be derived in the centre of mass of the n particles.

The angle Θ between the two particles, say n and n - 1, can be found from

$$\cos \Theta = \frac{\vec{p}_n \cdot \vec{p}_{n-1}}{p_n \, p_{n-1}} .$$

We will in the following, interpret R_n (before performing the integration) as a function of all the momenta $p_1, p_2 \ldots p_n$; thus we can also express R_n as a function of $\cos \Theta$, that is $R_n = R_n(0, E, \cos \Theta)$. The angular distribution function between the two particles is then given by

$$\frac{dR_n}{d(\cos \Theta)} .$$

Applying the same procedure as for the derivation of the recursion relation Eq. (10), we get

$$R_n(0,E, \cos \Theta) = \int \frac{d^3 p_n}{2E_n} \frac{d^3 p_{n-1}}{2E_{n-1}} R_{n-2} \; (0, \epsilon_{n-2}) \qquad (13a)$$

where

$$\epsilon^2_{n-2} = (E - E_n - E_{n-1})^2 - (\vec{p}_n + \vec{p}_{n-1})^2$$

$$= E^2 + m^2_n + m^2_{n-1} - 2\;(EE_n + EE_{n-1} - E_n E_{n-1} + p_n p_{n-1} \cos \Theta).$$

In polar coordinates, with Φ_n and Θ_n giving the direction of $\vec{p}_n$ and Φ and Θ the direction of $\vec{p}_{n-1}$ with respect to $\vec{p}_n$ (see Fig. 1), we can write

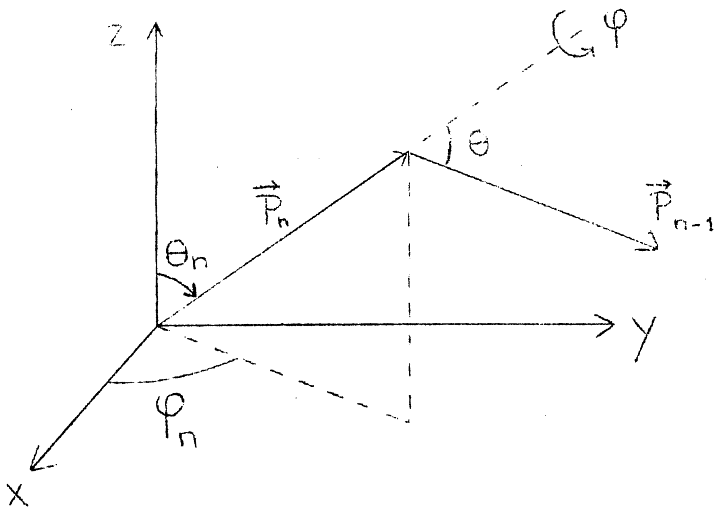


Fig. 1

$$d^3 p_n = p_n^2 \, dp_n \, d\Omega_n = p_n^2 \, dp_n \sin \Theta_n \, d\Theta_n \, d\Phi_n$$

$$d^3 p_{n-1} = p_{n-1}^2 \, dp_{n-1} \, d\Omega_{n-1} = p_{n-1}^2 \, dp_{n-1} \sin \Theta \, d\Theta \, d\Phi \, .$$

The integration over $d\Omega_n = \sin \Theta_n \, d\Theta_n \, d\Phi_n$ gives a factor 4π and the integration over $d\Phi$ a factor 2π, so that Eq. (13a) can be expressed as

$$\frac{dR_n(0,E, \cos \Theta)}{d(\cos \Theta)} = 2\pi^2 \int \frac{p_n^2 \, p_{n-1}^2}{E_n E_{n-1}} \, dp_n \, dp_{n-1} \, R_{n-2} \, (0, \epsilon_{n-2}) \, . \qquad (13b)$$

The integration limits of dp_{n-1} and dp_n will depend on $\cos \Theta$. In general for all possible values of $\cos \Theta$ the integration limits of p_n have to be equal to or greater than $p_n(\min)$ and equal to or less than $p_n(\max)$ where

$$p_n(\min) = 0$$

and

$$p_n(\max) = \frac{\{[E^2 - (m_1 + \ldots + m_{n-1} + m_n)^2]\,[E^2 - (m_1 + \ldots + m_{n-1} - m_n)^2]\}^{1/2}}{2E} \, .$$

The corresponding limits for p_{n-1} are found by replacing n by n - 1 in the above equation. The value for $p_n(\max)$ represents the configuration where all the first n- 1 particles are emitted with the same velocity (that is as one body) opposite to the particle n.

Let us now look in more detail at the integration limits as a function of cos Θ. If we first integrate over dp_{n-1}, we want the limits for fixed p_n and fixed cos Θ.

It is easily verified that p_{n-1} will now have its maximum value if the momentum $(\vec{p}_n + \vec{p}_{n-1})$ is compensated by one body of mass

$$M = \sum_{i=1}^{n-2} m_i .$$

This determines the maximum value of p_{n-1} for fixed p_n and cos Θ to be the positive root in the equation:

$$E_n + (m_{n-1}^2 + p_{n-1}^2)^{1/2} + \left[\left(\sum_{i=1}^{n-2} m_i\right)^2 + p_n^2 + p_{n-1}^2 + 2p_n p_{n-1} \cos \Theta \right]^{1/2} = E . \tag{14}$$

The positive solution of this equation is

$$p_{n-1} = \frac{-ap_n \cos \Theta + (E - E_n) \sqrt{a^2 - 4m_{n-1}^2 b}}{2b} \tag{15}$$

where

$$a = (E - E_n)^2 - \left(\sum_{i=1}^{n-2} m_i\right)^2 + m_{n-1}^2 - p_n^2$$

$$b = (E - E_n)^2 - p_n^2 \cos^2 \Theta .$$

The upper limit of p_n for fixed cos Θ can, in principle, be found if we interchange p_n and p_{n-1} in Eq. (15) and determine the value of p_{n-1} for which $\partial p_n / \partial p_{n-1} = 0$.

b) Three-body angular distribution

Using formula Eq. (13b) from the preceding paragraph, we find for the angular distribution between particles 2 and 3 from a 3-body phase space

$$\frac{dR_3(0,E,\ \cos\Theta)}{d(\cos\Theta)}\ \alpha \int \frac{p_2^2\ p_3^2}{E_2\ E_3}\ dp_2\ dp_3\ R_1(0,\epsilon) \tag{16}$$

where

$$\epsilon^2 = [E - (E_2 + E_3)]^2 - (\vec{p}_2 + \vec{p}_3)^2 \ .$$

We first find

$$R_1(0,\epsilon) = \int d^3p_1\ dE_1\ \delta^3(\vec{p}_1)\ \delta(E_1-\epsilon)\ \delta[E_1^2 - (p_1^2 + m_1^2)]$$

which upon integration over p_1 and E_1 gives

$$R_1(0,\epsilon) = \delta(\epsilon^2 - m_1^2)\ .$$

Thus

$$\frac{dR_3(0,E,\ \cos\Theta)}{d(\cos\Theta)} = \int \frac{p_2^2\ p_3^2}{E_2\ E_3}\ dp_2\ dp_3\ \delta(\epsilon^2 - m_1^2)$$

$$= \int \frac{p_2^2\ p_3^2}{E_2\ E_3}\ dp_2\ dp_3\ \delta\ \underbrace{[(E - E_2 - E_3)^2 - p_2^2 - p_3^2 - 2p_2p_3\ \cos\Theta - m_1^2]}_{A}$$

Integration over p_2 can be done easily here because of the δ function. We calculate

$$\frac{\partial A}{\partial p_2} = -2(E - E_3)\cdot\frac{p_2}{E_2} - 2p_3\ \cos\Theta$$

for the value of p_2 for which $A = 0$. This value of p_2 can be found from formula Eq. (15) if we put $n = 3$.

The integration over p_2 therefore gives

$$\frac{dR_3(0,E,\ \cos\Theta)}{d(\cos\Theta)} = \int \frac{p_2^2\ p_3^2}{E_3}\ dp_3\ \frac{1}{2p_2(E - E_3) + 2p_3\ E_2\ \cos\Theta} \tag{17}$$

where p_2 is given by Eq. (15) as a function of p_3. The final integration over p_3 cannot be performed exactly in the general case where all three particle masses are different and all different from zero.

V EFFECTIVE MASS

a) Definition and special case

The effective mass of two particles with masses m_1 and m_2 is defined as

$$M_{12}^2 = (E_1 + E_2)^2 - (\vec{p}_1 + \vec{p}_2)^2 \tag{18}$$

We will in this Chapter discuss the effective mass of a system of particles with momentum distributions determined from phase space. To illustrate a possible straightforward method, we will first calculate the effective mass of two particles from 3-body phase space.

In the preceding Chapter we evaluated the differential momentum spectrum of a random particle in 3-body phase space

$$\frac{dR_3}{dp_3} = \frac{\pi^2 p_3^2}{E_3} \frac{\{[E^2 - 2EE_3 + m_3^2 - (m_1 + m_2)^2]\,[E^2 - 2EE_3 + m_3^2 - (m_1 - m_2)^2]\}^{1/2}}{E^2 - 2EE_3 + m_3^2} \tag{19}$$

We are interested here in the differential distributiob dR_3/dM_{12} for the effective mass of particles 1 and 2.

Introducing energy and momentum conservation in Eq. (18) one gets

$$\begin{aligned} M_{12}^2 &= (E - E_3)^2 - p_3^2 \\ &= E^2 - 2EE_3 + m_3^2 \quad . \end{aligned} \tag{20}$$

From this follows by differentiation

$$M_{12}\ dM_{12} = -E\ dE_3 = -\frac{Ep_3}{E_3}\ dp_3 \quad .$$

Now, p_3 can be found from Eq. (20) as an explicit function of M_{12}

$$p_3 = \frac{\{[E^2 - (M_{12} + m_3)^2]\,[E^2 - (M_{12} - m_3)^2]\}^{1/2}}{2E} \quad .$$

We then get

$$\frac{dR_3}{dM_{12}} = \frac{dR_3}{dp_3}\frac{dp_3}{dM_{12}} = \frac{M_{12}E_3}{E p_3}\frac{dR_3}{dp_3} \quad \text{(dropping minus sign)}$$

and finally by the use of Eq. (20) and (19)

$$\frac{dR_3}{dM_{12}} = \frac{\pi^2 \left\{ \left[M_{12}^2 - (m_1+m_2)^2\right]\left[M_{12}^2 - (m_1-m_2)^2\right]\left[E^2 - (m_3+M_{12})^2\right]\left[E^2 - (m_3-M_{12})^2\right]\right\}^{1/2}}{2E^2 M_{12}} \tag{21}$$

which is the differential effective mass distribution of two particles from 3-body phase space. The lower and upper limits of M_{12} are easily found from the limits on p_3 given in Eq. (12). We get

$$M_{12}(\min) = m_1 + m_2$$

$$M_{12}(\max) = E - m_3 \quad .$$

With reference to the next paragraph which gives a general formula for the effective mass distribution, we want to re-write Eq. (21) in a form which reveals the features of the general formula.

We put

$$\frac{dR_3}{dM_{12}} = (2M_{12})\,\frac{\pi}{M_{12}}\,\frac{\left\{\left[M_{12}^2 - (m_1+m_2)^2\right]\left[M_{12}^2 - (m_1-m_2)^2\right]\right\}^{1/2}}{2M_{12}}$$

$$\times\,\frac{\pi}{E}\,\frac{\left\{\left[E^2 - (m_3+M_{12})^2\right]\left[E^2 - (m_3-M_{12})^2\right]\right\}^{1/2}}{2E} \quad .$$

Recalling formula Eq. (8) for the 2-body phase space we see that the expression above is a product of two different R_2's, thus

$$\frac{dR_3}{dM_{12}} = (2M_{12})\; R_2\,(0, M_{12}, m_1, m_2)\; R_2\,(0, E, m_3, M_{12})$$

where the first factor R_2 is the 2-body phase space for particles of masses m_1 and m_2 and total energy M_{12}. The last factor represents a system of masses m_3 and M_{12} with total energy E.

b) General formula

The effective mass of k particles selected from n body phase space ${}^{n}_{k}M$ is given by

$$ {}^{n}_{k}M^{2} = \left(\sum_{i=1}^{k} E_i\right)^2 - \left(\sum_{i=1}^{k} \vec{p}_i\right)^2 = \left(\sum_{i=1}^{k} q_i\right)^2 \tag{22} $$

or from energy and momentum conservation

$$ {}^{n}_{k}M^{2} = \left(E - \sum_{i=k+1}^{n} E_i\right)^2 - \left(\vec{P} - \sum_{i=k+1}^{n} \vec{p}_i\right)^2 $$

$$ = \left(Q - \sum_{i=k+1}^{n} q_i\right)^2 . $$

We have chosen here the k particles to be the first k when numbering the n particles in order 1,2 ... k, k+1 .. n. This convention does not restrict the generality of our evaluation. We would now like to find an expression

$$ f(\mu^2) = \frac{dR_n}{d({}^{n}_{k}M^{2})} = \frac{d}{d({}^{n}_{k}M^{2})} R_n \, (P,E,m_1, \; \dots \, m_n) $$

where $f(\mu^2)$ is the probability that the effective mass of the first k particles has the value μ. We have explicitly written for clarity that R_n is a function of all the masses of the n particles. Using the expression for R_n from formula Eq. (7b) we can write $f(\mu^2)$ as

$$ f(\mu^2) = \int \left[\prod_{i=1}^{n} d^4 q_i \; \delta(q_i^2 - m_i^2) \right] \delta^4 \left(\sum_{i=1}^{n} q_i - Q \right) \delta({}^{n}_{k}M^{2} - \mu^2) . \tag{23} $$

The δ function $\delta({}^{n}_{k}M^{2} - \mu^2)$ makes all contributions from phase space vanish except for the cases where ${}^{n}_{k}M^{2} = \mu^2$.

We will now try to transform Eq. (23) to a form where we can make use of the earlier developed formula for R_n and its recursion relation*. This will be more convenient for practical calculations.

Since

$$\int \delta^4 \left(\sum_{i=1}^{k} q_i - {}_k^n M \right) d^4 \, {}_k^n M = 1$$

for integrations including

$$\sum_{i=1}^{k} q_i = {}_k^n M$$

we can write

$$\delta^4 \left(\sum_{i=1}^{n} q_i - Q \right) = \delta^4 \left(\sum_{i=1}^{k} q_i + \sum_{i=k+1}^{n} q_i - Q \right)$$

$$= \int \delta^4 \left(\sum_{i=1}^{k} q_i - {}_k^n M \right) \delta^4 \left({}_k^n M + \sum_{i=k+1}^{n} q_i - Q \right) d^4 \, {}_k^n M \ .$$

When we put this expression into Eq. (23) we get

$$f(\mu^2) = \int \left[\prod_{i=1}^{k} d^4 q_i \, \delta(q_i^2 - m_i^2) \right] \left[\prod_{i=k+1}^{n} d^4 q_i \, \delta(q_i^2 - m_i^2) \right] \times$$

$$\delta^4 \left(\sum_{i=1}^{k} q_i - {}_k^n M \right) \delta^4 \left({}_k^n M + \sum_{i=k+1}^{n} q_i - Q \right) \delta \left({}_k^n M^2 - \mu^2 \right) d^4 \, {}_k^n M$$

$$= \int \left\{ \left[\prod_{i=1}^{k} d^4 q_i \, \delta(q_i^2 - m_i^2) \right] \delta^4 \left(\sum_{i=1}^{k} q_i - {}_k^n M \right) \right\}$$

$$\times \left\{ \left[\prod_{i=k+1}^{n} d^4 q_i \, \delta(q_i^2 - m_i^2) \right] d^4 \, {}_k^n M \, \delta \left({}_k^n M^2 - \mu^2 \right) \delta^4 \left({}_k^n M + \sum_{i=k+1}^{n} q_i - Q \right) \right\} .$$

Since with the introduced nomenclature we can write

$$R_k(0, {}_k^n M, m_1 \ \dots \ m_k) = \int \left[\prod_{i=1}^{k} d^4 q_i \, \delta(q_i^2 - m_i^2) \right] \delta^4 \left(\sum_{i=1}^{k} q_i - {}_k^n M \right)$$

*The following use of the recursion relation to calculate the effective mass distribution was, according to the author's knowledge, first made by A. Muller and A. Verglas in an internal report (1962) at Centre d'Etudes Nucleaires de Saclay, Paris.

and

$$R_{n-k+1}(0,E,m_{k+1}\ldots m_n,\mu) = \int \prod_{i=k+1}^{n} d^4 q_i\, \delta(q_i^2 - m_i^2)\, d^4\, {}_k^nM\, \delta({}_k^nM^2 - \mu^2) \times$$

$$\delta^4({}_k^nM + \sum_{i=k+1}^{n} q_i - Q)$$

we finally get

$$f(\mu^2) = R_k(0,\mu,m_1 \ldots m_k) \cdot R_{n-k+1}(0,E,m_{k+1}\ldots m_n,\mu)\,. \qquad (24)$$

This is a general formula for the effective mass of k particles from n body phase space.

We see that $f(\mu^2)$ is a product of two functions. The first, R_k, gives the probability that the first k particles have total energy μ in their centre of mass. The second factor R_{n-k+1} is the probability that the n-k particles plus one other particle with mass μ (equal to the effective mass of the first k particles) have total energy E in the centre of mass of the original n particles. In other words, $f(\mu^2)$ expresses the probability that all n particles have energy E and simultaneously the k first particles have energy μ.

Exercise

Make use of the rule

$$\frac{dR}{d\mu^2} = \frac{1}{2\mu}\frac{dR}{d\mu}$$

and derive from Eq. (24) the formula for effective mass distribution of two particles from 3-body phase space given in Eq. (21).

Compare this result with the curve for the (=$K\pi$) system given in Chapter VIII.

c) "Shape" of effective mass distributions

The purpose of this paragraph is simply to show that all effective mass distributions (the probability to find the effective mass μ between μ and $\mu + d\mu$) may be classified into a few groups of distributions; each group has a characteristic shape determined by the values of n and k. The general appearance of each group may be used in a qualitative check of a particular calculation of an effective mass distribution. A comparison of the different shapes for different final state configurations may also give an indication of, for instance, in which configuration it will be most fruitful to look for a predicted resonance, in order to minimize the background expected from phase space.

In general the effective mass distribution has zero probability for two values of the effective mass ${}^{n}_{k}M$

$$ {}^{n}_{k}M(\min) = m_1 + m_2 + \dots + m_k $$

$$ {}^{n}_{k}M(\max) = E - (m_{k+1} + \dots + m_n) \ . $$

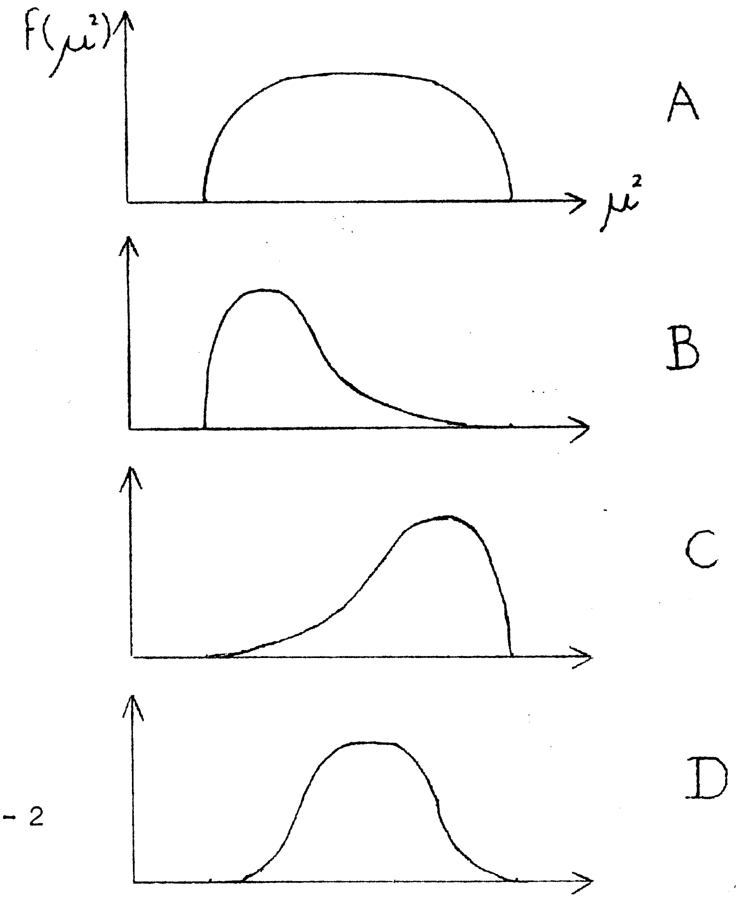
The tangents to the curve at these points (minimum and maximum) will to a large degree determine the shape of the distribution, and we have chosen to characterize the curve according to whether these tangents are horizontal or vertical. The following drawings illustrate the four different shapes one can have

n = 3 k = 2

n ≥ 4 k= 2

n ≥ 4 k = n- 1

n ≥ 5 3 ≤ k ≤ n- 2



Exercise

Show that the distribution of type A has vertical tangents at both the minimum and maximum value of ${}^{3}_{2}M$. To do this calculate $df(\mu^2)/d\mu$ from formula Eq. (21) and (24).

Show by the same method that the distribution of type B has vertical tangent at the minimum value of ${}^{n}_{2}M$.

VI EFFECT OF A RESONANCE ON THE EFFECTIVE MASS DISTRIBUTION

a) Statement of problem

We will in this Chapter consider a system where we observe in total n particles in the final state but where we also observe, or know there exists, a resonance between the k first particles so that ${}^{n}_{k}M = M^*$. For this system we ask the question: What is the effective mass distribution between any number of random particles in the final state? It is clear that the presence of a resonance between some of the particles will influence the effective mass distribution between all pairs or groups of particles in the system.

We will assume in the following discussion that only (n-k+ 1) particles, with masses $M^*, m_{k+1} \ldots m_n$, are produced in the original production process which can be well separated from the decay process $M^* \to m_1 + \ldots m_k$. This description may be meaningful in particular if the resonance has narrow width. To simplify the calculations we will also make the extreme assumption that the resonance has zero width and can be described by a δ function, $\delta({}^{n}_{k}M - M^*)$. It is clear that this assumption is in most cases quantitatively not correct, but it will, even for a broad resonance, describe in a qualitative way, the effect which the resonance has on the effective mass distribution of the particles.

Our problem can be separated into three cases according to the method of calculation of the effective mass.

(i) Calculation of effective mass of a group of particles none of which participate in the resonance. In this case we can use directly the earlier developed formulae applied to a (n- k+ 1) body final state, where one particle has mass M^*, that is

$$R_{n-k+1}(0, E, M^*, m_{k+1} \ldots m_n) .$$

(ii) Effective mass of a group of particles which all participate in the resonance. This can also be calculated from the formulae in Chapter V using a k body phase space with total energy M^*,

$$R_n(0, M^*, m_1, m_2 \ldots m_k) \; .$$

(iii) Effective mass of a group of particles, some of which participate in the resonance and some do not. This problem is more complex than (i) and (ii) and cannot in the general case be performed with the help of a recursion relation. We will in the following paragraphs discuss some very simple special cases. In these calculations we have again assumed a resonance of zero width. In principle the calculations might as well be performed with a resonance of finite width as for instance with a simple Breit Wigner shape

$$f({}^n_k M) = \text{const}\; \left[({}^n_k M - M^*)^2 + (\Gamma/2)^2 \right]^{-1} \; .$$

b) Three-body

We start with the invariant 3-body phase space formula

$$R_3 \;\alpha \int \frac{d^3 p_1}{E_1} \frac{d^3 p_2}{E_2} \frac{d^3 p_3}{E_3} \, \delta^3 (\vec{p}_1 + \vec{p}_2 + \vec{p}_3) \; \delta(E_1 + E_2 + E_3 - E) \; .$$

We want to find the distribution of effective mass of particles 1 and 2 (dR_3/dM_{12}) for the case where particles 2 and 3 are already in a resonant state. We will first assume the resonance has zero width and mass M^*. In that case we can write

$$\frac{dP_3}{dM_{12}} = \frac{d}{dM_{12}} \int \frac{d^3 p_1}{E_1} \frac{d^3 p_2}{E_2} \frac{d^3 p_3}{E_3} \, \delta(p) \; \delta(E) \; \delta(M_{23} - M^*) \; .$$

Integration over p_3 gives

$$\frac{dR_3}{dM_{12}} = \frac{d}{dM_{12}} \int \frac{d^3 p_1}{E_1} \frac{d^3 p_2}{E_2} \frac{1}{E_3 (p_1 p_2)} \, \delta(E_1 + E_2 + E_3 (p_1 p_2) - E) \; \delta(M_{23} - M^*)$$

Integration over angles gives (as demonstrated several times earlier)

$$\frac{dR_3}{dM_{12}} = \frac{d}{dM_{12}} \int \frac{p_1\,dp_1}{E_1}\,\frac{p_2\,dp_2}{E_2}\,\frac{p_1 p_2\,d(\cos\Theta)}{E_3(p_1 p_2)}\,\delta(E)\,\delta(M) \;.$$

Now for p_1 and p_2 constant we have from momentum conservation

$$p_3\,dp_3 = p_1\,p_2\,d(\cos\Theta)$$

and since

$$p_3\,dp_3 = E_3\,d\,E_3$$

we get by substitution and integration over E_3

$$\frac{dR_3}{dM_{12}} = \frac{d}{dM_{12}} \int d\,E_1\,d\,E_2\;\delta(M_{23} - M^*)$$

where momentum and energy conservation require that

$$E_2 = E - E_1 - E_3 = E - E_1 - \frac{E^2 - M_{12}^2 + m_3^2}{2E} = \frac{E^2 + M_{12}^2 - m_3^2}{2E} - E_1 \;.$$

For constant E_1 it follows that

$$dE_2 = \frac{M_{12}}{E}\,dM_{12} \;.$$

Substituting into the expression above and recalling that

$$M_{23} = \sqrt{E^2 + m_1^2 - 2E\,E_1}$$

we get

$$\frac{dR_3}{dM_{12}} = \int dE_1\,\frac{M_{12}}{E}\,\delta(\sqrt{E^2 + m_1^2 - 2EE_1} - M^*) \;.$$

Integration over E_1 gives

$$\frac{dR_3}{dM_{12}} = \frac{M_{12}}{E}\;\frac{\sqrt{E^2 + m_1^2 - 2E\,E_1}}{E}$$

for $\qquad E_1 = \dfrac{E^2 - M^{*2} + m_1^2}{2E} \;.$

Finally

$$\frac{dR}{dM_{12}} \propto \frac{M_{12}}{E} \cdot \frac{M^*}{E} \; . \qquad (25)$$

We need the maximum and minimum values of M_{12}. Due to the resonance $M_{23} = M^*$ particle one always has a fixed energy

$$E_1 = \frac{E^2 + m_1^2 - M^{*2}}{2E} \; .$$

We see that

$$M_{12}^2 = m_1^2 + m_2^2 + 2E_1 E_2 - 2p_1 p_2 \cos\Theta$$

is a minimum for $\cos\Theta = 1$. To find the value of E_2 for which this takes place we put

$$M_{12} \frac{\partial M_{12}}{\partial E_2} = E_1 - p_1 \cdot \frac{E_2}{p_2} = 0$$

and find

$$p_2 = \frac{m_2}{m_1} p_1$$

which gives

$$M_{12}^2(\min) = m_1^2 + m_2^2 + 2\frac{m_2}{m_1}(E_1^2 - p_1^2)$$

$$= (m_1 + m_2)^2 \; .$$

Similarly for $\cos\Theta = -1$ we find

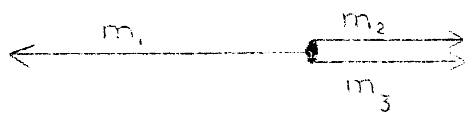
$$M_{12}^2(\max) = m_1^2 + m_2^2 + 2\frac{m_2}{m_1}(E_1^2 + p_1^2)$$

$$= m_1^2 + m_2^2 - 2m_1 m_2 + 4\frac{m_2}{m_1} E_1^2$$

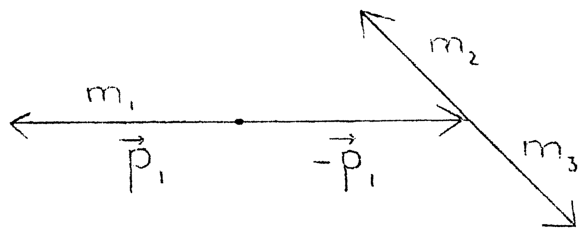
$$= (m_1 - m_2)^2 + \frac{m_2}{m_1} \frac{(E^2 + m_1^2 - M^{*2})^2}{E^2} \; .$$

Exercise

A 3-body final state with a resonance between two of the particles ($M_{23} = M^*$) can be thought of as an original 2-body decay



followed by a decay of the resonance



into two particles which in the M^* centre of mass are emitted with fixed energy. Make a Lorentz transformation of particle 1 to the M^* centre of mass, and from the fact that m_2 is emitted isotropically in this system, derive the effective mass distribution of M_{12} given in formula Eq. (25).

c) Four-body

We want to find the distribution dR_4/dM_{12} for the case where we have a resonance between particles 2 and 3. Assuming again that the resonance $M_{23} = M^*$ has zero width, we can write

$$\frac{dR_4}{dM_{12}} \propto \frac{d}{dM_{12}} \int \frac{d^3p_1}{E_1} \frac{d^3p_2}{E_2} \frac{d^3p_3}{E_3} \frac{d^3p_4}{E_4} \delta^3(\vec{p}_1 + \vec{p}_2 + \vec{p}_3 + \vec{p}_4)$$

$$\times \delta(E_1 + E_2 + E_3 + E_4 - E) \; \delta(M_{23} - M^*) \; .$$

Integration over p_4 gives

$$\frac{dR_4}{dM_{12}} = \frac{d}{dM_{12}} \int \frac{d^3p_1}{E_1} \frac{d^3p_2}{E_2} \frac{d^3p_3}{E_3} \frac{1}{E_4} \delta\left[E_1 + E_2 + E_3 + E_4(p_1\, p_2 p_3) - E\right] \delta(M_{23} - M^*)$$

for momentum balance.

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We introduce polar coordinates as illustrated in Fig. 2.

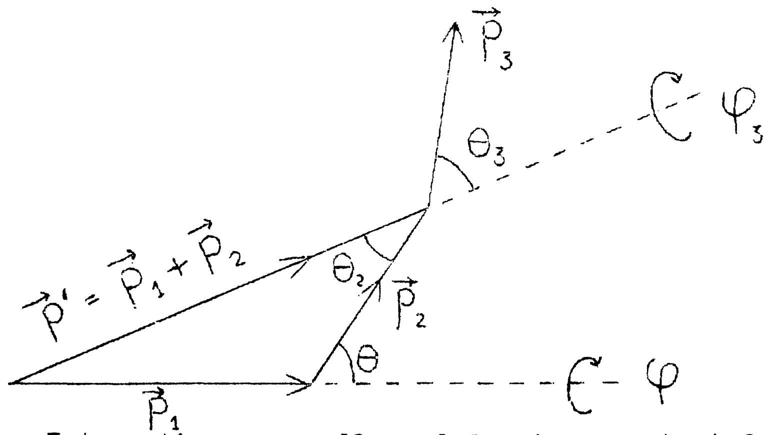


Fig. 2

Integrations over $d\Omega_1$ and $d\varphi$ give constant factors only; therefore

$$\frac{dR_4}{dM_{12}} \propto \frac{d}{dM_{12}} \int \frac{p_1^2 dp_1}{E_1} \frac{p_2^2 dp_2 \; d(\cos\Theta)}{E_2} \frac{p_3^2 dp_3 \; d(\cos\Theta_3)}{E_3} \frac{d\Phi_3 \quad 1}{E_4 \, (p_1 \, p_2 \, p_3)}$$

$$\times \; \delta(E_1 + E_2 + E_3 + E_4 - E) \; \delta(M_{23} - M^*) \; .$$

From momentum conservation we have

$$p_4^2 = (\vec{p}_1 + \vec{p}_2 + \vec{p}_3)^2 = p'^2 + p_3^2 + 2p' \, p_3 \cos\Theta_3$$

so that for constant p', p_3 and $\cos\Theta$

$$p_4 \; dp_4 = E_4 \; dE_4 = p' p_3 \; d(\cos\Theta_3) \; .$$

Substituting and integrating over E_4 one gets

$$\frac{dR_4}{dM_{12}} = \frac{d}{dM_{12}} \int \frac{p_1^2 dp_1}{E_1} \frac{p_2^2 dp_2 \; d(\cos\Theta)}{E_2} \frac{dE_3 \, d\Phi_3}{p'} \, \delta(M_{23} - M^*)$$

for $E_4 = E - E_1 - E_2 - E_3$ (energy conservation). If we express E_4 by p_4 from the expression above we get

$$p'^2 + p_3^2 + 2p_3 p' \cos\Theta_3 = (E - E_1 - E_2 - E_3)^2 - m_4^2$$

or when solved with respect to $\cos\Theta_3$

$$\cos\Theta_3 = \frac{(E - E_1 - E_2 - E_3)^2 - m_4^2 - p'^2 - p_3^2}{2p' p_3} . \qquad (26)$$

Formula Eq. (26) will be used later to find the integration limits of E_3. Since

$$M_{12}^2 = (E_1 + E_2)^2 - (\vec{p}_1 + \vec{p}_2)^2$$

$$= m_1^2 + m_2^2 + 2E_1 E_2 - 2p_1 p_2 \cos\Theta$$

we can for constant p_1 and p_2 put

$$M_{12}\, dM_{12} = p_1 p_2 d(\cos\Theta) .$$

If we also put $dp_1 = \frac{E_1}{p_1} dE_1$ etc. we get

$$\frac{dR_4}{dM_{12}} = M_{12} \int \frac{dE_1\ dE_2\ dE_3\ d\Phi_3}{p'} \delta(M_{23} - M^*) .$$

Interpreting the argument of the δ function as a function of Φ_3 we get by performing the integration over this variable

$$\frac{dR_4}{dM_{12}} = M_{12} \int \frac{dE_1\ dE_2\ dE_3}{p'} \frac{1}{\left|\frac{\partial(M_{23}-M^*)}{\partial\Phi_3}\right|_{(M_{23}-M^*) = 0}}$$

Here M_{23} is given by

$$M_{23}^2 = (E_2 + E_3)^2 - (\vec{p}_2 + \vec{p}_3)^2 = m_2^2 + m_3^2 + 2E_2 E_3 - 2p_2 p_3 \cos\Theta_{23}$$

where Θ_{23} (the angle in space between $\vec{p}_2$ and $\vec{p}_3$) is a function of Θ_2, Θ_3 and Φ_3. From spherical geometry

$$\cos\Theta_{23} = \cos\Theta_2 \cos\Theta_3 + \sin\Theta_2 \sin\Theta_3 \cos\Phi_3$$

Θ_3 is given by Eq. (26) and Θ_2 can be found from

$$\cos\Theta_2 = \frac{\vec{p}\,'\,\vec{p}_2}{p'\,p_2} = \frac{(\vec{p}_1+\vec{p}_2)\,\vec{p}_2}{p'\,p_2} = \frac{p_1\,p_2\cos\Theta + p_2^2}{p'\,p_2} \qquad (27)$$

$$= \frac{m_1^2 + m_2^2 - M_{12}^2 + 2E_1\,E_2 + 2p_2^2}{2p'\,p_2} \; .$$

By proper substitution it is therefore possible to express explicitly M_{23} as a function of Φ_3 and determine $\partial(M_{23}-M^*)/\partial\Phi_3$. This expression should then be evaluated for the Φ_3 that makes $M_{23} - M^* = 0$. We find

$$\cos\Phi_3 = \frac{m_2^2 + m_3^2 + 2E_2\,E_3 - 2p_2\,p_3\cos\Theta_2\cos\Theta_3 - M^{*2}}{2p_2\,p_3\sin\Theta_2\sin\Theta_3} \; .$$

Recalling that Θ_3 and Θ_2 by Eq. (26) and (27) are expressed as functions of E_1, E_2, E_3 and M_{12}, we see that we are left with a triple integral in E_1, E_2 and E_3. This integral has to be solved numerically. The integration limits can be found as follows. For fixed E_1 and E_2 are $E_3(\min)$ and $E_3(\max)$ determined by $M_{23} = M^*$ for $\cos\Theta_{23} = -1$ and $\cos\Theta_{23} = +1$ respectively, together with the possible restriction that from Eq. (26) $-1 \leq \cos\Theta_3 \leq +1$. Similarly the limits for E_2 with fixed E_1 are determined from $M_{12} = m_1^2 + m_2^2 + 2E_1\,E_2 - 2p_1\,p_2\cos\Theta$ by putting $\cos\Theta = \pm 1$ if for these values Eq. (27) satisfies $-1 \leq \cos\Theta_2 \leq +1$.

The lower and upper limits of E_1 are given by

$$E_1(\min) \geq m_1$$

and

$$E_1(\max) \leq \frac{\{[E^2 - (M^* + m_4 + m_1)^2]\,[E^2 - (M^* + m_4 - m_1)^2]\}^{1/2}}{2E} \; .$$

VII DALITZ PLOT

a) Contours of the Dalitz plot

We will discuss here in some more detail the 3-body phase space, and especially the properties of the Dalitz representation.

First we take the general case of three particles with different rest masses m_1, m_2 and m_3. In their centre of mass the particles are emitted with kinetic energies T_1, T_2 and T_3. The Dalitz representation is a scatter plot of the kinetic energies of any two of the particles, say T_1 and T_2, along the x and y axes of a Cartesian coordinate system. The kinematical limits of the reaction imposed by energy and momentum conservation will now confine the points to the area within a closed curve which touches the two axes, see Fig. 3.

The kinematical constraints are

$$E_3 = E - (E_1 + E_2) \qquad (E_1 = T_1 + m_1 \text{ etc.})$$
$$p_3^2 = p_1^2 + p_2^2 + 2p_1 p_2 \cos \Theta_{12} \quad (\Theta_{12} \text{ angle between } \vec{p}_1 \text{ and } \vec{p}_2) . \tag{28}$$

It is clear that for given T_1 and T_2 (or equivalent p_1 and p_2) we have also a fixed T_3. From the second equation (28) we then have determined Θ_{12}. This means that for given T_1 and T_2 we have a uniquely specified situation.

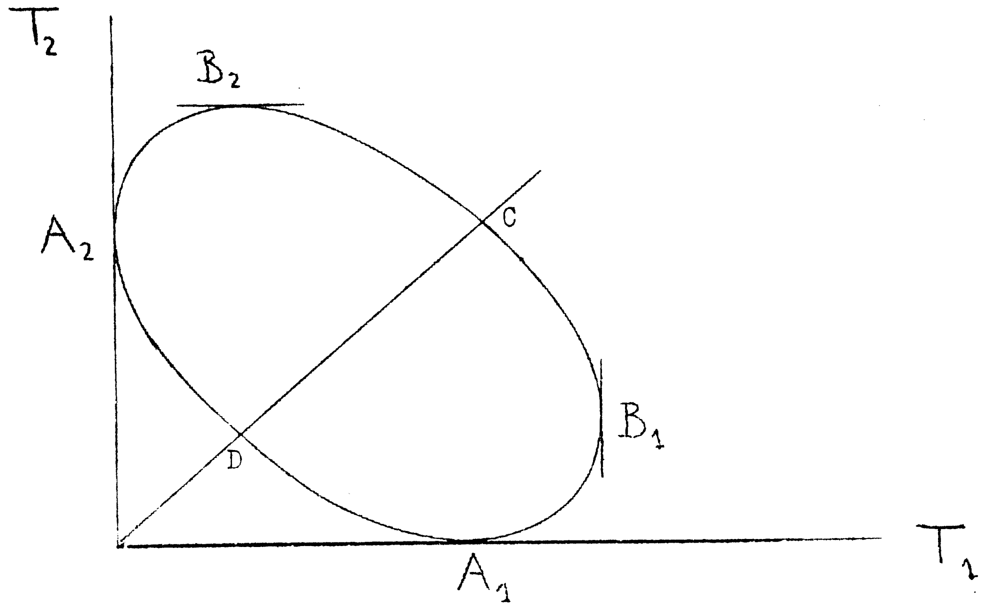


Fig. 3

In other words, each point on the Dalitz plot corresponds to a well-defined configuration in the final state.

The contour of the Dalitz plot represents the configurations where particles 1 and 2 are emitted parallel or antiparallel, that is for

$$\cos\Theta = a = \pm 1$$

where a = +1 for $\vec{p}_1$ and $\vec{p}_2$ parallel and a = -1 for $\vec{p}_1$ and $\vec{p}_2$ antiparallel, that is at the lower (A_1 D A_2) and upper (A_1 B_1 B_2 A_2) half of the contour of the Dalitz plot respectively.

From equation (28) above we can eliminate E_3 and P_3 and get

$$T_1 = \frac{\nu + a\sqrt{\nu^2 - uw}}{u} \qquad (29)$$

where

$$u = B^2 - 2ET_2$$
$$\nu = BC - (AB + C - 2m_1 m_2)\,T_2 + ET_2^2$$
$$w = (C - AT_2)^2$$
$$A = E - m_1$$
$$B = E - m_2$$
$$C = \tfrac{1}{2}\left[(E - m_1 - m_2)^2 - m_3^2\right]$$

Formula Eq. (29) gives the general relation between T_1 and T_2 along the contour of the Dalitz plot. We will look in more detail at some special cases along the contour and see what they mean physically.

First let the curve touch the T_1 and T_2 axes at points A_1 and A_2 respectively (Fig. 3). Point A_1 corresponds to the case where particle 2 has zero momentum, that is $\vec{p}_1 = -\vec{p}_3$. The value of T_1 at A is

$$T_1(A) = \frac{(E - m_1 - m_2)^2 - m_3^2}{2(E - m_2)} .$$

Similarly point A_2 corresponds to the case where particle 1 has zero momentum and $\vec{p}_2 = -\vec{p}_3$. $T_2(A)$ can be found by interchanging particles 1 and 2 above.

The straight line in Fig. 3 represents states where $|\vec{p}_1| = |\vec{p}_2|$. This line crosses the contour at points C and D. Point C corresponds to the state where $p_3 = 0$ and a value of T_1 given by

$$T_1(C) = \frac{(E - m_3 - m_1)^2 - m_2^2}{2(E - m_3)} .$$

We denote the maximum values of T_1 and T_2 by B_1 and B_2 respectively. Point B_1 corresponds to the situation where particles 2 and 3 are emitted parallel and with equal velocity, that is

$$T_1(\max) = \sqrt{m_1^2 + p_1^2(\max)}$$

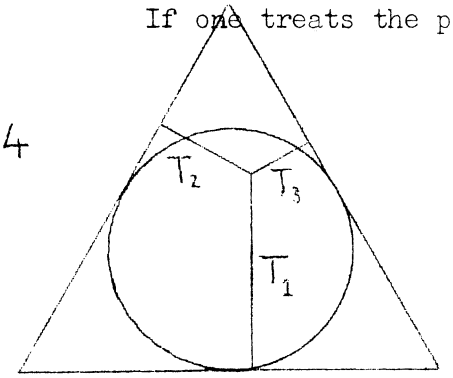
where $p_1(\max)$ is given by equation (12) in Chapter III.d.

Point B_2 is found in the same way by interchanging particles 1 and 2 above, and corresponds to the case where particles 2 and 3 are emitted parallel and with equal velocity but opposite to particle 1.

In the case where the three particles have equal masses as in the decay of the τ meson, the ω resonance etc. another coordinate system is mostly used. This representation is based on the fact, that from any point inside an equilateral triangle (Fig. 4) the sum of the distances to the sides is equal to the height of the triangle. One therefore plots the kinetic energies of the particles along the normals to the sides; then $T_1 + T_2 + T_3$ = Q value = height of triangle. Not all points inside the triangle are available because of momentum conservation.

