2% chance for ρ to be greater than the experimental value of 7 and that there is a 7% probability[20] that $\rho<1/7$. Thus there is a significant probability that a single measurement of the ratio could result in a value very different from unity. This leads us to conclude that although the measured value $\rho=7$ is somewhat surprising, it does not in any way establish the existence of a dependence of the strength function on spin; it merely suggests an area for further investigation.

[20] The probability pr[$\rho>\alpha$] is different from pr[$\rho<1/\alpha$] because of the difference in the skewness of the probability density functions for the two sums of reduced neutron widths that determine ρ. If both sums are governed by the same density function, or if these functions are different but symmetrical, pr[$\rho>\alpha$] is equal to pr[$\rho<1/\alpha$], of course.

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Can a Scalar Meson "Bootstrap" Itself?

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It is shown that although a scalar meson can be described by a renormalizable field theory, it is not possible for such a particle to "bootstrap" itself, because the force arising from the crossed channels is too great. This result is in accordance with the "bootstrap" philosophy that there should be only one solution of the S-matrix equations consistent with maximal analyticity of the second kind, and also indicates the need for symmetries in strong interactions.

INTRODUCTION

It has been proposed that there should be only one solution for the scattering matrix in strong interactions which is consistent with unitarity and maximal analyticity of the second kind.[1] All the poles should be continuable in angular momentum. This would mean that no experimental information need be included to derive the properties of all the observed particles, whether bound states or resonances. Alternatively it may be that it is necessary to know the masses and coupling constants of a certain number of "elementary" particles before the properties of the other particles can be derived from purely dynamical considerations.

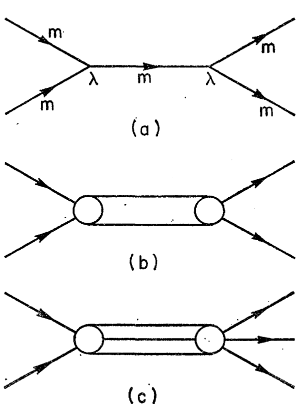

FIG. 1. The unitarity diagrams.

As yet we are unable to perform calculations that encompass all the known particles, and so no decision can be made in the matter, but one might be able to show that a set of particles other than those which have been observed can give rise to a self-consistent S matrix, i.e., can "bootstrap" themselves, and thus demonstrate the need for the inclusion of at least some experimental information in order to arrive at the solution corresponding to the real world.

Of course if the hypothetical set of particles is too complicated one is again unable to solve the S-matrix equations, but if only a single type of particle is considered the problem is quite tractable. The neutral scalar meson is a likely candidate, because it obeys the renormalizable Hurst-Thirring[2] field theory with an interaction Lagrangian

$$\mathcal{L}_I=\lambda\phi^3, \tag{1}$$

and thus is viable as an independent particle. Other renormalizable field theories involve the interaction of two different types of particles. It is certainly not obvious that a "bootstrap" solution can exist because, as we shall show, there are fewer free parameters than conditions to be satisfied, but of course this is also true of the hypothetical solution involving all the strongly interacting particles. In this article we shall try to

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[1] See, for example, G. F. Chew and S. C. Frautschi, Phys. Rev. Letters **7**, 395 (1961); F. Zachariasen and C. Zemach, Phys. Rev. **128**, 849 (1962); E. Abers and C. Zemach, *ibid.* **131**, 2305 (1963).

[2] N. N. Bogoliubov and D. V. Shirkov, *Introduction to the Theory of Quantized Fields* (Interscience Publishers, Inc., New York, 1959), p. 352.

discover whether it is possible for neutral scalar mesons alone to form a solution of the "bootstrap" equations, in contradiction to the "bootstrap" philosophy.

THE CROSSING SYMMETRIC SCATTERING AMPLITUDE

In our calculation we include forces from the exchange of a bound state [Fig. 1(a)], as well as two-particle states [Fig. 1(b)], but neglect forces from the exchange of three or more particles [Fig. 1(c)]. Correspondingly we neglect in the unitarity condition the contribution of intermediate states containing three or more particles whose thresholds lie at $9m^2$ and above.