If one treats the pions non-relativistically, one sees easily that the points have to be within the inscribed circle of the triangle. For more details about this special kind of Dalitz plot we refer to the original work of R.H. Dalitz (Proc. Phys. Soc. A69, 527 (1956), and Reports in Prog. in Phys. 20, 163 (1957)). For the relativistic case, see Fabri, Nuovo Cimento 11, 479 (1954).

Fig. 4



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Exercise

Place a Cartesian coordinate system with origin in the centre of the inscribed circle in Fig. 4 with y axis along T_1.

Show that

$$x = \frac{T_2 - T_3}{\sqrt{3}}$$

$$y = T_3 - \frac{Q}{3} \text{ (Q value = height of triangle)}$$

and prove that for non-relativistic particles (of equal mass) the constraint equations

$$T_1 + T_2 + T_3 = Q \quad \text{(Energy conservation)}$$

$$\begin{cases} \vec{p}_1 + \vec{p}_2 + \vec{p}_3 = 0 \\ \cos \Theta_{12} = \pm 1 \end{cases} \quad \text{(Momentum conservation for collinear particles)}$$

lead to

$$x^2 + y^2 = \left(\frac{Q}{3}\right)^2$$

which is the equation for the inscribed circle of the triangle.

Exercise

A resonance between particles m_1 and m_2 will give points clustering along a 45° line crossing the T_1 and T_2 axes on a T_1 T_2 plot. Derive the equation for this line for a given resonance mass $M_{12} = M^*$.

b) Distribution of points on Dalitz plot

We will show here one of the special advantages of the Dalitz representation, namely: equal areas on the Dalitz plot correspond to equal probabilities in the Lorentz invariant phase space. In other words, phase space predicts uniform population of points on a Dalitz plot.

The importance of this fact appears in practice when one plots the kinetic energies (for 3-body states) on a Dalitz plot to see if the points are equally distributed throughout the plot. If the points are clustered in certain regions on the plot or along certain lines, this indicates that (apart from experimental bias and statistical fluctuation) some final state interaction has affected the distribution. The density of points is proportional to the square of the invariant matrix element of the reaction.

We will now prove that phase space predicts uniform population of points. From formula Eq. (7) we get

$$R_3 = \int \frac{d^3 p_1}{2E_1} \frac{d^3 p_2}{2E_2} \frac{d^3 p_3}{2E_3} \delta(E_1 + E_2 + E_3 - E)\ \delta^3(\vec{p}_1 + \vec{p}_2 + \vec{p}_3)$$

which by integration over p_3 gives

$$R_3 = \int \frac{1}{8E_1\ E_2\ E_3} d\Omega_1\ p_1^2\ dp_1\ d\Omega_2\ p_2^2\ dp_2\ \delta(E_1 + E_2 + E_3 - E) .$$

Integrations over space angles other than Θ_{12} gives

$$R_3 = \int \frac{1}{8E_1\ E_2\ E_3} 4\pi\ p_1^2\ dp_1\ 2\pi\ d(\cos\Theta_{12}) p_2^2\ dp_2\ \delta(E_1 + E_2 + E_3 - E) .$$

Now, the space angle between $\vec{p}_1$ and $\vec{p}_2$ is for fixed p_1 and p_2 determined from momentum conservation

$$p_3^2 = p_1^2 + p_2^2 + 2p_1\ p_2\ \cos\Theta_{12}$$

so that

$$p_3\ dp_3 = p_1\ p_2\ d(\cos\Theta_{12}) .$$

Further: from

$$p^2 = E^2 - m^2$$

we find

$$p\ dp = E\,dE = E\ dT$$

which when substituted into the expression for R_3 above, gives

$$R_3 \propto \int dT_1\ dT_2\ dT_3\ \delta(E_1 + E_2 + E_3 - E) .$$

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Integration over T_3 finally gives

$$R_3 \propto \int dT_1 \; dT_2$$

which expresses the fact that the density of final states is proportional to the area in a T_1 T_2 plot.

Exercise

Show that for non-relativistic particles, that is for

$$T = \frac{p^2}{2m} ,$$

it is the non-invariant phase space ρ and not the invariant phase space R which is proportional to the area in a T_1 T_2 plot.

c) Effective mass plot

We showed in the preceding paragraph that equal areas on a Dalitz plot correspond to equal probabilities in phase space. We will now introduce another much used form of Dalitz plot, namely one where the effective mass squared of any two particles from 3-body final state is plotted along the x and y axes in a Cartesian coordinate system*.

We will now show that phase space predicts a uniform population of points on this plot as well.

We have

$$\begin{aligned} M_{12}^2 &= (E_1 + E_2)^2 - (\vec{p}_1 + \vec{p}_2)^2 \\ &= (E - E_3)^2 - p_3^2 \\ &= E^2 + m_3^2 - 2EE_3 \\ &= E^2 + m_3^2 - 2Em_3 - 2ET_3 \end{aligned}$$

* A useful representation for the masses of a pair of particles from a 4-body final state has recently been given by Goldhaber et al. in Phys. Rev. Letters 9, 330 (1962) and Physics Letters 6, 62 (1963).

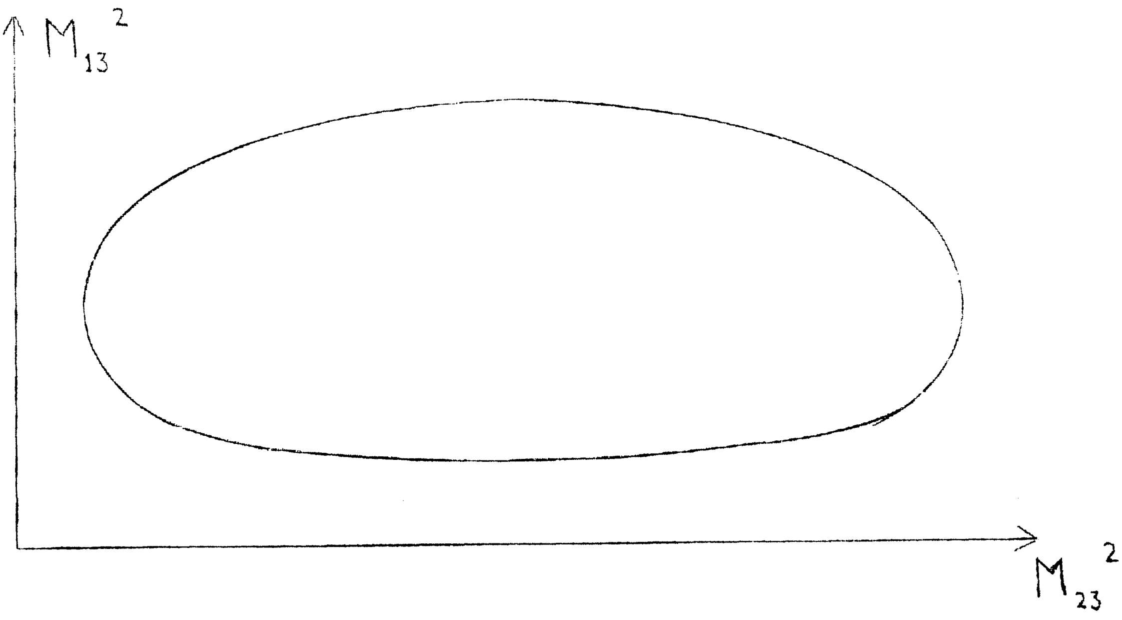

Fig. 5

We see that M_{12}^2 is a linear function of T_3, which by differentiation gives

$$d(M_{12}^2) \propto dT_3 .$$

This means that

$$d(M_{23}^2)\ d(M_{13}^2) \propto dT_1\ dT_2 \quad ,$$

or put in words: Areas on the M_{23}^2 M_{13}^2 plot are proportional to areas on the T_1 T_2 plot. From this follows that equal areas on the M^2 plot correspond to equal probabilities in phase space.

We may note that the minimum and maximum values of the invariant masses are given by

$$M_{12}(\text{min}) = m_1 + m_2$$

$$M_{12}(\text{max}) = E - m_3 \quad .$$

Similarly for M_{13}, by interchanging particle 2 and 3. The contours on the M^2 plot can in general be found by using the relation between T_1 and T_2 given by equation (29). For each value of T_2 we have $p_2 = \sqrt{T_2^2 + 2m_2 T_2}$ and

$M^2_{13} = (E - E_2)^2 - p^2_2$. From Eq. (29) we find the corresponding T_1 and can then calculate

$$M^2_{12} = (E_1 + E_2)^2 - (p^2_1 + p^2_2 \pm 2p_1\ p_2) \ .$$

Exercise

Show that if a resonance $M_{13} = M^*$ decays isotropically in space with respect to particle 2 (that is $dR/d \cos \Theta_{12}$ = constant) the points will be evenly distributed along the M^2_{12} axes on an M^2_{12} M^2_{13} plot.

We see also that formula Eq. (25)

$$\frac{dR}{dM_{12}} \propto M_{12} \text{ for } M_{13} = M^*$$

can be directly verified from the fact that the points are evenly distributed along the M_{12} axis on a M^2_{12} M^2_{13} plot.

Show that the three possible effective mass combinations are related to each other by the equation

$$M^2_{12} + M^2_{13} + M^2_{23} = E^2 + m^2_1 + m^2_2 + m^2_3 \ .$$

d) Effects of angular momentum conservation

We have stated in the preceding paragraphs that phase space predicts uniform population of points on a Dalitz plot. This statement is true, however, only to the extent to which one can neglect the influence of constraint equations imposed by angular momentum conservation. In production processes, for example, where one has a Q value high enough to expect some contribution from production amplitudes with angular momentum states greater than zero, one might expect these amplitudes to contribute differently to different final states, since not all final states otherwise allowed by phase space, will conserve angular momentum. The areas on the Dalitz plot corresponding to such states will then be depopulated. Since one expects the contribution from higher partial waves to increase with the Q value of the production process, the relative depopulation of points on the Dalitz plot will probably vary with the Q value in the production process. This makes it important in the study of particle resonances that the presence of an apparent "bump" on the phase space plot for one particular incident momentum is verified for other incident particle momenta.

The effects of angular momentum conservation in connection with the production of the Y_1^* (1385) in the reaction $K_2^0 + p \rightarrow \Lambda + \pi^+ + \pi^0$, has been discussed by R.K. Adair in Rev.Mod.Phys. 33, 406 (1961). A detailed quantitative estimate of the effect depends on the type of particles involved in the reaction, such as the values of the particle spins, isospins etc. We will give here a simple qualitative description of the angular momentum effect with reference to the Dalitz plot for particles of different masses.

The configuration of three particles in their centre of mass system can be specified in terms of two momenta; $\vec{p}$ the momentum of say, the third particle in the three particle rest system, and $\vec{q}$ the momentum of particle one or two in the centre of mass system of these two particles (see Fig. 6).

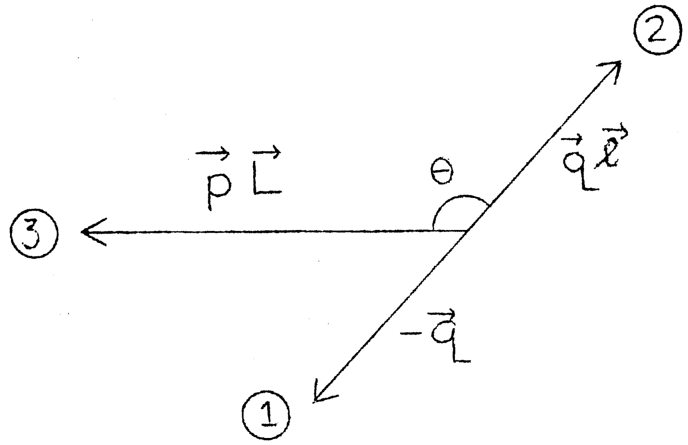

Fig. 6

Using this description of the three particle state we can express the total angular momentum ($\vec{J}$) of the three particles as the vector sum of two independent angular momenta $\vec{\ell}$ and $\vec{L}$ defined as follows: Particles 1 and 2 revolve about their mutual centre of mass with orbital angular momentum $\vec{\ell}$, while particles 1 and 2 together with particle 3 revolve around the 3-body centre of mass with orbital angular momentum $\vec{L}$. The total angular momentum of the 3-body system is then $\vec{J} = \vec{\ell} + \vec{L}$. For simplicity we will in the following assume all particles in initial and final states to have spin zero.

For a given total energy of the system, the total production amplitude (A) of a particular angular momentum state $\vec{J}$ can be expressed as a sum of partial amplitudes of different $\vec{\ell}$ and $\vec{L}$.

$$A = \Sigma\, a_{J\ell L} .$$

The summation is to be extended over all possible values of ℓ and L satisfying $\vec{\ell} + \vec{L} = \vec{J}$. The complex partial production amplitudes, a, are functions of the individual particle momenta, but are already integrated over space angles i.e. the total production intensity is given by $|A|^2 = \Sigma\, |a_{J\ell L}|^2$. The intensity of a specific partial wave can be expressed as

$$|a_{J\ell L}|^2 = K\, R_3(0,E)\, P_L P_\ell$$

where K contains the matrix element and is a function of the same variables as a; $R_3(0,E)$ is the invariant 3-body phase space integral, and P_L and P_ℓ are the angular momentum barrier factors for the orbital angular momentum of particle 3, and the two particle system 1 and 2 respectively. (The labelling of the particles as 1,2 or 3 is of course in arbitrary order and the labelling should be permuted in the summation above for the total production amplitude).

P_L and P_ℓ can be calculated for different values of linear momenta and orbital angular momentum of the system. For a simple qualitative estimate it suffices to remember that for fixed angular momentum J, P_J decreases with decreasing linear momentum of the particle so that for $J > 0$ is $P_J = 0$ for zero momentum. For fixed momentum, P_J decreases with increasing values of angular momentum. From this follows that contributions from partial waves with L or $\ell > 0$ diminish whenever one of the particle momenta is zero or whenever two particles touch in momentum space, e.g. have zero relative momentum. The points in phase space corresponding to these configurations can easily be found on the Dalitz plot.

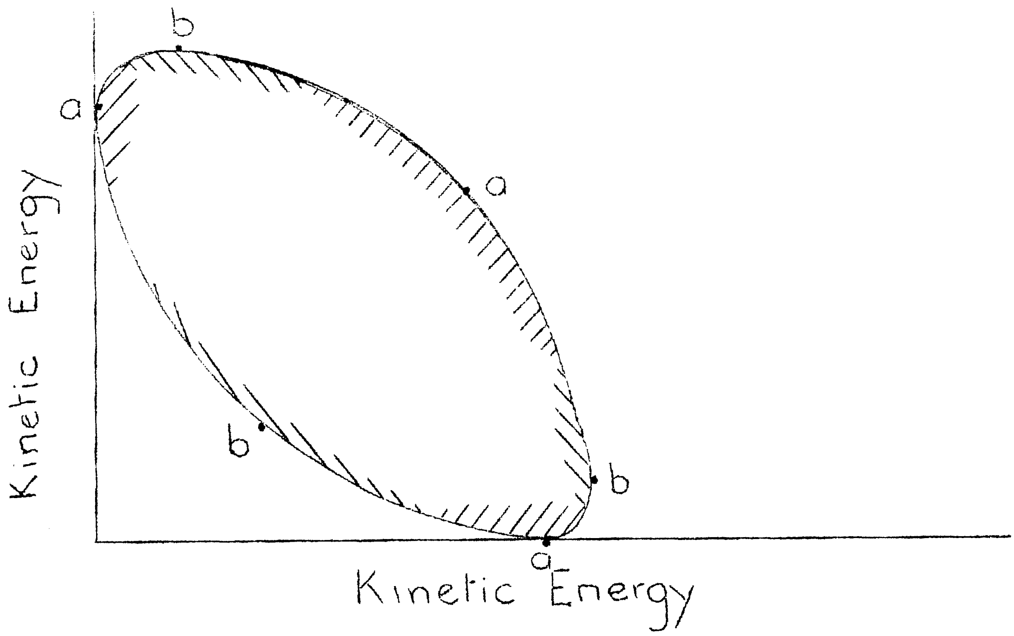

Fig. 7

With reference to Fig. 7 we schematically indicate the physical interpretation of a few special points on the Dalitz plot. We have not indicated the kinetic energy of which particle is plotted along the axes on Fig. 7, since the choice is arbitrary.

Points a represent states where one of the particles has zero momentum.

Points b represent states where two particles have equal momenta.

It follows therefore that partial amplitudes where L > 0, which we indicate by $a_{J\ell L>0}$, give no contribution around the points marked a, and partial amplitudes with $\ell > 0$ $(a_{J\ell>0 L})$ are zero around points b in Fig. 7. The size of the region which in this way will be depopulated, obviously depends on the absolute magnitude of the angular momentum vectors $\vec{\ell}$ and $\vec{L}$. For instance around points a we expect P_L to be small for

$$|\vec{p}| < \frac{L \cdot \hbar}{r}$$

where r is an effective radius, which we can take as the Compton wavelength of the pion. We find that partial amplitudes with L=1 are strongly reduced for $|\vec{p}| < 150$ MeV/c.

Similarly we expect p_ℓ to be small around the points b for $|\vec{q}| < 150 \cdot \ell$ MeV/c. Thus, to summarize, in the presence of angular momentum states J ≥ 1 one expects some depletion of events in the region of the points a and b in Fig. 7. These areas have been shaded to give a qualitative illustration of the effect.

So far we have seen that the requirement of angular momentum conservation gives rise to a centrifugal barrier effect which tends to depopulate the Dalitz plot in certain regions. The arguments are valid irrespective of the direction in space of the angular momentum vector.

We will now consider another effect of angular momentum conservation, mainly pertinent to production processes. This effect arises from the fact that the orbital angular momentum vector in the initial state is not arbitrarily distributed in space but confined to a plane perpendicular to the direction of the incoming particle. Even though the Dalitz plot representation does not contain any information regarding the orientation of the particles in final state with respect to the beam direction or production plane, we nevertheless find that the requirement of the angular momentum vector perpendicular to the beam will influence the distribution of the points on the Dalitz plot.

A convenient description of the angular distribution of the particles in the three particle system (see Fig. 6) is in terms of the distribution function dN/d cos Θ where the angle Θ is given by

$$\cos \Theta = \frac{\vec{p}\cdot\vec{q}}{pq} .$$

One particular advantage of this description is the following. Equal intervals along lines parallel to the axes on the Dalitz plot correspond to equal intervals of cos Θ. This means that if $\vec{q}$ is isotropically distributed in space with respect of $\vec{p}$ the Dalitz plot will be evenly populated with points along lines parallel to the axes and along lines at 45° with respect to the axes. (This can easily be proved by performing a Lorentz transformation of $\vec{q}$ and Θ to the three particle centre of mass system. (Cf. also exercise in Chapter VII.c).

Since $\vec{\ell}$ and $\vec{L}$ are randomly orientated in planes perpendicular to $\vec{q}$ and $\vec{p}$ respectively, an isotropic distribution of $\vec{q}$ with respect to $\vec{p}$ results in a uniform distribution of the space angle between $\vec{\ell}$ and $\vec{L}$. We will now show that the restriction on the direction (given by the vector sum of $\vec{J} = \vec{\ell} + \vec{L}$ perpendicular to the beam) leads to a non-uniform distribution in space between $\vec{q}$ and $\vec{p}$.

We introduce a Cartesian coordinate system with z axis in the beam direction and assume the physical system to exhibit rotational symmetry about this direction. With no loss of generality the y axis can therefore be taken in the direction of the total angular momentum $\vec{J}$. Angular momentum conservation now requires $\vec{L}$ and $\vec{\ell}$ to lie in a plane through the y axis. For all combinations of $\vec{L}$ and $\vec{\ell}$ satisfying

$$\vec{J} = \vec{L} + \vec{\ell}$$

obviously

$$j_z = L_z + \ell_z = 0 \ .$$

All plane angles β between $\vec{L}$ and $\vec{\ell}$ are equally probable. We will now seek the angular distribution between $\vec{p}$ and $\vec{q}$ for a fixed and arbitrary value of β, that is, we want the distribution function $dN(\beta)/d \cos \Theta$. We recall that $\vec{p}$ lies randomly in a plane (S in Fig. 8) perpendicular to $\vec{L}$ and $\vec{q}$ randomly in a plane (T) perpendicular to $\vec{\ell}$. We arbitrarily fix $\vec{p}$ to be perpendicular to the intersection line between the two planes. The direction of $\vec{q}$ is given by its angle γ with a plane containing $\vec{p}$ and $\vec{L}$, see Fig. 8. It then follows that the angle Θ in space between $\vec{q}$ and $\vec{p}$ is given by

$$\cos \Theta = \cos \gamma \cos (\pi - \beta)$$

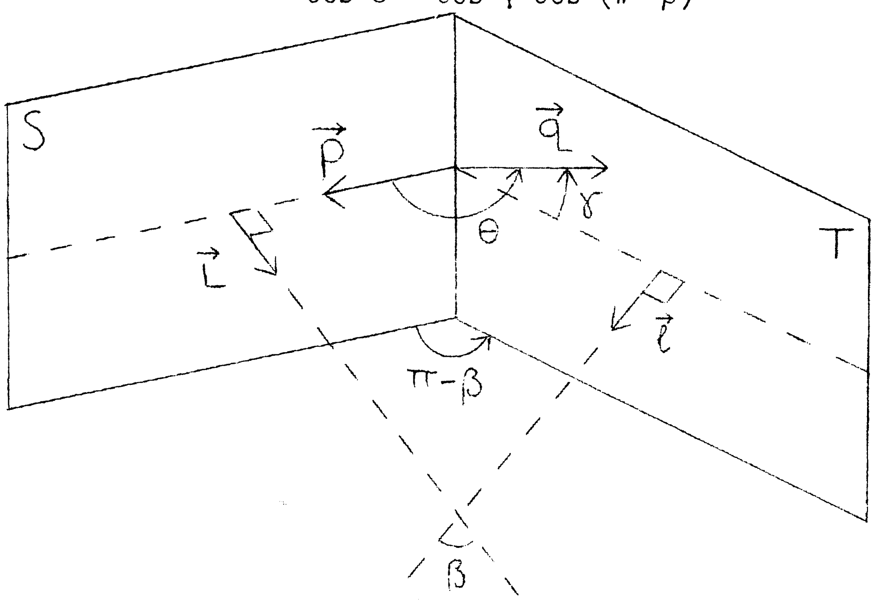

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Fig. 8

Since $\vec{q}$ can have any direction in T, $\frac{dN(\beta)}{d\gamma}$ = const, or equivalently

$$\frac{dN(\beta)}{d \cos \gamma} \propto \frac{1}{\sin \gamma}$$

we find

$$\frac{dN(\beta)}{d \cos \Theta} = \frac{dN(\beta)}{d \cos \gamma} \cdot \frac{d \cos \gamma}{d \cos \Theta} \propto \frac{1}{\sin \gamma} \cdot \frac{1}{\cos (\pi-\beta)} \quad .$$

Eliminating γ we get the distribution function for fixed β

$$\frac{dN(\beta)}{d \cos \Theta} \propto (\cos^2 \beta - \cos^2 \Theta)^{-1/2} \quad .$$

Finally, since all plane angles β have equal weights we find by integration over β

$$\frac{dN}{d \cos \Theta} \propto \int_0^{+\Theta} (\cos^2 \beta - \cos^2 \Theta)^{-1/2} \, d\beta \quad .$$

For numerical calculation this expression can be given a more convenient form on substituting

$$\sin \Theta = k \qquad \sin \beta = kt \ .$$

We then get a standard form of an elliptical integral of the first kind

$$\frac{dN}{d \cos \Theta} = f(k) \propto \int_0^1 \frac{dt}{\sqrt{1-t^2}\,\sqrt{1-k^2 t^2}}$$

which is tabulated in, for instance, Jahnke and Emde "Tables of Functions".

The distribution function is shown in Fig. 9.

We also see from Fig. 9 that dN/d cos Θ, which is symmetrical about cos Θ = 0, is strongly peaked forwards and backwards. This results in an uneven population of points on the Dalitz plot, i.e. there will be more points concentrated near the boundary than in the middle of the plot.

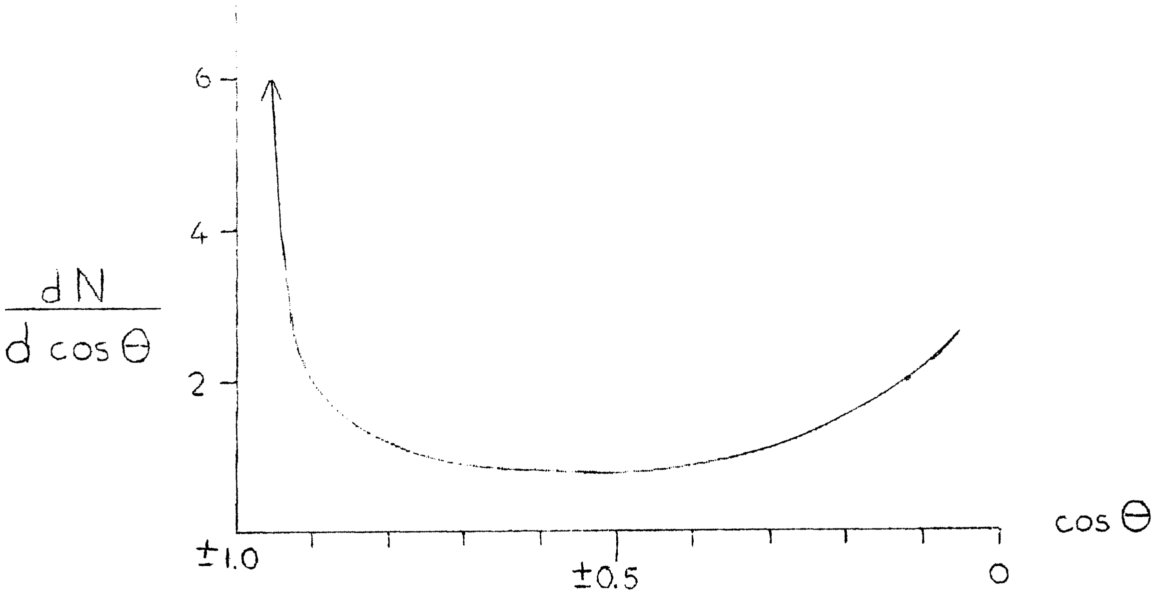


Fig. 9

In summing up we conclude: The effect of angular momentum barrier is to decrease the point density in special regions near the boundary of the Dalitz plot. The uneven distribution in space of the angular momentum vector will lead to decreasing point density around the centre of the plot. We stress, however, that these remarks are qualitative only. In a specific case, the effects will depend strongly on the particular type of particles in question due to the conservation of isospin and parity.

e) Dalitz-Stevenson plot

From a very detailed study of the distribution of points on a Dalitz plot Maglic and Stevenson et al. (Phys.Rev.Letters 7, 178 (1961), Phys.Rev. 125, 687 (1962) were able to determine the spin and parity for the three particle decay mode $\omega^0 \rightarrow \pi^+ + \pi^- + \pi^0$. Their elegant treatment is also applicable to other three particle decay modes proceeding via strong interactions. Stevenson et al. made some special assumptions about the form of the matrix element. This enabled them to perform a quantitative comparison between prediction and experimental data. Before reviewing their arguments, however, we will state some more general and qualitative arguments given by Dalitz which are valid irrespective of the particular form of the transition matrix element and which are valid for spin values less than three. [R.H. Dalitz: "Three Lectures on Elementary Particles" BNL 735 (1961) and "Strange Particles and Strong Interactions, Oxford University Press (1962)].

The foundation of the following discussion is the experimental fact that no charged state of the ω has been found. Consequently ω has isospin I = 0, i.e. the isospin wavefunction Φ_ω is a scalar.

We assume that the decay

$$\omega \rightarrow \pi^+ + \pi^- + \pi^0$$

proceeds via a strong interaction, i.e. isospin and parity are conserved. Therefore, the isospin wavefunction of the final state must also be a scalar. The only way to obtain a scalar quantity from the three isovector wavefunctions Φ_1, Φ_2 and Φ_3 of the pions, is to form a triple product

$$\Phi_\omega = \Phi_1 \cdot (\Phi_2 \times \Phi_3)$$

$\Phi_\omega = 0$ and the transition matrix element will vanish if two or more of the particles are equal. Thus, the $3\pi^0$ decay mode of the ω^0 is forbidden.

The triple product Φ_ω is antisymmetric in any pair of pions. Since the pions are Bose particles their total wavefunction (which can be expressed as a product of a space function and isospin function) must be symmetric. Thus, the space wavefunction of the ω must also be antisymmetric in any pair of pions. This fact has important consequences on the symmetry of the points on a Dalitz plot representing the decay configurations of the pions.

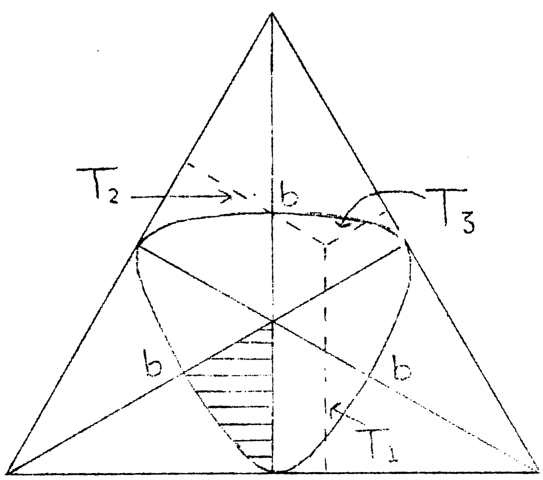


Fig. 10.

We consider in Fig. 10 a triangular plot (cf. Fig. 4 in Chapter VII.a) of the kinetic energies of the three pions. The contour deviates slightly from a circle because of the relativistic energies of the pions. From the antisymmetry of the space wavefunction in all pairs of pions it follows that the distribution of events is unchanged by a reflection across any one of the three symmetry axes of the triangle. This means that the distribution of points should be the same in all six sectors of the Dalitz plot; one of these sectors is shaded in Fig. 10. In a study of the statistical distribution one can, therefore, very conveniently, concentrate the points in a so called 6-fold Dalitz plot.

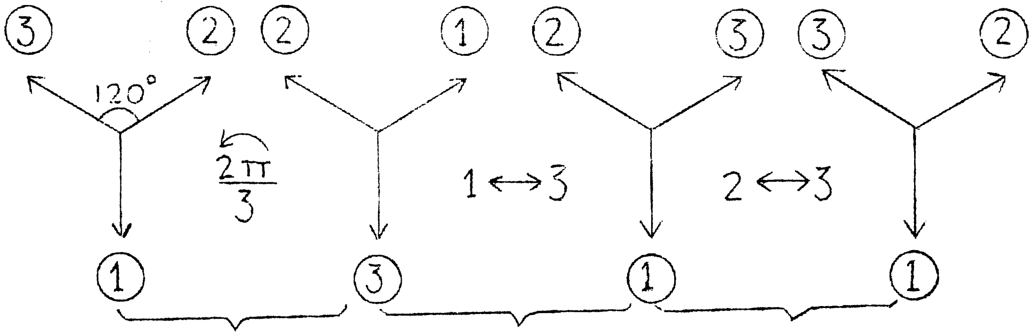
We will next consider some general features of the Dalitz plot distributions expected for different spin (J) and parity (w) assignments of the ω meson. (Sometimes we will denote the sign of the parity by a superscript to the spin value, e.g. J^+, J^-).

(i) The density of points will vanish whenever one of the pions has its maximum kinetic energy. This follows from the antisymmetry of the space wavefunction. An interchange of any two pions makes the transition matrix element vanish if two pions "touch" in momentum space. This happens when the third pion has its maximum energy which is indicated by the points b in Fig. 10. The above statement holds for all spin and parity assignments of the ω to be considered.

(ii) The density of points will vanish on the boundary of the plot for ω parity $w = (-1)^J$. The contour represents configurations where the three pions are emitted collinearly and their directions can therefore be specified by a single vector. The space wavefunction will be a spherical harmonic $Y_J^m(\cos\Theta)$, which has parity $(-1)^J$. Since the intrinsic parity of each pion is (-1), the total parity of the collinear configuration is $(-1)^3$. $(-1)^J = -(-1)^J$. Hence, if the ω parity is $(-1)^J$, the matrix element must vanish on the boundary of the Dalitz plot.

(iii) The density of points will vanish at the centre of the plot if the ω parity is even. The centre of the plot represents configurations where the pions have equal energy and their directions of motion make angles of 120° with each other. Therefore, if the three particle system is first rotated 120° around an axis (N) normal to its plane it can be returned to its initial position by interchanging the particles 1 → 3 and 2 → 3.

The operations are schematically illustrated below.



$$\exp\left(\frac{2\pi}{3}\,mi\right) \qquad (-1) \qquad (-1)$$

If the angular momentum is quantized along N with a magnetic quantum number m, the rotation corresponds to an operation

$$\exp\left(\frac{2\pi}{3}\,mi\right) .$$

Each of the two successive interchanges corresponds to multiplication of the space wavefunction with a factor (-1). Since the system is restored to its initial position we must have

$$\exp\left(\frac{2\pi}{3}\,mi\right) \cdot (-1)(-1) = +1 .$$

For $J < 3$ (recall that $m \leq J$) this equation can only be satisfied for $m = 0$. This means that $m = 0$ is the only possibility for a non-vanishing matrix element of the symmetrical configuration. This information will now be used when we perform the following operations.

We reflect the symmetrical configuration with respect to the origin and return it to the initial state by rotating the system 180° about N (the normal to the plane of the system). The effect of the first operation is to multiply the space wavefunction by $(-1)^3 \cdot w$ (w = intrinsic parity of the ω) whereas the second operation for $m = 0$ leaves the wavefunction unchanged. Thus, we require

$$(-1)^3\, w(+1) = +1 .$$

This means that the matrix element must vanish at the symmetry point if the ω parity is even. This statement is valid for all $J < 3$.

(iv) The density of points will vanish at the centre of the plot if J is even. To show this we rotate the configuration at the symmetry point about an axis along the direction of motion of one pion, say pion 3, and return the system to the initial state by the interchange $1 \leftrightarrow 2$.

The first operation corresponds to an operator

$$\exp (i\pi\ J_y)$$

where the y axis is oriented along the direction of motion of particle 3. Now, since $m = 0$, J_y can also be quantized. Therefore the rotation multiplies the wavefunction by $(-1)^J$. The second operation (interchange $1 \leftrightarrow 2$) changes the wavefunction by a factor (-1). In total we must have

$$(-1)^J(-1) = +1 \ .$$

Thus, the matrix element has to be equal to zero at the centre of the plot if J is even. This statement is also valid for all $J < 3$.

The above qualitative but general considerations are sufficient to distinguish between the possible spin and parity assignments of the ω meson if $J < 3$. A 0^+ state of the ω decaying via the mode $\omega \rightarrow \pi^+ + \pi^- + \pi^0$ violates the conservation of parity. The other possible states are $0^-, 1^-, 1^+, 2^-$ and 2^+. From (iii) and (iv) follows that the states 1^+ and $0^-, 2^+, 2^-$ respectively, would all demand zero point density at the centre of the plot. The experimental distribution on the 6-folded Dalitz plot in Fig. 11 clearly reveals that the density of points does not vanish near the centre of the plot. Thus, of all the admissible spin parity assignments for $J < 3$ we are left with only one possibility, 1^-, which from (ii) should require a depopulation of events near the contour. This is in agreement with the observed distribution in Fig. 11.

We conclude therefore, that the ω meson probably has $J = 1$ and $w = -1$. This was already shown by Stevenson et al. in their original paper, and we now proceed to discuss their assumptions about the transition matrix element and the quantitative comparison they made on the basis of the assumptions.

The transition matrix element of the ω decay is most conveniently analysed in terms of the vectors $\vec{p}$ and $\vec{q}$ introduced in Fig. 6. We recall that $\vec{p}$ is the momentum of one pion (3) in the 3-body centre of mass system and $\vec{q}$ the momentum of one of the other pions (2) in the centre of mass of these two pions (1 and 2).

No rigorous derivation of the different matrix elements will be attempted here. Our aim is simply to present some physical arguments from which one can understand the qualitative form of the simplest matrix elements derived from different tentative spin and parity assignments of the ω. The implicit assumption being that the wavelengths of the decay pions are large compared to the interaction dimension of the decay which presumably is of the order of the Compton wavelength of the ω. Consequently the momentum dependence of the transition matrix element will be determined mainly by the coefficients for the centrifugal barrier penetration and will for L and ℓ greater or equal to one be of the form

$$M = \langle \psi_f | H_{int} | \psi_i \rangle \propto \frac{(pR)^L}{(2L+1)!!} \frac{(qR)^\ell}{(2\ell+1)!!}$$

where R is the radius of interaction. We note that the momentum dependence of M is independent of R, so that in order to study the variation of M from the Dalitz plot it suffices to write

$$M \propto p^L q^\ell \; .$$

One general remark can be made about the possible values of ℓ. A reversal of $\vec{q}$ which multiplies the wavefunction by $(-1)^\ell$ corresponds to an interchange of the two particles (1 ↔ 2 in Fig. 6) which changes the wavefunction by (-1). Thus we must have

$$(-1)^\ell \cdot (-1) = +1 \; .$$

It follows that only odd values of ℓ are allowed. Therefore only $\ell = 1$ will be considered in the following discussion of the three possible matrix elements for ω spin less or equal to one.

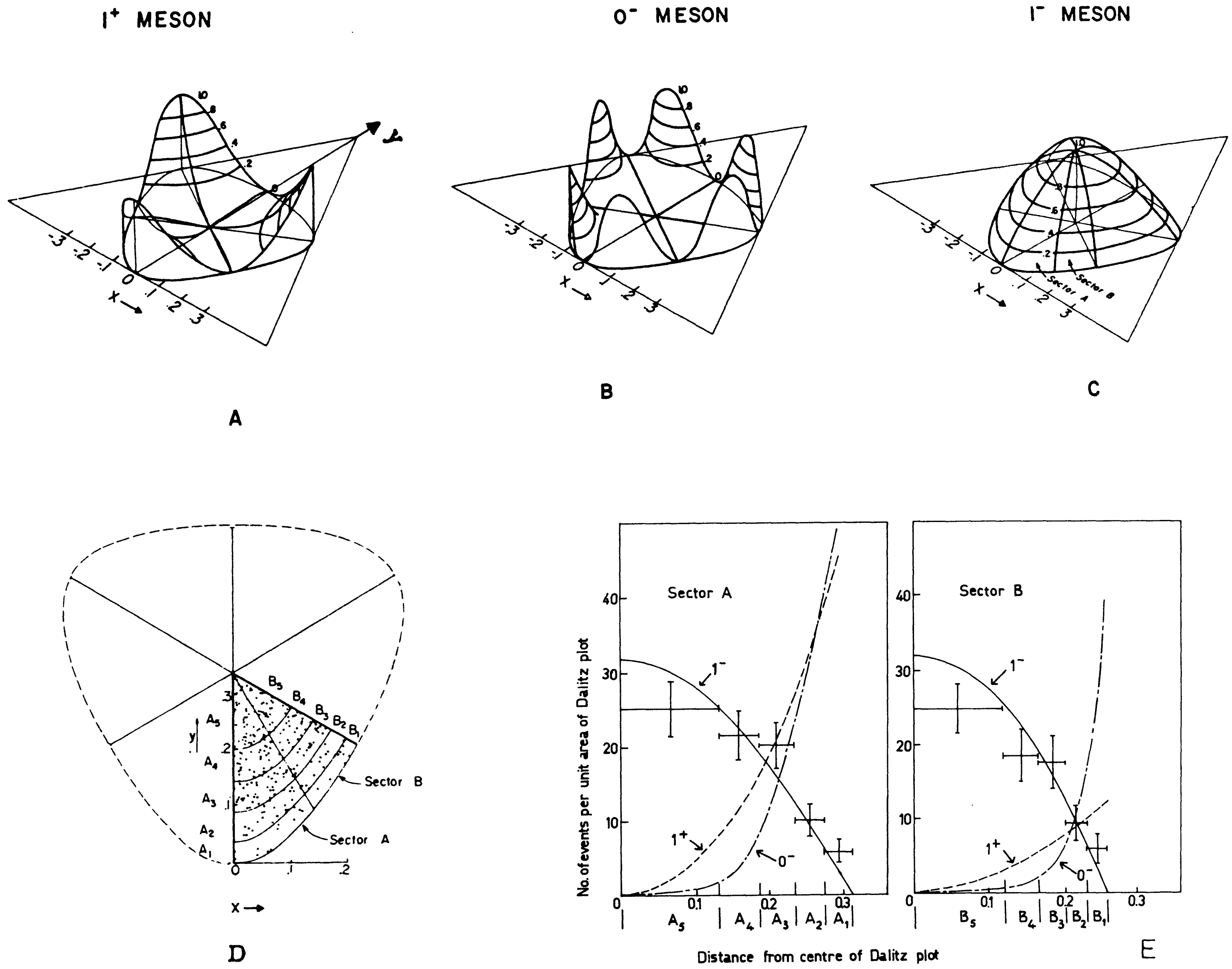

1+ MESON
0- MESON
1- MESON
A
B
C
Sector B
Sector A
D
Sector A
Sector B
No. of events per unit area of Dalitz plot
Distance from centre of Dalitz plot
E

Fig. 11

0^- meson ($L = \ell = 1$). Since the intrinsic parity of the three pions is $(-1)^3 = -1$ the 0^- meson should require a scalar transition matrix element. The only scalar quantity which is odd with respect to an interchange of any two pions (say 1 and 2) is a scalar product of the form $\vec{p}\cdot\vec{q}$ (interchange of particles 1 and 2 corresponds to a reversal of $\vec{q}$).

We will now express this scalar product in terms of the centre of mass energies of the pions. In the non-relativistic limit we can write

$$E_1 = \frac{\left(-\frac{\vec{p}}{2} - \vec{q}\right)^2}{2m} + m$$

$$E_2 = \frac{\left(-\frac{\vec{p}}{2} + \vec{q}\right)^2}{2m} + m$$

where E_1 and E_2 are the total energies of pions 1 and 2 in the 3-body centre of mass system. From these two equations follows

$$\vec{p}\cdot\vec{q} = m(E_1 - E_2) .$$

Now, since the matrix element (M) should be symmetric in the labels of all three pions, it will, for a 0^- meson be of the form

$$M(0^-) = (E_1 - E_2)\ (E_2 - E_3)\ (E_3 - E_1) .$$

We see that $M(0^-)$ vanishes whenever two of the pions have equal energies, that is along all three symmetry lines of the Dalitz plot. In particular the density of points will be very low where the symmetry lines intersect, that is, in the centre of the plot.

1^- meson. The matrix element describing the transition from a vector meson state 1^- to the three pion state must have the properties of an axial vector (pseudovector). For the decay with $L = \ell = 1$, $M(1^-)$ must therefore be of the form $\vec{p}\times\vec{q}$. We note that this matrix element is also odd for an interchange of two particles (reversal of $\vec{q}$). Since $\vec{p}_3 = \vec{p}$ and $\vec{p}_2 = -\vec{p}/2 + \vec{q}$ etc.
M can be expressed in terms of the momenta of the particles in their overall centre of mass system as

$$M(1^-) = \vec{p}_1 \times \vec{p}_2 + \vec{p}_2 \times \vec{p}_3 + \vec{p}_3 \times \vec{p}_1$$

where M is made symmetrical in the labels of all three pions.

We see that $M(1^-) = 0$ whenever the pions are emitted collinearly; that is the density of points vanishes on the boundary of the Dalitz plot.

<u>1^+ meson</u>. The transition matrix element must in this case have the properties of a vector. The simplest decay of a 1^+ ω meson is by the emission of one s-wave pion ($L = 0$) and two pions in a relative p-state ($\ell = 1$). Then the matrix element will be of the form $E_3 \vec{q}$. Since

$$\vec{q} = \frac{\vec{p}_3}{2} + \vec{p}_2$$

we have from momentum conservation

$$\vec{q} = - \tfrac{1}{2}(\vec{p}_1 - \vec{p}_2) \ .$$

Thus, a matrix element symmetrical in the labels of all pions and odd for the interchange of any two pions will be proportional to

$$M(1^+) = E_3(\vec{p}_1 - \vec{p}_2) + E_1(\vec{p}_2 - \vec{p}_3) + E_2(\vec{p}_3 - \vec{p}_1) \ .$$

This matrix element vanishes whenever any two pions have the same momentum, that is at the points b in Fig. 10. It also vanishes at the symmetry point of the Dalitz plot where $E_1 = E_2 = E_3$.

The variation in the point density on the Dalitz plot for the different matrix elements can be illustrated on a three dimensional plot - referred to as Dalitz-Stevenson plot (see Fig. 11) - where the height above the Dalitz plot is proportional to the square of the matrix element. Due to the finite width of the resonances (i.e. the variation of the Q value from event to event) it is most meaningful to make the Dalitz plot in terms of the normalized variables T_1/Q, T_2/Q and T_3/Q. In a Cartesian coordinate system with x axis and origin as indicated in Fig. 11 we now have

$$x = \frac{T_2 - T_3}{\sqrt{3}\ Q}$$

$$y = \frac{T_3}{Q} \ .$$

Figures 11 A,B, and C show isometric graphs of $|M(1^+)|^2$, $|M(0^-)|^2$ and $|M(1^-)|^2$ respectively. The maximum height above the plane is arbitrarily chosen as unity and the contours have been drawn at 0.2 intervals. For comparison with experimental data the contours are projected onto the plane of the Dalitz plot. Because of the symmetry of the plot referred to earlier, it suffices to make the projection in one sixth of the Dalitz plot. Such a projection is shown for the 1^- meson in Fig. 11 D.

The matrix elements for the 1^+ and 0^- meson show considerable and different azimuthal variation. For instance $M(0^-) = 0$ for $x=y=0$ whereas $M(1^+)$ is large at the same point. The 6-folded Dalitz plot has therefore more or less arbitrarily been divided into two sectors A and B. The contours (for constant $|M|^2$) in turn divide each sector into sub-areas A_1 ... A_5 and B_1 ... B_5 as illustrated in the 1^- case in Fig. 11 D. Finally, Fig. 11 E displays the number of events found within each area A_1, A_2 etc., as well as the theoretical curves of $|M|^2$ (obtained by azimuthal integration from Fig. 11 A, Band C) versus the distance from the centre of the Dalitz plot. (This plot is also generally referred to as Stevenson plot). It is evident that the 1^- assignment to the ω meson is in perfect agreement with the experimental result. On the other hand, a 0^- or a 1^+ assignment would contradict the experimental data for both sectors A and B.

With reference to the general remarks made in the preceding paragraph about the effect of angular momentum barrier on the distribution of points on the Dalitz plot, one comment should be made. It was shown that the contribution from partial waves with $L > 0$ would diminish whenever one of the particles in final state had zero linear momentum. In the case of the 1^+ ω meson, however, $|M(1^+)|^2$ has its maximum value at $y = 0$ (Fig. 11 A). The reason for this, is, that the matrix element is calculated for $L = 0$ only. Inclusion of higher L values would still give finite values of $M(1^+)$ for $y = 0$, but M would show a decrease for $T_3 \to 0$ due to smaller contribution from $L > 0$ states.

7863/p/cm

VIII. EXAMPLES

a) Branching ratio in π decay

One has observed two 2-body decay modes of the π^+ meson

$$\pi^+ \to \mu^+ + \nu$$

$$\pi^+ \to e^+ + \nu \ .$$

If the matrix element of the transition is the same in both reactions the branching ratio is determined by phase space. We will calculate here the branching ratio predicted both from Lorentz non-invariant and Lorentz invariant phase space. In the centre of mass of the pion we have

$$\begin{aligned} E &= 139.59 \text{ MeV} \\ E_\mu &= 109.78 \text{ MeV} \\ p_\mu &= 29.81 \text{ MeV/c} \\ E_e &= 69.80 \text{ MeV} \\ p_e &= 69.79 \text{ MeV/c} \ . \end{aligned}$$

Using formulae (6) and (8) these values give the following phase space predictions for the branching ratio

$$\frac{\rho_2(\mu\nu)}{\rho_2(e\nu)} = 0.427 \text{ (non-invariant)}$$

$$\frac{R_2(\mu\nu)}{R_2(e\nu)} = 0.207 \text{ (invariant)}$$

We notice firstly a marked difference between prediction from non-invariant and invariant phase space. Secondly we have a large discrepancy between the predicted values and the experimental result.

Experimental branching ratio

$$\left(\frac{\mu\nu}{e\nu}\right) \approx 10^4 \ .$$

From this discrepancy follows that there is no reason to believe that the matrix element is the same for the two decay modes. On the contrary, we expect from the two component theory of the neutrino that the matrix element will be very sensitive to the velocity of the emitted lepton. Since the positively charged leptons are preferentially emitted with positive helicity, the decay rates of $\pi^+ \to \mu^+ + \nu$ and $\pi^+ \to e^+ + \nu$ are reduced by factors $(1-{}^v\mu/c)$ and $(1-{}^ve/c)$ respectively. Disregarding phase space this gives a branching ratio of the order of 10^4 [see for instance A. Lundby, Progress in Elementary Particle and Cosmic Ray Physics V, 1 (1960)].

b) Effective mass distributions

To illustrate the use of formulae (21) and (24) and to show the appearance of some effective mass distributions, consider the reaction

$$K^- + N \to K^{\overset{0}{+}} + \Xi^- + n\pi$$

which has recently been studied at CERN (Belliere et al. Physics Letters 6, 316 (1963) and the Sienna Conference 1962).

The reactions were produced in a heavy liquid bubble chamber filled with CF_3Br by a separated K^- beam with an average momentum in the chamber of about 3.4 GeV/c. A K^- interaction with a single nucleon at rest corresponds to a total energy in the centre of mass of 2.75 GeV. Since in the heavy liquid the nucleons are bound in a nucleus, the centre of mass energy will be spread out due to the Fermi momentum of the nucleons. The effect of the Fermi momentum on the phase space distribution of the $\Xi\pi$ mass is illustrated in Fig. 12 for the case where only one pion is produced (3-body final state). We have calculated here from Eq. (21) the $\Xi\pi$ mass distribution for the case of a nucleon at rest (No Fermi) and for the two rather extreme cases that the target nucleon moves parallel to the beam and with a momentum of 200 MeV/c towards or away from the beam particle. These extremes correspond to a centre of mass energy of 3.01 GeV and 2.52 GeV respectively. For the cases with two or three pions (4 or 5-body final state) we have used Eq. (24) to calculate the $\Xi\pi$ mass distribution for a target nucleon at rest only.

The experimental values of the $\Xi\pi$ effective masses are presented as histograms in Fig. 13 and Fig. 14 for events containing 1,2 and 3 pions.

The $\Xi^- \pi^-$ system which has a third component of isotopic spin $I_z = -3/2$ and therefore $I \geq 3/2$ is presented in Fig. 13 and the $\Xi^- \pi^+$ and $\Xi^- \pi^0$ mass values which have $I \geq 1/2$ are combined in Fig. 14.

The curves in Fig. 13 and Fig. 14 are phase space curves constructed from Fig. 12 for interactions with target nucleons at rest, and are simply a super position of 3,4 and 5-body phase space weighted according to the respective number of events observed.

There seems to be good agreement between the experimental $\Xi^- \pi^-$ mass distribution and the predicted phase space curve. This is rather remarkable since we have neglected all effects from Fermi motion of the target nucleons, secondary interaction in the target nucleus of primary and secondary particles etc. On the other hand the phase space curve in Fig. 14 seems to fit the experimental histogram rather poorly.

The above qualitative statements should be expressed in a more objective and quantitative way by performing a goodness of fit test of the distributions. To do this we divide the mass values of Fig. 13 into four groups with approximately equal number of events in each group. Recalling the definition of X^2 of a variable x

$$X^2 = \sum_i \frac{[x_i \text{ (experimental)} - x_i \text{ (theoretical)}]^2}{x_i \text{ (theoretical)}}$$

we find from Fig. 13 $X^2 \approx 3$ for three degrees of freedom. This gives a probability of about 40%, so that our experimental mass distribution should have $X^2 > 3.0$. In other words, if our sample of events is taken from a universe which follows the laws of phase space there is 40% probability that new measurements performed on an equivalent number of events will give a distribution which deviates even more from the theoretical curve.

Correspondingly we find from Fig. 14 $X^2 \approx 10$ for four degrees of freedom, i.e. a probability of about 2%. It is therefore rather unlikely that the $\Xi^- \pi^0_+$ mass distribution follows phase space.

We interpret that data in Fig. 14 as a likely production of two $\Xi\pi$ resonances; one is the well-established Ξ_δ with mass 1.53 GeV, the other is a possible resonance at about 1.75 GeV.

The phase space effective mass distributions of the $\Xi\pi\pi$ system from the same 4 and 5-body final states are shown in Fig. 15.

The effect of a resonance on the effective mass distribution is illustrated in Fig. 16 for the following reactions

$$K^- + N \to K^* + \Xi^-$$
$$\hookrightarrow K + \pi$$

and

$$K^- + N \to K^* + \Xi^- + \pi^0$$
$$\hookrightarrow K^0 + \pi^+ \quad .$$

Fig. 16 illustrates both the $\Xi^-\pi^0$ and $\Xi^-\pi^+$ effective mass distribution from the last configuration.

c) Angular distribution

To illustrate the use of the formulae evaluated in Chapter IV we have calculated the angular distribution of two pions from the τ-decay. This example is particularly simple since as a good approximation we can do the calculation for three non-relativistic particles of equal mass. The result is shown in Fig. 17.

Exercise

Show that for three non-relativistic particles of equal mass, the integration limits of p_3 in Eq. (17) will be given by

$$p_3(\text{min}) = 0$$

$$p_3(\text{max}) = \left(\frac{4mE}{4 - \cos^2\Theta} \right)^{1/2}$$

where m = mass of any of the three particles. Show that the angular distribution is independent of the mass of the particles and of the total energy of the system E.

d) Dalitz plot

In Fig. 18 is drawn the contour of a $T_\pi + T_\pi -$ Dalitz plot for the reaction

$$K^- + p \to \Lambda + \pi^+ + \pi^-$$

with total energy in the centre of mass of about 2.02 GeV. This reaction has been studied at CERN in a 30 cm hydrogen bubble chamber (Cooper et al. 1963 Sienna International Conference on Elementary Particles) using a separated K^- beam with momentum 1.45 GeV/c.

Fig. 18 shows the result from an analysis of 582 events, and reveals an example of a non-uniform distribution of points. There is a marked clustering of points for both a constant T_{π^-} and a constant T_{π^+} value indicating the production of $\Lambda\pi^+$ and $\Lambda\pi^-$ resonances respectively, i.e. $Y_1^{*\pm}(1385)$.

Figure 18 shows also a tendency that the points within a resonance band may not be evenly distributed along the axis. For the Y_1^{*+} there seems to be more points for high T_{π^-} values etc. One explanation of this is some production of the ρ^o meson. We leave as an exercise the finding of the band on the plot within which one should expect points due to the meson resonance $\rho^o \rightarrow \pi^+ + \pi^-$.

Acknowledgement

The author would like to thank his colleagues at CERN and Oslo for many valuable comments. In particular cand. real. A.G. Frodesen is acknowledged for many clarifying discussions and suggestions of improvement.

7863/p/cm

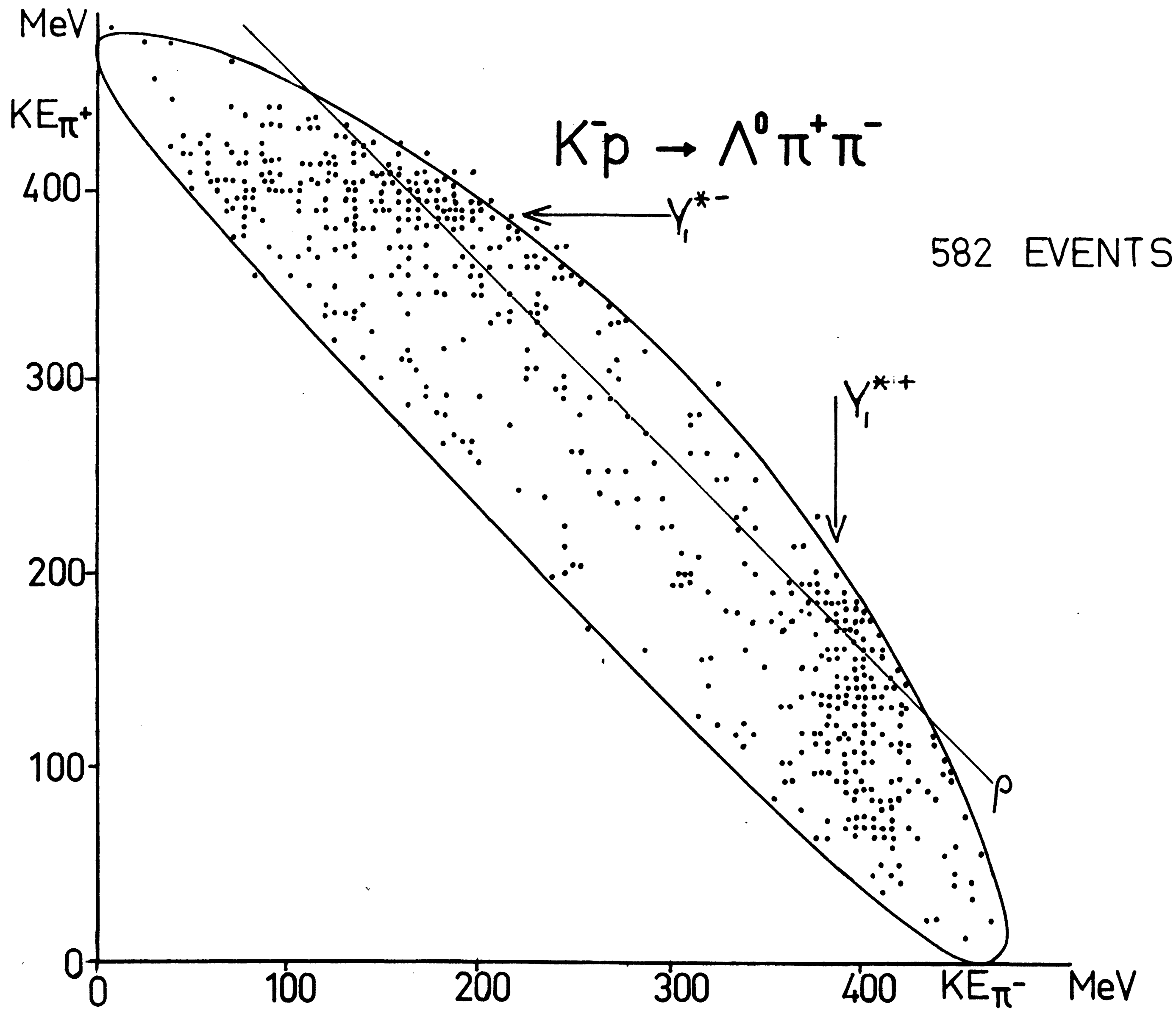

MeV
KE_{π^+}
400
300
200
100
0
0
100
200
300
400
KE_{π^-}
MeV
$K^-p \rightarrow \Lambda^0\pi^+\pi^-$
Y_1^{*-}
Y_1^{*+}
582 EVENTS
ρ

Fig.18

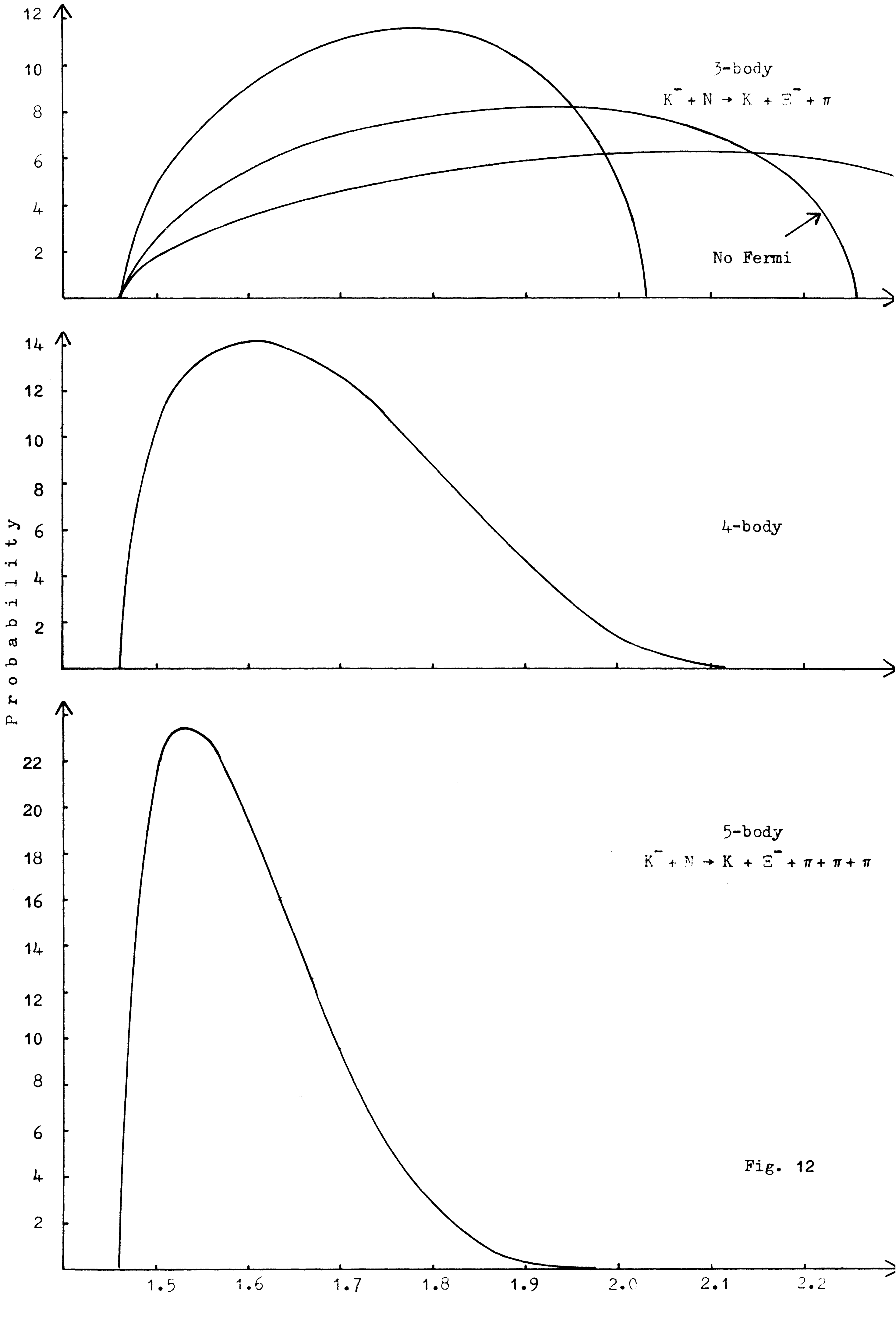
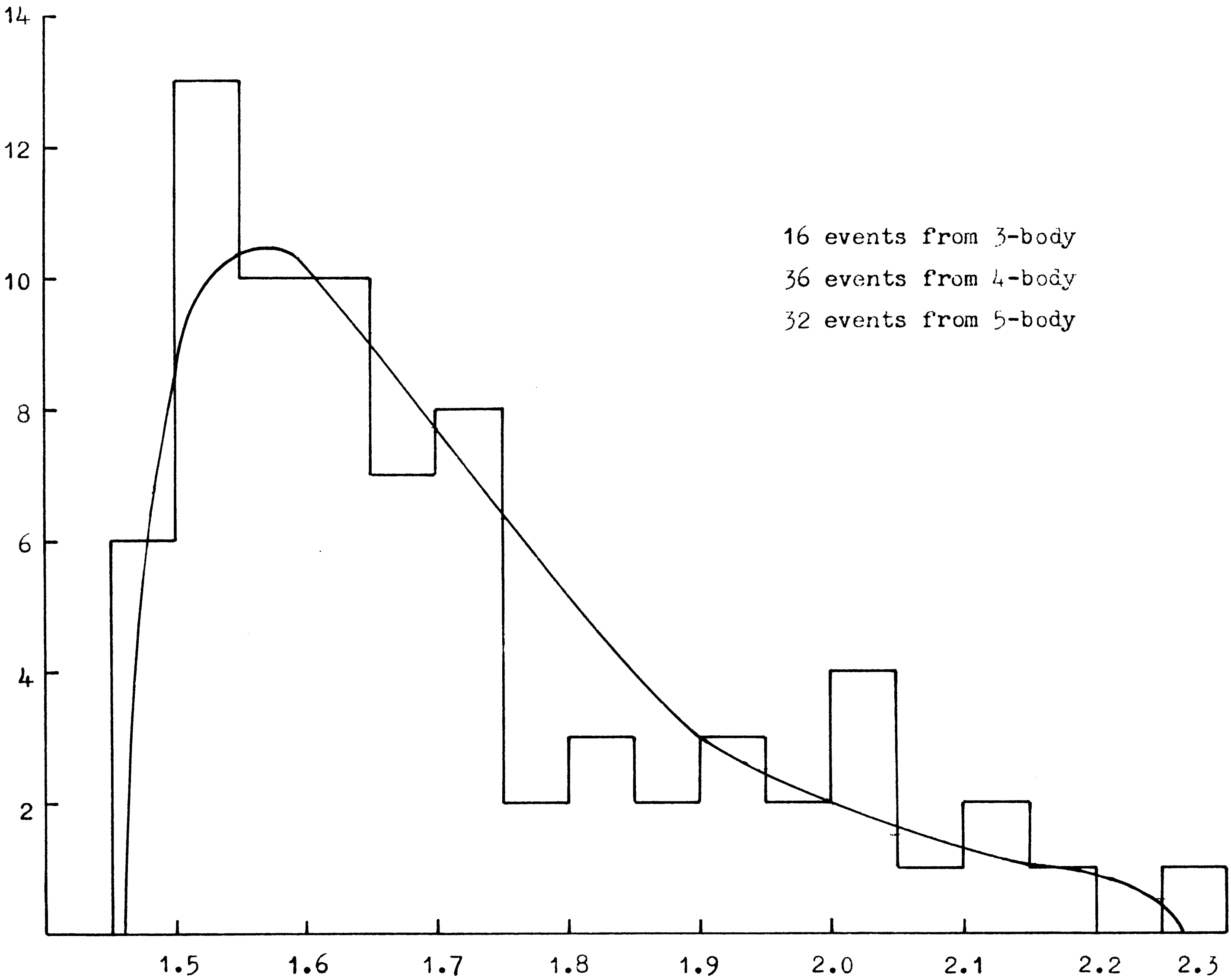

3-body
K⁻ + N → K + Ξ⁻ + π
No Fermi
4-body
5-body
K⁻ + N → K + Ξ⁻ + π + π + π
Probability
1.5
1.6
1.7
1.8
1.9
2.0
2.1
2.2

Fig. 12



Effective Mass $\Xi^-\pi^-$ in GeV

Fig. 13

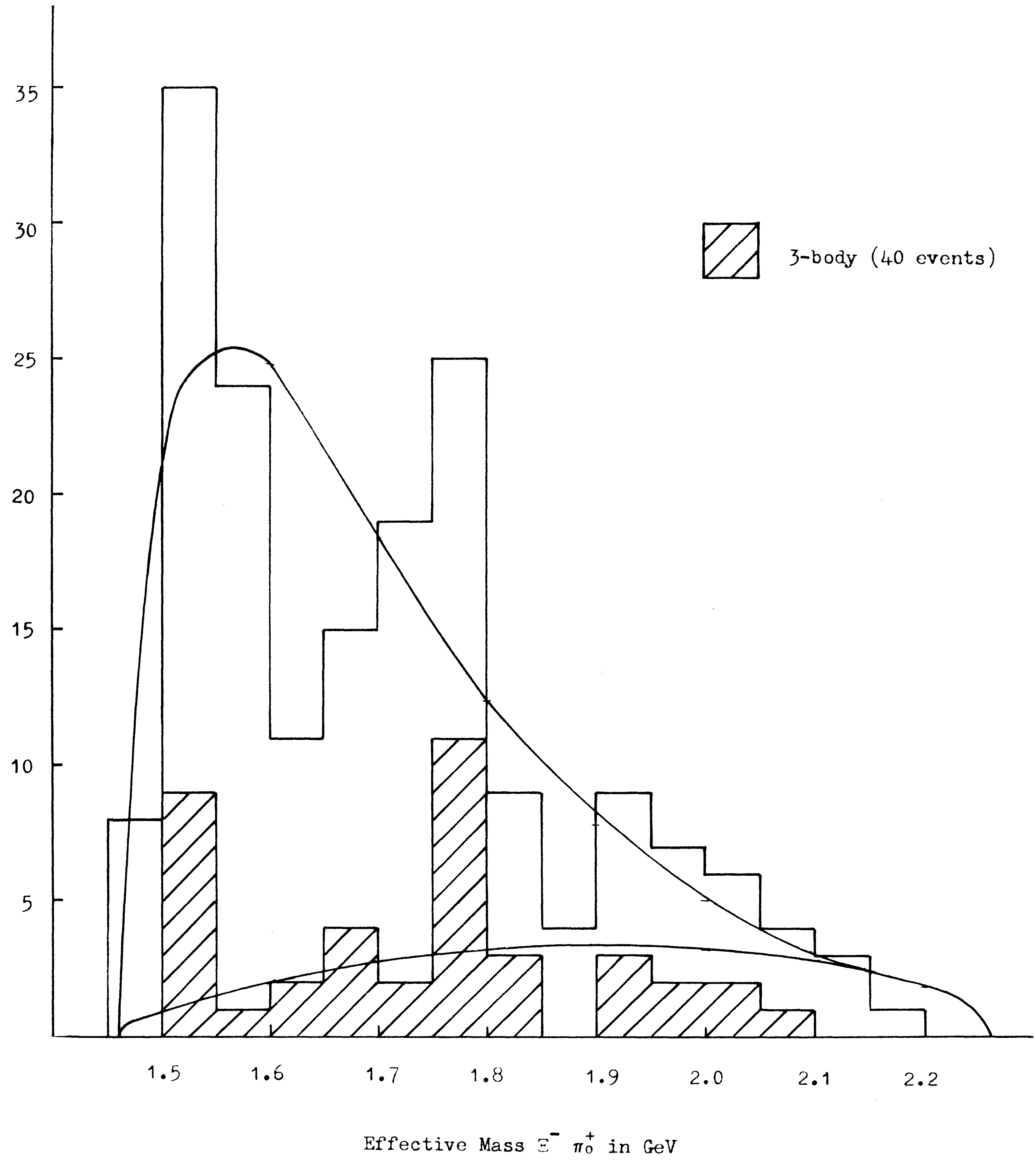
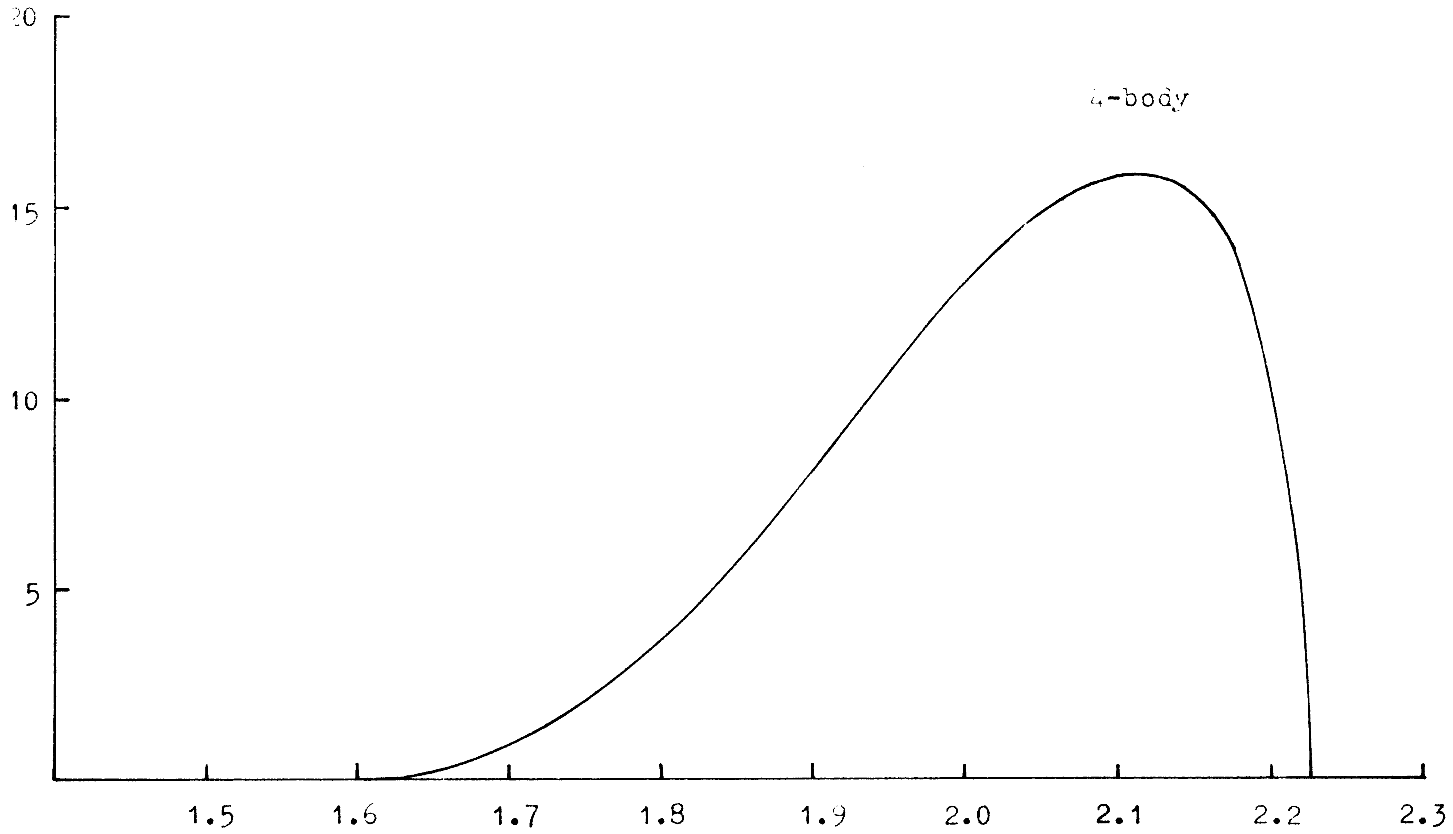
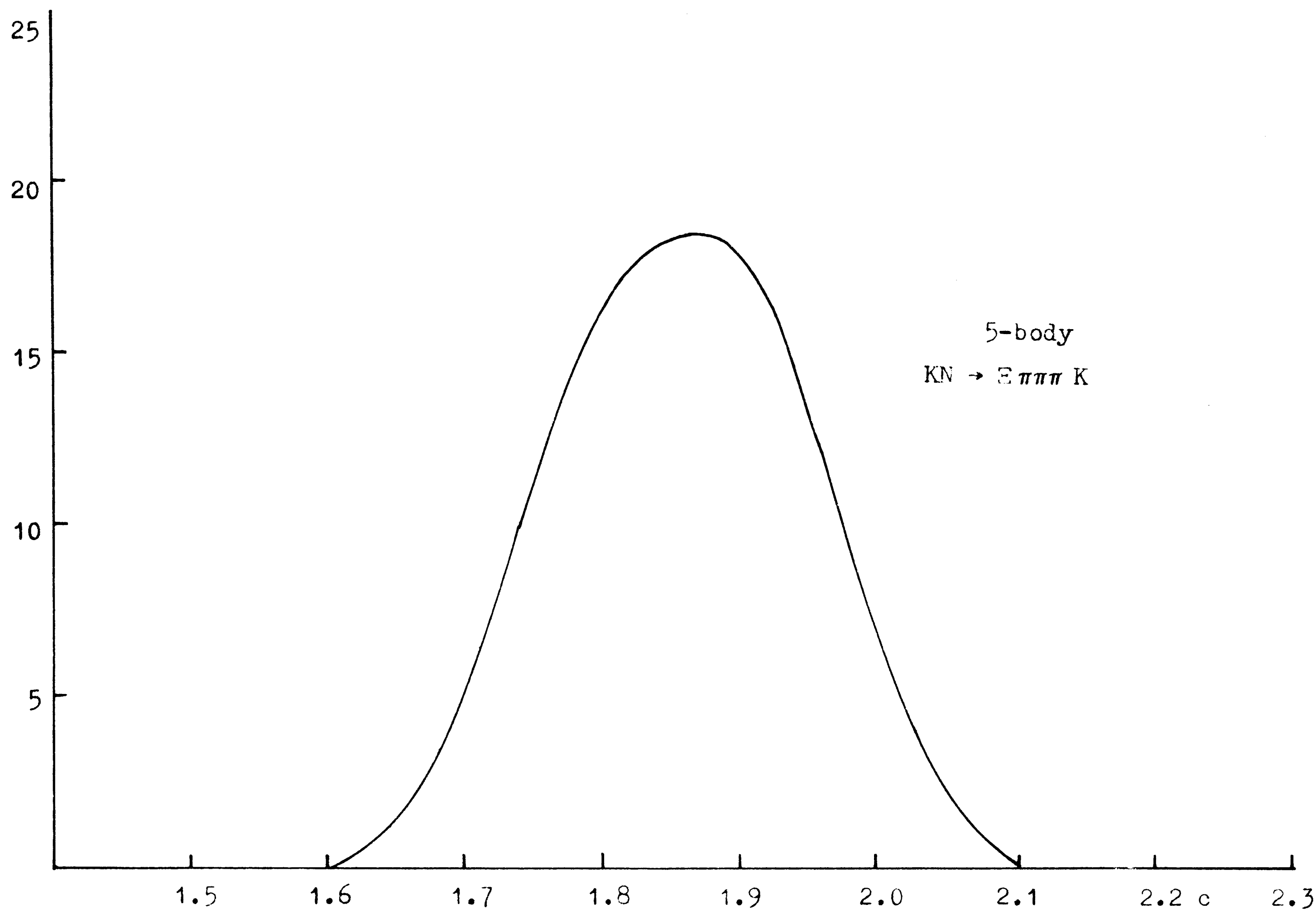

35
30
25
20
15
10
5
3-body (40 events)
1.5
1.6
1.7
1.8
1.9
2.0
2.1
2.2
Effective Mass $\Xi^- \pi_o^+$ in GeV

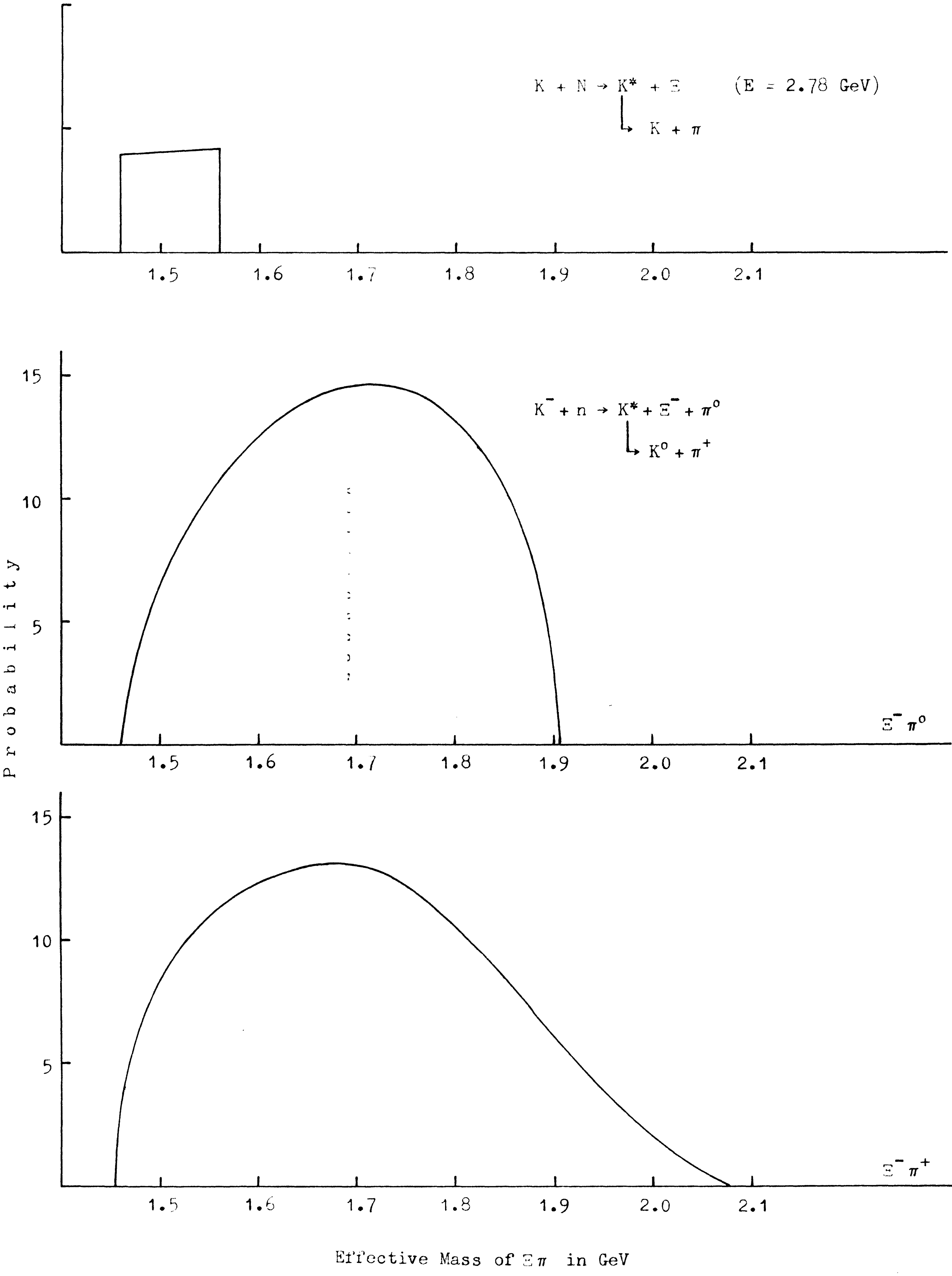
Fig. 14





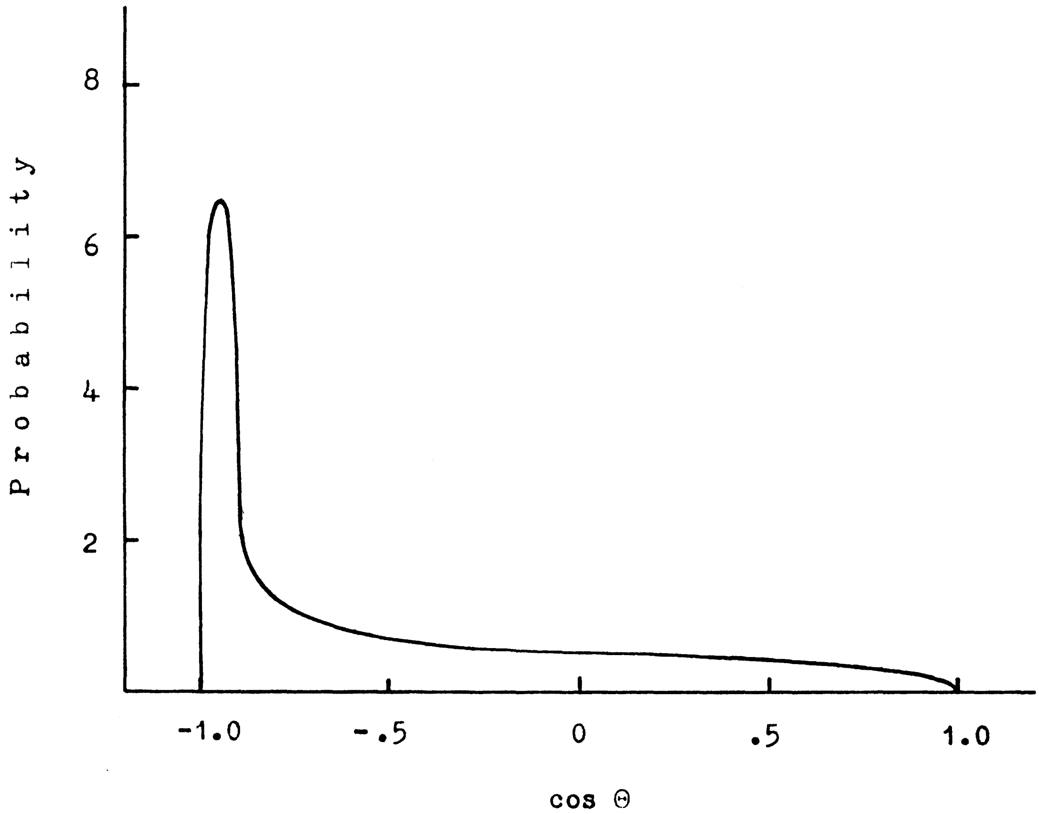
Effective Mass of Ξ ππ in GeV

Fig. 15



K + N → K* + Ξ (E = 2.78 GeV)
K + π
1.5
1.6
1.7
1.8
1.9
2.0
2.1
15
10
5
Probability
K⁻ + n → K* + Ξ⁻ + π⁰
K⁰ + π⁺
Ξ⁻ π⁰
Ξ⁻ π⁺
Effective Mass of Ξπ in GeV

Fig. 16



Angular Distribution in τ decay

Fig. 17

APPENDIX

We define the Dirac δ function by the equations

$$\left.\begin{array}{lll} \delta(x) = 0 & \text{for } x \neq 0 & \\ \lim\limits_{x\to 0} \delta(x) \to \infty & \text{in such a way that } \int \delta(x)\, dx = 1 \end{array}\right\} \qquad \text{(i)}$$

for all integrations enclosing the origin.

From this definition it follows that

$$\int_a^b f(x)\, \delta(x-c)\, dx = \begin{cases} f(c) & \text{for } a < c < b \\ 0 & \text{for } c < a \text{ or } c > b\,. \end{cases} \qquad \text{(ii)}$$

Correspondingly if we define a three-dimensional δ function $\delta(\vec{r})$ as

$$\delta(\vec{r}) = \delta(x)\, \delta(y)\, \delta(z)$$

the integration over a volume V gives

$$\int_V f(\vec{r})\, \delta(\vec{r}-\vec{r}_0)\, d\vec{r} = \begin{cases} f(\vec{r}_0) & \text{if } \vec{r}_0 \text{ lies inside V} \\ 0 & \text{if } \vec{r}_0 \text{ lies outside V.} \end{cases}$$

In the case where the argument of the δ function is itself a function, like $\delta(\Phi(x))$, we find by substitution

$$\int \delta[\Phi(x)]\, dx = \int \delta(y)\, \frac{dy}{|\Phi'(x)|}$$

where the absolute value is necessary to ensure that $dx = \frac{dy}{|\Phi'(x)|}$ is always positive.