In the Mandelstam representation the amplitude may be written

$$A(s,\cos\theta)=\frac{g}{m^2+2q_s^2(1+\cos\theta)}+\frac{g}{m^2+2q_s^2(1-\cos\theta)}+\frac{1}{\pi}\int dt'\frac{A_t(s,t',4m^2-s-t')}{t'+2q_s^2(1+\cos\theta)}+\frac{1}{\pi}\int du'\frac{A_u(s,4m^2-s-u',u')}{u'+2q_s^2(1-\cos\theta)}, \quad (2)$$

where s, t, and u are the usual squares of the four-momentum invariants, $q=\lambda^2$, and A_t and A_u are the absorptive parts in the t and u channels. θ is the scattering angle, and g_s is the center-of-mass momentum in the s channel:

$$s=4(q_s^2+m^2)\,;\; t=-2q_s^2(1+\cos\theta)\,;\; u=-2q_s^2(1-\cos\theta).$$

If maximal analyticity of the second kind is assumed, the pole in s is contained in the integrals over A_t and A_u.

We define the partial-wave amplitude

$$A_l(s)=\tfrac{1}{2}\int_{-1}^{1} d(\cos\theta)P_l(\cos\theta)A(s,\cos\theta). \quad (3)$$

Substituting (2) in (3), we find that the s-wave amplitude has, in addition to the fixed s singularities of $A(s,\cos\theta)$ shown in Fig. 2, branch points at $s=3m^2$ from the pole terms, and at $s=0$ from the elastic t and u thresholds. The inelastic thresholds in the crossed channels would give rise to branch points at $-5m^2$, but again we neglect these. The singularities of A_0 are shown in Fig. 3.

The imaginary part of A_0 along these left-hand cuts is

$$\mathrm{Im}A_0=2\pi g/(s-4m^2)\,,\quad \text{for}\quad 0<s<3m^2$$

$$\mathrm{Im}A_0=\frac{2\pi g}{s-4m^2}-\frac{2}{s-4m^2}\int_{-4q_s^2}^{0} A_t(t,s)dt,\ \text{for}\ s<0, \quad (4)$$

where we remember that $A_s=A_t=A_u$ for our symmetrical problem.

If we are to have a self-consistent solution, these

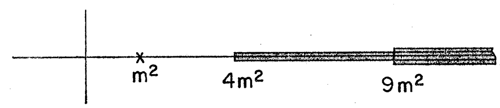

FIG. 2. Singularities of the scattering amplitude in the s channel.

left-hand cuts should provide an attractive force sufficient to produce the bound state at $s=m^2$. Since we have the parameter g at our disposal, it will be possible to choose it such that this condition is satisfied, keeping $A_s=A_t$. However, self-consistency also demands that the residue of the pole be equal to the value of g which we have used. There is no guarantee *a priori* that this can be so, because we have no further free parameters in the problem. It will be realized that the mass m is not a parameter because it serves only to define the size of our energy unit. There is only one dimensionless free parameter, g/m^2, but there are two criteria to be satisfied: crossing symmetry of both the pole positions and its residue.

THE N/D EQUATIONS

The problem may conveniently be solved following the method of Chew and Mandelstam.[3]

We define the amplitude to be

$$A_0(s)=N(s)/D(s)\,, \quad (5)$$

where $N(s)$ has the left-hand cuts of A_0, and $D(s)$ has the right-hand unitary cut.

$$\mathrm{Im}N(s)=D(s)\mathrm{Im}A_0(s)\quad \text{for}\quad s<3m^2. \quad (6)$$

Along the right-hand unitary cut we may write

$$A_0(s)=e^{i\delta(s)}\sin\delta(s)/\rho(s)\,, \quad (7)$$

where $\delta(s)$ is the phase shift, and $\rho(s)=[(s-4m^2)/s]^{1/2}$; comparing real and imaginary parts, we have

$$\mathrm{Im}(1/A_0)=-\rho(s)\,, \quad (8)$$

or

$$\mathrm{Im}D=-\rho(s)N(s)\quad \text{for}\quad s>4m^2. \quad (9)$$

Thus we may write dispersion relations for N and D:

$$N(s)=\frac{1}{\pi}\int_{-\infty}^{3m^2}\frac{D(s')\mathrm{Im}A_0(s')}{s'-s}ds'\,, \quad (10)$$

$$D(s)=1-\frac{1}{\pi}\int_{4m^2}^{\infty}\frac{\rho(s')N(s')}{s'-s}ds'\,, \quad (11)$$

where we have normalized D to 1 and N to 0 at infinity, selecting the solution without poles in N or D prescribed by second-degree analyticity. Substituting (10) in (11) and integrating over the right-hand cut, we

[3] G. F. Chew and S. Mandelstam, Phys. Rev. **119**, 467 (1960); G. F. Chew, S. Mandelstam, and H. P. Noyes, *ibid.* **119**, 478 (1960).