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From the above follows

$$\int \delta[\Phi(x)]\ dx = \frac{1}{|\Phi'(x_0)|} \text{ if } \Phi(x_0) = 0 . \qquad \text{(iii)}$$

The equation (iii) can also be generalized, using Eq. (ii), as

$$\int g(x)\ \delta[\Phi(x)]\ dx = \frac{g(x_0)}{|\Phi'(x_0)|} \text{ if } \Phi(x_0) = 0 .$$

This rule is valid for all functions $g(x)$ which are continuous at $x = x_0$.

SYMBOLS USED

$\vec{P}$ = Momentum of the system in initial state.

E = Total energy of the system in initial state. It will be evident from the context whether this is in the laboratory or centre of mass system.

E_i = Total energy of i'th particle in final state.

m_i = Rest mass of i'th particle.

T_i = Kinetic energy of i'th particle in final state.

$\vec{p}_i$ = Momentum of i'th particle in final state.

M_{ij} = Effective mass of two particles i and j.

${}^n_kM$ = Effective mass of the first k particles taken from a system of n particles.

q_i = Four vector = $(\vec{p}_i, E_i)$ of length $q_i^2 = E_i^2 - p_i^2$.

DETERMINATION OF THE Ξ DECAY PARAMETERS

W. Koch,
Track Chamber Division, CERN

Let us consider a two-body decay of a particle ($A \rightarrow B + C$). If the decay interaction is strong (e.g. $\rho \rightarrow 2\pi$) parity is conserved, i.e.

$$\xi_A = \xi_B\ \xi_C\ (-1)^L \tag{1}$$

where the ξ's are the instrinsic parities of the particles involved, L the angular momentum of the final state, and $(-1)^L$ its contribution to the parity. As can be seen from Eq. (1), only either odd or even L are allowed for the final state. If, however, as is usual in weak interactions, parity is not conserved, relation (1) breaks down and both even <u>and</u> odd angular momenta are permitted, as far as the total spin can be conserved.

We choose the Ξ decay

$$\Xi \rightarrow \Lambda + \pi$$

as a specific example to discuss how the wave function describing a non-leptonic weak two-body decay can be found experimentally. It will turn out that this wave function (of the decay products) is most conveniently described in terms of the so-called "decay parameters" which are more closely related to measurable quantities than the wave equation's amplitudes themselves. We shall assume that both Ξ and Λ have spin ½ (well established now). The π has spin 0. The following considerations, though presented for spin ½ particles can easily be generalized for higher spins.

As initial state we choose a Ξ in its c.m. with the spin pointing into the positive z direction of the reference frame:

Initial state:

$$\overset{z}{\uparrow} \quad \underset{\Xi}{\Uparrow} \quad \Psi_{\Xi} = s_{1/2} \, .$$

Now we ask for all possible final states allowed by angular momentum conservation. Only the spin angular momentum part of the wave function need be considered. Angular momentum conservation allows only the following spin configurations of the decay products:

Final state:

$$\underset{\Lambda}{\Uparrow} \qquad \Psi_1 = A_s Y_0^0 \, s_{1/2} = A_s \cdot 1 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\overset{z}{\uparrow}\Theta \; \underset{\Lambda}{\Uparrow} \xrightarrow{\ell=1} \Psi_2 = A_p \left(-\sqrt{\frac{1}{3}}\right) Y_1^0 \, s_{1/2} = A_p \left(-\sqrt{\frac{1}{3}}\right) (\sqrt{3} \cos \Theta) \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\underset{\Lambda}{\Downarrow} \uparrow \ell=1 \quad \Psi_3 = A_p \sqrt{\frac{2}{3}} \, Y_1^1 \, s_{-1/2} = A_p \sqrt{\frac{2}{3}} \left(-\sqrt{\frac{3}{2}} \sin \Theta \, e^{i\varphi}\right) \begin{pmatrix} 0 \\ 1 \end{pmatrix} .$$

Here the complex numbers A_s and $-A_p$ are the amplitudes of the S and P angular momentum state. We use $-A_p$ instead of A_p as P wave amplitude to obtain the sign convention of Teutsch et al.[1]. The Y's are orbital angular momentum eigenfunctions and

$$s_{1/2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } s_{-1/2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

are the Pauli spin functions. The factors $(-\sqrt{1/3})$ and $\sqrt{2/3}$ are Clebsch-Gordon coefficients arising from the decomposition of the initial spin $1/2$ state into two spins, $1/2$ and 1:

$$s_{1/2} = -\sqrt{1/3} \; Y_1^0 \, s_{1/2} + \sqrt{2/3} \; Y_1^1 \, s_{-1/2} \, .$$

The total final-state wave function Ψ is, of course, a linear superposition of the three possible spin configurations:

$$\Psi = \Psi_1 + \Psi_2 + \Psi_3 = \left(A_s + A_p \cos\Theta\right)\begin{pmatrix}1\\0\end{pmatrix} + \left(A_p e^{i\varphi} \sin\Theta\right)\begin{pmatrix}0\\1\end{pmatrix} \qquad (2)$$

and its conjugate

$$\Psi^* = \left(A_s + A_p \cos\Theta\right)^*\begin{pmatrix}1\\0\end{pmatrix} + \left(A_p e^{i\varphi} \sin\Theta\right)^*\begin{pmatrix}0\\1\end{pmatrix} .$$

The angular distribution of the decay as measured in the Ξ c.m. system is given by

$$\frac{dN}{d\cos\Theta} = \Psi^*\Psi = |A_s + A_p \cos\Theta|^2 + |A_p e^{i\varphi} \sin\Theta|^2 . \qquad (3)$$

With the general relation between two complex numbers

$$|A+B|^2 = |A|^2 + |B|^2 + 2\mathrm{Re}\,(A^*B) .$$

Eq. (3) can be written as

$$\begin{aligned}\frac{dN}{d\cos\Theta} = \Psi^*\Psi &= |A_s|^2 + |A_p|^2 \cos^2\Theta + 2\mathrm{Re}(A_s{}^*A_p)\cos\Theta \\ &\quad + |A_p|^2 \sin^2\Theta \\ &= |A_s|^2 + |A_p|^2 + 2\mathrm{Re}(A_s^*A_p)\cos\Theta .\end{aligned}$$

Now let us normalize the wave function so that

$$|A_s|^2 + |A_p|^2 = 1 \qquad (4)$$

This gives

$$\frac{dN}{d\cos\Theta} = 1 + \underbrace{2\mathrm{Re}(A_s^*A_p)}_{\alpha}\cos\Theta . \qquad (5)$$

$2\mathrm{Re}\,(A_s^*A_p)$ is called the α-decay parameter. If $\alpha = 2\mathrm{Re}\,(A_s^*A_p) \neq 0$, an up-down asymmetry is observed in the decay which directly indicates parity non-conservation; so α can be regarded a measure for parity violation.

Now we introduce two other decay parameters, which in a similar way to α are related to some angular distributions to be discussed below. We define

$$\left.\begin{aligned} \alpha &= 2\mathrm{Re}(A_s^*A_p) \\ \beta &= 2\mathrm{Im}(A_s^*A_p) \\ \gamma &= |A_s|^2 - |A_p|^2 \end{aligned}\right\} \qquad (6)$$

Not all the authors use the same sign convention. Ours agrees with that assumed in the article of Teutsch et al.[1]. For example Crawford[2] defines $\alpha = -2\mathrm{Re}(A_s^*A_p)$.

One may easily verify that

$$\alpha^2 + \beta^2 + \gamma^2 = 1 \qquad (7)$$

Two of the decay parameters determine the absolute value of the third one, but not its sign. Further, we notice that α, β, γ are invariant under a phase transformation

$$\begin{aligned} A_s &\to A_s\, e^{i\varphi} \\ A_p &\to A_p\, e^{i\varphi} . \end{aligned}$$

So two decay parameters and the sign of the third will completely determine the final-state wave function; that means, A_s and A_p which are subject to normalization condition (4) will be known up to a common irrelevant phase factor $e^{i\varphi}$.

α and β may be written in the following way:

$$\alpha = 2\mathrm{Re}(A_s^* A_p) = 2\mathrm{Re}\left(|A_s||A_p| e^{-i\varphi_s + i\varphi_p}\right)$$

$$= 2\ |A_s||A_p| \cos\ (\varphi_p - \varphi_s) \tag{8}$$

$$\beta = 2\mathrm{Im}(A_s^* A_p) = 2\mathrm{Im}\left(|A_s||A_p| e^{-i\varphi_s + i\varphi_p}\right)$$

$$= 2\ |A_s||A_p| \sin\ (\varphi_p - \varphi_s)\ .$$

$\varphi_p - \varphi_s = \varphi_{rel}$ is the relative phase between the S and P waves. Now, as shown in Appendix II, if there are no final-state interactions, time reversal invariance requires β to be 0. As a consequence of Eq. (6), $\varphi_{rel} = 0$ in this case, which means that A_s and A_p are relatively real, and $\alpha = 2|A_s||A_p|$.

So far we have seen that the α parameter can be determined from the decay angular distribution of a polarized Ξ. In the following we shall investigate the polarization of the daughter Fermion Λ. It will turn out that the polarization of the Λ is related to β and γ so that these can be determined by polarization measurements.

Before going into details of our problems we shall briefly recall how the polarization of a spin ½ particle is described quantum mechanically.

The most general pure spin state of a spin ½ particle is

$$\Psi = a\ s_{1/2} + b\ s_{-1/2}$$

$$= a \begin{pmatrix} 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{9}$$

with a and b being complex numbers normalized to 1

$$|a|^2 + |b|^2 = 1\ . \tag{10}$$

a and b are related to the direction in which the spin of the particle is actually pointing, or more precisely the direction for which the spin expectation value is 1.

The spin expectation values along the x,y,z axis are

$$\begin{aligned} \langle J_x \rangle &= \Psi^* J_x \Psi \quad ; \quad J_x = \tfrac{1}{2}\sigma_x \\ \langle J_y \rangle &= \Psi^* J_y \Psi \quad ; \quad J_y = \tfrac{1}{2}\sigma_y \\ \langle J_z \rangle &= \Psi^* J_z \Psi \quad ; \quad J_z = \tfrac{1}{2}\sigma_z \end{aligned} \tag{11}$$

where the σ's are the Pauli spin matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The polarization vector for spin ½ particles is

$$\vec{P} = 2 \langle \vec{J} \rangle = \Psi^* \vec{\sigma} \Psi \tag{11a}$$

Applying (11a) to our general state (9) gives

$$P_x = \left[a^* \begin{pmatrix} 1 \\ 0 \end{pmatrix} + b^* \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right] \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \left[a \begin{pmatrix} 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right]$$

$$= \left[a^* \begin{pmatrix} 1 \\ 0 \end{pmatrix} + b^* \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right] \left[a \begin{pmatrix} 0 \\ 1 \end{pmatrix} + b \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right]$$

$$\underline{P_x = ab^* + a^*b = 2\mathrm{Re}(a^*b)}$$

$$P_y = \left[a^* \begin{pmatrix} 1 \\ 0 \end{pmatrix} + b^* \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right] \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \left[a \begin{pmatrix} 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right]$$

$$= \left[a^* \begin{pmatrix} 1 \\ 0 \end{pmatrix} + b^* \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right] \left[i a \begin{pmatrix} 0 \\ 1 \end{pmatrix} + (-i)\, b \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right] \tag{12a}$$

$$= -ia^*b + iab^* = i(ab^* - a^*b)$$

$$\underline{P_y = i(ab^* - a^*b) = 2\mathrm{Im}(a^*b)}\ . \tag{12b}$$

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In the same way one finds

$$\underline{P_z = a^*a - b^*b = |a|^2 - |b|^2} \ . \tag{12c}$$

From (12) follows

$$P_x^2 + P_y^2 + P_z^2 = 1 \ .$$

So the polarization is 1 for the direction

$$\vec{P} = 2\mathrm{Re}(a^*b)\ \hat{e}_x + 2\mathrm{Im}(a^*b)\ \hat{e}_y + (|a|^2 - |b|^2)\ \hat{e}_z$$

where the $\hat{e}$ are unit vectors in the x,y,z direction.

(Conversely one can show that if the spin points in a direction defined by the polar angles Θ,Φ, the corresponding spin state is

$$\Psi(\Theta,\Phi) = \underbrace{-\sin\left(\tfrac{1}{2}\Theta\right)}_{a} \begin{pmatrix}1\\0\end{pmatrix} + \underbrace{e^{i\Phi}\cos\left(\tfrac{1}{2}\Theta\right)}_{b} \begin{pmatrix}0\\1\end{pmatrix} \ .$$

a and b are of course defined only up to a common phase factor $e^{i\varphi}$.)

Now let us apply these relations to calculate the polarization direction of the Λ from the Ξ decay. The problem will be treated here in a purely non-relativistic way. However, in Appendix I it will be shown how the non-relativistic formulae are to be interpreted in the relativistic case.

The final state wave function (2)

$$\Psi = \left(A_s + A_p \cos\Theta\right)\begin{pmatrix}1\\0\end{pmatrix} + \left(A_p\, e^{i\varphi} \sin\Theta\right)\begin{pmatrix}0\\1\end{pmatrix}$$

contains of course all necessary information about the Λ spin. In order to obtain the standard form (9) of a spin $\frac{1}{2}$ state, the coefficients of the spin function have to be normalized:

$$\Psi_\Lambda = \frac{\Psi}{|\Psi|^2} = \underbrace{\frac{A_s + A_p \cos\Theta}{1+\alpha\cos\Theta}}_{a}\begin{pmatrix}1\\0\end{pmatrix} + \underbrace{\frac{A_p e^{i\varphi}\sin\Theta}{1+\alpha\cos\Theta}}_{b}\begin{pmatrix}0\\1\end{pmatrix} . \qquad (13)$$

So the spin state of the Λ, characterized by a and b, depends on its emission direction Θ,Φ in the Ξ system. As in our problem only the z direction in the Ξ system is fixed one can, without lack of generality, neglect the φ dependence by choosing $\varphi = 0$, $e^{i\varphi} = 1$. With this choice we have bound the Λ emission direction to the z - x plane. To calculate the polarization direction of the Λ we apply relations (12) to the Λ spin state (13):

$$\begin{aligned}(1+\alpha\cos\Theta)\, P_{\Lambda x} &= 2\mathrm{Re}\,[(A_s^* + A_p^*\cos\Theta)\,(A_p\sin\Theta)] \\ &= 2\mathrm{Re}\,[A_s^*A_p\sin\Theta + |A_p|^2\cos\Theta\sin\Theta] \\ &= \underline{\alpha\sin\Theta + 2|A_p|^2\cos\Theta\sin\Theta}\end{aligned} \qquad (14a)$$

$$\begin{aligned}(1+\alpha\cos\Theta)\, P_{\Lambda y} &= 2\mathrm{Im}\,[(A_s^* + A_p^*\cos\Theta)\,(A_p\sin\Theta)] \\ &= 2\mathrm{Im}(A_s^*A_p\sin\Theta) + \underbrace{2\mathrm{Im}(A_p^*A_p)}_{=0}\cos\Theta\sin\Theta \\ &= \underline{\beta\sin\Theta}\end{aligned} \qquad (14b)$$

$$\begin{aligned}(1+\alpha\cos\Theta)\, P_{\Lambda z} &= |A_s + A_p\cos\Theta|^2 - |A_p\sin\Theta|^2 \\ &= |A_s|^2 + |A_p|^2\cos^2\Theta + 2\mathrm{Re}(A_s^*A_p)\cos\Theta \\ &\quad - |A_p|^2\sin^2\Theta \\ &= |A_s|^2 - |A_p|^2 + 2|A_p|^2\cos^2\Theta + 2\mathrm{Re}(A_s^*A_p)\cos\Theta \\ &= \underline{\gamma + \alpha\cos\Theta + 2|A_p|^2\cos^2\Theta} .\end{aligned} \qquad (14c)$$

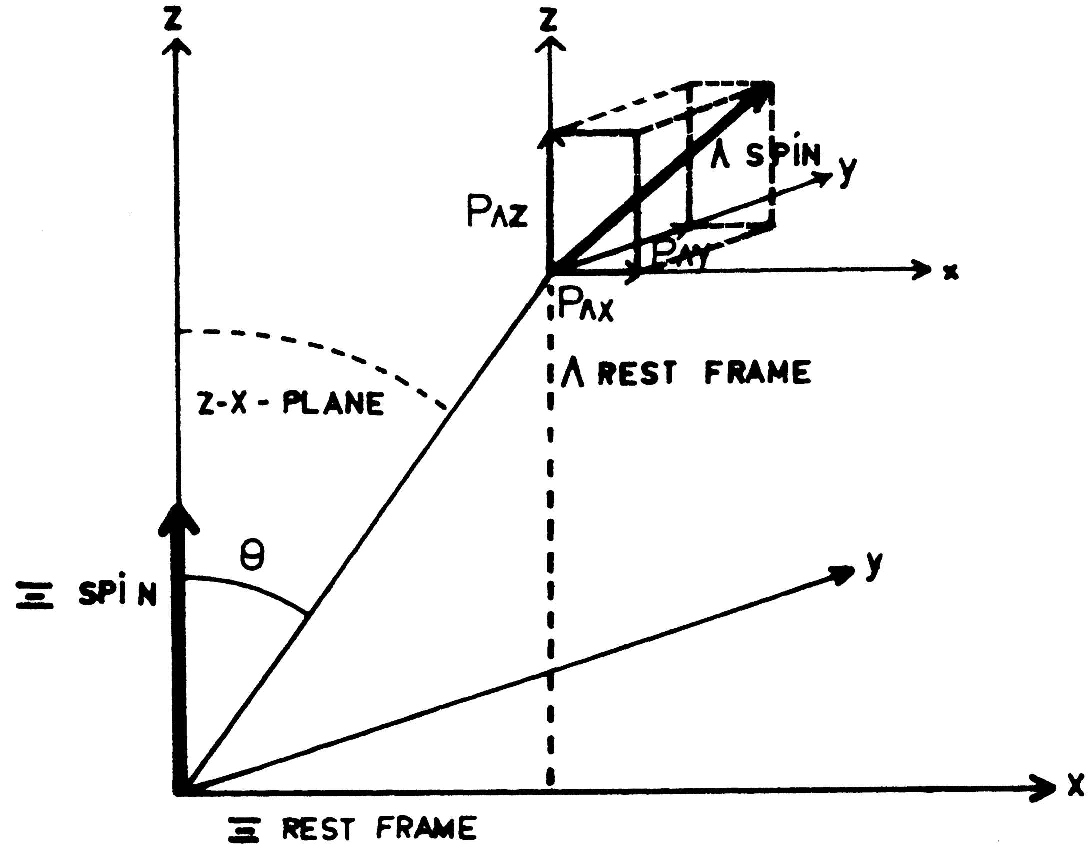

FIG. 1.

These relations describing the dependence of the Λ polarization on the emission angle θ hold in this form only if the Ξ is completely polarized. They have to be modified for a Ξ polarization $P < 1$. Naively one might assume that then the Λ polarization is weakened by a factor P, but in reality it is more complicated:

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We consider a source of Ξ's where N(↑) particles have their spins along the positive z axis and N(↓) particles in the opposite direction. Then the polarization of the source is

$$P = \frac{N(\uparrow) - N(\downarrow)}{N(\uparrow) + N(\downarrow)} .$$

Now we fix a direction Θ in space and let either of the two sorts of particles decay along this direction.

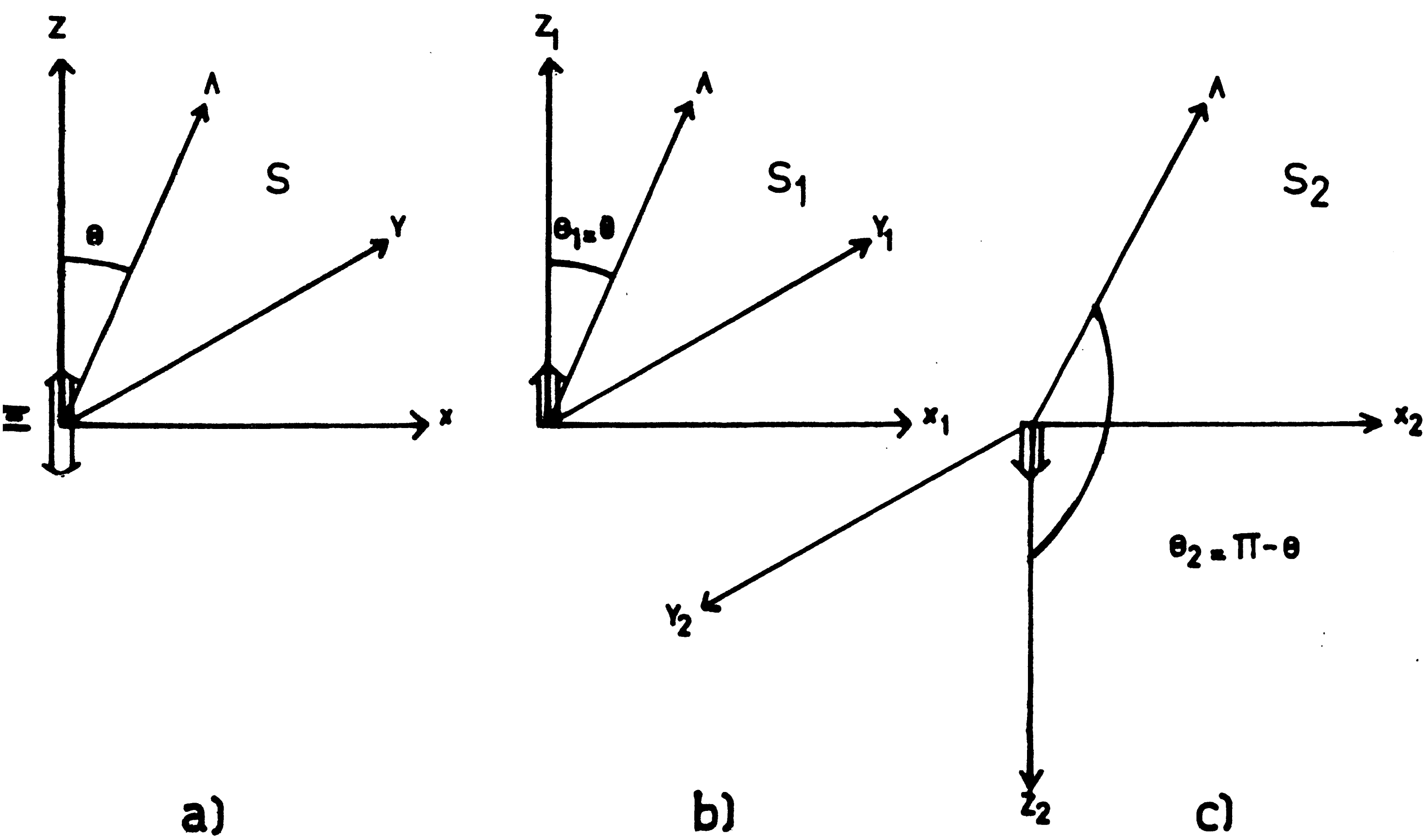

FIG. 2.

To get the Λ spin expectation values for a <u>fixed direction in space</u> (Fig. 2a) we apply formulae (14) to both the Ξ spin states (Fig. 2b) and (Fig. 2c) and add their contributions weighted by $N(\uparrow)/(N(\uparrow)+N(\downarrow))$ and $N(\downarrow)/(N(\uparrow)+N(\downarrow))$ respectively. The system S in which the direction Θ in space is defined coincides with the system S_1 in which the spin-up Ξ's were analysed. To apply formulae (14) also to Ξ's with spin down (with respect to S) we introduce a new coordinate system S_2 (Fig. 2c) in which again the Ξ spin points along a positive z_2 axis. The configuration of S_2 is translated into $S \equiv S_1$ by the transformation:

$$\left.\begin{aligned} y_2 &= -y \\ z_2 &= -z \\ \Theta_2 &= \pi - \Theta \end{aligned}\right\}$$

Thus for Ξ's with spin down in S, relations (14) are modified in the following way

$$\begin{aligned} P_y &\to -P_y \\ P_z &\to -P_z \\ \cos\Theta &\to -\cos\Theta \end{aligned}$$

If corresponding terms in (14) have the same sign for both Ξ spin directions they remain unchanged in the corresponding expressions for the mixture; if they change sign they are weakened by a factor P_Ξ. Therefore the Λ polarization for a Ξ source with polarization P_Ξ is

$$\left.\begin{aligned} P_{x\Lambda} &= \frac{\alpha \sin\Theta + 2P_\Xi\,|A_p|^2 \cos\Theta \sin\Theta}{1 + P_\Xi \alpha \cos\Theta} \\ P_{y\Lambda} &= \frac{P_\Xi \beta \sin\Theta}{1 + P_\Xi \alpha \cos\Theta} \\ P_{z\Lambda} &= \frac{P_\Xi \gamma + \alpha\cos\Theta + 2P_\Xi |A_p|^2 \cos^2\Theta}{1 + P_\Xi \alpha \cos\Theta} \end{aligned}\right\} \qquad (15)$$

For a source with P< 1 the decay angular distribution is of course different from Eq. (4); it becomes

$$\frac{dN}{d\cos\Theta} = 1 + P\alpha \cos\Theta \qquad (16)$$

Let us transform the Λ polarization to another coordinate system defined in Fig. 3 by the unit vectors $\hat{e}_1, \hat{e}_2, \hat{e}_3$.

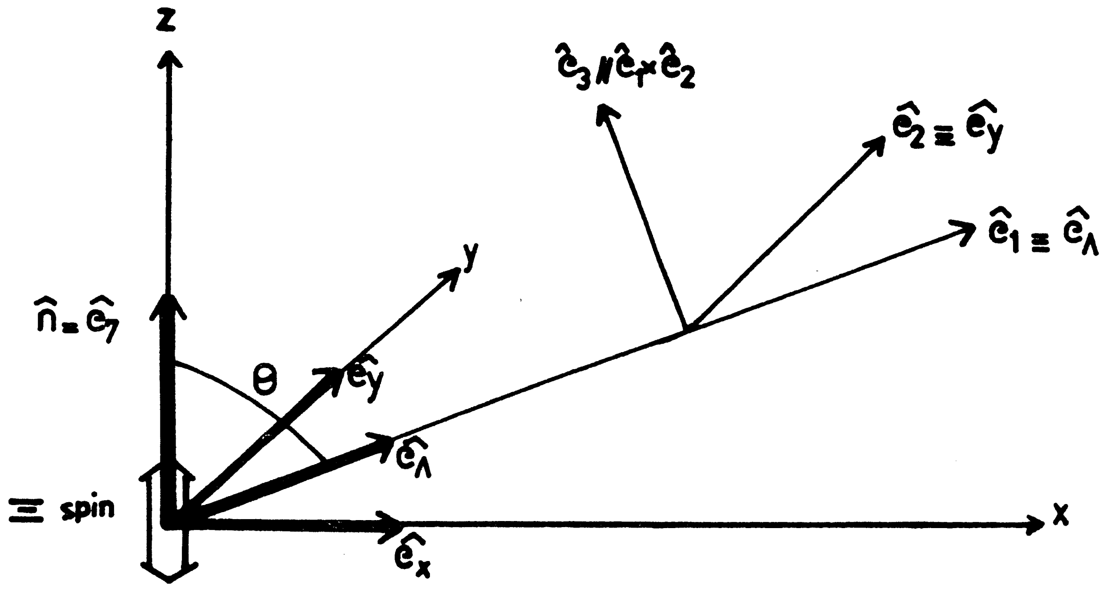

FIG. 3.

$\hat{n} \parallel z$		direction of Ξ polarization
$\hat{e}_1 = \hat{e}_\Lambda$		direction of Λ in Ξ rest frame
$\hat{e}_2 \parallel \hat{n} \times \hat{e}_1 \parallel y$		in our case
$\hat{e}_3 \parallel \hat{e}_1 \times \hat{e}_2$		

The Λ polarization components along $\hat{e}_1, \hat{e}_2, \hat{e}_3$ are

$$\left.\begin{aligned} P_{1\Lambda} &= P_{x\Lambda} \sin\Theta + P_{z\Lambda} \cos\Theta \\ &= \frac{P_\Xi \cos\Theta + \alpha}{1 + P\alpha\cos\Theta} \\ P_{2\Lambda} &= P_{y\Lambda} \\ &= \frac{P_\Xi \beta \sin\Theta}{1 + P\alpha\cos\Theta} \\ P_{3\Lambda} &= P_{z\Lambda} \sin\Theta - P_{x\Lambda} \cos\Theta \\ &= \frac{P_\Xi \gamma \sin\Theta}{1 + P\alpha\cos\Theta} \quad . \end{aligned}\right\} \qquad (17)$$

It is interesting to notice that even completely unpolarized Ξ's will lead to a Λ polarization of magnitude α in the direction $\hat{e}_1 = \hat{e}_\Lambda$.

As relation (17) involves the decay parameters, it is clear that they can be determined from polarization measurements. In order to stress the close relation between the Λ polarization and the decay parameters, let us rewrite Eq. (17) for a Λ emission angle $\Theta = 90°$:

$$\left.\begin{aligned} P_{1\Lambda} &= P_{x\Lambda} = \alpha \\ P_{2\Lambda} &= P_{y\Lambda} = P_{\Xi}\beta \\ P_{3\Lambda} &= P_{z\Lambda} = P_{\Xi}\gamma \end{aligned}\right\} \quad \text{for } \Theta = \pi/2$$

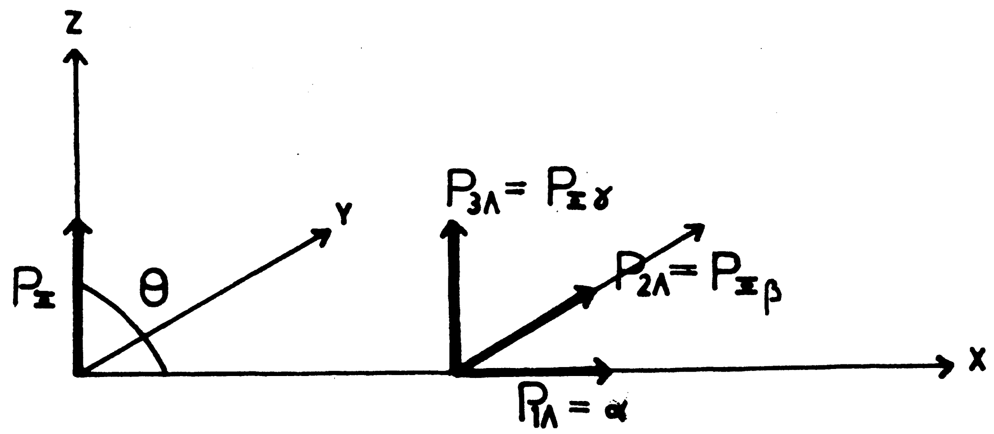

FIG. 4.

Usually a polarization analysis is performed by scattering experiments. Here, however, we are in a favourable position, because the Λ in turn has a parity violating decay with a large asymmetry parameter $\alpha_\Lambda \approx 0.62$[4]. So the asymmetry of the ($\Lambda \rightarrow p + \pi^-$) decay can be used as polarization analyser.

From theory we know that the decay angular distribution of a polarized sample is of the form

$$\frac{dN}{d\cos\Theta} = 1 + \alpha_\Lambda P_\Lambda \cos\Theta_p = 1 + a\cos\Theta_p \qquad (18)$$

with respect to some direction $\hat{A}$.

Let us now try to find the direction $\hat{A}$ and the asymmetry parameter a for an experimentally observed sample of Λ decays. To this purpose we first study the following problem. What does the angular distribution Eq. (18) look like with respect to another direction $\hat{B}$ in terms of the polar angle ϑ and the azimuth α?

Angular distributions may be regarded as density functions on the unit sphere, so our porblem consists only in an angle transformation. In the given density function

$$\frac{dN}{d\cos\Theta} = F_A(\Theta,\Phi)$$

we have to substitute the new angles ϑ,α:

$$F_A(\Theta,\Phi) = F_A(\Theta(\vartheta,\alpha),\ \Phi(\vartheta,\alpha)) = F_B(\vartheta,\alpha)$$

As our F_A as defined in Eq. (18) does not depend on Φ, we need only Θ as a function of ϑ and α. Without lack of generality we choose the unit vector $\hat{B}$ as coinciding with the z direction and $\hat{A}$ lying in the z-x plane.

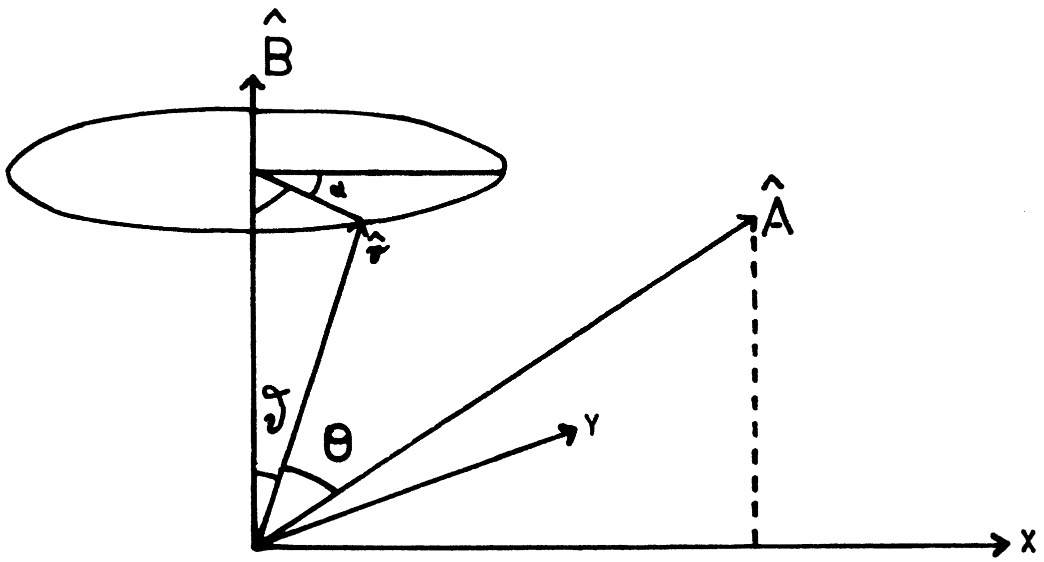

FIG. 5.

The vector $\hat{r}$ in Fig. 5 defines an arbitrary point on the unit sphere. Writing $\hat{r}$ as

$$\hat{r} = \sin\vartheta \cos\alpha \, \hat{e}_x - \sin\vartheta \sin\alpha \, \hat{e}_y + \cos\vartheta \, \hat{e}_z$$

one immediately obtains cos Θ as

$$\cos\Theta = \hat{A}\hat{r} = \sin\vartheta \cos\alpha \, A_x + \cos\vartheta \, A_z \ .$$

Substitution of cos Θ in

$$F_A(\Theta) = 1 + a \cos\Theta$$

gives

$$F_B(\vartheta,\alpha) = 1 + a[\sin\vartheta \cos\alpha \, A_x + \cos\vartheta \, A_z]$$

Averaging over the azimuth α leads to

$$\langle F_B \rangle_\alpha = \frac{1}{2\pi} \int_0^{2\pi} F_B(\vartheta, \alpha)\, d\alpha$$

$$= 1 + a\, A_z \cos\vartheta$$

or, as $\hat{B}$ is parallel to the z direction

$$\langle F_B \rangle_\alpha = 1 + a(\hat{A}\hat{B}) \cos\vartheta. \tag{19}$$

Now we apply this result to our sample of Λ decays which has the angular distribution

$$\frac{dN}{d\cos\Theta} = 1 + \alpha_\Lambda P_\Lambda \cos\Theta_p$$

with respect to the polarization direction $\hat{P}_\Lambda$. If one now introduces a coordinate system defined by three unit vectors $\hat{e}_1, \hat{e}_2, \hat{e}_3$ and calculates the decay angular distributions with respect to these directions, one obtains

$$\left.\begin{aligned} \frac{dN}{d\cos\vartheta_1} &= 1 + \alpha_\Lambda P_{\Lambda 1} \cos\vartheta_1 \\ \frac{dN}{d\cos\vartheta_2} &= 1 + \alpha_\Lambda P_{\Lambda 2} \cos\vartheta_2 \\ \frac{dN}{d\cos\vartheta_3} &= 1 + \alpha_\Lambda P_{\Lambda 3} \cos\vartheta_3 \end{aligned}\right\} \tag{20}$$

where $P_{\Lambda 1}$, $P_{\Lambda 2}$, $P_{\Lambda 3}$ are the components of the Λ polarization $\vec{P}_\Lambda$ in the directions $\hat{e}_1, \hat{e}_2, \hat{e}_3$. Thus, if α_Λ is known, the magnitude and direction of the Λ polarization can be determined by measuring the asymmetry coefficients of the angular distributions (20) with respect to the axes $\hat{e}_1, \hat{e}_2, \hat{e}_3$ of an arbitrary coordinate system. After having shown how to use the Λ decay as polarization analyser, we now apply the results to the Λ's produced in the Ξ decay.

In Eq. (17) we calculated the three components of the Λ polarization as a function of the Ξ decay angle Θ. Furthermore the coordinate system $\hat{e}_1, \hat{e}_2, \hat{e}_3$, in which the Λ polarization was defined, depends on the Λ emission direction $\hat{e}_1$ so that the polarization of every Λ is defined in a different coordinate system.

For Λ's emitted under a fixed angle Θ one will obtain decay distributions with respect to the three directions $\hat{e}_k$ which are according to Eq. (20) of the form

$$\frac{dN}{d\cos\vartheta_k} = 1 + \alpha_\Lambda P_{\Lambda k}(\Theta) \cos\vartheta_k \tag{21}$$

where ϑ_k is the angle between the direction $\hat{e}_k$ and the decay-proton in the Λ rest frame. Adding up corresponding distributions (21) for all Ξ decay angles Θ, will give the resulting distributions

$$\frac{dN}{d\cos\vartheta_k} = 1 + \alpha_\Lambda \langle P_{\Lambda k}\rangle \cos\vartheta_k$$

where $\langle P_{\Lambda k}\rangle$ is the statistical average of the polarization components $P_{\Lambda k}(\Theta)$ over the Ξ decay angle Θ. So with the Ξ decay angular distribution

$$f(\Theta) = \tfrac{1}{2}(1 + \alpha_\Xi P_\Xi \cos\Theta)$$

the averages $\langle P_{\Lambda k}\rangle$ are

$$\langle P_{\Lambda k}\rangle = \int P_{\Lambda k}(\Theta)\, f(\Theta)\, d\Omega$$

$$= \int_{-1}^{+1} P_{\Lambda k}(\Theta) \cdot \tfrac{1}{2}(1 + \alpha_\Xi P_\Xi \cos\Theta)\, d\cos\Theta \,.$$

With the expressions (17) for $P_{\Lambda k}(\Theta)$ one finds in particular

$$\langle P_{\Lambda 1}\rangle = \tfrac{1}{2}\int_{-1}^{+1} (P_\Xi \cos\Theta + \alpha_\Xi)\, d\cos\Theta = \alpha_\Xi$$

$$\langle P_{\Lambda 2}\rangle = \tfrac{1}{2}\int_0^\pi P_\Xi \beta_\Xi \sin^2\Theta\, d\Theta = \tfrac{1}{2} P_\Xi \beta_\Xi \tfrac{1}{2}(\Theta - \sin\Theta\cos\Theta)\Big/_0^\pi$$

$$= \frac{\pi}{4} P_\Xi \beta_\Xi$$

and correspondingly

$$\langle P_{\Lambda 3} \rangle = \frac{\pi}{4} P_{\Xi} \gamma_{\Xi}$$

so the distributions Eq. (22) are explicitly

$$\frac{dN}{d\cos\vartheta_1} = 1 + \alpha_{\Xi} \alpha_{\Lambda} \cos\vartheta_1 \quad ; \quad \cos\vartheta_1 = \hat{e}_p \hat{e}_1$$

$$\frac{dN}{d\cos\vartheta_2} = 1 + \frac{\pi}{4} P_{\Xi} \beta_{\Xi} \alpha_{\Lambda} \cos\vartheta_2 \quad ; \quad \cos\vartheta_2 = \hat{e}_p \hat{e}_2 \qquad (25)$$

$$\frac{dN}{d\cos\vartheta_3} = 1 + \frac{\pi}{4} P_{\Xi} \gamma_{\Xi} \alpha_{\Lambda} \cos\vartheta_3 \quad ; \quad \cos\vartheta_3 = \hat{e}_p \hat{e}_3 \; .$$

Furthermore one has the Ξ decay distribution Eq. (16)

$$\frac{dN}{d\cos\Theta} = 1 + P_{\Xi} \alpha_{\Xi} \cos\Theta \quad ; \quad \cos\Theta = \hat{e}_1 \hat{n} \; . \qquad (25a)$$

In the definitions of the angles $\hat{e}_p$ is the emission direction of the p in the Λ rest frame and $\hat{n}$ is the direction of the Ξ polarization. The $\hat{e}_k$ have been defined before as

$$\hat{e}_1 = \hat{e}_{\Lambda}$$

$$\hat{e}_2 = \frac{\hat{n} \times \hat{e}_1}{|\hat{n} \times \hat{e}_1|}$$

$$\hat{e}_3 = \frac{\hat{e}_1 \times \hat{e}_2}{|\hat{e}_1 \times \hat{e}_2|}$$

Relations, Eq. (25) have first been given by Teutsch et al.[1].

How is the situation experimentally? If the Ξ's produced in some reaction, e.g.

$$K^- + p \rightarrow \Xi^- + K^+ + n\pi$$

are unpolarized, one can, nevertheless determine the α_{Ξ} parameter from the asymmetry coefficient $a_1 = \alpha_{\Xi} \alpha_{\Lambda}$ in the first angular distribution Eq. (25)

if α_Λ is known from another experiment ($\alpha_\Lambda = 0.62 \pm 0.02$)[4]. For the other decay parameters, polarized Ξ's are required. Polarized fermions can be produced either in elastic scattering or in inelastic two-body reactions, such as

$$K^- + p \rightarrow \Xi^- + K^+$$

where the polarization has to be normal to the production plane, i.e. parallel to the direction

$$\hat{n} = \hat{K}^- \times \hat{\Xi}^- .$$

Any other direction is forbidden by parity conservation. The polarization depends on the production angle $\Theta_{pr.}$ in the c.m. of the colliding particles. $P_\Xi(\Theta_{pr.})$ tends to 0 for $\Theta_{pr.} \rightarrow 0°, 180°$. If the average Ξ polarization in an angle interval $\Theta_1 < \Theta_{pr.} < \Theta_2$

$$\langle P_\Xi \rangle = \int_{\Theta_1}^{\Theta_2} P_\Xi(\Theta_{pr.}) \left(\frac{dN(\cos \Theta_{pr.})}{d \cos \Theta_{pr.}} \right) . \; d \cos \Theta_{pr.}$$

is different from 0, the method described above can be applied. The four asymmetry coefficients in the angular distributions Eq. (25),

$$\begin{aligned} a_1 &= \alpha_\Xi \, \alpha_\Lambda \\ a_2 &= \frac{\pi}{4} P_\Xi \, \beta_\Xi \, \alpha_\Lambda \\ a_3 &= \frac{\pi}{4} P_\Xi \, \gamma_\Xi \, \alpha_\Lambda \\ a_4 &= P_\Xi \, \alpha_\Xi \end{aligned} \qquad (26)$$

are determined experimentally which allows the solving of relations Eq. (26) for $P_\Xi, \alpha_\Xi, \beta_\Xi, \gamma_\Xi$, if α_Λ is known. Actually α, β, γ are overdetermined because they have to obey the relation $\alpha^2 + \beta^2 + \gamma^2 = 1$. Therefore a bestfit can be obtained in minimizing the expression

$$M = \Sigma \left(\frac{a_k - a_{k\,best}}{\Delta a_k} \right)^2 = Min$$

with the constraint

$$\alpha^2 (a_{k\,best}) + \beta^2 (a_{k\,best}) + \gamma^2 (a_{k\,best}) = 1 .$$

Here the a_k are the experimentally observed asymmetry coefficients Eq. (26) with the errors Δa_k and, $a_{k,best}$ are the corresponding bestfitted values. The decay parameters α,β,γ can be expressed as functions of $a_{k,best}$ and so the constraint can be written in terms of $a_{k,best}$. If the polarization changes very rapidly with the production angle so that no large enough angle interval can be found with average polarization $\neq 0$, another method can be applied, as was done by Alvarez et al.[3]. From a partial wave analysis of the production angular distribution, the Ξ polarization $P_\Xi(\Theta_{pr.})$ may be derived. The decay parameters are determined by a maximum likelihood treatment of the observed events. This works in the following way. Firstly one writes down for each event, the probability W_i to occur in all its essential details (containing information about α,β,γ), as it actually had occurred in the chamber. The W_i will contain α,β,γ as unknowns. The requirement that the product of all W_i's becomes a maximum

$$\Pi W_i(\alpha,\beta,\gamma) = \text{Max}$$

together with the constraint

$$\alpha^2 + \beta^2 + \gamma^2 = 1$$

determines α,β,γ.