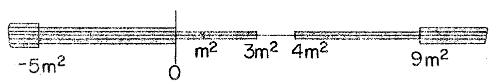

FIG. 3. Singularities of $A_0(s)$.

obtain

$$D(s)=1+\frac{1}{\pi^2}\int_{-\infty}^{3m^2} K(s,s')\operatorname{Im}A_0(s')D(s')ds', \quad (12)$$

where

$$K(s,s')=\frac{2}{s'-s}\left[\left(\frac{s-4m^2}{s}\right)^{1/2}\ln\left(\frac{(s-4m^2)^{1/2}+s^{1/2}}{2m}\right)-\left(\frac{s'-4m^2}{s'}\right)^{1/2}\ln\left(\frac{(s'-4m^2)^{1/2}+s'^{1/2}}{2m}\right)\right].$$

If $\operatorname{Im}A(s')$ vanishes as $s'\to-\infty$, this is a Fredholm equation for D which can be solved providing that we know A_t in Eq. (4). Along the right-hand cut $\operatorname{Re}(1/A)=\operatorname{Re}D/N$, and comparing real and imaginary parts in Eq. (8), we find

$$\operatorname{Im}A_0(s)=\frac{\rho(s)}{\rho^2(s)+(\operatorname{Re}D/N)^2} \quad \text{for} \quad s>4m^2. \quad (13)$$

If we suppose that the imaginary part of the amplitude is contained wholly in the s wave, we can identify

$$\operatorname{Im}A_0(s)=A_s(s,t). \quad (14)$$

The validity of this approximation will be discussed later. Remembering that A_s should be equal to A_t, we now have a means of calculating the second term on the right-hand side of formula (5) in a self-consistent manner. One easily verifies that the resulting $\operatorname{Im}A_0(s)$ vanishes as $s\to-\infty$.

We first take $\operatorname{Im}A_0$ to be given just by the first term of (4), and solve (12) for D. With this solution we solve (10) for N, and then obtain A_s from (13). Substituting this value of A_s in Eq. (4), we can repeat the cycle, and continue until self-consistency is achieved.

The equations were solved on a computer, using the transformed variable $x^2=-(4m^2/s-4m^2)$ for the integral equation, so that the range of integration in (12) becomes $x=0$ to 2. In this range, 50 mesh points were taken, and the equation was solved by matrix inversion. Five cycles were required to produce self-consistency between the elastic discontinuities in the crossed and direct channels. Various values of g were tried until a solution with a zero in $D(s)$ at $s=m^2$ was obtained. This gives an amplitude which is also self-consistent as regards the pole positions in the crossed and direct channels, and fixes the value of g. A graph of the solution is given in Fig. 4.

It only remains to discover whether the residue of this direct channel pole is equal to g. Now $1/A(s)=D(s)/N(s)$, and expanding about $s=m^2$, and remembering that $D(m^2)=0$, we have

$$\frac{1}{A}=(s-m^2)\frac{dD(s)/ds|_{m^2}}{N(m^2)}+\cdots,$$

so that the residue is

$$g'=\frac{N(m^2)}{dD(s)/ds|_{m^2}}. \quad (15)$$

If we have found a bootstrap solution, g' will be equal to g. In fact we find that to get a direct channel pole at m^2 requires $g/m^2=9.3$, but that in this case $g'/m^2=67$. This discrepancy is so great that it is most unlikely that it could be rectified by improving the approximations.

One approximation has been to use Eq. (14), whereas the full expression is

$$A_s(s,t)=\sum_{l=0}^{\infty}(2l+1)\operatorname{Im}A_l(s)P\left(1+\frac{t}{2q_s^2}\right), \quad (16)$$

though this is not convergent for large $|t|$. Because of parity, there is no coupling of the even and odd partial waves, so the lowest neglected wave is the D wave. We can estimate its order of magnitude by taking the D-wave Born term generated by the crossed channel poles,

$$B_2(s)=\frac{g}{2q_s^2}\left[Q_2\left(1+\frac{m^2}{2q_s^2}\right)-Q_2\left(-1-\frac{m^2}{2q_s^2}\right)\right]$$

and applying elastic unitarity,

$$\operatorname{Im}A_2(s)=\frac{\rho(s)}{\rho^2(s)+(1/B_2)^2}. \quad (17)$$

We find that $\operatorname{Im}A_2$ is only a few percent of $\operatorname{Im}A_0$ in the region between the threshold and the inelastic threshold, although it rises to nearly one third of its value at $s=15\,m^2$. However, it is the low-energy region just above threshold which is important for the integral in Eq. (5), except for large negative s. [Remember that A_t is obtained from (16) by interchanging s and t.] The D-wave contribution to the total force should thus be small, and even if there is some additional force it is unlikely that it will change the ratio of g to g' greatly. An indication is given by the fact that if we solve the problem including only the force from the poles and not