What does W_i look like? It contains two factors corresponding to the two successive decays.

1) The probability that the Λ of the Ξ decay includes the angle Θ_Λ with the Ξ polarization direction $\hat{P}_\Xi$:

$$F_1 = 1 + P_\Xi\ \alpha_\Xi(\hat{P}_\Xi\hat{\Lambda})\ ; \quad \cos\Theta_\Lambda = (\hat{P}_\Xi\hat{\Lambda})\ .$$

2) The probability that the proton of the Λ decay includes the angle Θ_p with the Λ polarization direction $\hat{P}_\Lambda$:

$$F_2 = 1 + P_\Lambda\ \alpha_\Lambda(\hat{P}_\Lambda\hat{p})\ ; \quad \cos\Theta_p = (\hat{P}_\Lambda\hat{\Lambda})\ .$$

Here P_Ξ is a function of the production angle $\Theta_{pr.}$

$$\vec{P}_\Xi = \vec{P}_\Xi(\Theta_{pr.})$$

and $\vec{P}_\Lambda$ is the Λ polarization, depending, according to relations Eq. (17), on the Λ emission angle Θ_Λ, on $\alpha_\Xi, \beta_\Xi, \gamma_\Xi$ and P_Ξ:

$$\vec{P}_\Lambda = \vec{P}_\Lambda(\Theta_\Lambda, \alpha_\Xi, \beta_\Xi, \gamma_\Xi, P_\Xi) \ .$$

So the maximum likelihood function

$$M = \prod_{i=1}^{N} W_i = \prod_{i=1}^{N} F_{1i} \ F_{2i}$$

depends on the decay parameters which can be calculated by maximizing M.

The latest results about the Ξ decay parameters are summarized in the following Table. The data of Alvarez et al.[3] was obtained with the overall maximum likelihood method described before. As no resulting Ξ polarization was observed, $P_\Xi(\Theta_{pr.})$ was determined from a partial wave analysis. The other two groups (UCLA, EP+) found a resulting polarization and applied the analysis method proposed by Teutsch et al.[1]

Table I

Ξ decay parameters

Lab.	Ref.	α_{Ξ^-}	β_{Ξ^-}	γ_{Ξ^-}
LRL	(5)	-0.41 ± 0.08	$+0.08 \pm 0.26$	$+0.91$
UCLA	(5)	-0.64 ± 0.13	$+0.65 \pm 0.16$	$+0.41 \pm 0.28$
EP+ ...	(6)	-0.44 ± 0.11	$-0.24^{+0.57}_{-0.50}$	$+0.87^{+0.05}_{-0.28}$

APPENDIX I

RELATIVISTIC GENERALIZATION OF THE RESULTS

When calculating the Λ spin state as a function of the emission angles Θ,φ in the Ξ rest frame

$$\Psi_\Lambda = a(\Theta,\varphi)\begin{pmatrix}1\\0\end{pmatrix} + b(\Theta,\varphi)\begin{pmatrix}0\\1\end{pmatrix}$$

we tacitly assumed that the Λ spin function Ψ_Λ refers to the Λ rest frame. In fact the spin has a simple physical meaning only in the rest frame of a particle where the eigenfunctions $|j,m\rangle$ are defined. In the spin description we did not distinguish between the Ξ system where the decay orbital angular momentum is strictly defined and the Λ rest system where the Λ spin is strictly defined. It is clear therefore that our treatment is typically non-relativistic.

Let us now try to find a relativistic generalization.

For this purpose it is necessary to construct spin states for particles in motion.

The relativistic decay state of the Ξ in its rest system can be written as a linear superposition of states

$$|L,M,s,m\rangle = \int d\Omega_{\vec{p}}\, Y_L^M(\Omega_{\vec{p}})\ \varphi_{\vec{p}}\ \Gamma_{\vec{p}}\ \chi_s^m \qquad (A.1)$$

where the quantities on the right side are defined as follows:

$\Omega_{\vec{p}}$: angles $\Theta_{\vec{p}},\Phi_{\vec{p}}$, denoting the direction of the Λ momentum $\vec{p}$ in the Ξ rest system.

$d\Omega_{\vec{p}}$: $d\cos\Theta_{\vec{p}}\, d\Phi_{\vec{p}}$.

$Y_L^M(\Omega_{\vec{p}})$: decay orbital angular momentum state with angular momentum L and z component M.

$\varphi_{\vec{p}}$: plane wave state $e^{i\vec{p}\vec{x}}$ for Λ (and π) in Ξ rest system where Λ has momentum $\vec{p}$ and π has $-\vec{p}$.

χ_s^m : spin eigenstate of Λ in its own rest system which is so defined that its coordinate axes are parallel to those in the Ξ rest system. s denotes the Λ spin, in its z component.

If for example the Λ spin is ½, $\chi_{1/2}^{1/2}$ and $\chi_{1/2}^{-1/2}$ can be described by the Dirac spinors in the particle rest system

$$\chi_{1/2}^{1/2} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \chi_{1/2}^{-1/2} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$\Gamma_{\vec{p}}$: is a matrix that transforms the Λ spin state from the Λ rest system to the Ξ rest system. It corresponds to the Lorentz transformation $L_{\vec{p}}$ which brings the Λ rest system to the Ξ rest system:

$$L_{\vec{p}} : \begin{cases} \vec{x}' = \vec{x} + \beta\gamma \left[\frac{\gamma}{\gamma+1} \vec{\beta}\vec{x} - t \right] \\ t' = \gamma \left[t - \vec{\beta}\vec{x} \right] \quad \text{with} \quad \gamma = \frac{1}{\sqrt{1-\beta^2}} \end{cases} \qquad (A.2)$$

where $\vec{\beta}$ is the velocity of the Λ in the Ξ rest system. $\Gamma_{\vec{p}}$ reduces to the unit matrix in the non-relativistic case.

In the following we shall show that the rotation properties of the relativistic states $|L,M,s,m\rangle$ are independent of the Λ momentum $\vec{p}$, i.e. they transform under rotations in the Ξ rest system as the corresponding non-relativistic states. Once this is proved, one may argue as follows: All angular momentum operators J_x, J_y, J_z are proportional to operators of infinitesimal rotations about the x,y,z axes. The commutation relations between J_x, J_y, J_z define the entire spin-angular momentum algebra. As in our problem the relativistic states Eq. (A.1) transform under rotations as the corresponding non-relativistic states, our non-relativistic treatment of the Ξ decay, which considers only angular momentum in the Ξ rest system is formally identical to the relativistic treatment when expressing the Ξ decay state as a linear superposition of the states $|L,M,s,m>$ in Eq. (A.1) So it remains to show that the rotation transformations of these states are independent of the Λ momentum $\vec{p}$.

Apply a rotation R in the Ξ rest system to the state Eq. (A.1)

$$R|L,M,s,m> = \int R\, d\Omega_{\vec{p}} \left(R\ Y_L^M(\Omega_{\vec{p}})\ \varphi_{\vec{p}}\right)\left(R\ \Gamma_{\vec{p}} \chi_s^m\right) .$$

The factor $Y_L^M(\Omega_{\vec{p}})\,\varphi_{\vec{p}}$ is a plane wave of amplitude $Y_L^M(\Omega_{\vec{p}})$ running in the direction of $\vec{p}$. Rotating this configuration gives another plane wave $\varphi_{R\vec{p}}$ of the same amplitude but running in the direction of $R\vec{p}$, i.e. $Y_L^M(\Omega_{\vec{p}})\varphi_{R\vec{p}}$. Therefore we have

$$R|L,M,s,m> = \int d\Omega_{R\vec{p}}\ Y_L^M(\Omega_{\vec{p}})\,\varphi_{R\vec{p}}\left(R\ \Gamma_{\vec{p}}\chi_s^m\right)$$

where we have replaced $d\Omega_{\vec{p}}$ by $d\Omega_{R\vec{p}}$ (one has $d\Omega_{\vec{p}} = d\Omega_{R\vec{p}}$).

Below we show that

$$R\Gamma_{\vec{p}} = \Gamma_{R\vec{p}}\ R \qquad (A.3)$$

i.e. instead of first transforming the Λ spin state to the Ξ system and then rotating it, one can first rotate the Λ spin state in the Λ rest system and then transform it to the Ξ rest system along the rotated direction $R\vec{p}$. With Eq. (A.3) one can write

$$R|L,M,s,m> = \int d\Omega_{R\vec{p}}\ Y_L^M(\Omega_{\vec{p}})\,\varphi_{R\vec{p}}\,\Gamma_{R\vec{p}}\ R\chi_s^m \quad .$$

Substituting $\vec{p}$ for $R\vec{p}$ gives

$$R|L,M,s,m> = \int d\Omega_{\vec{p}}\ Y_L^M\ (\Omega_{R^{-1}\vec{p}})\ \varphi_{\vec{p}}\ \Gamma_{\vec{p}}\ R\chi_s^m \quad .$$

In terms of the D matrices for finite rotations of non-relativistic spin states one can write

$$Y_L^M(\Omega_{R^{-1}\vec{p}}) = \sum_{M'} D^{(L)}_{MM'}(R)\, Y_L^{M'}(\Omega_{\vec{p}}) \quad \text{and}$$

$$R\chi_s^m = \sum_{m'} D^{(s)}_{mm'}(R)\, \chi_s^{m'}$$

and consequently:

$$R|L,M,s,m\rangle = \sum_{M'm'} D^{(L)}_{MM'}(R)\, D^{(s)}_{mm'}(R)\, |L,M',s,m'\rangle \quad , \tag{A.4}$$

just as in the non-relativistic case. So it is shown that the relativistic states transform under rotations in the Ξ rest system as non-relativistic spin states.

We still have to prove the relation

$$R\Gamma_{\vec{p}} = \Gamma_{R\vec{p}}\, R \tag{A.3}$$

which is **essential for** the simple transformation property Eq. (A.4) of the states $|L,M,s,m\rangle$. Eq. (A.3) is equivalent to the relation

$$R\, L_{\vec{p}} = L_{R\vec{p}}\, R \tag{A.5}$$

where $L_{\vec{p}}$ is the Lorentz transformation Eq. (A.2)

$RL_{\vec{p}}$ is

$$\vec{x}' = R\vec{x} + R\vec{\beta}\gamma\left[\frac{1}{\gamma+1}(\vec{\beta}\vec{x}) - t\right]$$

$$t' = \gamma\left[t - (\vec{\beta}\vec{x})\right]$$

$L_{R\vec{p}}R$ is

$$\vec{x}' = R\vec{x} + R\vec{\beta}\gamma\left[\frac{1}{\gamma+1}(R\vec{\beta}.R\vec{x}) - t\right]$$

$$t' = \gamma\left[t - (R\vec{\beta}.R\vec{x})\right]$$

Hence

$$RL_{\vec{p}} = L_{R\vec{p}}R$$

follows from $(R\vec{\beta}.R\vec{x}) = (\vec{\beta}\vec{x})$.

Although we have now proved that our original treatment of the Ξ decay is formally correct for the relativistic case also, we have still to re-interpret the results in terms of the relativistic decay states, Eq. (A.1). According to the definition of these states the Λ spin eigenstates χ_s^m refer to that Λ rest system which results from the Ξ rest system by applying a Lorentz transformation $L_{-\vec{p}}$ (defined in Eq. (A.2)) to the Ξ rest system. Therefore to every Λ emission direction there corresponds a different Λ rest system, in which the Λ polarization and its decay angular distribution are exactly described by the non-relativistic formulae derived in this paper.

In this connection it has to be stressed that in the relativistic case the Λ rest system defined as above by a Lorentz translation applied to the Ξ CM system is only one among an infinite number of possible rest systems, differing from each other by rotations of the coordinate system. For instance if one transforms the Λ from the Ξ CM first to the laboratory system and from there by a translation to its rest system, this rest system will, in general, be different from the rest system defined above, by a rotation which tends to zero for decreasing relative velocities. This rotation is a purely kinematical effect known as "Thomas precession". It comes from the fact that translations are not a subgroup of the Lorentz group in contrast to the Galilei group. That means the product of two Lorentz translations is in general a translation coupled with a rotation.

To compare experiment with theory one has to transform experimental quantities such as polarization or angular distribution, to the system in which they have been calculated theoretically. In our problem of the decay parameters, we are, for instance, interested in the angular distribution of the protons from the Λ decay with respect to the Λ direction of flight <u>measured in that Λ rest frame which is defined by the Ξ rest frame and the Λ momentum $\vec{p}_\Lambda$ in this system</u>. Experimentally everything is observed in the laboratory, so to compare with the theoretical angular - or polarization distribution, one has firstly to transform the Λ <u>and the decay proton</u> to the Ξ rest frame, and from there, the proton to the Λ rest frame defined as above.

Let us finally calculate the rotation which results from two successive Lorentz translations.

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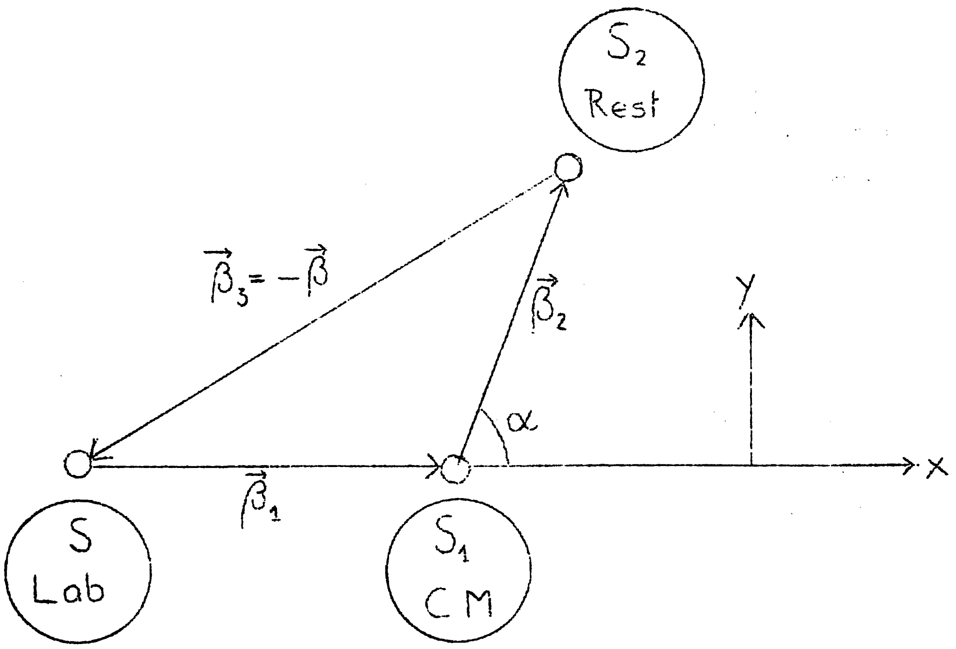

Fig. e

Assume system S_1 moves relative to system S with a velocity β_1 in x direction. System S_2 moves relative to system S_1 with a velocity $\vec{\beta}_2$ lying in the x-y plane and including an angle α with the x axis. System S moves with the velocity $\vec{\beta}_3 = -\vec{\beta}$ relative to S_2 where $\vec{\beta}$ is the velocity resulting from the addition of $\vec{\beta}_1$ and $\vec{\beta}_2$

$$\vec{\beta} = \frac{\vec{\beta}_1 + \vec{\beta}_2 \gamma_2 \left[\frac{\gamma_2}{\gamma_2 + 1} (\vec{\beta}_1 \vec{\beta}_2) + 1 \right]}{\gamma_2 \left[1 + \vec{\beta}_1 \vec{\beta}_2 \right]} \tag{A.6}$$

In connection with our problem, systems S, S_1 and S_2 can be identified with the laboratory system, the Ξ CM system, and the Λ rest system respectively.

The Lorentz translation

$T(\vec{\beta}_1)$ transforms S into S_1

$T(\vec{\beta}_2)$ transforms S_1 into S_2

so the product transformation

$L = T(\vec{\beta}_2)\, T(\vec{\beta}_1)$ transforms S into S_2 .

What we wish to show is that

$$L = T(\vec{\beta}_2)\ T(\vec{\beta}_1)$$

can be written as the product of a translation $T(\vec{\beta})$ and a rotation $R(\omega)$:

$$R(\omega)\ T(\vec{\beta}) = T(\vec{\beta}_2)\ T(\vec{\beta}_1)$$

or

$$R(\omega) = T(\vec{\beta})^{-1}\ T(\vec{\beta}_2)\ T(\vec{\beta}_1)$$

or, as

$$\vec{\beta} = -\vec{\beta}_3$$

$$R(\omega) = T(\vec{\beta}_3)\ T(\vec{\beta}_2)\ T(\vec{\beta}_1)\ . \qquad \text{(A.7)}$$

Let us represent the T's as matrices operating on four vectors:

$$T(\vec{\beta}_1) = \begin{bmatrix} \gamma_1 & 0 & 0 & i\beta_1\gamma_1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta_1\gamma_1 & 0 & 0 & \gamma_1 \end{bmatrix}$$

$$T(\vec{\beta}_2) = \begin{bmatrix} 1+(\gamma_2-1)\cos^2\alpha & (\gamma_2-1)\sin\alpha\cos\alpha & 0 & i\beta_2\gamma_2\cos\alpha \\ (\gamma_2-1)\sin\alpha\cos\alpha & 1+(\gamma_2-1)\sin^2\alpha & 0 & i\beta_2\gamma_2\sin\alpha \\ 0 & 0 & 1 & 0 \\ -i\beta_2\gamma_2\cos\alpha & -i\beta_2\gamma_2\sin\alpha & 0 & \gamma_2 \end{bmatrix}$$

$T(\vec{\beta}_3)$ is a matrix corresponding to $T(\vec{\beta}_2)$ but containing instead of β_2, γ_2

$$\beta_3 = \frac{1}{\gamma_3}\sqrt{\gamma_3^2 - 1}$$

$$\gamma_3 = \gamma_1\gamma_2\left[1 + \beta_1\beta_2\cos\alpha\right]$$

and according to Eq. (A.6) instead of $\sin\alpha$, $\cos\alpha$

$$\sin\varphi = -\gamma_1 \frac{\beta_2\gamma_2 \sin\alpha \left[\frac{\gamma_2}{\gamma_2+1}\beta_1\beta_2 \cos\alpha + 1\right]}{\sqrt{\gamma_3^2 - 1}}$$

$$\cos\varphi = -\gamma_1 \frac{\beta_1 + \beta_2\gamma_2 \cos\alpha \left[\frac{\gamma_2}{\gamma_2+1}\beta_1\beta_2 \cos\alpha + 1\right]}{\sqrt{\gamma_3^2 - 1}}$$

Working out the product matrix

$$R(\omega) = T(\vec{\beta}_3)\ T(\vec{\beta}_2)\ T(\vec{\beta}_1)$$

one finds that it has the structure

$$R(\omega) = \begin{bmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

This Lorentz matrix can only effect a rotation about the z axis by an angle ω with

$$a_{11} = \cos\omega \qquad a_{12} = -\sin\omega$$
$$a_{21} = \sin\omega \qquad a_{22} = \cos\omega\ .$$

The element a_{12} can be written in the form

$$a_{21} = \sin\omega = \beta_1\beta_2 \sin\alpha\ \gamma_1\gamma_2 \frac{1+\gamma_1+\gamma_2+\gamma_3}{(1+\gamma_1)(1+\gamma_2)(1+\gamma_3)};\ \gamma_3 = \gamma$$

or in an arbitrary coordinate system

$$\sin\omega = \vec{\beta}_1 \times \vec{\beta}_2\ \gamma_1\gamma_2 \frac{1+\gamma_1+\gamma_2+\gamma}{(1+\gamma_1)(1+\gamma_2)(1+\gamma)} \qquad \text{(A.8)}$$

Applied to our special case, $\vec{\beta}_1$ is the Ξ velocity in the laboratory, $\vec{\beta}_2$ is the Λ velocity in the Ξ system and $\vec{\beta}$ is the Λ velocity in the laboratory. The result means the following: When transforming the decay proton directly from the laboratory to the Λ rest system one has afterwards to rotate the proton momentum vector by an angle ω defined by Eq. (A.8) in the right-handed sense around the axis $\vec{\beta}_1 \times \vec{\beta}_2$ in order to make it coincide with the proton momentum vector which was transformed via the Ξ system to the Λ rest frame.

Generalized treatments of the relativistic spin problem can be found in articles by H.P. Stapp[7], M.I Shirokov[8], N. Jacob and G.C. Wick[9] and N. Jacob[10].

APPENDIX II

CONSEQUENCES OF TIME REVERSAL INVARIANCE AND FINAL STATE INTERACTIONS

As already mentioned time reversal invariance predicts a zero β decay parameter in the case where strong final state interactions do not modify the outgoing wave. For this theorem we want to give here, only an intuitive argument which is essentially due to Crawford[11]. A more generalized and rigorous treatment of these problems can be found in an article by Lee and Yang[12].

In the Ξ decay the β parameter is proportional to a Λ polarization perpendicular to the Ξ polarization direction $\hat{P}_{\Xi}$ and the Λ emission direction $\hat{\Lambda}$. We shall assume a Ξ decay state as indicated in Fig. a) with a Λ polarization in the direction

$$\hat{P}_{\Xi} \times \hat{\Lambda}$$

and shall show that this configuration leads to a contradiction with time reversal invariance.

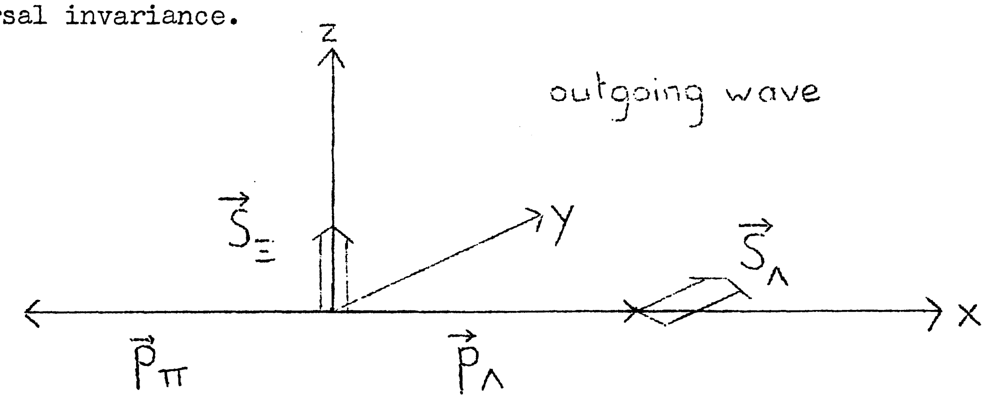

Fig. a)

In Fig. a the Ξ is polarized in the z direction, the Λ emission direction is the x axis and the Λ is assumed to be polarized in the y direction. Under the time reversal operation T momenta and spins change sign

$$T\vec{p} = -\vec{p}$$
$$T\vec{S} = -\vec{S} \ .$$

The corresponding time reversed state of Fig. a) is indicated in Fig. b).

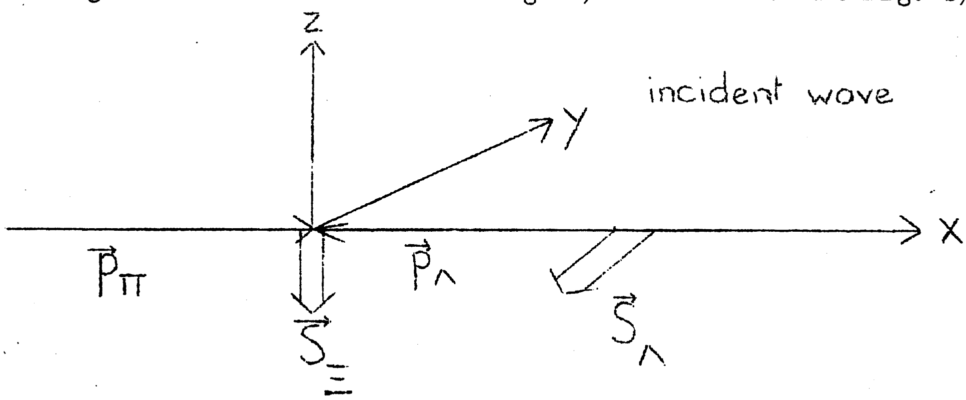

Fig. b)

The time reversed incident state Fig. b) can give rise to a weak scatter. Due to the low coupling constant, weak interactions are in general, sufficiently well described in first order perturbation theory which does not contain spin-orbit couplings. As a consequence no spinflip can occur so that in our case after the weak scatter the Λ has still the same spin direction. After the scatter we have the configuration of Fig. c).

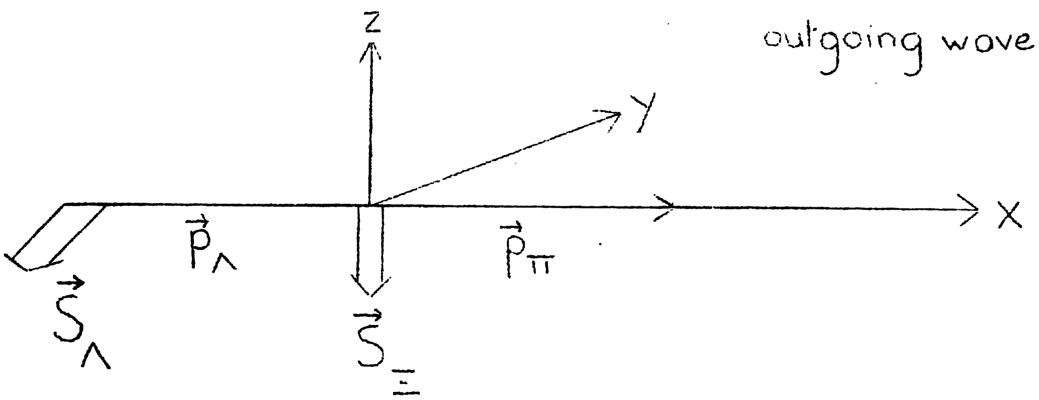

Fig. c)

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Rotating the configuration in Fig. c) by 180° about the z axis and by 180° about the x axis leads to the situation in Fig. d).

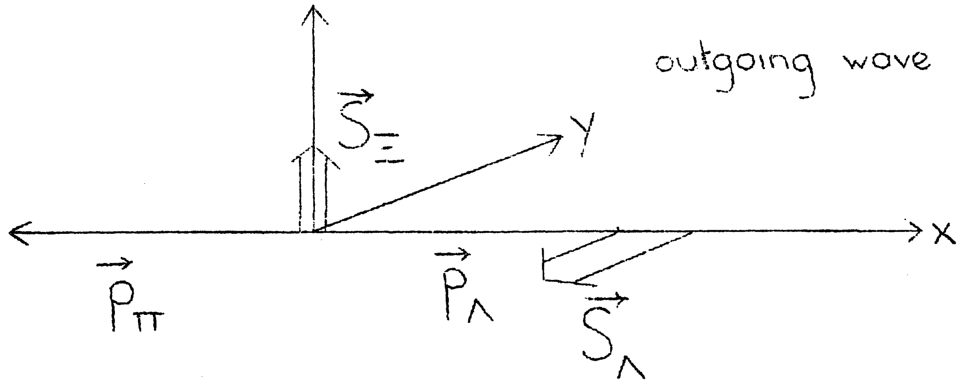

Fig. d

From state a) we came to state d) by the time reversal operation and a weak scatter that did not change spin and momentum directions. Now the principle of time reversal invariance postulates that, to every state there exists another time reversed state, which also satisfies the equations of motion. So if state a) satisfies the equations of motion of the Ξ decay, the time reversed state d) should also satisfy them. When starting from a Ξ in a pure spin state after the decay the Λ has also to be in a pure spin state i.e. completely polarized in a definite direction. As states a) and the time reversed state d) have opposite Λ polarizations but the same Ξ polarization, they cannot both satisfy the Hamiltonian of the Ξ decay. As a consequence a non-vanishing Λ polarization in the direction $\vec{S}_\Xi \times \vec{p}_\Lambda$ violates the time reversal invariance of the weak decay interaction.

By the same kind of argumentation one can generalize our result and show that in purely weak interactions all scalar or pseudoscalar observables that change sign under the time reversal operation, have zero expectation value. Such observables are for example:

$$\vec{S}_1 \cdot (\vec{S}_2 \times \vec{S}_3)$$
$$\vec{P}_1 \cdot (\vec{P}_2 \times \vec{P}_3)$$
$$\vec{S}_1 \cdot (\vec{S}_2 \times \vec{P}) .$$

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On the other hand, strong final state interactions can flip the Λ spin and produce a $\beta \neq 0$. By the strong final state interactions the phases of the S and P wave amplitudes are shifted

$$A_s \rightarrow A_s' = A_s\, e^{i\delta_s}$$
$$A_p \rightarrow A_p' = A_p\, e^{i\delta_p} .$$

As shown before A_s and A_p are relatively real if time reversal invariance holds. So in the case of final state interactions the observed decay parameters are

$$\alpha = 2 \text{ Re } A_s'^* A_p' = 2 \text{ Re}(A_s^* A_p) \cos (\delta_p - \delta_s)$$
$$\beta = 2 \text{ Im } A_s'^* A_p' = 2 \text{ Re}(A_s^* A_p) \sin (\delta_p - \delta_s)$$
$$\gamma = |A_s'|^2 - |A_p'|^2 .$$

If the phase shifts are known the influence of the final state interaction can be subtracted.

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SOME METHODS OF SPIN DETERMINATION OF ELEMENTARY PARTICLES AND RESONANCES

W. Koch,
Track Chamber Division, CERN.

I. RECAPITULATION OF QUANTUM MECHANICAL SPIN DESCRIPTION

This chapter is mainly intended to supply the reader with the necessary formulae used later on. As most of them can be found in familiar textbooks, such as Shiff, Condon and Shortley, Messiah etc., almost no derivations or proofs will be given.

a) Operators and Eigenfunctions

In analogy to the classical definition of angular momentum

$$\vec{L} = \vec{r} \times \vec{p}$$

quantum mechanical spin operators, J_x, J_y, J_z for the components of the total angular momentum, and $\vec{J}^2$ for the square, can be defined which obey the commutation relations:

$$[J_x, J_y] = i\hbar J_z,\ [J_y, J_z] = i\hbar J_x,\ [J_z, J_x] = i\hbar J_y\ . \qquad \text{(I.1)}$$

From this follows

$$[\vec{J}^2, J_{x,y,z}] = 0\ .$$

As all the spin components commute with $\vec{J}^2$ but do not commute with one another, only one component can have simultaneous eigenstates with $\vec{J}^2$. It is customary to choose representations where J_z and $\vec{J}^2$ have the same eigenstates.

Eigenstates and eigenvalues of the spin operators are defined by the eigenvalue equations

$$\begin{aligned} \vec{J}^2\ \Psi_J &= C_{\vec{J}^2}\ \Psi_J \\ J_z\ \Psi_J &= C_{J_z}\ \Psi_J\ . \end{aligned} \qquad \text{(I.2)}$$

Applying the commutation relations to these operator equations one finds as possible eigenvalues for $\vec{J}^2$ and J_z

$$C_{\vec{J}^2} = \hbar j(j+1), \quad j = \tfrac{1}{2}, 1, \tfrac{3}{2}, 2 \ldots \qquad \text{(I.3)}$$
$$C_{J_z} = \hbar m \qquad m = -j, -j+1, \ldots +j.$$

Only integer or half integer values are allowed for j, the (quantum number of) total angular momentum. Each angular momentum j has (2j+1) possible integer or half integer values for the magnetic quantum number m which runs from m = -j to m = +j.

At the same time the eigenvalue equations define an eigenfunction

$$\Psi_J = \Psi(j,m)$$

for every pair j,m, so that the solution of Eq. (I.2) is

$$\vec{J}^2\ \Psi(j,m) = \hbar j(j+1)\ \Psi(j,m)$$
$$J_z\ \Psi(j,m) = \hbar m \Psi(j,m) \qquad \text{(I.4)}$$
$$j = \tfrac{1}{2}, 1, \tfrac{3}{2} \ldots, \quad m = -j, -j+1 \ldots +j .$$

So far the symbol $\Psi(j,m)$ is a rather abstract object, a so-called "basis vector in Hilbert space", specifying an eigenstate of angular momentum. Instead of $\Psi(j,m)$ we shall use as well the Dirac ket vector notation

$$|j,m\rangle \equiv \Psi(j,m) .$$

Once a specific spin representation is chosen, $|j,m\rangle$ will be a well defined mathematical object, e.g. a column vector or a spherical harmonic.

<u>Examples</u>

$$\psi(\tfrac{1}{2},\tfrac{1}{2}) = |\tfrac{1}{2},\tfrac{1}{2}\rangle = \begin{bmatrix}1\\0\end{bmatrix}$$

$$\psi(\tfrac{3}{2},-\tfrac{1}{2}) = |\tfrac{3}{2},-\tfrac{1}{2}\rangle = \begin{bmatrix}0\\0\\1\\0\end{bmatrix}$$

$$\varphi(1,1) = |1,1\rangle = \begin{bmatrix}1\\0\\0\end{bmatrix} = Y_1^1 = -\frac{1}{\sqrt{4\pi}}\sqrt{\tfrac{3}{2}}\ \sin\Theta\ e^{i\varphi}$$

For any basis vector $\psi(j,m)$ a conjugate vector

$$\psi^*(j,m) \equiv \langle j,m|$$

can be defined together with an operation

$$(\psi^*(j_1,m_1),\ \psi(j_2,m_2)) \equiv \langle j_1 m_1 | j_2 m_2 \rangle$$

which we call the scalar product of two states. This product is 1 for identical basis vectors and 0 for different basis vectors:

$$\langle j_1,m_1 | j_2 m_2 \rangle = \delta_{j_1 j_2}\ \delta_{m_1 m_2}\ . \tag{I.5}$$

This property is called orthonormality. It can be shown that the $|j,m\rangle$ form a complete orthonormal set of basis vectors.

In the matrix representation the scalar product of two states is the scalar product of two vectors.

$$\langle j_1,m_1 | j_2,m_2 \rangle = \begin{bmatrix} a_1^* \\ a_2^* \\ \cdot \\ \cdot \\ a_\mu^* \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ b_\mu \end{bmatrix} \tag{I.6}$$

$$= \sum_{k=1}^{\mu} a_m^* b_k = \delta_{j_1 j_2}\ \delta_{m_1 m_2}$$

(Note: Column vectors representing conjugate states should always be regarded as row vectors in the conventional mathematical meaning.)
In terms of spherical harmonics $Y_j^m(\Theta,\varphi)$, the scalar product of two states is an integral over the total solid angle

$$\langle j_1,m_1 | j_2 m_2 \rangle = \int Y_{j_1}^{m_1\,*}\, Y_{j_2}^{m_2}\ d\Omega = \delta_{j_1 j_2}\ \delta_{m_1 m_2} \tag{I.7}$$

where Y* is the complex conjugate of Y.

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b) States and Matrix Elements

Consider a quantum mechanical system of spin j. The most general pure state is by definition a linear superposition of basis vectors belonging to the same j:

$$\Psi_j = |j\rangle = \sum_{m=-j}^{m=+j} a_m |j,m\rangle \tag{I.8}$$

and the conjugate state

$$\Psi_j^* = \langle j| = \sum_{m=-j}^{m=j} a_m^* \langle j,m| \; .$$

The a_m are complex numbers and a_m^* their complex conjugates. It is useful to consider only normalized states with

$$\sum_{m=-j}^{j} a_m^* a_m = \sum_{m=-j}^{j} |a_m|^2 = 1 \; . \tag{I.9}$$

From the orthonormality Eq. (I.5) and the normalization condition Eq. (I.9), it follows that the scalar product of the state $|j\rangle$ with itself $\langle j|j\rangle$ is 1.

$$\langle j|j\rangle = \sum_{m=-j}^{j} a_m^* a_m \underbrace{\langle j,m|j,m\rangle}_{1} = 1 \; . \tag{I.10}$$

So far we have considered only pure states which are described by a linear superposition of basis vectors. This mode of description implies that the maximum information possible in a quantum mechanical framework is actually available. There also exist, however, "mixed states" with less than maximum information which are statistical mixtures of pure states.

Example

In a system of spin $\frac{1}{2}$ particles, a particle is with a probability of p_1 in the pure state ψ_1 and a probability of p_2 in ψ_2. A description of this state could be e.g.

$$\left\{\begin{array}{ll} \psi_1 = |\tfrac{1}{2},\tfrac{1}{2}> & p_1 = 0.3 \\ \psi_2 = \frac{1}{\sqrt{2}}|\tfrac{1}{2},\tfrac{1}{2}> + \frac{i}{\sqrt{2}}\,|\tfrac{1}{2},-\tfrac{1}{2}> & p_2 = 0.7 \end{array}\right\}$$

Mixed states will be treated in detail in connection with the density matrix.

From the general concept of quantum mechanics it is well known how to calculate expectation values of physical observables. For every observable q a Hermitian operator Q can be defined. Then the expectation value of a physical observable q in a system specified by a state vector $|a>$ is

$$<Q> = <a|Q|a> \, . \qquad (I.11)$$

This means: Apply the operator Q to the state $|a>$ which will give you a new state $|b>$

$$|b> = Q|a> \, .$$

Then form the scalar product of $|a>$ and $|b>$

$$<Q> = <a|b> = <a|\,Q|\,a> \, .$$

In terms of spin eigenstates we have the following relations:

$$\begin{aligned} |a> &= \Sigma a_m |\,j,m> \\ <a| &= \Sigma a_m^* <j,m| \\ |b> &= Q|a> = \Sigma b_m |\,j,m> \end{aligned} \qquad (I.12)$$

$$<Q> = <a|Q|a> \; = <a|b> = \Sigma a_m^* b_m \underbrace{<jm|\,jm>}_{1} \; .$$

This can be written in a more convenient way:

$$<a|Q|a> = \left\langle \Sigma a_m^* <j,m|Q|\Sigma a_k|jk> \right\rangle$$

$$= \sum_{m,k} a_m^* a_k <j,m|Q|j,k> \qquad (I.13)$$

where the quantities

$$<j,m|Q|j,k>$$

are the so-called "matrix elements" of the operator Q. They are defined in the same way as the expectation value $<Q>$ in Eq. (I.11) but now connecting two eigenstates $|j,m>$ and $|j,k>$.

For the operators $J_x, J_y, J_z, \vec{J}^2$ these matrix elements can be derived from the commutation relations. The result is:

$$\begin{aligned}
<j,m+1|J_x|j,m> &= \tfrac{1}{2}\sqrt{(j-m)(j+m+1} \\
<j,m-1|J_x|j,m> &= \tfrac{1}{2}\sqrt{(j+m)(j-m+1)} \\
<j,m+1|J_y|j,m> &= -\tfrac{1}{2}i\sqrt{(j-m)(j+m+1)} \\
<j,m-1|J_y|j,m> &= \tfrac{1}{2}i\sqrt{(j+m)(j-m+1)} \\
<j,m|J_z|j,m> &= m \\
<j,m|\vec{J}^2|j,m> &= j(j+1) \ .
\end{aligned} \qquad (I.14)$$

All other matrix elements are 0. The factor $\hbar$ on the right side has been omitted. As is seen from Eq. (I.14) the J_x and J_y matrix elements are connected by the relation

$$<j,m'|J_y|j,m> = e^{i\frac{\pi}{2}(m-m')} <j,m|J_x|jm'> \ . \qquad (I.15)$$

Let us now calculate the spin expectation values in x,y,z direction for a general pure state of spin j

$$|j> = \sum_{m=-j}^{j} a_m|j,m> \ .$$

From the matrix elements given in Eq. (I.14) and the definition of an expectation value Eq. (I.13) follows immediately

$$\langle j|J_z|j\rangle = \langle J_z\rangle = \sum_{m=-j}^{+j} m|a_m|^2 \; . \tag{I.16}$$

Furthermore

$$\langle j|J_x|j\rangle = \langle J_x\rangle =$$

$$= \sum_{m=-j}^{j-1} a^*_{m+1}\, a_m \langle j,m+1|J_x|j,m\rangle + \sum_{m=-j}^{j-1} a^*_m\, a_{m+1} \langle j,m|J_x|j,m+1\rangle$$

$$= \sum_{m=-j}^{j-1} a^*_{m+1}\, a_m \tfrac{1}{2}\sqrt{(j-m)(j+m+1)} + \sum_{m=-j}^{j-1} a^*_m\, a_{m+1} \tfrac{1}{2}\sqrt{(j-m)(j+m+1)}$$

$$= \tfrac{1}{2} \sum_{m=-j}^{j-1} (a^*_m\, a_{m+1} + a_m\, a^*_{m+1}) \sqrt{(j-m)(j+m+1)}$$

$$= \sum_{m=-j}^{j-1} \mathrm{Re}(a^*_{m+1}\, a_m) \sqrt{(j-m)(j+m+1)} \; .$$

A similar calculation yields a corresponding expression for $\langle J_y\rangle$. So the expectation values for the three spin components are:

$$\langle J_x\rangle = \sum_{m=-j}^{j-1} \mathrm{Re}(a^*_{m+1}\, a_m) \sqrt{(j-m)(j+m+1)}$$

$$\langle J_y\rangle = \sum_{m=-j}^{j-1} \mathrm{Im}(a^*_{m+1}\, a_m) \sqrt{(j-m)(j+m+1)} \tag{I.17}$$

$$\langle J_z\rangle = \sum_{m=-j}^{j+j} m|a_m|^2 \; .$$

Let us now introduce the concept of the polarization vector $\vec{P}(P_x, P_y, P_z)$.

Definition

$$P_x = \frac{<J_x>}{j} \; , \; P_y = \frac{<J_y>}{j} \; , \; P_z = \frac{<J_z>}{j} \qquad (I.18)$$

$$\text{or} \quad \vec{P} = \frac{1}{j} <\vec{J}> .$$

This means, the component of the polarization vector in a certain direction is given by the spin expectation value in this direction, divided by the maximum spin expectation value, which is m = j.

Inspecting the expressions for $<J_x>$, $<J_y>$ in Eq. (I.17) one states that only the interference of adjacent eigenstates

$$a_m | j,m> \; , \; a_{m+1} | j,m+1>$$

produces polarization in x or y direction. For example the states

$$\psi = \frac{1}{\sqrt{2}} |1,1> + \frac{1}{\sqrt{2}} |1,-1> \quad \text{or}$$

$$\psi = \frac{1}{\sqrt{2}} |\tfrac{3}{2}, \tfrac{3}{2}> + \frac{1}{\sqrt{2}} |\tfrac{3}{2}, -\tfrac{1}{2}>$$

have $\qquad P_x, P_y = 0$

so at least one pair of neighbouring amplitudes a_m in the expression for the general pure state $|j>$ has to be different from 0 in order to give non-vanishing polarization in x or y direction. This criterion for polarization $P_x, P_y \neq 0$, however, is only necessary but not sufficient, as the following example shows.

Consider the state

$$\psi = e^{i\varphi} \left[\frac{1}{\sqrt{3}} |1,1> + \frac{1}{\sqrt{3}} |1,0> - \frac{1}{\sqrt{3}} |1,-1> \right].$$

Applying Eq. (I.17) yields

$$\langle J_x \rangle = \underbrace{\mathrm{Re}(e^{-i\varphi}\, e^{+i\varphi})}_{1} \left\{ -\frac{1}{3}\underbrace{\sqrt{(1+1)(1-1+1)}}_{\sqrt{2}} + \frac{1}{3}\underbrace{\sqrt{(1-0)(1+0+1)}}_{\sqrt{2}} \right\} = 0$$

$$\langle J_y \rangle = 0 \quad \text{because} \quad \mathrm{Im}(e^{-i\varphi}\, e^{+i\varphi}) = 0$$

$$\langle J_z \rangle = \frac{1}{3} \cdot 1 + \frac{1}{3}(-1) = 0 \ .$$

As we shall often deal with spin ½ particles in the following, let us re-write relations Eq. (I.17) for j = ½.

General j = ½ state: $|{\textstyle\frac{1}{2}}\rangle = a|{\textstyle\frac{1}{2}},{\textstyle\frac{1}{2}}\rangle + b|{\textstyle\frac{1}{2}},-{\textstyle\frac{1}{2}}\rangle$

with $|a|^2 + |b|^2 = 1$

$$\begin{aligned} P_x &= 2\langle J_x \rangle = 2\mathrm{Re}\ a^*b \\ P_y &= 2\langle J_y \rangle = 2\mathrm{Im}\ a^*b \\ P_z &= 2\langle J_z \rangle = |a|^2 - |b|^2 \end{aligned} \qquad \text{(I.19)}$$

One can easily verify

$$|\vec{P}|^2 = (2\mathrm{Re}(a^*b))^2 + (2\mathrm{Im}(a^*b))^2 + (|a|^2 - |b|^2)^2 = 1 \ .$$

As a consequence, pure spin ½ states, e.g. nucleons or electrons in pure states, are always completely polarized. This is an exclusive property of spin ½ which is clearly no longer true for higher spins. Generally valid, also for spin ½, is the following statement:

Completely polarized are only the states

$$\psi = |j,j\rangle \text{ and } \psi = |j,-j\rangle$$

and those which result from them by rotation of the coordinate system. All other pure states are only partially polarized or have no vector polarization at all, as for instance the integer spin states $|j,0\rangle$.

On the other hand, for $j > \frac{1}{2}$, all incompletely polarized pure spin states or even such with 0 vector polarization, have a so-called tensor polarization or spin alignment, where alignment means here every spin configuration which is not invariant under rotations and thus gives rise to detectable anisotropies. Forinstance in the eigenstates $|j,m\rangle$ the spin is aligned along cones around the z axis.

Let us anticipate the corresponding properties of mixed spin states, which will become clearer in connection with the density matrix. There again spin $\frac{1}{2}$ states are an exception. Mixed states with $j > \frac{1}{2}$ can be spin aligned without having any vector polarization, for instance an equal mixture of the states

$$|\tfrac{3}{2},\tfrac{1}{2}\rangle \quad \text{and} \quad |\tfrac{3}{2},-\tfrac{1}{2}\rangle .$$

In the case of spin $\frac{1}{2}$ particles, however, no mixture with resulting polarization 0 is aligned.

c) The Matrix Representation

For a fixed j one can enter the matrix elements of the operator Q

$$Q_{mm'} = \langle j,m|\, Q\,|\, jm' \rangle$$

into a $(2j+1)$ dimensional square matrix. This is called the matrix representation of the operator Q in the system of basis vectors $|j,m\rangle$. We shall choose the convention where the indexing runs from the highest to the lowest value of m. So, if we write A_{ik} for the members of a matrix

$$A_{11} = Q_{j,j}$$
$$A_{12} = Q_{j,j-1}$$
$$A_{22} = Q_{j-1,j-1}$$

the matrix has the structure

	m′	j	(j-1)	(j-2)	(-j)	
m	k \ i	1	2	3	(2j+1)	
j	1	A_{11}	A_{12}	A_{13}	$A_{1,2j+1}$	
(j-1)	2	A_{21}	A_{22}	A_{23}	$A_{2,2j+1}$	
(j-2)	3	A_{31}	A_{32}	A_{33}	$A_{3,2j+1}$	(I.20)
.	.					
.	.					
-j	(2j+1)	$A_{2j+1,1}$	$A_{2j+1,2}$	$A_{2j+1,3}$	$A_{2j+1,2j+1}$	

Let us write down in this way the matrices of the operators J_x, J_y, J_z. For a given j, the matrix elements are defined in relations Eq. (I.14).

$j = \tfrac{1}{2}$

$$J_x = \tfrac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad J_y = \tfrac{1}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad J_z = \tfrac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & -i \end{bmatrix}$$

$j = 1$

$$J_x = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad J_y = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix} \quad J_z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \qquad \text{(I.21)}$$

$j = \tfrac{3}{2}$

$$J_x = \tfrac{1}{2}\begin{bmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{bmatrix} \quad J_y = \tfrac{1}{2}\begin{bmatrix} 0 & -\sqrt{3}i & 0 & 0 \\ \sqrt{3}i & 0 & -2i & 0 \\ 0 & 2i & & -\sqrt{3}i \\ 0 & 0 & \sqrt{3}i & 0 \end{bmatrix} \quad J_z = \tfrac{1}{2}\begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}$$

The corresponding matrix for $\vec{J}^2$ is always given by a $(2j+1)$ dimensional unit matrix multiplied by $j(j+1)$

$$\vec{J}^2 = j(j+1)\,\mathbb{1}_{2j+1} \,.$$

If we now represent the basis vector $|j,m\rangle$ by a $(2j+1)$ dimensional column vector of the form

$$\begin{array}{c|c|} j & 0 \\ j-1 & 0 \\ j-2 & 0 \\ \cdot & \cdot \\ \cdot & \cdot \\ m & 1 \\ \cdot & \cdot \\ \cdot & \cdot \\ -j & 0 \end{array}$$

containing 1 in the place corresponding to m and 0's everywhere else, it can be immediately verified, that the matrix element

$$\langle j,m|Q|jm'\rangle$$

is given by an expression of the form

$$\underset{\langle j,m|}{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}} \underset{Q}{\begin{bmatrix} \\ \\ Q_{mm'} \\ \\ \\ \end{bmatrix}} \underset{|j,m'\rangle}{\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ \cdot \\ 0 \end{bmatrix}}$$

where the first vector should operate as a row vector, that means: apply the matrix $(Q_{mm'})$ to the vector corresponding to $|j,m'\rangle$ which gives you a new column vector. Then take the scalar product between this and the column vector corresponding to $|j,m\rangle$.

The linear combination of two states

$$\psi = a|j,m\rangle + b|j,m'\rangle$$

can clearly be written as

$$\psi = a \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ b \\ a \\ 0 \\ 0 \end{bmatrix} .$$

The most general pure state

$$|j> = \sum_{m=-j}^{+j} a_m |j,m> \quad \text{is now given by}$$

$$\begin{bmatrix} a_j \\ a_{j-1} \\ \cdot \\ \cdot \\ \cdot \\ a_{-j} \end{bmatrix} .$$

Example

Calculate the spin expectation value $<J_x>$ of a spin $3/2$ particle in the state

$$\psi = \frac{1}{\sqrt{2}} |\tfrac{3}{2},\tfrac{3}{2}> + \frac{1}{\sqrt{2}} |\tfrac{3}{2},\tfrac{1}{2}>$$

$$\psi = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \\ 0 \end{bmatrix} .$$

With the matrices Eq. (I.21) we find

$$\langle J_x \rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \\ 0 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \\ 0 \end{bmatrix} = \frac{1}{2} \cdot \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} \quad & & \quad \\ & & \\ & & \\ & & \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} \sqrt{3} \\ \sqrt{3} \\ 2 \\ 0 \end{bmatrix} = \frac{1}{4} \cdot 2\sqrt{3} = \frac{1}{2}\sqrt{3}$$

d) <u>Spherical Harmonics</u>

Whereas the matrix representation is applicable to all integer and half integer spin states, an analytic representation can be found only for integer spins. In the abstract operator representation, the eigenstates are found from pure operator algebra, defined by the commutation relations. In the analytic representation one does not make use of the commutation relations at all (though they are of course still valid) but only solves the eigenvalue equation defined by differential spin operators. The solutions of these partial differential equations are the spherical harmonics which can represent only the eigenstates of systems with integer spin.

The angular momentum operators are in terms of polar coordinates

$$\begin{aligned} L_x &= i\hbar\left(\sin\varphi \frac{\partial}{\partial\Theta} + \cot\Theta \cos\varphi \frac{\partial}{\partial\varphi}\right) \\ L_y &= i\hbar\left(-\cos\varphi \frac{\partial}{\partial\Theta} + \cot\Theta \sin\varphi \frac{\partial}{\partial\varphi}\right) \\ L_z &= -i\hbar \frac{\partial}{\partial\varphi} \\ \vec{L}^2 &= -\hbar^2 \left[\frac{1}{\sin\Theta} \frac{\partial}{\partial\Theta} \left(\sin\Theta \frac{\partial}{\partial\Theta}\right) + \frac{1}{\sin^2\Theta} \frac{\partial^2}{\partial\varphi^2}\right] \end{aligned} \qquad \text{(I.22)}$$

The eigenvalue equations

$$\vec{L}^2 \ \psi = C_{\vec{L}^2}\psi \quad \text{and} \quad L_z \ \psi = C_{L_z}\psi$$

are

$$-\left[\frac{1}{\sin\Theta}\frac{\partial}{\partial\Theta}\left(\sin\Theta\frac{\partial}{\partial\Theta}\right)+\frac{1}{\sin^2\Theta}\frac{\partial^2}{\partial\varphi^2}\right]\psi = \ell(\ell+1)\ \psi$$

$$-\ i\frac{\partial}{\partial\varphi}\ \psi = m\psi \ . \tag{I.23}$$

Solutions $\psi_{\ell m}$ $(\Theta\varphi)$ exist for integer ℓ and m

$$(\ell = 0,1,2,3,m, \quad m = -\ell,-\ell+1, \ \dots \ \ell)$$

$$\psi_{\ell m}(\Theta,\varphi) = Y_\ell^m(\Theta,\varphi) = \Theta_{\ell,m}(\Theta)\cdot\Phi_m(\varphi) \ . \tag{I.24}$$

The function $\Theta_{\ell m}$ can be calculated by the recursion formula

$$\Theta_{\ell m} = (-1)^\ell \sqrt{\frac{(2\ell+1)}{2}\frac{(\ell+m)!}{(\ell-m)!}}\ \frac{1}{2^\ell \ell!}\ \frac{1}{\sin^m\Theta}\ \frac{d^{(\ell-m)}}{(d\cos\Theta)^{(\ell-m)}}(\sin^{2\ell}\Theta) \tag{I.25}$$

or in terms of the associated Legendre functions $P_\ell^m(\cos\Theta)$

$$\Theta_{\ell m} = (-1)^m \sqrt{\frac{(2\ell+1)}{2}\frac{(\ell-|m|)!}{(\ell+|m|)!}}\ P_\ell^m(\cos\Theta) \ . \tag{I.26}$$

The functions Φ_m are

$$\Phi_m = \frac{1}{\sqrt{2\pi}}\ e^{im\varphi} \ . \tag{I.27}$$

The Y_ℓ^m have the property of orthonormality:

$$\int_0^{2\pi}\int_0^{\pi} Y_{\ell_1}^{m_1 *}\ Y_{\ell_2}^{m_2}\ \underbrace{\sin\vartheta\ d\vartheta\ d\varphi}_{d\Omega} = \delta_{\ell_1\ell_2}\ \delta_{m_1 m_2} \ . \tag{I.28}$$

Furthermore the following relations are valid

$$Y_\ell^{-m} = (-1)^m\ Y_\ell^{m*} \tag{I.29}$$

$$\sum_{m=-\ell}^{+\ell} Y_\ell^{m*} Y_\ell^m = 1 \ .$$

Table I

Spherical harmonics in the form

$$Y_\ell^m = \sqrt{\frac{1}{4\pi}} \sqrt{(2\ell+1)\frac{(\ell-|m|)!}{(\ell+|m|)!}}\, P_\ell^m(x)\, e^{i\varphi} \quad ; \quad x = \cos\Theta$$

$\ell = 0$ $$Y_0^0 = \sqrt{\frac{1}{4\pi}}$$

$\ell = 1$ $$Y_1^0 = \sqrt{\frac{1}{4\pi}}\sqrt{3}\,x$$

$$Y_1^{\pm1} = \mp\sqrt{\frac{1}{4\pi}}\sqrt{\frac{3}{2}}\sqrt{1-x^2}\; e^{\pm i\varphi}$$

$\ell = 2$ $$Y_2^0 = \sqrt{\frac{1}{4\pi}}\sqrt{5}\cdot \tfrac{1}{2}(3x^2-1)$$

$$Y_2^{\pm1} = \mp\sqrt{\frac{1}{4\pi}}\sqrt{\frac{5}{6}}\cdot 3x\sqrt{1-x^2}\; e^{\pm i\varphi}$$

$$Y_2^{\pm2} = \sqrt{\frac{1}{4\pi}}\sqrt{\frac{5}{24}}\cdot 3(1-x^2)\; e^{\pm2i\varphi}$$

$\ell = 3$ $$Y_3^0 = \sqrt{\frac{1}{4\pi}}\sqrt{7}\cdot \tfrac{1}{2}(5x^3-3x)$$

$$Y_3^{\pm1} = \mp\sqrt{\frac{1}{4\pi}}\sqrt{\frac{7}{12}}\cdot \tfrac{3}{2}(5x^2-1)\sqrt{1-x^2}\; e^{\pm i\varphi}$$

$$Y_3^{\pm2} = \sqrt{\frac{1}{4\pi}}\sqrt{\frac{7}{120}}\cdot 15x(1-x^2)\; e^{\pm2i\varphi}$$

$$Y_3^{\pm3} = \mp\sqrt{\frac{1}{4\pi}}\sqrt{\frac{7}{720}}\cdot 15(1-x^2)\sqrt{1-x^2}\; e^{\pm3i\varphi}$$

Table I contains the spherical harmonics for $\ell = 0$ to 3.

Let us briefly sketch how to calculate spin expectation values in this representation. Take as example the state

$$\psi = \sqrt{1/2}\,|1,1> + \sqrt{1/2}\,|1,0> = \sqrt{1/2}\; Y_1^1 + \sqrt{1/2}\; Y_1^0$$

and calculate the angular momentum component in x direction. This is done by performing the integral

$$<L_x> = \int \psi^* L_x \psi \, d\Omega$$

$$= \int_0^{2\pi}\int_0^{\pi} 1/2\,(Y_1^{1*} + Y_1^{0*}) \left[i \left(\sin\varphi \frac{\partial}{\partial\Theta} + \cot\Theta \cos\varphi \frac{\partial}{\partial\varphi} \right] (Y_1^1 + Y_1^0) \sin\Theta d\Theta . 2\pi d\right.$$

This is a tiresome job, but it gives of course the same result as the matrix method

$$<L_x> = \begin{bmatrix} \sqrt{1/2} \\ \sqrt{1/2} \\ 0 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{1/2} \\ \sqrt{1/2} \\ 0 \end{bmatrix} = 1/2\sqrt{1/2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$= 1/2\sqrt{1/2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \sqrt{1/2} \; .$$

Clearly the matrix method is a much more efficient tool to calculate spin components than the analytic method.

Nevertheless in the spin analysis we shall always deal with spherical harmonics when analysing angular distributions of decays. This can be best explained by an example. Let us assume a particle of spin 1, for instance a ρ meson, is produced in some reaction with its spin completely polarized along a direction z. So the ρ is in the spin state $|1,1>$.

Afterwards the particle decays into 2π mesons which both have spin 0. The decaying system is described by a wave function

$$\psi(\vec{r}_1, \vec{r}_2)$$

where $\vec{r}_1, \vec{r}_2$ are the position vectors of the two particles. As a decay in two particles is collinear in the c.m. system we can write

$$\psi(\vec{r}_1, \vec{r}_2) = R(r)\ f(\Theta,\varphi)$$

where Θ,φ indicate the emission direction and $R(r)$ is the expression for an outgoing wave

$$R(r) = \frac{e^{ikr}}{r} .$$

As the spin of the two daughter pions is 0, the orbital angular momentum state of the final 2π system is the same as the spin state $|1,1>$ of the mother particle. The angular part of the wave function is therefore the spherical harmonic $Y_1^1(\Theta,\varphi)$:

$$\psi = \frac{e^{ikr}}{r}\ Y_1^1(\Theta,\varphi) .$$

The probability of finding one of the outgoing particles at a point with coordinates (r,Θ,φ) is clearly

$$\psi^*\psi = \frac{1}{r^2}\ f^*f = \frac{1}{r^2}\ Y_1^{1*}Y_1^1$$

therefore the angular distribution of the decay is

$$\frac{dN}{d\Omega} = f^*(\Theta,\varphi)\ f(\Theta,\varphi) \qquad (I.30)$$

$$= Y_1^{1*}Y_1^1 = \frac{1}{4\pi}\cdot\frac{3}{2}\sin^2\Theta .$$

e) Vector Coupling

In this section we deal with the following two problems:

1) Consider two systems in the eigenstates $|j_1, m_1>$ and $|j_2, m_2>$. Together they form one state specified by

$$|j_1 j_2 m_1 m_2> .$$

Express this state in terms of possible eigenstates $|J,M>$ of the resulting angular momentum. The result is of the form

$$|j_1 j_2 m_1 m_2> = \sum_{JM}\sum C_{JM}|J,M> \tag{I.31}$$

2) Consider a system in the eigenstate $|J,M>$. Decompose this system into two sub-systems of well specified spins j_1 and j_2 respectively. Express the state $|J,M>$ in terms of eigenstates $|j_1 j_2 m_1 m_2>$:

$$|J,M> = \sum_{m_1}\sum_{m_2} C_{m_1 m_2}|j_1 j_2 m_1 m_2> . \tag{I.32}$$

The rules of angular momentum combination require

$$m_1 + m_2 = M \tag{I.33}$$

and allow for given j_1 j_2, the following values for the total spin J

$$\begin{aligned} J = & |j_1 - j_2| \\ & |j_1 - j_2| + 1 \\ & \cdot \\ & \cdot \\ & j_1 + j_2 \end{aligned}$$

The condition Eq. (I.33) reduces the two summation indices in Eq. (I.3·) and (I.32) to one index only.

The C_{JM} and $C_{m_1m_2}$ are the so-called vector coupling or Clebsch-Gordan coefficients. We shall use for them the notation

$$C_{JM} = (j_1 j_2 \; J \; M | j_1 j_2 \; m_1 m_2)$$
$$C_{m_1m_2} = (j_1 j_2 \; m_1 m_2 | j_1 j_2 JM) \; .$$

Let us re-write the coupling relations in this notation:

$$|j_1 j_2 m_1 m_2 > = \sum_{J=|j_1-j_2|}^{j_1+j_2} |JM> (j_1 j_2 JM | j_1 j_2 m_1 m_2)$$

$$|JM> = \sum_{m_1=-j_1}^{+j_1} |j_1 j_2 m_1 \; M-m_1 > (j_1 j_2 m_1 \; M-m_1 | j_1 j_2 JM). \qquad (I.34)$$

We shall use a representation where all vector coupling coefficients are real and where

$$(j_1 j_2 JM | j_1 j_2 m_1 m_2) = (j_1 j_2 m_1 m_2 | j_1 j_2 JM) \; .$$

Furthermore we have the relations

$$\sum_{JM} (j_1 j_2 m_1 m_2 | j_1 j_2 JM) (j_1 j_2 JM | j_1 j_2 m_1' m_2') = \delta_{m_1 m_1'} \; \delta_{m_2 m_2'}$$

$$\sum_{m_1 m_2} (j_1 j_2 JM | j_1 j_2 m_1 m_2)(j_1 j_2 m_1 m_2 | j_1 j_2 J'M') = \delta_{JJ'} \; \delta_{MM'} \; . \qquad (I.35)$$

Formulae to calculate general Clebsch-Gordan coefficients are rather complicated so that we prefer to copy the tables[25] for the most frequent types of spin coupling (Table II). The different nomenclature in Table II is related to ours by

$$U_{j_1}^{m_1} V_{j_2}^{m_2} \equiv |j_1 j_2 m_1 m_2 > \quad \text{and} \quad W_J^M \equiv |JM> \; .$$

$\underline{D_l \times D_{1/2}}$

	$W^m_{l+1/2}$	$W^m_{l-1/2}$
$U_l^{m+1/2}\ V_{1/2}^{-1/2}$	$\sqrt{\dfrac{l-m+1/2}{2l+1}}$	$\sqrt{\dfrac{l+m+1/2}{2l+1}}$
$U_l^{m-1/2}\ V_{1/2}^{1/2}$	$\sqrt{\dfrac{l+m+1/2}{2l+1}}$	$-\sqrt{\dfrac{l-m+1/2}{2l+1}}$

$\underline{D_l \times D_1}$

	W^m_{l+1}	W^m_l	W^m_{l-1}
$U_l^{m+1}\ V_1^{-1}$	$\sqrt{\dfrac{(l-m+1)(l-m)}{(2l+1)(2l+2)}}$	$\sqrt{\dfrac{(l-m)(l+m+1)}{l(2l+2)}}$	$\sqrt{\dfrac{(l+m)(l+m+1)}{(2l)(2l+1)}}$
$U_l^m\ V_1^0$	$\sqrt{\dfrac{2(l+m+1)(l-m+1)}{(2l+1)(2l+2)}}$	$\sqrt{\dfrac{m}{l(l+1)}}$	$-\sqrt{\dfrac{2(l-m)(l+m)}{(2l)(2l+1)}}$
$U_l^{m-1}\ V_1^1$	$\sqrt{\dfrac{(l+m+1)(l+m)}{(2l+1)(2l+2)}}$	$-\sqrt{\dfrac{(l+m)(l-m+1)}{l(2l+2)}}$	$\sqrt{\dfrac{(l-m)(l-m+1)}{(2l)(2l+1)}}$

$\underline{D_l \times D_{3/2}}$

	$W^m_{l+3/2}$	$W^m_{l+1/2}$	$W^m_{l-1/2}$	$W^m_{l-3/2}$
$U_l^{m+3/2}\ V_{3/2}^{3/2}$	$\sqrt{\dfrac{(l-m-1/2)(l-m+1/2)(l-m+3/2)}{(2l+1)(2l+2)(2l+3)}}$	$\sqrt{\dfrac{3(l-m-1/2)(l-m+1/2)(l+m+3/2)}{(2l+3)(2l+1)(2l)}}$	$\sqrt{\dfrac{3(l-m+1/2)(l-m+3/2)(l-m-1/2)}{(2l-1)(2l+1)(2l+2)}}$	$\sqrt{\dfrac{(l+m-1/2)(l+m+1/2)(l+m+3/2)}{(2l-1)(2l)(2l+1)}}$
$U_l^{m+1/2}\ V_{3/2}^{-1/2}$	$\sqrt{\dfrac{3(l+m+3/2)(l-m+1/2)(l-m+3/2)}{(2l+1)(2l+2)(2l+3)}}$	$(l+3m+3/2)\sqrt{\dfrac{(l-m+1/2)}{(2l+3)(2l+1)(2l)}}$	$-(l-3m-1/2)\sqrt{\dfrac{(l+m+1/2)}{(2l-1)(2l+1)(2l+2)}}$	$-\sqrt{\dfrac{3(l-m-1/2)(l+m-1/2)(l+m+1/2)}{(2l-1)(2l)(2l+1)}}$
$U_l^{m-1/2}\ V_{3/2}^{1/2}$	$\sqrt{\dfrac{3(l+m+3/2)(l+m+1/2)(l+m+3/2)}{(2l+1)(2l+2)(2l+3)}}$	$-(l-3m+3/2)\sqrt{\dfrac{(l+m+1/2)}{(2l+3)(2l+1)(2l)}}$	$-(l+3m-1/2)\sqrt{\dfrac{(l-m+1/2)}{(2l+3)(2l+1)(2l+2)}}$	$\sqrt{\dfrac{3(l-m-1/2)(l-m+1/2)(l+m-1/2)}{(2l-1)(2l)(2l+1)}}$
$U_l^{m-3/2}\ V_{3/2}^{3/2}$	$\sqrt{\dfrac{(l+m+3/2)(l+m+1/2)(l+m-1/2)}{(2l+1)(2l+2)(2l+3)}}$	$-\sqrt{\dfrac{3(l+m-1/2)(l+m+1/2)(l-m+3/2)}{(2l+3)(2l+1)(2l)}}$	$\sqrt{\dfrac{3(l-m+1/2)(l-m+3/2)(l+m-1/2)}{(2l-2)(2l+2)(2l+2)}}$	$-\sqrt{\dfrac{(l-m-1/2)(l-m+1/2)(l-m+3/2)}{(2l-1)(2l)(2l+1)}}$

TABLES II

f) Rotations of the coordinate system

Consider a scalar field

$$\psi = \psi(x,y,z)$$

defined in the coordinate system $S(x,y,z)$. Rotate the coordinate system by an angle α around the z axis, in the negative sense, to a new frame $S'(x',y',z')$. The coordinate transformation is

$$\begin{aligned} x &= x' \cos\alpha + y' \sin\alpha \\ y &= -x' \sin\alpha + y' \cos\alpha \qquad \text{(I.36)} \\ z &= z' \end{aligned}$$

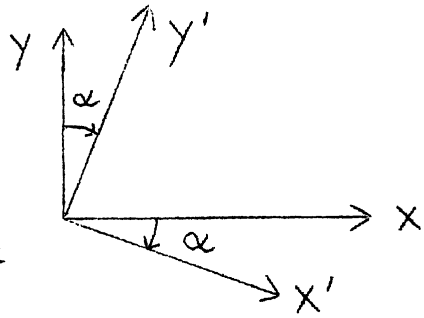

Fig.1

Now keep the field ψ fixed in space and express it in terms of x',y',z'. This is simply done by inserting the substitution Eq. (I.36) in $\psi(x,y,z)$. So

$$\psi(x,y,z) \rightarrow \psi'(x',y',z') \ .$$

An infinitesimal transformation by an angle $d\alpha$ leads to corresponding relations:

$$\begin{aligned} x &= x' + y'd\alpha = x' + dx' \\ y &= y' - x'd\alpha = y' + dy' \\ z &= z' \qquad\quad\;\; = z' \end{aligned}$$

and

$$\begin{aligned} \psi(x,y,z) \rightarrow \psi'(x',y',z') &= \psi(x' + dx', y' + dy', z') = \\ &= \psi(x',y',z') + \frac{\partial\psi}{\partial x'}\, dx' + \frac{\partial\psi}{\partial y'}\, dy' \\ &= \psi(x',y',z') + \frac{\partial\psi}{\partial x'}\,(+y'd\alpha) + \frac{\partial\psi}{\partial y'}\,(-x'd\alpha) \\ &= \psi(x',y',z') + \frac{d\psi}{d\alpha}\cdot d\alpha \end{aligned}$$

where the derivative $d\psi/d\alpha$ is

$$\frac{d\psi}{d\alpha} = \left(-x'\frac{\partial}{\partial y'} + y'\frac{\partial}{\partial x'}\right)\psi(x',y',z') = R_z\psi$$

The infinitesimal rotation operator around the z axis

$$R_z = -\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$$

is proportional to the operator of the angular momentum in z direction

$$L_z = i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$$

so that the relations hold

$$R_z = \frac{i}{\hbar} L_z \qquad (I.37)$$

and for rotations around the x and y axis:

$$R_x = -\frac{i}{\hbar} L_x \; ; \; R_y = -\frac{i}{\hbar} L_y \, .$$

The operators $R_{x,y,z}$ were defined as rotating the coordinate system in the mathematically negative sense. This is equivalent to keeping the coordinate system fixed and rotating ψ in the positive sense.

A finite rotation by an angle α can be constructed by repeated application of the infinitesimal operator. From the infinitesimal rotation

$$\psi \rightarrow \psi' = \psi + d\psi = (1 + Rd\alpha)\psi$$

one constructs the finite rotation

$$\psi \rightarrow \psi' = D(\alpha)\psi = \lim_{n\rightarrow\infty} \left(1 + R\,\frac{\alpha}{n}\right)^n \psi$$

$$= e^{R\alpha}\psi = \left(\sum_0^\infty \frac{1}{n!}\,(R\alpha)^n\right)\psi$$

so

$$D(\alpha) = e^{R\alpha} = \sum_0^\infty \frac{1}{n!}\,(R\alpha)^n \qquad (I.38)$$

is the operator of a finite rotation by the angle α.

The operators $R_{x,y,z}$ and the corresponding finite rotation operators $D_{x,y,z}(\alpha)$ can clearly be applied to the wave functions of integer spin representations. For half integer spin states where only the matrix representation is available, one has to construct matrix operators. It can be shown that for any integer or half integer spin state with total spin J relations Eq. (I.37) become

$$R_x = -\frac{i}{\hbar} J_x \; ; \; R_y = -\frac{i}{\hbar} J_y \; ; \; R_z = -\frac{i}{\hbar} J_z \qquad (I.39)$$

independently of a specific representation. From Eqs. (I.39) and (I.38) one constructs the finite rotation operators for rotations around the x,y,z axis

$$D_x(\alpha) = e^{-\frac{i}{\hbar}J_x}, \; D_y(\alpha) = e^{-\frac{i}{\hbar}J_y}, \; D_z(\alpha) = e^{-\frac{i}{\hbar}J_z} . \qquad (I.40)$$

It is customary to describe the most general rotation as a product of three rotations: a first rotation around the z axis by an angle γ, a second rotation about the y axis by an angle β and a third rotation again around the z axis by an angle α

$$\begin{aligned} D(\alpha,\beta,\gamma) &= D_z(\alpha)\, D_y(\beta)\, D_z(\gamma) \\ &= e^{-\frac{i\alpha}{\hbar}J_z}\, e^{-\frac{i\beta}{\hbar}J_y}\, e^{-\frac{i\gamma}{\hbar}J_z} . \end{aligned} \qquad (I.41)$$

In the following $\hbar$ will always be put equal to 1.

Let us now rotate an eigenstate $|j,m\rangle$ around the z axis. The resulting spin state ψ' is

$$\begin{aligned} \psi' &= D_z(\alpha)|j,m\rangle \\ &= e^{-i\alpha J_z}|j,m\rangle = \sum_{o}^{\infty} \frac{1}{n!} (-i\alpha J_z)^n |j,m\rangle . \end{aligned}$$

As J_z is diagonal

$$J_z^n|j,m> = m^n|j,m>$$

and

$$\psi' = \sum_0^\infty \frac{1}{n!}(-i\alpha M)^n|j,m> .$$

Therefore

$$D_z(\alpha)|j,m> = e^{-i\alpha M}|j,m> . \qquad (I.42)$$

Rotations around the y axis are more complicated, because J_y is not a diagonal operator. Let us first take the spin state $|\frac{1}{2},\frac{1}{2}>$

$$D_y^{(1/2)}(\beta)|\tfrac{1}{2},\tfrac{1}{2}> = \sum_0^\infty \frac{1}{n!}(-i\beta J_y)^n|\tfrac{1}{2},\tfrac{1}{2}> . \qquad (I.43)$$

From the matrix representation of J_y Eq. (I.21) one verifies immediately

$$J_y|\tfrac{1}{2}, \pm\tfrac{1}{2}> = \pm\frac{i}{2}|\tfrac{1}{2}, \mp\tfrac{1}{2}> .$$

From this follows

$$J_y^n|\tfrac{1}{2}, +\tfrac{1}{2}> = (\tfrac{1}{2})^n|\tfrac{1}{2},\tfrac{1}{2}> \qquad \text{for n even}$$

$$= (\tfrac{1}{2})^n i|\tfrac{1}{2},-\tfrac{1}{2}> \qquad \text{for n odd} .$$

So one can split the summation in Eq. (I.43):

$$D_y^{(1/2)}(\beta)|\tfrac{1}{2},\tfrac{1}{2}> = \sum_{k=0}^{\infty} \frac{1}{(2k)!}\left(-\frac{i\beta}{2}\right)^{2k}|\tfrac{1}{2},\tfrac{1}{2}>$$

$$+ \sum_{k=0}^{\infty}\frac{1}{(2k+1)}\left(\frac{-i\beta}{2}\right)^{2k+1} i|\tfrac{1}{2},-\tfrac{1}{2}> \qquad (I.44)$$

$$= \cos\left(\frac{\beta}{2}\right)|\tfrac{1}{2},\tfrac{1}{2}> + \sin\frac{\beta}{2}|\tfrac{1}{2},-\tfrac{1}{2}> .$$

Correspondingly one finds

$$D_y^{(1/2)}(\beta)\,|1/2,-1/2> \;=\; -\sin\frac{\beta}{2}\,|1/2,1/2> \;+\; \cos\frac{\beta}{2}\,|1/2,-1/2> \;. \tag{I.45}$$

The coefficients of the eigenstates in Eqs. (I.44) and (I.45) are the matrix elements

$$<1/2,m|\,D_y^{(1/2)}(\beta)\,|1/2,m'>$$

which build up the y rotation matrix for spin ½

$$D_y^{(1/2)}(\beta) = \begin{bmatrix} \cos\frac{\beta}{2} & -\sin\frac{\beta}{2} \\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{bmatrix} \tag{I.46}$$

Applied to a pure state it rotates the state by β around the y axis, so that the z direction moves towards the x direction. Rotation around the x axis is effected by the matrix

$$D_x^{(1/2)}(\beta) = \begin{bmatrix} \cos\frac{\beta}{2} & -i\sin\frac{\beta}{2} \\ -i\sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{bmatrix}$$

Here the y direction moves towards the z direction.

In a similar way one constructs the y rotation matrix for spin 1

$$D_y^{(1)}(\beta) = \begin{bmatrix} \frac{1}{2}(1+\cos\beta) & -\sqrt{\frac{1}{2}}\,\sin\beta & \frac{1}{2}(1-\cos\beta) \\ \sqrt{\frac{1}{2}}\,\sin\beta & \cos\beta & -\sqrt{\frac{1}{2}}\,\sin\beta \\ \frac{1}{2}(1-\cos\beta) & \sqrt{\frac{1}{2}}\,\sin\beta & \frac{1}{2}(1+\cos\beta) \end{bmatrix} \tag{I.47}$$

The matrix elements $D_y(\beta)_{mm'}$ obey the symmetry relation $D_y(\beta)_{mm'} = D_y(-\beta)_{m'm}$. The y rotation matrices have a particularly simple structure for rotations of 90°. Therefore it is sometimes advantageous to decompose an arbitrary rotation $D_y(\beta)$ in rotations around the z axis and 90° rotations around y. $D_y(\beta)$ may be written as

$$D_y(\beta) = D_z\left(\frac{\pi}{2}\right) D_y\left(\frac{\pi}{2}\right) D_z\ (\beta)\ \ D_y\left(-\frac{\pi}{2}\right) D_z\left(-\frac{\pi}{2}\right) \qquad (I.48)$$

$D_y(-\pi/2)\ D_z(-\pi/2)$ transforms the y axis into the z direction. Then $D_z(\beta)$ effects a rotation by β around the z axis. Finally the z axis is transformed back into the y axis by $D_z(\pi/2)\ D_y(\pi/2)$.

Table III gives the matrices $D_y^{(j)}(\pi/2)$ for $j = ½, 1, \tfrac{3}{2}, 2$.

Table III

y rotation matrices for $\beta = \pi/2$.

$$D_y^{(1/2)}(\pi/2) = \begin{bmatrix} \sqrt{1/2} & -\sqrt{1/2} \\ \sqrt{1/2} & \sqrt{1/2} \end{bmatrix} \qquad D_y^{(1)}(\pi/2) = \begin{bmatrix} 1/2 & -\sqrt{1/2} & 1/2 \\ \sqrt{1/2} & 0 & -\sqrt{1/2} \\ 1/2 & \sqrt{1/2} & 1/2 \end{bmatrix}$$

$$D_y^{(3/2)}(\pi/2) = 1/2 \begin{bmatrix} \sqrt{1/2} & -\sqrt{3/2} & \sqrt{3/2} & -\sqrt{1/2} \\ \sqrt{3/2} & -\sqrt{1/2} & -\sqrt{1/2} & \sqrt{3/2} \\ \sqrt{3/2} & \sqrt{1/2} & -\sqrt{1/2} & -\sqrt{3/2} \\ \sqrt{1/2} & \sqrt{3/2} & \sqrt{3/2} & \sqrt{1/2} \end{bmatrix}$$

$$D_y^{(2)}(\pi/2) = 1/2 \begin{bmatrix} 1/2 & -1 & \sqrt{3/2} & -1 & \sqrt{1/2} \\ 1 & -1 & 0 & 1 & -1 \\ \sqrt{3/2} & 0 & -1 & 0 & \sqrt{3/2} \\ 1 & 1 & 0 & -1 & -1 \\ 1/2 & 1 & \sqrt{3/2} & 1 & 1/2 \end{bmatrix}$$

Example

Construct a spin ½ state completely polarized in the direction (Θ,φ).

$$\psi(\Theta,\varphi) = D_z(\varphi)\, D_y(\Theta)\, |1/2, 1/2 >$$

$$= \begin{bmatrix} e^{-\frac{i\varphi}{2}} & 0 \\ 0 & e^{\frac{i\varphi}{2}} \end{bmatrix} \begin{bmatrix} \cos\frac{\Theta}{2} & -\sin\frac{\Theta}{2} \\ \sin\frac{\Theta}{2} & \cos\frac{\Theta}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} e^{-\frac{i\varphi}{2}} \cos\frac{\Theta}{2} & -e^{-\frac{i\varphi}{2}} \sin\frac{\Theta}{2} \\ e^{\frac{i\varphi}{2}} \sin\frac{\Theta}{2} & e^{\frac{i\varphi}{2}} \cos\frac{\Theta}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} e^{-i\varphi} \cos\frac{\Theta}{2} \\ e^{i\varphi} \sin\frac{\Theta}{2} \end{bmatrix}$$

$$= e^{-\frac{i\varphi}{2}} \cos\frac{\Theta}{2}\, |1/2, 1/2 > + e^{\frac{i\varphi}{2}} \sin\frac{\Theta}{2}\, |1/2, -1/2 > .$$

Exercise

Let ê be a fixed direction in space. Show that the operator

$$\vec{\sigma}\, \hat{e}$$

when applied to a pure state of spin ½ effects a rotation of 180° around the axis ê. For the proof, it is convenient to express the rotation matrices in terms of $\sigma_x, \sigma_y, \sigma_z$ and the unit matrix $\mathbb{1}$.

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$$D_x^{(1/2)}(\alpha) = \cos\frac{\alpha}{2}\mathbb{1} - i \sin\frac{\alpha}{2}\sigma_x$$

$$D_y^{(1/2)}(\alpha) = \cos\frac{\alpha}{2}\mathbb{1} - i \sin\frac{\alpha}{2}\sigma_y$$

$$D_z^{(1/2)}(\alpha) = \cos\frac{\alpha}{2}\mathbb{1} - i \sin\frac{\alpha}{2}\sigma_z$$

[References: (1), (2), (3)]

II THE ADAIR ANALYSIS

a) Spin Determination of Fermions

This type of spin analysis, theoretically at least, is most adequate for the spin determination of fermions (half integer spin) produced in two body reactions of the type

$$B_1(0) + F_1(1/2) \rightarrow B_2(0) + F_2(n/2)$$

where B_1 and B_2 are spinless bosons, F_1 is a fermion of spin $1/2$ and F_2 is the fermion whose spin $n/2$ should be determined. To be specific we consider the spin determination of the Λ produced in the reaction

$$\pi^- + p \rightarrow K^o + \Lambda \ .$$

Let us discuss the possible angular momentum configurations of initial and final state in the c.m. The z axis is chosen to lie in the direction of the incident particle.

Initial state

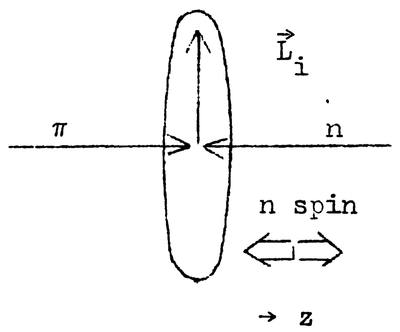

Fig. 2

From classical arguments the initial orbital angular momentum $\vec{L}_i$ has to be aligned in a plane normal to the incident z direction. Quantum mechanical argument: Developing an incident plane wave in terms of angular momentum eigenfunctions $|\ell,m\rangle$ the series contains only the functions $|\ell,0\rangle = Y_\ell^0$ which corresponds to 0 angular momentum in z direction. - The protons are assumed to be unpolarized and can be described by an equal statistical mixture of two states with opposite polarization in +z and -z direction, therefore the initial spin state is described by two equally probable wave functions

$$\psi_{1\,in} = \sum_\ell a_\ell Y_\ell^0 |1/2, 1/2\rangle \quad \psi_{2\,in} = \sum a_\ell Y_\ell^0 |1/2, -1/2\rangle \ .$$

The general final state is of much more complicated structure. Let us discuss a specific example. Consider only one incident partial wave $\ell = 1$ and assume the Λ has spin $J_\Lambda = 3/2$. So the initial states are

$$\psi_{1\,in} = Y_1^0 |1/2, 1/2\rangle \quad \psi_{2\,in} = Y_1^0 |1/2, -1/2\rangle \ .$$

The total angular momentum of the system J_{tot} is, according to the vector coupling rules, a linear superposition of $J_{tot} = 1/2$ and $3/2$ states. The possible values for the orbital angular momentum of the final state L_{fin} range from $|J_{tot} - J_\Lambda|$ to $|J_{tot} + J_\Lambda|$, in our case from 0 to 3. Therefore the most general final state for our assumption is

$$\begin{aligned}
\psi_{1fin} = \ & \alpha_{0,1/2}\, Y_0^0 |3/2, 1/2\rangle && \Big\} \ L_{fin} = 0 \\
& + \alpha_{1,3/2}\, Y_1^{-1} |3/2, 3/2\rangle && \\
& + \alpha_{1,1/2}\, Y_1^0 |3/2, 1/2\rangle && \Big\} \ L_{fin} = 1 \\
& + \alpha_{1,-1/2}\, Y_1^1 |3/2, -1/2\rangle && \\
& + \alpha_{2,3/2}\, Y_2^{-1} |3/2, 3/2\rangle && \\
& + \alpha_{2,1/2}\, Y_2^0 |3/2, 1/2\rangle && \Big\} \ L_{fin} = 2 \\
& + \alpha_{2,-1/2}\, Y_2^1 |3/2 -1/2\rangle && \\
& + \alpha_{2,-3/2}\, Y_2^2 |3/2 -3/2\rangle && \\
& + \text{corresponding terms for } L_{fin} = 3 .
\end{aligned} \qquad \text{(II.1)}$$

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The α are some complex coefficients depending on the interaction mechanism. The Y_ℓ^m describe the final angular momentum state and therefore will determine the decay angular distribution, as shown in Chapter I.d*.

Now we look at some properties of the spherical harmonics:

$$\text{For } \cos\Theta = \pm 1 \begin{cases} Y_\ell^m = 0 & \text{for } m \neq 0 \\ Y_\ell^o = 1 & . \end{cases}$$

This is because the Y_ℓ^m always contain a factor $\sin^m\Theta$.