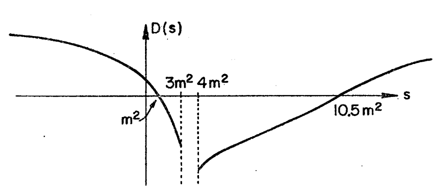

FIG. 4. The solution for $D(s)$.

from the S-wave elastic discontinuity, though g has to be increased to 16.5, g' becomes 105, and the large discrepancy is maintained.

DISCUSSION OF THE RESULTS

Of course if we had included a pole in the equation for N, writing (10) in the form

$$N(s)=\frac{1}{\pi}\int_{-\infty}^{3m^2}\frac{D(s')\mathrm{Im}A_0(s')}{s'-s}ds'+\frac{g}{m^2-s},$$

and normalized $D(s)$ to unity at $s=m^2$, it would have been possible to impose complete crossing symmetry for any value of g which is not so great as to produce a zero of D. This introduction of a "*CDD*" pole[4] corresponds to treating the meson as an elementary particle, and gives the solution corresponding to the $\lambda\phi^3$ field theory. Our solution of the N/D equations gives a boundstate pole which lies on a Regge trajectory, and is the solution corresponding to maximal analyticity of the second kind.

Implicit in the calculation is the assumption that this trajectory, Fig. 5, is the leading trajectory. It is the fact that $\alpha(t)<0$ for $t<0$ which assures the required asymptotic behavior of the kernel in (12), and enables us to avoid a cutoff parameter.[5] Through crossing symmetry the total cross section goes to zero at high energy as $s^{\alpha(0)-1}$ and the low-energy elastic S wave dominates the dynamics. However, it could be that the meson is not on the leading trajectory, but that the high-energy behavior is controlled by one or more higher trajectories. For example, in Fig. 6 we show a Pomeranchuk trajectory (which would give a constant total cross section at high energy), the scalar meson being associated with a secondary trajectory. Such a solution, if it is possible, would probably contain a spin-2 (D-wave) resonance, and high-energy effects would be crucial. But if our neglect of this possibility is justified we have shown that a scalar meson cannot "bootstrap" itself.

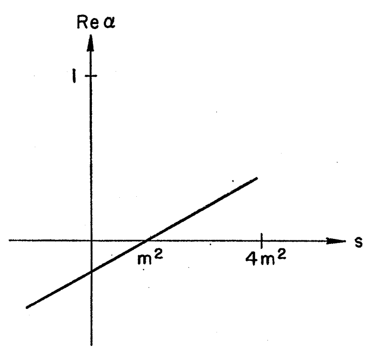

FIG. 5. The Regge trajectory.

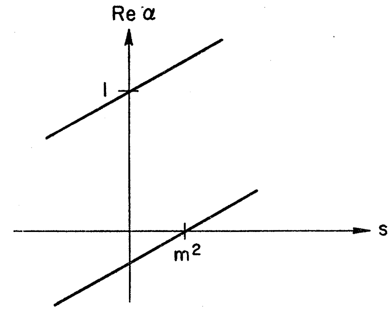

FIG. 6. Pomeranchuk and meson trajectories.

A final point which may be remarked is that we are unable to obtain a self-consistent solution because the force from the crossed channels is too great ($g<g'$). However, if instead of a single particle we had a set forming a representation of some symmetry group, the crossed and direct channels would be related by a crossing matrix, and only some fraction of the strong s-wave force would be available to any given two-particle channel. Thus for SU_2 the isotopic-spin crossing matrix is[3]

$$\beta_{II'}=\begin{bmatrix}1/3 & 1 & 5/3\\ 1/3 & 1/2 & -5/6\\ 1/3 & -1/2 & 1/6\end{bmatrix},$$

and so for our problem with $I=0$, the contribution from $I'=0$ would be only one-third as great. Higher symmetries give smaller fractions $[1/(n^2-1)$ for $SU_n]$,[6] and our results perhaps indicate the need for there to be such symmetries if "bootstrap" solutions are to be obtained.

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[5] R. Omnes, Phys. Rev. **133**, B 1543 (1964).

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