Applying this to our problem we immediately notice that when selecting only Λ's emitted at 0^o or 180^o we fix $\cos\Theta$ to ± 1 in the expression for the final spin state Eq. (II.1) which then reduces to

$$\psi_{1fin} = \sum_L \alpha_{L,\frac{1}{2}} |{}^3\!/\!_2, {}^1\!/\!_2 > .$$

That means, Λ's selected in this way are in the pure spin state $|{}^3\!/\!_2, {}^1\!/\!_2>$. This is clearly valid for any superposition of initial angular momenta L_{in}. For the general spin J_Λ the final spin states of Λ's emitted at 0^o and 180^o would be of course

$$\psi_{1fin} = |J_\Lambda, {}^1\!/\!_2 > ; \ \psi_{2fin} = |J_\Lambda, -{}^1\!/\!_2 > .$$

Let us now explain this result by a more intuitive pseudo-classical argument: When selecting events with Λ emission at 0^o or 180^o we fix also the final angular momentum $\vec{L}_{fin}$ to the plane perpendicular to the z direction.

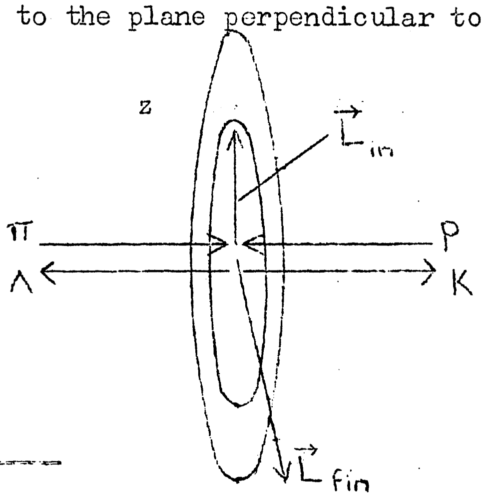

Fig. 3

* It should be mentioned that for a fixed L_{in} only, either even or odd, L_{fin} are allowed because the interaction is strong and conserves parity, however this is irrelevant for our discussion.

Therefore the angular momentum flips in a plane perpendicular to the z direction. This cannot affect the spin z component of the total angular momentum $M_{tot} = +\frac{1}{2}, -\frac{1}{2}$. As a consequence the outgoing Λ has spin z component $+\frac{1}{2}, -\frac{1}{2}$ regardless of its spin J_Λ. The two final spin states of the Λ, $|J, \frac{1}{2}>$ and $|J, -\frac{1}{2}>$, give rise to specific angular decay distribution depending only on J.

Consider a Λ in the state $|\frac{3}{2}, \frac{1}{2}>$ decaying into p and π. Spin conservation allows the $(p\pi)$ system to have orbital angular momentum $\ell = 1$ and $\ell = 2$. If parity is not conserved, both $P(\ell=1)$ and $D(\ell=2)$ waves will contribute to the final state. Decomposing the Λ spin state $|\frac{3}{2}, \frac{1}{2}>$ into the orbital angular momentum part and the proton spin part of the final state, gives according to the vector coupling relations described in Chapter I.e.

$$\psi = P\left\{\sqrt{\tfrac{2}{3}}\, Y_1^0 |\tfrac{1}{2}, \tfrac{1}{2}> + \sqrt{\tfrac{1}{3}}\, Y_1^1 |\tfrac{1}{2} - \tfrac{1}{2}>\right\} + D\left\{-\sqrt{\tfrac{2}{5}}\, Y_2^0 |\tfrac{1}{2}, \tfrac{1}{2}> + \sqrt{\tfrac{3}{5}}\, Y_2^1 |\tfrac{1}{2} - \tfrac{1}{2}>\right\} \quad \text{(II.2)}$$

P and D are complex numbers representing the P and D wave amplitudes which depend on the decay mechanism. Normalization is assumed

$$|P|^2 + |D|^2 = 1 .$$

We re-write ψ:

$$\psi = \left(\sqrt{\tfrac{2}{3}}\, PY_1^0 - \sqrt{\tfrac{2}{5}}\, DY_2^0\right)\begin{pmatrix}1\\0\end{pmatrix} + \left(\sqrt{\tfrac{1}{3}}\, PY_1^1 + \sqrt{\tfrac{3}{5}}\, DY_2^1\right)\begin{pmatrix}0\\1\end{pmatrix}$$

and ψ^*:

$$\psi^* = \left(\sqrt{\tfrac{2}{3}}\, P^*Y_1^0{}^* - \sqrt{\tfrac{2}{5}}\, D^*Y_2^0{}^*\right)\begin{pmatrix}1\\0\end{pmatrix} + \left(\sqrt{\tfrac{1}{3}}\, P^*Y_1^1{}^* + \sqrt{\tfrac{3}{5}}\, D^*Y_2^1{}^*\right)\begin{pmatrix}0\\1\end{pmatrix}$$

The angular distribution of the decay is

$$\frac{dN}{d\Omega} = \psi^*\psi = \tfrac{2}{3}|P|^2|Y_1^0|^2 + \tfrac{2}{5}|D|^2|Y_2^0|^2 + 2\sqrt{\tfrac{2}{3}}\sqrt{\tfrac{2}{5}}\, \mathrm{Re}(P^*DY_1^0{}^*Y_2^0)$$
$$+ \tfrac{1}{3}|P|^2|Y_1^1|^2 + \tfrac{3}{5}|D|^2|Y_2^1|^2 + 2\sqrt{\tfrac{1}{3}}\sqrt{\tfrac{3}{5}}\, \mathrm{Re}(P^*DY_1^1{}^*Y_2^1) .$$

To simplify expressions we omit the factor $\sqrt{1/4\pi}$ in the spherical harmonics. As a consequence, all calculated distributions $\frac{dN}{d\cos\Theta}$ will be normalized to 2. With $x = \cos\Theta$ one finds

$$\psi^*\psi = \tfrac{2}{3}|P|^2\, 3x^2 + \tfrac{2}{5}|D|^2 . 5 . \tfrac{1}{4}(9x^4 - 6x^2 + 1)$$

$$+ 2\sqrt{\tfrac{2}{3}}\sqrt{\tfrac{2}{5}} . \mathrm{Re}(P^*D) . \sqrt{3}x\sqrt{5} . \tfrac{1}{2}(3x^2 - 1)$$

$$+ \tfrac{1}{3}|P|^2 . \tfrac{3}{2}(1 - x^2) + \tfrac{3}{5}|D|^2 . \tfrac{5}{6} . 9x^2(1 - x^2)$$

$$+ 2\sqrt{\tfrac{1}{3}}\sqrt{\tfrac{3}{5}} . \mathrm{Re}(P^*D\sqrt{\tfrac{3}{2}}\sqrt{1 - x^2}\, e^{-i\varphi}\sqrt{\tfrac{5}{6}}\, 3x\sqrt{1 - x^2}\, e^{i\varphi}) .$$

After a straightforward calculation this reduces to

$$\psi^*\psi = \frac{dN}{d\cos\Theta} = \tfrac{1}{2}(1 + 3x^2)\;(1 + 2\mathrm{Re}(P^*D)x)\;;\; x = \cos\Theta .$$

The other Λ state $|\tfrac{3}{2} - \tfrac{1}{2}>$ leads to

$$\frac{dN}{d\cos\Theta} = \tfrac{1}{2}(1 + 3x^2)\;(1 - 2\mathrm{Re}(P^*D)x)$$

so the resultant distribution is

$$\frac{dN}{d\cos\Theta} = \tfrac{1}{2}(1 + 3x^2)\;;\; x = \cos\Theta . \qquad \text{(II.3)}$$

In the same way one derives angular distributions for higher fermion spins. They are given in Table IVa.

Table IVa

Adair - distributions for half integer spins

$$J = \tfrac{1}{2} \quad \frac{dN}{d\cos\Theta} = 1$$

$$J = \tfrac{3}{2} \quad = \tfrac{1}{2} + \tfrac{3}{2}\, x^2$$

$$J = \tfrac{5}{2} \quad = \tfrac{3}{4} - \tfrac{3}{2}\, x^2 + \tfrac{15}{4}\, x^4$$

$$J = \tfrac{7}{2} \quad = \tfrac{9}{16} + \tfrac{45}{16}\, x^2 - \tfrac{165}{16}\, x^4 + \tfrac{175}{16}\, x^6$$

If the fermions spin is $n/2$, the highest power in the $\cos\Theta$ distribution is $n - 1$.

b) Spin Determination of Bosons

Let us now apply the Adair analysis to a case where both the particles in the final state have spin, and one is a boson. Consider the process

$$\pi^+ + p \rightarrow \rho^+ + p \ .$$

ρ stands for any boson with integer spin J. When selecting events where the ρ is emitted into the forward or backward direction, we fix the z component of the final orbital angular momentum to 0. The combined spin state of ρ and p is described by

$$\psi = |J_\rho J_p M_\rho M_p>$$

where J_ρ is the ρ spin, $J_p = \frac{1}{2}$ the proton spin and M_ρ, M_p the corresponding z components. The z component of the total spin has to be conserved. Thus the initial proton spin state

$$|\tfrac{1}{2},\tfrac{1}{2}>_p \quad \text{leads to } \psi_1 = \alpha_1 |J_\rho, \tfrac{1}{2}, 0, \tfrac{1}{2}> + \beta_1 |J_\rho, \tfrac{1}{2}, 1, -\tfrac{1}{2}>$$

$$\text{and} \quad |\tfrac{1}{2},-\tfrac{1}{2}>_p \quad \text{leads to } \psi_2 = \alpha_2 |J_\rho, \tfrac{1}{2}, 0, -\tfrac{1}{2}> + \beta_2 |J_\rho, \tfrac{1}{2}, -1, \tfrac{1}{2}>$$

for the final states. The amplitudes α and β refer to non-spinflip and spinflip production. If Y_J^M describes the spin state of the ρ the two possible final states can be written as

$$\psi_1 = \alpha_1 Y_J^0 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta_1 Y_J^1 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\psi_2 = \alpha_2 Y_J^0 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \beta_2 Y_J^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \ .$$

With $\alpha_1^2 = \alpha_2^2 = \alpha^2$ and $\beta_1^2 = \beta_2^2 = \beta^2$

the angular distribution of the ρ decay is

$$\psi^*\psi = \tfrac{1}{2}(\psi_1^*\psi_1 + \psi_2^*\psi_2) = |\alpha|^2 |Y_J^0|^2 + |\beta|^2 |Y_J^1|^2 .$$

Assume α and β normalized

$$|\alpha|^2 + |\beta|^2 = 1$$

and call $|\beta|^2 = r$, the fraction of the spin flip cross section, then

$$\frac{dN}{d\cos\Theta} = |Y_J^0|^2 + r\,(|Y_J^1|^2 - |Y_J^0|^2) . \qquad \text{(II.4)}$$

Table IVb gives the angular distribution for J = 0,1,2.

<u>Table IVb</u>

Adair distributions for integer J. r is the spin flip ratio. $x = \cos\Theta$

$$J = 0 \quad \frac{dN}{d\cos\Theta} = 1$$

$$J = 1 \quad = 3x^2 + r(3/2 - 9/2\ x^2)$$

$$J = 2 \quad = 5/4 - 15/2\ x^2 + 45/4\ x^4 + r(-5/4 + 15\ x^2 - 75/4\ x^4)$$

The highest power in the cos Θ distribution is 2J. Observe, however, that an appropriate r can make the highest power term vanish. For instance r = 2/3 for J = 1 yields an isotropic distribution.

c) <u>Choice of Acceptance Angle</u>

The decay distributions are strictly valid only for particles emitted at 0° or 180°. In practice however, one has to choose a finite acceptance angle. How large may it be so that the decay distribution is still significant for the spin of the produced particle? Take the case of a fermion. Here the requirement is of course, that nearly all produced particles are in the states $|J,\frac{1}{2}>$ or $|J,-\frac{1}{2}>$. Regarding the example Eq. (II.1) we see that all other states $|J,m>$, $m \neq \frac{1}{2},-\frac{1}{2}$ are associated with production orbital momentum states $Y_\ell^m(m\neq 0)$. The $Y_\ell^m(m\neq 0)$ are 0 at 0° and 180°, which is the crucial requirement for the Adair analysis to work, and rise with $\sin^m \Theta$. So the cutoff angle should be reasonably chosen smaller or equal to the first maximum of all substantially contributing $Y_\ell^m(m\neq 0)$. Looking at tables of the adjoint Legendre functions e.g. in Jahnke-Emde, one finds that for fixed ℓ, Y_ℓ^1 reaches the maximum first and in the same time covers a larger area from 0° to Θ_{max} than the other Y_ℓ^m of the same ℓ. So Y_ℓ^1 is the most dangerous term for the Adair analysis for a given ℓ. Θ_{max} is roughly $1/\ell$, so the highest ℓ contributing substantially to the production - call it ℓ_{max} - determines the cutoff angle which corresponds to the first maximum of $Y^1_{\ell_{max}}$. Thus we have

$$\Theta_{cutoff} \approx \frac{1}{\ell_{max}} .$$

If one assumes that the range of the interaction is given by the Compton wavelength of the π

$$\lambda_\pi = \frac{\hbar}{m_\pi c} , \qquad \text{(II.5)}$$

an estimate of ℓ_{max} can easily be obtained regarding λ_π as the maximum impact parameter. Applying the classical definition of angular momentum one finds the relation

$$\hbar\ell_{max} = p_{inc}^{CM}\,\lambda_\pi = p_{inc}^{CM} \cdot \frac{\hbar}{m_\pi C} \quad \text{or}$$
$$\ell_{max} = \frac{p_{inc}^{CM}}{m_\pi C} \qquad \text{(II.6)}$$

p_{inc}^{CM} = momentum of incident particle in the c.m. system

d) Examples of Application

The Adair analysis was originally proposed[5] for the spin determination of strange particles. Applied to Λ and Σ hyperons produced in two body reactions, it gave the result that the Λ and Σ spins are $\frac{1}{2}$ [6]. Later on it was widely used in connection with excited baryon states and meson-meson resonances[7] though other methods to be discussed later, often proved more efficient because they either permit larger samples of events or because the Adair analysis is inadequate to the problem. There are, in fact, cases where the Adair analysis is bound to fail, namely in all reactions having a zero production amplitude in forward and backward direction. It was L. Stodolsky and J.J. Sakurai[8] who pointed out that the Adair analysis is not applicable if particles are produced in a certain channel of vector meson (ρ, K^*, spin 1 bosons) exchange where the amplitude is zero for production angles $\Theta = 0, \pi$.

For instance the reactions

$$\pi^+ + p \rightarrow N^{*++}_{3/2} + \pi^0$$

$$K^+ + p \rightarrow N^{*++}_{3/2} + K^0$$

$$\pi^+ + p \rightarrow Y^{*+}_1 + K^+$$

$$K^- + p \rightarrow Y^{*+}_1 + \pi^-$$

are most likely to proceed via ρ or K^* exchange, because π or K exchange are forbidden by parity. In fact it seems that in none of these reactions the Adair analysis has given satisfactory results for the spin determination of the baryons[8].

An important result of Stodolsky's and Sakurai's calculations was that the decay distribution of an $N^*_{3/2}$ produced in the most probable $M1 \rightarrow p_{3/2}$ transition of vector meson exchange should be of the form

$$\frac{dN}{d\cos\Theta} = \tfrac{1}{2}\left[1 + 3(\hat{n}\hat{\pi})^2\right]$$

where $\hat{n}$ is the production normal and $\hat{\pi}$ the direction of the π in the N^* decay measured in the N^* c.m.

This angular distribution corresponds to the eigenstate

$$\psi_{N^*} = |{}^3\!/_2, {}^1\!/_2 >$$

with respect to the production normal. Let us see what the angular distribution would be if measured with respect to the Adair direction. Firstly we rotate the state ψ_{N^*} by 90° so that the beam direction becomes the spin quantization direction. Applying the y rotation matrix of Table III

$$\psi'_{N^*} = D_y(\pi/2)\psi_{N^*} = \sqrt{{}^1\!/_8} \begin{bmatrix} 1 & -\sqrt{3} & \sqrt{3} & -1 \\ \sqrt{3} & -1 & -1 & \sqrt{3} \\ \sqrt{3} & 1 & -1 & -\sqrt{3} \\ 1 & \sqrt{3} & \sqrt{3} & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\psi'_{N^*} = \sqrt{{}^1\!/_8}\,[-\sqrt{3}\,|{}^3\!/_2, {}^3\!/_2 >' - |{}^3\!/_2, {}^1\!/_2 >' + |{}^3\!/_2, -{}^1\!/_2 >' + \sqrt{3}\,|{}^3\!/_2 - {}^3\!/_3 >']$$

where the states $|\quad >'$ are quantized along the beam direction. Averaging over the azimuthal angular distribution we can neglect the interference of different eigenstates $|\quad >'$, so that the resulting angular distribution is given by

$$\frac{dN}{d\cos\Theta} = {}^3\!/_4\,(\psi^*\psi)_{{}^3\!/_2, \pm{}^3\!/_2} + {}^1\!/_4\,(\psi^*\psi)_{{}^3\!/_2, \pm{}^1\!/_2}$$

where the two terms may be taken from Table VI. So

$$\frac{dN}{d\cos\Theta} = {}^3\!/_4\,({}^3\!/_2 - {}^3\!/_2 x^2) + {}^1\!/_4\,({}^1\!/_2 + {}^3\!/_2 x^2) = {}^5\!/_4\,(1 - {}^3\!/_5 x^2)$$

whereas the expected Adair distribution is ${}^1\!/_2\,(1 + 3x^2)$.

The Adair analysis offers no possibility to determine the parity of a baryon resonance of spin J decaying into a fermion of spin ${}^1\!/_2$ and a spinless meson. As seen in the example of a spin ${}^3\!/_2$ particle, the decay angular distribution does not depend on whether the decay proceeds via a P or a D wave. This feature was generalized in the theorem of Minami[13].

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For the Adair case it has the consequence that the resulting decay angular distribution of the two equally populated states $|J,\frac{1}{2}>$ and $|J,-\frac{1}{2}>$ is the same for the two possible orbital angular momenta $L_1 = J - \frac{1}{2}$, $L_2 = J + \frac{1}{2}$.

In addition the daughter fermions show no polarization at all if the decay is strong which means that no interference between L_1 and L_2 is possible. This can be seen from the polarization distributions calculated in Chapter IV, Table VI. Thus a polarization measurement of the daughter fermions of Adair selected events gives no answer about the parity either.

Problem

Discuss the Adair analysis of a spin $\frac{5}{2}$ particle A strongly decaying into a spin $\frac{3}{2}$ particle B and a π meson. Try to find a method to determine the relative parity A,B.

III THE DENSITY MATRIX

a) The General Formalism

In practice it is very rare that one deals with particles in pure spin states. Mostly one has statistical mixtures of pure states.

Take as an example, the hypothetical case of a system of spin $\frac{1}{2}$ particles where $\frac{1}{3}$ is completely polarized in z direction, the other $\frac{2}{3}$ in x direction. Furthermore for an individual particle it is not known to which category it belongs. It has probabilities of $p_1 = \frac{1}{3}$, $p_2 = \frac{2}{3}$ and of being in one or the other categories. Nevertheless it is possible to calculate expectation values of physical observables. The natural procedure is to calculate the expectation values for both the contributing states and to add them up with their relative probabilities.

There is a convenient mathematical tool to do this, namely, the formalism of the density matrix.

For a pure spin state

$$\psi = \sum_m a_m |j,m>$$

the expectation value of an operator Q is according to Eq. (I.15)

$$\langle Q \rangle = \sum_{m,k} a_m^* \, a_k \, \langle j,m | Q | j,k \rangle$$

$$= \sum_{m,k} a_m^* \, a_k \, Q_{m,k}$$

where $\langle j,m| Q | j,k\rangle$ is the matrix element $Q_{m,k}$ of the operator Q.

For a statistical mixture of states $\psi^{(i)}$ with probabilities $p^{(i)}$ the expectation value is

$$\langle Q \rangle = \sum_i p^{(i)} \langle Q \rangle^{(i)} = \sum_i p^{(i)} \left\{ \sum_{m,k} a_m^{*(i)} a_k^{(i)} \, Q_{mk} \right\} .$$

Defining the density matrix ρ as

$$\rho_{km} = \sum_i p^{(i)} a_m^{*(i)} a_k^{(i)} \tag{III.1}$$

$\langle Q \rangle$ can be written as

$$\left| \begin{aligned} \langle Q \rangle &= \sum_{m,k} Q_{mk} \, \rho_{km} = \sum_m (Q\rho)_{mm} \\ &= \mathrm{Tr}(Q\rho) \quad = \mathrm{Tr}(\rho Q) \end{aligned} \right| \tag{III.2}$$

Eq. (III.2) can be regarded as a definition of the density matrix and also connects it to measurable quantities $\langle Q \rangle$.

Let us quote some important properties of the density matrix which can easily be verified by the reader.

1) It is hermitian

$$\rho_{mk} = \rho_{km}^*$$

2) The sum of the diagonal elements is 1

$$\mathrm{Tr}(\ \rho) = \mathrm{Tr}(\rho) = \sum_m \rho_{mm} = 1$$

3) The diagonal elements are positive

$$\rho_{mm} \geq 0$$

4) $\mathrm{Tr}(\rho^2) = \Sigma_{mk} |\rho_{mk}|^2 \leq 1$

5) If the density matrix describes a mixture of N pure states

$$N\,\mathrm{Tr}(\rho^2) \geq \mathrm{Tr}(\rho) = 1$$

6) Under a unitary transformation U of the system of orthogonal basis vectors the density matrix transforms as an operator

$$\rho' = U\rho U^{-1} .$$

Expectations values $< Q >$ stay invariant under this transformation.

$$< Q' > = \mathrm{Tr}(Q'\rho') = \mathrm{Tr}(UQU^{-1}\,U\rho U^{-1})$$
$$= \mathrm{Tr}(UQ\rho U^{-1}) = \mathrm{Tr}(Q\rho) = < Q >$$

because the trace is invariant under unitary transformations.

7) A system of spin j has $2j+1 = N$ orthogonal basis vectors. The $N \times N$ complex elements of the density matrix correspond to $2N^2$ real parameters which are subject to different constraints:

a) hermiticity $\rho_{mk} = \rho^*_{km}$ gives N^2 constraints,
b) one constraint on the trace

$$\sum_1^N \rho_{mm} = 1$$

so there are left

$$n = 2N^2 - N^2 - 1 = N^2 - 1$$

real parameters which determine the density matrix.

8) As the density matrix can be regarded as a hermitian operator there exists always a unitary transformation that makes the density matrix diagonal. This transformation corresponds to a new choice of basis vectors. As a consequence a pure spin state can always be described by a density matrix having all elements 0 but one diagonal element which is 1.

As an example let us construct the density matrix for the system defined at the beginning of this Chapter.

$$\psi_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad ; \; p_1 = 1/3$$

$$\psi_2 = \sqrt{1/2}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \sqrt{1/2}\begin{bmatrix} 0 \\ 1 \end{bmatrix} ; \; p_2 = 2/3$$

In the terminology of Eq. (III.1) we have

$$a^{(1)}_{1/2} = 1 \qquad a^{(2)}_{1/2} = \sqrt{1/2}$$

$$a^{(1)}_{-1/2} = 0 \qquad a^{(2)}_{-1/2} = \sqrt{1/2}$$

$$p_1 = 1/3 \qquad p_2 = 2/3$$

and calculate

$$\rho = 1/3 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 2/3 \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 2/3 & 1/3 \\ 1/3 & 1/3 \end{bmatrix}$$

The polarization of this mixed state is

$$\langle P_x \rangle = \mathrm{Tr}(P_x \rho) = \mathrm{Tr}(\sigma_x \rho) =$$

$$= \mathrm{Tr}\left\{ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2/3 & 1/3 \\ 1/3 & 1/3 \end{bmatrix} \right\} = \mathrm{Tr}\begin{bmatrix} 1/3 & 1/3 \\ 2/3 & 1/3 \end{bmatrix} = 2/3$$

Correspondingly one finds

$$\langle P_y \rangle = 0 \quad ; \quad \langle P_z \rangle = 1/3$$

as was expected from the construction of the example.

The density matrix of spin $1/2$ particles depends on $2^2 - 1 = 3$ real parameters, so it is presumably possible to express the density matrix in terms of the three polarization components P_x, P_y, P_z. To this end one expands the 2×2 density matrix in terms of the unit matrix $\mathbb{1}$ and the matrices $\sigma_x, \sigma_y, \sigma_z$ which are all linearly independent. From the condition

$$\vec{P} = \tfrac{1}{2} \quad \langle\vec{J}\rangle = \langle\vec{\sigma}\rangle = \mathrm{Tr}(\vec{\sigma}\rho) \tag{III.3}$$

one calculates ρ in terms of $\vec{\sigma}$

$$\rho = \tfrac{1}{2} \ (\mathbb{1} + P_x\sigma_x + P_y\sigma_y + P_z\sigma_z)$$

$$= \tfrac{1}{2} \ (\mathbb{1} + \vec{P}\vec{\sigma}) = \tfrac{1}{2} \begin{bmatrix} 1 + P_z & P_x - iP_y \\ P_x + iP_y & 1 - P_z \end{bmatrix} \tag{III.4}$$

One verifies immediately that Eq. (III.4) satisfies Eq. (III.3). So the measurement of the three polarization components of spin ½ particles completely determines the spin density matrix of the system.

Generally, for systems with spin j, having $N = 2j + 1$ eigenstates, $N^2 - 1$ linear independent observables will determine the density matrix.

b) Random Mixture of Spin States

One may specify a random mixture of states with spin j by requiring that all possible pure states in the mixture are equally probable, or that the probability of the mixture to be in a definite pure spin state ψ_j is the same for all possible states $\psi_j^{(i)}$. This probability is

$$p = \psi_j^* \ \rho \ \psi_j \tag{III.5}$$

where the state ψ_j is represented by a column vector

$$\psi_j = \begin{bmatrix} a_1 \\ \vdots \\ a_m \\ \vdots \\ a_n \end{bmatrix}$$

and ψ_j by the corresponding row vector. The expression Eq. (III.5) is independent of the particular ψ_j (i.e. the normalized coefficients a_m)

only if ρ is a multiple of the unit matrix. Because of the normalization requirement of the density matrix, all diagonal elements have the value

$$\rho_{mm} = \frac{1}{2j+1} \quad .$$

Therefore a random mixture of spin j particles is described by an equal mixture of all eigenstates $|j,m\rangle$.

This special structure of the density matrix for the random spin state can also be derived by the requirement that it be invariant under any rotation D:

$$\rho' = D\rho D^{-1} = \rho \quad . \qquad \text{(III.6)}$$

As the D matrices are irreducible representations of the rotation group, and thus do not contain invariant subspaces, Eq. (III.6) is valid for arbitrary D's only if ρ is a multiple of the unit matrix.

c) Density Matrix of Particles Produced in Strong 2-body Reactions

We consider the reaction

$$A + B \rightarrow C + D$$

where all the particles may have different and arbitrary spins but particles A and B are required to represent a random mixture of pure spin states, e.g. unpolarized protons. Now take out of the mixture a certain pure initial state Φ_i described by

$$\Phi_i = \psi_i(\vec{r})\ S_A\ S_B$$

where $\psi(\vec{r})$ is the space part of the wave function and S_A and S_B are pure spin states of particles A and B. The space part is an incident wave in x direction

$$\psi(\vec{r}) = e^{ipx} \ .$$

Apply to Φ_i the parity operation P which effects a reflection at the origin:

$$\vec{r} \rightarrow -\vec{r} \ .$$

P acts on the constituents of Φ_i in the following way:

$$P\psi(\vec{r}) = e^{-ipx}$$

$$PS_A = \xi_A S_A$$

$$PS_B = \xi_B S_B$$

where ξ_A and ξ_B are the intrinsic parities of particles A and B. So

$$P\Phi_i = \xi_A\, \xi_B\, e^{-ipx}\, S_A\, S_B\ .$$

Now we rotate the state $P\Phi_i$ by 180° around the z axis, which is chosen to be the production normal $\hat{n}$

initial state final state

$$D_z(\pi)P\Phi_i = \xi_A\, \xi_B\, D_z(\pi)\, e^{-ipx}\, S_A\, S_B\ .$$

For the special choice of the rotation axis the operation $D_z(\pi)$ is with respect to $\psi_i(\vec{r})$ equivalent to the parity operation P. So we have

$$D_z(\pi)P\Phi_i = \psi_i(\vec{r}) \left[\xi_A D_z(\pi) S_A\right]\left[\xi_B D_z(\pi) S_B\right]\ .$$

The transformed spin states

$$S_A{}' = \xi_A D_z(\pi) S_A$$

$$S_B{}' = \xi_B D_z(\pi) S_B$$

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must be contained in the initial random mixture with equal probability as S_A and S_B. Therefore

$$D_z(\pi)P\Phi_i$$

is another initial pure state with the same probability as Φ_i. As this is true for any pure initial state we have shown that the random initial state is invariant under the operation

$$D_z(\pi)P \ .$$

Now, if this symmetry property is not changed by the interaction, i.e. if parity is conserved in the interaction, the final mixed state must also be invariant under $D_z(\pi)P$.

This requirement implies that for every production angle the spin density matrices of particle C and D are invariant under a rotation of 180° about the production normal, and thus imposes a special structure on the density matrix. The density matrix transforms as an operator, so one requires

$$\rho' = D_z(\pi) \ \rho \ D_z^{-1}(\pi) = \rho$$

or

$$\rho D_z(\pi) = D_z(\pi)\rho$$

$D_z(\pi)$ being a diagonal matrix, if z is the spin quantization direction we have for one matrix element ρ_{km}

$$\rho_{km} D_{mm} = D_{kk} \rho_{km}$$

$$\rho_{km} e^{-im\pi} = e^{-ik\pi} \rho_{km}$$

$$\rho_{km} = e^{i(m-k)\pi} \rho_{km} \ .$$

As a consequence, all matrix elements ρ_{km} with $|m-k| = 2n+1$ $(n = 0,1,2 \ldots)$ are 0 so that the density matrix shows a checker board pattern[9])

$$\rho = \begin{bmatrix} x & 0 & x & 0 \\ 0 & x & 0 & x \\ x & 0 & x & 0 \\ 0 & x & 0 & x \end{bmatrix}$$

Now it is easy to show that, as a corollary to this more general result, particles produced in strong 2-body reactions from random initial spin states can have a vector polarization only in the direction of the production normal (z direction), the spin expectation values $\langle J_x \rangle$, $\langle J_y \rangle$ being 0.

This can be immediately verified, recalling that in the J_x, J_y matrices (cf. Eq. (I.21) only the elements $C_{k,k+1}$ are different from 0; so

$$\langle J_{x,y} \rangle = \mathrm{Tr}\,(J_{x,y}\rho)$$

$$= \mathrm{Tr}\left\{ \underbrace{\begin{bmatrix} 0 & x & 0 & 0 \\ x & 0 & x & 0 \\ 0 & x & 0 & x \\ 0 & 0 & x & 0 \end{bmatrix}}_{J_{x,y}} \underbrace{\begin{bmatrix} x & 0 & x & 0 \\ 0 & x & 0 & x \\ x & 0 & x & 0 \\ 0 & x & 0 & x \end{bmatrix}}_{\rho} \right\} = 0$$

[References: (4), (10), (11), (12)]

IV ANGULAR AND POLARIZATION DISTRIBUTION WITH RESPECT TO THE PRODUCTION NORMAL

As discussed in the preceding Chapter, parity conservation in strong interactions allows particles produced in 2-body reactions to be polarized only in the direction of the production normal. Thus in view of the spin determination it seems promising to study angular decay distributions with respect to this direction; for if the particles are polarized or aligned along the production normal, the observed decay distribution may allow a statement about the spin.

We take the production normal as spin quantization direction z and consider for the moment only spin states corresponding to the diagonal elements of the density matrix.

First consider integer spin particles in the state $|J,M\rangle$ decaying into two spinless bosons. This case refers to 2-body decays of meson-meson resonance. The angular distributions clearly are

$$\frac{dN}{d\cos\Theta} = Y_J^{M*} Y_J^M \ . \qquad (IV.1)$$

The results for the same integer J values are listed in Table V.

Table V

Decay distributions of spin J bosons in state $|J,M\rangle$ into two spinless particles. $x = \cos\Theta$

State	$\frac{dN}{d\cos\Theta}$
$\|0,0\rangle$	1
$\|1,0\rangle$	$3x^2$
$\|1\pm1\rangle$	$3/2 - 3/2\,x^2$
$\|2,0\rangle$	$5/4 - 30/4\,x^2 + 45/4\,x^4$
$\|2,\pm1\rangle$	$15/2\,x^2 - 15/2\,x^4$
$\|2,\pm2\rangle$	$15/8 - 15/4\,x^2 + 15/8\,x^4$

Observe, that according to the spherical harmonic addition theorem Eq. (I.29) an equal mixture of all states from $|J,-J>$ to $|J,+J>$ results in an isotropic angular distribution. Furthermore, even if there is a resultant polarization, the angular distribution might still add up to isotropy as for instance in the case where only states from $|J,0>$ to $|J,J>$ are present with equal weights for $|J,1>$ to $|J,J>$ and half that weight for $|J,0>$. This also follows directly from the addition theorem Eq. (I.29), observing that $|J,M>$ and $|J,-M>$ lead to identical distributions.

Let us now turn to baryons decaying into a spin $\frac{1}{2}$ fermion and a spinless meson, as

$$Y^* \to \Lambda + \pi \quad .$$

We confine ourselves here to parity conserving decays. Take for example a Y^* in the spin state $|\frac{3}{2},\frac{3}{2}>$. It can decay via a P wave or a D wave, depending on its parity.

a) P wave decay:

$$|\tfrac{3}{2},\tfrac{3}{2}> \to \psi = Y_1^1 \, |\tfrac{1}{2},\tfrac{1}{2}> \quad .$$

The angular distribution of the decay is

$$\psi^*\psi = |Y_1^1|^2 = \tfrac{3}{2} - \tfrac{3}{2}x^2 \quad .$$

Now consider the polarization of the Λ. It is clearly 1 in z direction for all decay angles because the Λ is in the eigenstate $|\frac{1}{2},\frac{1}{2}>$.

Instead of regarding the polarization $\vec{P}$ of the daughter fermion itself, we shall use the quantity

$$\psi^*\psi \; \vec{P}(\Theta) = \frac{dN}{d \cos \Theta} \, \vec{P}(\Theta)$$

because this expression is often less complicated than $\vec{P}(\Theta)$. Therefore for our case

$$\psi^*\psi \; P_z = \tfrac{3}{2} - \tfrac{3}{2}x^2 \quad .$$

b) D wave decay

$$|{}^3\!/_2,{}^3\!/_2\rangle \rightarrow \psi = -\sqrt{{}^1\!/_5}\; Y_2^1\,|{}^1\!/_2,{}^1\!/_2\rangle + \sqrt{{}^4\!/_5}\; Y_2^2\,|{}^1\!/_2,-{}^1\!/_2\rangle$$

$$\psi = \sqrt{{}^1\!/_6}\; 3x\sqrt{1-x^2}\; e^{i\varphi}\begin{pmatrix}1\\0\end{pmatrix} + \sqrt{{}^1\!/_6}\,.\,3(1-x^2)\; e^{2i\varphi}\begin{pmatrix}0\\1\end{pmatrix}$$

$$= a\begin{pmatrix}1\\0\end{pmatrix} + b\begin{pmatrix}0\\1\end{pmatrix}$$

The polarization in z,x,y direction is

$$\psi^*\psi\; P_z = |a|^2 - |b|^2 = {}^3\!/_2 x^2\;(1-x^2) - {}^3\!/_2(1-2x^2+x^4)$$

$$= {}^3\!/_2(-1+3x^2-2x^4)$$

$$\psi^*\psi\; P_x = 2\mathrm{Re}(a^*b) = 3x(1-x^2)^{3/2}\;\cos\varphi$$

$$\psi^*\psi\; P_y = 2\mathrm{Im}(a^*b) = 3x(1-x^2)^{3/2}\;\sin\varphi$$

so the total polarization has no φ dependence and the radial component is

$$\psi^*\psi\; P_r = 3x(1-x^2)^{3/2}$$

One may as well split the polarization $\vec{P}$ into a longitudinal component $P_{\parallel}$ parallel to the emission direction of the Λ and a transversal component $P_{\perp}$ perpendicular to $P_{\parallel}$ and lying in the plane defined by the z axis (production normal) and the emission direction of the Λ in the Y* system.

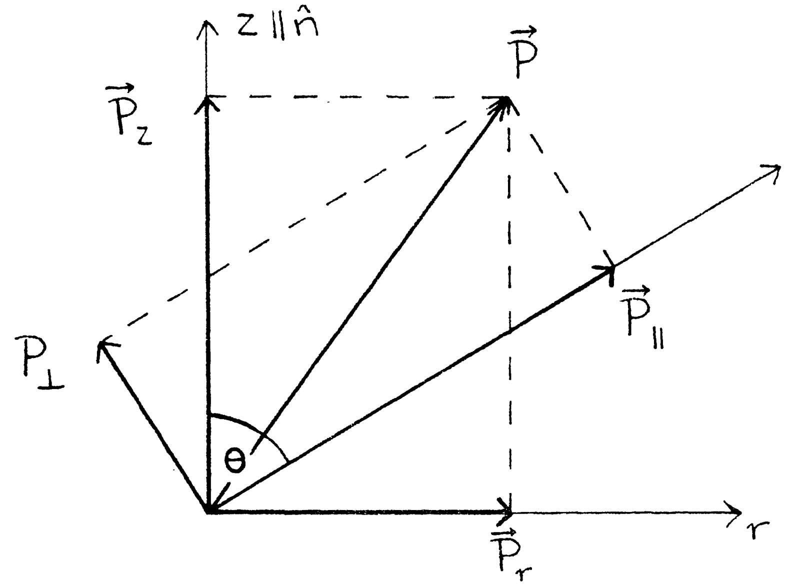

Fig. 4

8) $\psi^*\psi\, P_z$ is an even polynomial in $\cos\Theta$.

9) $\psi^*\psi\, P_r$ has the form $f(\cos^2\Theta)\cos\Theta\sin\Theta$, i.e. an odd function of $\cos\Theta$.

10) The polarization component $P_{\parallel}$ in the fermion emission direction does not depend on the parity of the state $|J,M\rangle$.

11) The perpendicular polarization component $P_{\perp}$ of the state $|JM\rangle$ changes sign for the opposite parity state.

12) $\psi^*\psi\, P_{\parallel}$ has the form $f_1(\cos^2\Theta)\cos\Theta$.

13) $\psi^*\psi\, P_{\perp}$ has the form $f_2(\cos^2\Theta)\sin\Theta$.

14) If a state $|JM\rangle$ leads to a certain polarization distribution in the z direction $P_z^+(\Theta)$, the opposite parity state shows the same polarization distribution with respect to the magic direction et v.v.

$$P_z^+(\Theta) = P_m^-(\Theta)\,.$$

Let us say a few words about the experimental situation. From the above results it follows that the spin of a particle can be determined, at least in principle, from the decay angular distribution, if the contributing states do not happen to add up to isotropy. If the sample of events is large enough it is advantageous to consider separate subsamples corresponding to different production angles, because for small intervals of production angles the spin density matrix of the produced particle is less likely to contain the sums over many pure states which could average out the angular distribution to isotropy. Remember that for a fixed production angle the spin density matrix contains, e.g. for the reaction $\pi + p \rightarrow Y^* + K$, only two pure states corresponding to the two pure initial spin states of the nucleons which build up the unpolarized target.

Because of the Minami ambiguity[13] the decay angular distribution does not give any information about the parity of the state i.e. whether the decay proceeds over a wave with even or odd angular momentum. The polarization distribution of the daughter fermion however, does depend on the parity of the decaying state. In the example of the decay $Y^* \rightarrow \Lambda + \pi$ the polarization of the Λ can conveniently be detected by its decay asymmetry, as discussed in connection with the decay parameters of the Ξ. How can the polarization functions

$$\frac{dN}{d\cos\Theta} P$$

listed in Table VI be obtained experimentally? As shown in the paper about the Ξ decay parameters, the decaying pion from the Λ has an angular distribution with respect to any polarization component $\vec{P}_c$

$$\frac{dN}{d\cos\Theta_{\pi c}} = N \, ½ \left[1 - \alpha_\Lambda (\vec{P}_c \hat{\pi}) \right] \qquad \text{(IV.3)}$$

where α_Λ is the α decay parameter of the Λ, $\hat{\pi}$ is a unit vector indicating the π direction in the Λ-CM and

$$(\vec{P}_c \hat{\pi}) = P_c \cos\Theta_{\pi c} \; .$$

So, fitting the measured values $\cos\Theta_{\pi c}$ for the Λ polarization components under consideration (P_z, P_r or $P_{\parallel}, P_{\perp}$) to the distribution Eq. (IV.3) one obtains P_c (α_Λ is known and = -0.62). This has to be done for different intervals of cos Θ (Θ = emission angle of Λ in Y* system in the meaning of Table VI) in order to obtain the polarization functions $\frac{dN}{d\cos\Theta} P_c$. In the limit of large statistics the fit of Eq. (IV.3) is effected very easily:

Calculate the average $\cos\Theta_\pi$:

$$\langle\cos\Theta_\pi\rangle = \int_{-1}^{+1} ½ \left[1 - \alpha_\Lambda P \cos\Theta_\pi \right] \cos\Theta_\pi \, d\cos\Theta_\pi$$

$$\langle\cos\Theta_\pi\rangle = ⅓ \, \alpha_\Lambda P$$

$$P = \frac{3}{\alpha_\Lambda} \langle\cos\Theta_\pi\rangle$$

$$= \frac{3}{\alpha_\Lambda} \frac{1}{N} \left\{ \sum_1^N \cos\Theta_{\pi i} \right\} .$$

The polarization functions in Table VI can now be written

$$\psi^*\psi\ P(\Theta) = \frac{N_\Theta}{N_{tot}} P = \frac{N_\Theta}{N_{tot}} \cdot \frac{3}{\alpha_\Lambda} \frac{1}{N_\Theta} \left\{ \sum_1^{N_\Theta} \cos \Theta_{\pi_i} \right\}$$

$$\psi^*\psi\ P(\Theta) = \frac{1}{N_{tot}} \cdot \frac{3}{\alpha_\Lambda} \left\{ \sum_1^{N_\Theta} \cos \Theta_{\pi_i} \right\} \qquad \text{(IV.4)}$$

where N_Θ is the number of events lying in a finite interval $\Delta \cos \Theta$ and N_{tot} is the total number of events. Observe that the distribution Eq. (IV.4) is normalized to 1 whereas all $\cos \Theta$ distributions in the Tables are normalized to 2.

This type of analysis was applied, e.g. by Shafer et al.[14] to determine the spin and parity of the 1385 MeV Y_1^* resonance produced in the reaction

$$K^- + p \rightarrow Y^{\pm *} + \pi^{\mp} \rightarrow \Lambda + \pi^+ + \pi^- .$$

They obtained for the angular distribution

$$\frac{dN}{d \cos \Theta} = 1 + (0.69 \pm 0.22)\ (\hat{\Lambda}\hat{n})^2$$

and for the Λ polarization components in the normal and magic direction

$$\frac{dN}{d \cos \Theta} P_n = -1 + 3.5\ (\hat{\Lambda}\hat{n})^2$$

$$\frac{dN}{d \cos \Theta} P_m = 1 - 9.7\ (\Lambda \hat{n})^2 + 11.2\ (\hat{\Lambda}\hat{n})^4 .$$

Comparing these results to the theoretical distributions in Table VI the conclusion was that the most likely spin parity assignment is $P_{3/2}$ i.e.

$$J = 3/2,\ \text{P wave decay.}$$

It should be emphasized that the theoretical derivations of angular and polarization distributions were based on the implicit assumption that particles should have a definitive spin and should decay free from interaction with other particles. In practice however, when dealing with resonances decaying by strong interaction, there is always an inseparable non-resonant background containing other angular momentum states which might interfere with the resonant state and considerably modify the pure resonance effects in angular and polarization distributions.

Adair[15] for instance, has shown that the above mentioned results of Shafer et al. for the Y_1^* could be explained as well by the spin parity assignment $P_{1/2}$ assuming a specific background interference with other angular momentum states. Without going into further details of the background problem it should be mentioned that background interference is most likely to change rapidly with the c.m. energy of the production process. Therefore if several analyses at different production energies give the same spin parity assignment for a resonance, there is little chance that background interference imitates the same wrong results in all cases.

Up to now we have only considered the decay of spin eigenstates corresponding to the diagonal elements of the density matrix, which are the only contributing ones, if averaging is done over the azimuth φ of the decay distribution around the production normal. However, it should be borne in mind, that an experimental bias in the decay azimuth φ will, in general, influence the observed polar decay distribution $dN/d\cos\Theta$. All this applies as well to events chosen at a fixed production angle as to a sample corresponding to a finite angular region.

Let us now investigate what one might learn from the azimuthal dependence of the decay angular distribution. Such φ dependences result from non diagonal elements in the spin density matrix of the decaying particle. As discussed in Chapter III the parity conserving production process allows only matrix elements of the form

$$C^*_m C_{m+2k} \quad \text{and} \quad C^*_{m+2k} C_m, \; k = 1,2,3 \ldots$$

corresponding to pure states of the form

$$\psi = \ldots + C_m | j,m> + C_{m+2k} | j,m+2> + \ldots$$

Take the simple case of a state with integer j decaying into two spin 0 mesons. For simplicity we confine ourselves to the superposition of only two eigenstates $|j,m>$ and $|j,m+2>$ which are in our case two spherical harmonics Y_j^m and Y_j^{m+2}. Thus ψ has the form

$$\psi = C_m \, f_1(\Theta) e^{im\varphi} + C_{m+2k} \, f_2(\Theta) e^{i(m+2k)\varphi}$$

$$\frac{dN}{d\Omega} = \psi^* \psi = |C_m|^2 \; f_1(\Theta)^2 + |C_{m+2k}|^2 \; f_2(\Theta)^2$$

$$+ 2f_1(\Theta) \; f_2(\Theta) \; \mathrm{Re}\left(C_m^* C_{m+2k} \; e^{\,ik\varphi} \right) . \qquad \text{(IV.5)}$$

With the substitution

$$C_m^* C_{m+2k} = |C_m^* C_{m+\ k}| e^{i\delta}$$

the angular distribution obtains the form

$$\frac{dN}{d\Omega} = F_1(\Theta) + F_2(\Theta) \; \cos \; (2k\varphi + \delta) \qquad \text{(IV.6)}$$

Exercise

Prove the following theorem:

Assume the angular momentum state with arbitrary J

$$\psi = C_{m_1} |j,m_1> + C_{m_2} |j,m_2>$$

performs a parity violating decay into two particles of arbitrary spin j_1, j_2. The resulting angular distribution can always be written as

$$\frac{dN}{d\Omega} = F_1(\Theta) + F_2(\Theta) \; \cos \; [(m_1 - m_2)\; \varphi + \delta] \; .$$

As seen from formula Eq. (IV.6) the angular distribution is invariant under rotation of 180° around the z aixs. This is just the property the density matrix was required to have, because of parity conservation.

For $k = 1$, $\delta = 0$ the second term in Eq. (IV.6) would give rise to a φ distribution sketched in Fig. 6 as a polar diagram

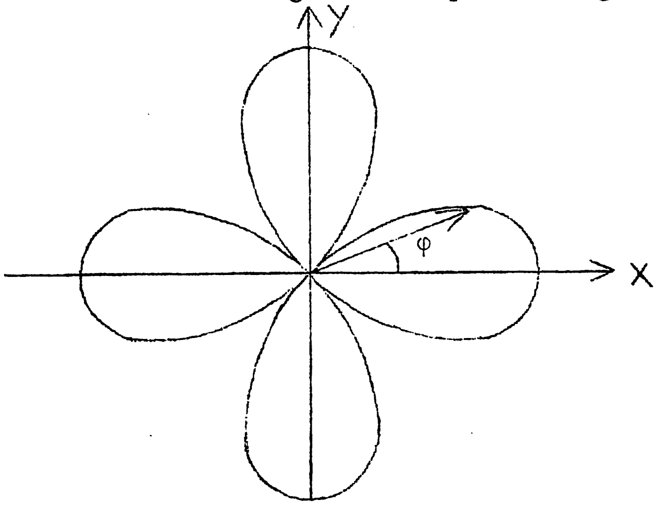

Fig.6

Terms of the form $\cos\left[(2k+1)\varphi\right]$ give corresponding polar diagrams but with an odd number of leaves so that the configuration is not invariant under rotations of 180°.

These properties of the azimuthal distributions may sometimes be of use as additional evidence in the spin determination. In particular a spin $\frac{1}{2}$ particle produced in a strong interaction should not show any azimuthal dependence in the decay distribution, even if it is a P wave decay, because the density matrix for spin $\frac{1}{2}$ does not contain any term $C^*_m C_{m+2}$. On the other hand a spin $\frac{3}{2}$ particle could show a similar decay pattern as indicated in Fig. 6, resulting from the terms $C^*_{3/2} C_{-1/2}$ and $C^*_{-3/2} C_{1/2}$.

V LEE-YANG TEST FUNCTIONS[18]

Let us now investigate parity non-conserving decays of particles in spin eigenstates with the production normal (z axis) as spin quantization direction. As shown before, any statistical mixture of spin eigenstates reduces to the diagonal elements of the density matrix if, in angular and polarization distributions, averaging over the azimuth is done. The case for spin $\frac{1}{2}$ is treated in the paper about the Ξ decay parameters. As for higher spins the mathematical procedure is essentially the same as for the spin $\frac{1}{2}$ case, no explicit calculations will be carried through here, but to indicate the procedure once more, take the case of a particle in the spin state $|\frac{3}{2},\frac{3}{2}>$. The decay into a fermion and spinless boson leads to the final state wave function

$$\psi = P\left\{Y_1^1 \begin{pmatrix}1\\0\end{pmatrix}\right\} + D\left\{-\sqrt{\tfrac{1}{5}}\; Y_2^1 \begin{pmatrix}1\\0\end{pmatrix} + \sqrt{\tfrac{4}{5}}\; Y_2^2 \begin{pmatrix}1\\0\end{pmatrix}\right\}$$

$$= \underbrace{\left\{PY_1^1 - \sqrt{\tfrac{1}{5}}\; DY_2^1\right\}}_{a}\begin{pmatrix}1\\0\end{pmatrix} + \underbrace{\left\{\sqrt{\tfrac{4}{5}}\; DY_2^2\right\}}_{b}\begin{pmatrix}0\\1\end{pmatrix}$$

$$= a\begin{pmatrix}1\\0\end{pmatrix} + b\begin{pmatrix}0\\1\end{pmatrix} \;.$$

The P and D wave amplitudes are normalized

$$|P|^2 + |D|^2 = 1 \;.$$

The decay angular distribution is

$$\psi^*\psi = |a|^2 + |b|^2$$

and the polarization of the final state fermion is expressed by

$$P_x = 2\mathrm{Re}(a^*b)$$

$$P_y = 2\mathrm{Im}(a^*b)$$

$$P_z = |a|^2 - |b|^2 \ .$$

Working out these expressions leads to the results in Table VII. There the polarization of the daughter fermion is given in a coordinate system defined in the following way. With the quantization direction $\hat{n}$ and the fermion emission direction $\hat{e}_1$ (in the c.m. system of the decaying particle) we define the unit vectors

$$\hat{e}_2 = \frac{\hat{n} \times \hat{e}_1}{|\hat{n} \times \hat{e}_1|}$$

$$\hat{e}_3 = \frac{\hat{e}_1 \times \hat{e}_2}{|\hat{e}_1 \times \hat{e}_2|}$$

as indicated in Fig. 7.

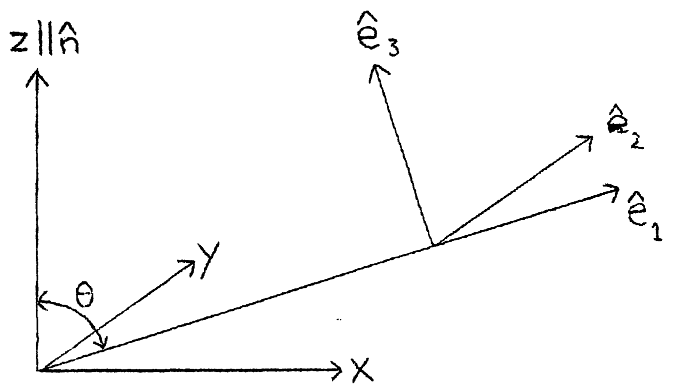

Fig.7

Table VII

Angular and polarization distributions of parity non-conserving fermion decays. Initial state: $|j,m\rangle$. $x = \cos\Theta$. $\psi^*\psi = dN/d\cos\Theta$.

ϵ = polarization of initial eigenstates $\begin{matrix} \epsilon = 1 & \text{for } m > 0 \\ \epsilon = -1 & \text{for } m < 0 \end{matrix}$

$|\frac{1}{2}, \pm\frac{1}{2}\rangle$

$$\psi^*\psi = 1 - \epsilon[2\,\mathrm{Re}(S^*P)\,x]$$

$$\psi^*\psi P_{\parallel} = \epsilon[x] - 2\,\mathrm{Re}(S^*P)$$

$$\psi^*\psi P_2 = -\epsilon[2\,\mathrm{Im}(S^*P)\sqrt{1-x^2}]$$

$$\psi^*\psi P_{\perp} = \epsilon[(|S|^2 - |P|^2)\sqrt{1-x^2}]$$

$|\frac{3}{2}, \pm\frac{3}{2}\rangle$

$$\psi^*\psi = \frac{3}{2} - \frac{3}{2}x^2 - \epsilon\left[2\,\mathrm{Re}(P^*D)\,x\left(\frac{3}{2} - \frac{3}{2}x^2\right)\right]$$

$$\psi^*\psi P_{\parallel} = \epsilon\left[x\left(\frac{3}{2} - \frac{3}{2}x^2\right)\right] - 2\,\mathrm{Re}(P^*D)\left(\frac{3}{2} - \frac{3}{2}x^2\right)$$

$$\psi^*\psi P_2 = -\epsilon\left[2\,\mathrm{Im}(P^*D)\sqrt{1-x^2}\left(\frac{3}{2} - \frac{3}{2}x^2\right)\right]$$

$$\psi^*\psi P_{\perp} = \epsilon\left[\left(|P|^2 - |D|^2\right)\sqrt{1-x^2}\left(\frac{3}{2} - \frac{3}{2}x^2\right)\right]$$

$|\frac{3}{2}, \pm\frac{1}{2}\rangle$

$$\psi^*\psi = \frac{1}{2} + \frac{3}{2}x^2 - \epsilon\left[2\,\mathrm{Re}(P^*D)\,x\left(-\frac{5}{2} + \frac{9}{2}x^2\right)\right]$$

$$\psi^*\psi P_{\parallel} = \epsilon\left[x\left(-\frac{5}{2} + \frac{9}{2}x^2\right)\right] - 2\,\mathrm{Re}(P^*D)\left(\frac{1}{2} + \frac{3}{2}x^2\right)$$

$$\psi^*\psi P_2 = -\epsilon\left[2\,\mathrm{Im}(P^*D)\sqrt{1-x^2}\left(-\frac{1}{2} + \frac{9}{2}x^2\right)\right]$$

$$\psi^*\psi P_{\perp} = \epsilon\left[\left(|P|^2 - |D|^2\right)\sqrt{1-x^2}\left(-\frac{1}{2} + \frac{9}{2}x^2\right)\right]$$

Table VII (continued)

$|\frac{5}{2}, \pm\frac{5}{2}>$

$$\psi^*\psi = \{1 - \epsilon[2\ \mathrm{Re}(D^*F)\ x]\}\ \frac{15}{8}\ (1 - x^2)^2$$

$$\psi^*\psi P_{/\!/} = \{\epsilon[x] - 2\ \mathrm{Re}(D^*F)\}\ \frac{15}{8}\ (1 - x^2)^2$$

$$\psi^*\psi P_2 = -\ \epsilon[\ 2\ \mathrm{Im}(D^*F)\sqrt{1 - x^2}\ ]\ \frac{15}{8}\ (1 - x^2)^2$$

$$\psi^*\psi P_\perp = \epsilon[\ (|D|^2 - |F|^2)\sqrt{1 - x^2}\ ]\ \frac{15}{8}\ (1 - x^2)^2$$

$|\frac{5}{2}, \pm\frac{3}{2}>$

$$\psi^*\psi = \{1 + 15\ x^2 - \epsilon[2\ \mathrm{Re}(D^*F)\ x\ (-\ 9 + 25\ x^2)]\}\ \frac{3}{8}\ (1 - x^2)$$

$$\psi^*\psi P_{/\!/} = \{\epsilon[x(-\ 9 + 25\ x^2)] - 2\ \mathrm{Re}(D^*F)\ (1 + 15\ x^2)\}\ \frac{3}{8}\ (1 - x^2)$$

$$\psi^*\psi P_2 = -\ \epsilon[2\ \mathrm{Im}\ (D^*F)\sqrt{1 - x^2}(-\ 1 + 25\ x^2)]\ \frac{3}{8}\ (1 - x^2)$$

$$\psi^*\psi P_\perp = \epsilon[(|D|^2 - |F|^2)\sqrt{1 - x^2}\ (-\ 1 + 25\ x^2)]\ \frac{3}{8}\ (1 - x^2)$$

$|\frac{5}{2}, \pm\frac{1}{2}>$

$$\psi^*\psi = \frac{3}{4} - \frac{6}{4}\ x^2 + \frac{15}{4}\ x^4 - \epsilon\left[2\ \mathrm{Re}(D^*F)\left(\frac{15}{4} - \frac{78}{4}\ x^2 + \frac{75}{4}\ x^4\right)\right]$$

$$\psi^*\psi P_{/\!/} = \epsilon\left[x\left(\frac{15}{4} - \frac{78}{4}\ x^2 + \frac{75}{4}\ x^4\right)\right] - 2\ \mathrm{Re}(D^*F)\left(\frac{3}{4} - \frac{6}{4}\ x^2 + \frac{15}{4}\ x^4\right)$$

$$\psi^*\psi P_2 = -\ \epsilon\left[2\ \mathrm{Im}(D^*F)\sqrt{1 - x^2}\left(\frac{3}{4} - \frac{42}{4}\ x^2 + \frac{75}{4}\ x^4\right)\right]$$

$$\psi^*\psi P_\perp = \epsilon\left[(|D|^2 - |F|^2)\sqrt{1 - x^2}\left(\frac{3}{4} - \frac{42}{4}\ x^2 + \frac{75}{4}\ x^4\right)\right]$$

The polarization components in the directions $\hat{e}_1, \hat{e}_2, \hat{e}_3$ are

$$P_1 = P_{/\!/}$$
$$P_2 = P_2$$
$$P_3 = P_{\perp}$$

where $P_{/\!/}$ and $P_{\perp}$ have the same meaning as in the preceding Chapter.

The symbol ϵ in Table VII is +1 for an initial state $|J,M\rangle$ with $M > 0$ and -1 for $M < 0$, so for a statistical mixture of two states ($M > 0$)

$|J,M\rangle$ with probability p_1

and $|J,-M\rangle$ " " p_2

ϵ gets the meaning of the polarization of the parent eigenstate:

$$\epsilon = p_1 - p_2 \ .$$

When putting one of the two contributing decay amplitudes equal to 0, one gains back, of course, the corresponding distributions for parity conserving decays as listed in Table VI.

For an arbitrary spin state $|J,M\rangle$ of the parent particle the angular and polarization distributions may all be written in the form (cf. Table VII for spin ½ 3/2)

$$\begin{aligned}
\psi^*\psi &= f_1(x^2) - \epsilon\, 2\mathrm{Re}\left(A_L^* A_{L+1}\right) x\, f_2(x^2) \\
\psi^*\psi P_{/\!/} &= \epsilon[x\, f_2(x^2)] - 2\mathrm{Re}\left(A_L^* A_{L+1}\right) f_1(x^2) \\
\psi^*\psi P_2 &= -\epsilon\left[2\mathrm{Im}\left(A_L^* A_{L+1}\right)\sqrt{1-x^2}\; f_3(x^2)\right. \\
\psi^*\psi P_{\perp} &= \epsilon\left[\left(|A_L|^2 - |A_{L+1}|^2\right)\sqrt{1-x^2}\; f_3(x^2)\right.
\end{aligned} \qquad (V.1)$$

The A_L, A_{L+1} are the decay wave amplitudes with $L = J - 1/2$, and the $f_{1,2,3}(x^2)$ polynomials in x^2. $f_1(x^2)$ is identical with the angular distribution of the corresponding parity conserving decay.

Note for

$$|M| = J \quad f_1 \equiv f_2 \equiv f_3 \ .$$

The terms containing A_L, A_{L+1} are called decay parameters.

We define

$$\begin{aligned} \alpha &= 2\mathrm{Re}(A_L^* A_{L+1}) \\ \beta &= 2\mathrm{Im}(A_L^* A_{L+1}) \\ \gamma &= |A_L|^2 - |A_{L+1}|^2 \end{aligned} \qquad (V.2)$$

$$\text{with} \quad \alpha^2 + \beta^2 + \gamma^2 = 1 \ .$$

(In the paper concerning the Ξ decay parameters α and β, though being defined as in Eq. (V.2), enter with opposite sign into the corresponding expressions for the angular distribution and the polarization. This is due to another sign convention in the decay wave function which corresponds to that adopted by Teutsch et al.[16] and Ticho[17]. The formulae given there, and here, become identical under the transformation $A_{L+1} \rightarrow -A_{L+1}$).

The fact that the term containing $\alpha = 2\mathrm{Re}(A_L^* A_{L+1})$ in the expression for $\psi^*\psi P_{//}$ does not depend on the polarization ϵ and has as factor $f_1(x^2)$, the parity conserving decay distribution yields an easy method for determining α experimentally for arbitrary spin: Calculate $P_{//}$ averaged over all emission angles for an arbitrary state $|J,M\rangle$ with polarization ϵ.

$$P_{//} = \frac{1}{\psi^*\psi}\left\{\epsilon[x \ f_2(x^2)] + \alpha \ f_1(x^2)\right\}$$

$$\langle P_{//}\rangle = 1/2 \int_{-1}^{+1} P_{//} \ \psi^*\psi \ dx$$

$$\langle P_{//}\rangle = 1/2 \int_{-1}^{+1} \left\{\epsilon[x \ f_2(x^2)] + \alpha \ f_1(x^2)\right\} dx$$

$$= \alpha \ 1/2 \int_{-1}^{+1} f_1(x^2) \ dx \ .$$

$f_1(x^2)$, however, is always the parity conserving distribution normalized to 2, so

$$<P_{\parallel}> = \alpha \quad . \tag{V.3}$$

This is true for any initial state $|J,M>$ therefore, also for any statistical mixture of states, averaged over the azimuth φ. This averaging, however, is done automatically because the above relation is independent of the coordinate system, in particular it no longer contains the original spin quantization direction $\hat{n}$.

It is shown therefore, that the method for determination of α described in the paper about the Ξ decay, remains valid for a Ξ spin higher than $\frac{1}{2}$.

However the relations given there for the determination of the β and γ parameters are no longer true for spin > $\frac{1}{2}$. Corresponding relations for higher spins would contain, not only the polarization, but also the population of the different eigenstates.

Now let us turn to the spin determination based on the decay angular distribution. The distribution observed experimentally is, of course, a statistical superposition of the distributions $(\psi^*\psi)_{J,M}$ referring to eigenstates $|J,M>$. So

$$\frac{dN}{d\cos\Theta} = \sum_M p_{J,M} \, (\psi^*\psi)_{J,M}$$

where $p_{J,M}$ is the population probability of the state $|J,M>$. According to the general structure of $(\psi^*\psi)_{J,M}$, of Eq. (V.1),

$$(\psi^*\psi)_{J,M} = F_{J,M}(x) + \epsilon\alpha \; G_{J,M}(x)$$

the angular distribution can be written as

$$\frac{dN}{d\cos\Theta} = \sum_{M>0} (p_{J,M} + p_{J,-M}) \; F_{J,M}(x) + \sum_{M>0} (p_{J,M} - p_{J,-M}) \; \alpha \; G_{J,M}(x) \tag{V.4}$$

A necessary condition for a certain spin J hypothesis is, that the observed angular distribution has the structure of Eq. (V.4). A straightforward test method would consist in fitting the experimental distribution to the theoretical expression Eq. (V.4) and finding the unknown parameters $p_{J,M}$, α which have to satisfy the relations

$$\sum_M p_{JM} = 1$$

$$0 \leq p_{JM} \leq 1 \qquad \text{(V.5)}$$

$$-1 \leq \alpha \leq 1 \quad .$$

Unfortunately, if α is a priori unknown, only

$$(p_{J,M} + p_{J,-M}) \quad \text{and} \quad \alpha \; (p_{JM} - p_{J,-M})$$

can be determined in the best fit, so that the only test relations are

$$p_{J,M} + p_{J-M} \geq | \; \alpha \; (p_{JM} - p_{J=M}) | \qquad \text{(V.5a)}$$

for all $M > 0$.

Instead of the fitting procedure, one could as well expand theoretical and experimental distributions in a series of functions and express the test conditions as relations between expansion coefficients. It is of great practical advantage to choose a series expansion of orthogonal functions, because then the expansion coefficients of an experimental distribution are, at least for large statistics, proportional to sample averages of the corresponding orthogonal functions. This may be seen as follows. Expand a given distribution $f(x)$ in orthogonal functions $P_\ell(x)$ satisfying the orthogonality conditions:

$$\int_a^b P_\ell^*(x) \; P_m(x) \; dx = C_m \; \delta_{\ell m}$$

$$C_m \neq 0$$

e.g. for Legendre polynomials $C_m = \frac{2}{2m+1}$ when integrated between -1 and +1 .

From the expansion

$$f(x) = \sum_{\ell} a_{\ell} P^{*}_{\ell}(x)$$

one calculates the averaged $P_m(x)$:

$$\langle P_m \rangle = \int_a^b P_m(x)\ f(x)\ dx = \int_a^b P^{*}_m(x)\ P_m(x)\ dx$$

$$= C_m\ a_m$$

so the expansion coefficient a_m is

$$a_m = \frac{1}{C_m} \langle P_m \rangle \ .$$

For an experimental distribution $f(x_i)$ the sample average is of course

$$\langle P_m \rangle = \frac{1}{N} \sum_{i=1}^{N} P_m(x_i) \, .$$

It should be mentioned however, that the coefficients a_m obtained in this way, are in general, different from the corresponding maximum likelihood solution for a_m in the case of finite statistics.

As in our problem, all distributions are polynomials, in cos Θ one chooses, of course, the Legendre polynomials as an appropriate set of orthogonal functions. Consider the case $J = 3/2$. The expected angular distribution is according to Eq. V.4)

$$\psi^{*}\psi = \frac{dN}{d\cos\Theta} = \sum_{M=1/2}^{3/2} (p_M + p_{-M})\ F_{3/2\ M}(x) \qquad \text{(V.6)}$$

$$+ \sum_{M=1/2}^{3/2} (p_M - p_{-M})\ \alpha G_{3/2,M}(x)$$

From Table VII we have, when normalizing $\psi^*\psi$ to 1

$$F_{3/2,3/2} = \tfrac{1}{2}\ (\tfrac{3}{2} - \tfrac{3}{2}x^2) \qquad G_{3/2,3/2} = \tfrac{1}{2}\ (-\tfrac{3}{2}x + \tfrac{3}{2}x^3)$$

$$F_{3/2,1/2} = \tfrac{1}{2}\ (\tfrac{1}{2} + \tfrac{3}{2}x^2) \qquad G_{3/2,1/2} = \tfrac{1}{2}\ (\tfrac{5}{2}x - \tfrac{9}{2}x^3) \tag{V.7}$$

so

$$\psi^*\psi = \left(p_{3/2} + p_{-3/2}\right) \tfrac{1}{2} \left(\tfrac{3}{2} - \tfrac{3}{2}x^2\right) + \left(p_{3/2} - p_{-3/2}\right) \alpha\tfrac{1}{2}\left(-\tfrac{3}{2}x + \tfrac{3}{2}x^3\right)$$

$$+ \left(p_{3/2} + p_{-3/2}\right) \tfrac{1}{2} \left(\tfrac{1}{2} + \tfrac{3}{2}x^2\right) + \left(p_{1/2} - p_{-1/2}\right) \alpha\tfrac{1}{2}\left(\tfrac{5}{2}x - \tfrac{9}{2}x^3\right) \tag{V.8}$$

Now one can easily evaluate the averages

$$<P_\ell> = \int_{-1}^{+1} \psi^*\psi\ P_\ell(x)\ dx\ .$$

The result is

$$<P_0> = 1$$

$$<P_1> = \tfrac{3}{15}\ \alpha \left(p_{3/2} - p_{-3/2}\right) - \tfrac{1}{15}\ \alpha \left(p_{1/2} - p_{-1/2}\right)$$

$$<P_2> = -\tfrac{1}{5} \left(p_{3/2} + p_{-3/2}\right) + \tfrac{1}{5} \left(p_{1/2} + p_{-1/2}\right) \tag{V.9}$$

$$<P_3> = \tfrac{3}{35}\ \alpha \left(p_{3/2} - p_{-3/2}\right) - \tfrac{9}{35}\ \alpha \left(p_{1/2} - p_{-1/2}\right)$$

All higher $<P_\ell>$ with $\ell > 3$ are 0 because the highest power in $\psi^*\psi$ is x^3.
The test conditions Eq. (V.5a)

$$1 > \left(p_{3/2,M} + p_{3/2,-M}\right) > \left|\alpha \left(p_{3/2,M} - p_{3/2,-M}\right)\right|;\ M = \tfrac{3}{2}, \tfrac{1}{2} \tag{V.9a}$$

can now be written in terms of expansion coefficients, i.e. in averages of Legendre polynomials. From Eq. (V.9) follows

$$\left[\tfrac{1}{2} - \tfrac{5}{2}\ \langle P_2 \rangle\right] \geq \left|-\tfrac{9}{2}\ \langle P_1 \rangle + \tfrac{7}{6}\ \langle P_3 \rangle\right|$$

$$\left[\tfrac{1}{2} + \tfrac{5}{2}\ \langle P_2 \rangle\right] \geq \left|-\tfrac{3}{2}\ \langle P_1 \rangle - \tfrac{7}{2}\ \langle P_3 \rangle\right| \ . \qquad \text{(V.10)}$$

An inequality of the type

$$a \geq |b|$$

can be replaced by two inequalities simultaneously valid

$$a \geq b \quad \text{and} \quad a \geq -b\ .$$

Applying this to the test conditions Eq. (V.10) one receives four inequalities

$$\langle T_{3/2,3/2} \rangle \equiv \langle 9P_1 + 5P_2 - \tfrac{7}{3}P_3 \rangle \leq 1$$

$$\langle T_{3/2-3/2} \rangle \equiv \langle -9P_1 + 5P_2 + \tfrac{7}{3}P_3 \rangle \leq 1$$

$$\langle T_{1/2,+1/2} \rangle \equiv \langle 3P_1 - 5P_2 + 7P_3 \rangle \leq 1 \qquad \text{(V.11)}$$

$$\langle T_{3/2-1/2} \rangle \equiv \langle -3P_1 - 5P_2 - 7P_3 \rangle \leq 1$$

The $T_{J,M}$ are the famous Lee-Yang test functions[18] for spin $\tfrac{3}{2}$. The inequalities Eq. (V.11) represent <u>necessary</u> conditions for the hypothesis that the particle has spin $\tfrac{3}{2}$. They are clearly not a sufficient criterion because a flat angular distribution which corresponds to an equal mixture of all states $|J,M\rangle$ would automatically satisfy the inequalities, whatever J may be. To further clarify this point, one immediately verifies that an observed angular distribution of the form

$$\frac{dN}{d\cos\Theta} = \tfrac{1}{2}(1 + a\cos\Theta)$$

gives

$$\langle T_{3/2,\pm 3/2} \rangle = \pm 3a$$
$$\langle T_{3/2,\pm 1/2} \rangle = \pm a \qquad \text{(V.12)}$$

which is 0 for an isotropic distribution. Thus the proof of a certain spin hypothesis has to consist in the rejection of all other hypotheses. In practice one starts the rejection with high spins, which elementary particles are less likely to have, e.g. with a spin hypothesis of $J = 5/2$. The corresponding test functions may be derived in an analogous way. They are

$$T_{5/2,5/2} = 15P_1 + 25/4 P_2 - 35/4 P_3 - 9/2 P_4 + 11/10 P_5$$
$$T_{5/2,-5/2} = -15P_1 + 25/4 P_2 + 35/4 P_3 - 9/2 P_4 - 11/10 P_5$$
$$T_{5/2,3/2} = 9P_1 - 5/4 P_2 + 49/4 P_3 + 27/2 P_4 - 11/2 P_5$$
$$T_{5/2,-3/2} = 9P_1 - 5/4 P_2 - 49/4 P_3 + 27/2 P_4 + 11/2 P_5 \qquad \text{(V.13)}$$
$$T_{5/2,1/2} = 3P_1 - 5P_2 + 7P_3 - 9P_4 + 11P_5$$
$$T_{5/2,-1/2} = 3P_1 - 5P_2 - 7P_3 - 9P_4 - 11P_5$$

If $J = 5/2$ fails, one tries $J = 3/2$, if $J = 3/2$ is rejected one tests the $J = 1/2$ hypothesis with the test functions

$$T_{1/2,1/2} = 3P_1$$
$$T_{1/2,-1/2} = -3P_1 \ . \qquad \text{(V.14)}$$

As the decay angular distribution of a spin $1/2$ particle is

$$\frac{dN}{d \cos \Theta} = 1/2 \, (1 + P\alpha \cos \Theta)$$

where P is the polarization and α the α decay parameter, the averages of the test functions are simply

$$\langle T_{1/2,\pm 1/2} \rangle = \pm P\alpha \ . \qquad \text{(V.15)}$$

By this procedure it was possible to exclude for the Λ[19] and, with rather high probability, also for the Ξ hyperon[17], all spins higher than $\frac{1}{2}$. It should be emphasized, however, that this method need by no means always be successful, even if statistical errors are negligibly small. This is immediately evident for a completely inorientated sample of particles, but it is for instance, also true for a completely polarized sample of spin $\frac{1}{2}$ particles with

$$\alpha \leq 2\mathrm{Re}(S^*P) < \tfrac{1}{3}.$$

According to Eq.(V.12) the averages of the $J = \frac{3}{2}$ test functions would all be < 1 so that no decision between spin $\frac{1}{2}$ and $\frac{3}{2}$ is possible. In this case only the absence or presence of a $\cos^2 \Theta$ term in the angular distribution, could decide between the two possibilities.

As a matter of fact, the larger the absolute value of the asymmetry parameter α, the more spin hypotheses can be excluded for a given sample of particles. This follows from the relation

$$-\frac{1}{2J+1} \leq \langle \cos \Theta \rangle \leq \frac{1}{2J+1} \quad \text{or} \quad -1 \leq (2J+1) \langle \cos \Theta \rangle \leq 1 \qquad (V.16)$$

which is implicitly contained in the test functions and which was proved by Lee and Yang[18] for arbitrary spin J. $\langle \cos \Theta \rangle$ is clearly a measure for the asymmetry of the distribution and therefore strongly dependent on α. Thus according to Eq. (V.16) the larger $|\langle \cos \Theta \rangle|$ is, the more spin hypotheses can be excluded.

In the case where α is known, the test condition Eq. (V.9a) can be sharpened:

$$1 > \left(p_{3/2,M} + p_{3/2-M}\right) \geq \left|p_{3/2,M} - p_{3/2,-M}\right| \; ; \; M = \tfrac{3}{2}, \tfrac{1}{2}. \qquad (V.17)$$

Expressing these inequalities with relations Eq. (V.9), in terms of $\langle P_\ell \rangle$ leads to a similar set of test functions which differ from the original ones, only by the factor $1/\alpha$ in front of all odd P_ℓ. This applies to all test functions of arbitrary spin. These modified functions together with the independently determined α parameters were used by Ticho et al.[17] to exclude spins $3/2$ for the Ξ hyperon.

In practice one always deals with finite statistics, so that the statistical errors in the averaged test functions must be taken into account. For a finite sample of N events the average T is

$$<T> = \frac{1}{N} \sum_i T_i$$

with the statistical error

$$\pm \Delta <T> = \pm \frac{1}{\sqrt{N}} \left[\frac{1}{N} \Sigma\ T_i^2 - \left(\frac{1}{N} \Sigma\ T_i \right)^2 \right] .$$

CONCLUDING REMARKS

Finally we should like to draw the reader's attention to a few more sophisticated spin test methods which have not been presented in this paper. However, after becoming acquainted with the spin analyses discussed here it should not be too difficult to understand these methods in their original versions.

M. Peshkin[23] has proposed a method for the spin determination of fermions which is similar to that of Lee and Yang but which uses also the azimuthal dependence of the decay angular distribution for special production processes. If the fermion F is produced in a 2-body reaction of the type

$$B_1 + N \rightarrow F + B_2$$

where B_1 and B_2 are spinless bosons and N is a nucleon then the spin density matrix of F contains for a fixed production angle only two pure spin states which are connected to each other by a reflection and a 180^0 rotation about the production normal. This special structure of the density matrix imposes certain spin dependent conditions on the decay angular distribution in addition to those of Lee and Yang. This method is strictly applicable only to particle samples with the same production angle.

G.F. Wolters has derived sets of test relations for strong fermion decays [21] and strong boson decays[22] using only the Θ dependence of the decay angular distributions. For fermion decays the diagonal spin density matrix elements can be expressed in terms of the even parts of the Lee-Yang test functions. The test conditions themselves are then formulated as relations between the density matrix elements. An analogous method is applied to bosons which decay into two spinless particles.

The most general method of fermion spin determination is that of N. Byers and S. Fenster[24] and an equivalent type of analysis proposed by N. Ademollo and R. Gatto[26]. Here the maximum information available from a sample of fermion decays, namely the decay angular distribution and the polarization distributions of the daughter fermions of spin $\frac{1}{2}$, are exhausted in a systematic way. Angular and polarization distributions are expanded into spherical harmonics. The spin density matrix for a certain spin hypothesis can be expressed in terms of the expansion coefficients obtained experimentally. An overall fit of the observed angular and polarization distributions to the constraints imposed by theory can decide between different spin-parity hypotheses. For weak decays this method will give the decay parameters as well. Ademollo and Gatto consider further constraints the density matrix is subject to for the Peshkin case discussed above. The method of Byers and Fenster was applied for the spin-parity determination of the Ξ^* [29].

Recently M. Peshkin[27] has presented a spin-parity determination method for bosons which, in a simplified version is due to A. Bohr[28]. It is applicable to bosons X produced in parity conserving reactions of the type

$$B_1 + B_2 \to X + B_3$$

and decaying into two particles

$$X \to B_4 + C \; .$$

All B's stand for spinless bosons, C can either be a spinless particle or a γ. The virtue of the proposed method is its simplicity: all information about the spin and parity of X is contracted in the highest possible expansion coefficient of the angular distribution about the production normal. In the special case under consideration this coefficient cannot accidentally be 0.

APPENDIX

MINAMI AMBIGUITY

If an exited baryon of spin J decays into a spin ½ baryon and a spinless meson the decay wave has either L = J + ½ or L = J - ½ depending on the parity of the excited baryon. The decay angular distribution is the same for both the cases. For this theorem we want to give a proof which is due to Dyson and Nambu[20].

The most general parent spin state is

$$\psi = \sum_M a_M |J,M\rangle \ . \qquad (A.1)$$

The decay state ψ_L is a sum of states ψ_L^M

$$\psi_L^M = \sum_M (L, \tfrac{1}{2}, M-m, m|L, \tfrac{1}{2}, J, M) \ Y_L^{M-m}(\vec{\Omega}) \, |\tfrac{1}{2}, m\rangle \qquad (A.2)$$

$$\psi_L = \sum_M a_M \, \psi_L^M \ . \qquad (A.3)$$

L can be either J + ½ or J - ½ . $\vec{\Omega}$ is a unit vector in the direction specified by (Θ,φ), the arguments of the spherical harmonics.

The decay angular distribution is

$$\frac{dN}{d\Omega}(\Theta,\varphi) = \psi_L^* \psi_L = |\psi_L|^2 \ .$$

Let us construct an operator which changes the parity of the decay state when applied to ψ_L. Such an operator is

$$O = \vec{\sigma} \cdot \vec{\Omega}$$

where the unit vector $\vec{\sigma}$ contains as x,y,z components, the Pauli spin operators $\sigma_x, \sigma_y, \sigma_z$. When applied to ψ_L, $\vec{\sigma}$ clearly acts only on the Pauli spinors

$|1/2, m\rangle$. The operator $\vec{\sigma}.\vec{\Omega}$ is odd under space inversion ($\vec{\sigma}$ = axial vector $\vec{\Omega}$ = polar vector) and thus changes the parity when applied to a state. It is invariant under rotations and thus leaves the angular momentum of the state unchanged. Furthermore $\vec{\sigma}\vec{\Omega}$ is unitary and Hermitian.

Therefore the decay state

$$\psi'_L = (\vec{\sigma}\vec{\Omega})\,\psi_L$$

has parity opposite to that of ψ_L. Both ψ_L and ψ'_L correspond to the same parent state ψ_J.

The angular distribution for the opposite parity state ψ'_L is

$$\frac{dN'}{d\Omega}(\Theta,\varphi) = \psi'^*_L\psi'_L = \psi^*_L\,(\vec{\sigma}\vec{\Omega})^*(\vec{\sigma}\vec{\Omega})\,\psi_L \qquad (A.4)$$

where the asterisk indicates the Hermitian conjugate.

As $\vec{\sigma}\,\vec{\Omega}$ is a unitary operator i.e. $(\vec{\sigma}\vec{\Omega})^* = (\vec{\sigma}\vec{\Omega})^{-1}$

$$\frac{dN'}{d\Omega}(\Theta,\varphi) = \psi'^*_L\psi'_L = \psi^*_L\psi_L$$

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ERRATA

Page	Appears as :	Should be :
28	... $M_{23} = M^*$ particle one ...	... $M_{23} = M^*$ particle 1 ...
28	$M_{12} \frac{\partial M_{12}}{\partial E_2} = E_1 - p_1 \cdot \frac{E_2}{p} = 0$	$M_{12} \frac{\partial M_{12}}{\partial E_2} = E_1 - p_1 \cdot \frac{E_2}{p_2} = 0$
32	$\cos \Theta_2 = \frac{\vec{p}' \ \vec{p}_2}{p' \ p_2} = \frac{(\vec{p}_1 + \vec{p}_2) \ \vec{p}_2}{p' \ p_2}$ ...	$\cos \Theta_2 = \frac{\vec{p}' \cdot \vec{p}_2}{p' \ p_2} = \frac{(\vec{p}_1 + \vec{p}_2) \cdot \vec{p}_2}{p' \ p_2}$
40	... axes on an $M_{12}^2 \ M_{13}^2$ plot.	... axis on a $M_{12}^2 \ M_{13}^2$ plot.
49	... 6-fold Dalitz plot.	... 6-folded Dalitz plot.
59	... and the Sienna Conference 1962.	... and the Sienna Conference 1963.
81	However, in Appendix it will ...	However, in Appendix I it will ...
82	... emission direction Θ, Φ in the Ξ system.	... emission direction Θ, φ in the Ξ system.
85	weighted by $N(\uparrow)/(N(\uparrow) + N(\downarrow)$ ...	weighted by $N(\uparrow)/(N(\uparrow) + N(\downarrow))$ ...
88 (Fig. 4)		
	$P_{3\Lambda} = P_{\Xi\gamma}$ and $P_{2\Lambda} = P_{\Xi\beta}$	$P_{3\Lambda} = P_{\Xi}\gamma$ and $P_{2\Lambda} = P_{\Xi}\beta$
90 (Fig. 5)		
	$\hat{\vartheta}$	$\hat{r}$
98	s denotes the Λ spin, in its z component.	s denotes the Λ spin, m its z component.
100	$RL_{\vec{p}}$ is ...	$RL_{\vec{p}}$ is $\begin{cases} \ldots \\ \ldots \end{cases}$
	$L_{R\vec{p}}R$ is ...	$L_{R\vec{p}}R$ is $\begin{cases} \ldots \\ \ldots \end{cases}$

7863/p/mh

Page	Appears as :	Should be :
102	Fig. e	Fig. 6
104	The element a_{12} can be ...	The element a_{21} can be ...
105	N. Jacob and G.C. Wick[9]) and N. Jacob[10]).	M. Jacob and G.C. Wick[9]) and M. Jacob[10]).
111	1) E.C.G. Sudarsham, 6) L. Janneau, D. Marellet,	1) E.C.G. Sudarshan, 6) L. Jauneau, D. Morellet,
115	$= \sum_{k=1}^{\mu} a_m^* b_k = \delta_{j_1 j_2} \delta_{m_1 m_2}$	$= \sum_{k=1}^{\mu} a_k^* b_k = \delta_{j_1 j_2} \delta_{m_1 m_2}$
119	$\langle J_z \rangle = \sum_{m=-j}^{j+j} m \lvert a_m \rvert^2 .$	$\langle J_z \rangle = \sum_{m=-j}^{j} m \lvert a_m \rvert^2 .$
128	$\ldots P_\ell^m(x)\, e^{i\varphi}$;	$\ldots P_\ell^m(x)\, e^{im\varphi}$;
136	$R_z = \frac{i}{\hbar} L_z$	$R_z = - \frac{i}{\hbar} L_z$
147	With $\alpha_1^2 = \alpha_2^2 = \alpha^2$ and $\beta_1^2 = \beta_2^2 = \beta^2$	With $\lvert\alpha_1^2\rvert = \lvert\alpha_2^2\rvert = \lvert\alpha^2\rvert$ and $\lvert\beta_1^2\rvert = \beta_2^2\rvert = \lvert\beta^2\rvert$
150	... with excited baryon states and meson-meson resonances[7]),	... with excited baryon states[30]) and meson-meson resonances[7,31,32]),
161	The results for the same integer J values are listed in Table V.	The results for some integer J values are listed in Table V.
	...$\lvert J,M\rangle$ into two spinless particles.	...$\lvert J,M\rangle$ decaying into two spinless particles.

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Page	Appears as :	Should be :
168	...(α_Λ is known and = - 0.62).	...(α_Λ is known and = + 0.62).
177		
177	(cf. Table VII for spin $1/2, 3/2$)	(cf. Table VII for spin $1/2, 3/2, 5/2$)
180	$p_{J,M} + p_{J-M} \geq \mid \alpha \; (p_{JM} - p_{J=M}) \mid$	$1 > p_{J,M} + p_{J-M} \geq \mid \alpha \; (p_{JM} - p_{J=M}) \mid$
	Clebsch-Gordon	Clebsch-Gordan

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CERN 64-13
28 February 1964

PROCEEDINGS OF THE 1964 EASTER SCHOOL FOR PHYSICISTS

Using the CERN Proton Synchrotron
and Synchro-Cyclotron

Herceg-Novi, May 18 - 31, 1964

by invitation of the

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