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Dalitz-Plot Analysis Including Duality*

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Using fairly general parametrizations of the Veneziano type for the decay amplitude, we study the annihilations of $\bar{N}N$ at rest (in the $T=1$ channel) into three pions. Very good agreement is obtained for the well-known $\bar{p}n$ channel in terms of four essential parameters. A previous model due to Lovelace is found to be inaccurate in fitting the data, and therefore his phenomenological interpretation is changed. If our predictions for other decay channels are confirmed, the generally accepted decoupling of the ρ in $\bar{N}N$ $(T=1)$ is probably incorrect. It is suggested that our procedure is the right one for parametrizing decay amplitudes in order to check if duality is a true property of strong interactions.

RECENTLY, Veneziano[1] was able to summarize the information from the finite-energy sum rules[2] by writing down closed forms for hadronic scattering amplitudes which explicitly take into account a great deal of dynamical information such as analyticity, crossing symmetry, Regge behavior (based on families of linear-rising parallel trajectories), and Regge-resonance duality. However, even for the simplest processes there is an infinity of solutions,[3,4] presumably because of the lack of unitarity.

A new type of application was proposed by Lovelace. In a remarkable paper,[5] he suggested that the best opportunity to study interference among resonances in different channels is offered by nucleon-antinucleon annihilation at rest into three pions. Here we study this possibility systematically and suggest a new way to parametrize decay amplitudes. In the present approach to Dalitz-plot analysis, the decay amplitude is generally written

$$A(s,t,u)=(\text{spin factor})\times[(\text{Breit-Wigner forms})+(\text{polynomial background})]. \quad (1)$$

We propose to replace the second set of parentheses with suitable combinations of forms of the type

$$A(s,t)=\sum_{n=0}\sum_{m=0}^{n} c_{nm}\frac{\Gamma(n-\alpha_s)\Gamma(n-\alpha_t)}{\Gamma(n+m-\alpha_s-\alpha_t)}, \quad (2)$$

where α_i is the trajectory function in channel i, and the range of n is to be chosen according to the case. These forms are very general and satisfy all the requirements stated at the beginning. By combining them suitably, one can construct proper isospin-carrying amplitudes, and once the trajectories are known (as they are already for most reactions), one can proceed to study a given decay. Moreover, the formula is flexible enough to allow for decoupling of individual states from the given external particles without affecting the other states—even those lying on the same trajectory. In fact, by adjusting the c_{nm}'s, one can obtain the most general imaginary part in the narrow-resonance limit.[6]

We restrict ourselves to decays of $N\bar{N}$ systems at rest in the ${}^1S_0$, $T=1$ state. This is the case for $\bar{p}n\to\pi^{+,0}\pi^{-,0}\pi^-$, $\bar{p}p\to 3\pi^0$, while the separation of the initial state in $T=0$ and $T=1$ for $\bar{p}p\to\pi^+\pi^-\pi^0$ is more or less available from present experiments. Therefore, the initial state we consider has the quantum numbers of a heavy pion. As first described by Shapiro and Yellin,[7] the $\pi\pi\to\pi''\pi''$ scattering amplitude, in a theory *à la* Veneziano, is written

$$\begin{aligned} A_s^0&=\tfrac{3}{2}[A(s,t)+A(s,u)]-\tfrac{1}{2}A(t,u),\\ A_s^1&=A(s,t)-A(s,u),\\ A_s^2&=A(t,u). \end{aligned} \quad (3)$$

Here, A_s^I is the isospin-I amplitude in the s channel, and $A(s,t)$ is given by (2), with the coefficients c_{nm} yet to be determined. The form $A(s,t)$ contains the ρ and the f^0 degenerate trajectories, as needed for the internal consistency of Eqs. (3); this is in reasonable agreement with independent experimental data on those two trajectories. The forms (3) for the amplitudes are uniquely derived from crossing symmetry, assuming the absence of isospin-2 resonances. Possible Pomeranchuk contributions are tentatively neglected.

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[1] G. Veneziano, Nuovo Cimento 57, 190 (1968).
[2] M. Ademollo, H. R. Rubinstein, G. Veneziano, and M. Virasoro, Phys. Rev. 176, 1904 (1968), and references therein.
[3] M. Virasoro, Phys. Rev. 177, 2309 (1969); S. Mandelstam, Phys. Rev. Letters 21, 1724 (1968). See also D. Atkinson, K. Dietz, and J. Honerkamp, University of Bonn Report (unpublished).
[4] G. Altarelli and H. R. Rubinstein, Phys. Rev. 178, 2165 (1969).
[5] C. Lovelace, Phys. Letters 28B, 265 (1968).
[6] For a discussion of this point in a similar case, see Ref. 4. Note also that terms of the form $[\Gamma(p-\alpha_s)\Gamma(q-\alpha_t)/\Gamma(l-\alpha_s-\alpha_t)+\alpha_s\leftrightarrow\alpha_t]$ for $A(s,t)$ would not be linearly independent for those in (2).
[7] J. Shapiro and J. Yellin, UCRL University of California Radiation Laboratory Report No. 18500 (unpublished); J. A. Shapiro, Phys. Rev. 179, 1345 (1969).

The various decay rates are

$$R(\bar{p}n \to \pi^+\pi^-\pi^-) = N|A(s,t)|^2, \tag{4a}$$

$$R(\bar{p}p_{T=1} \to \pi^+\pi^-\pi^0) = R(\bar{p}n \to \pi^-\pi^0\pi^0) = \tfrac{1}{4}N|A(t,u) - A(s,t) - A(s,u)|^2, \tag{4b}$$

$$R(\bar{p}p \to \pi^0\pi^0\pi^0) = \tfrac{1}{4}N|A(s,t) + A(s,u) + A(t,u)|^2. \tag{4c}$$

After integration over the phase space allowed by Bose statistics, we get the following relations among total rates:

$$R(\bar{p}p_{T=1} \to \pi^+\pi^-\pi^0) = 2R(\bar{p}n \to \pi^-\pi^0\pi^0) = 2[R(\bar{p}n \to \pi^+\pi^-\pi^-) - R(\bar{p}p \to \pi^0\pi^0\pi^0)], \tag{5}$$

$$\frac{1}{2} \leq \frac{R(\bar{p}p_{T=1} \to \pi^+\pi^-\pi^0)}{R(\bar{p}n \to \pi^+\pi^-\pi^-)} \leq 2. \tag{6}$$

These relations are general and merely follow from isospin invariance and Bose statistics for the pions.[8]

To proceed further, we need to specify $A(s,t)$. Lovelace assumed $A(s,t)$ of the form corresponding to $c_{11}=1$, all other $c_{nm}=0$, with

$$\alpha_s = 0.483 + 0.885s + i0.280(s - 4m_\pi^2)^{1/2}\theta(s - 4m_\pi^2). \tag{7}$$

The reasons that led him to this choice were simplicity and the result that the ρ does not couple in $N\bar{N}$ annihilations from singlet states at rest. This conclusion is based on Breit-Wigner fits that require the existence of isospin-2 unobserved states[9] and complicated three-pion final-state interactions.[10] Since we think these assumptions are not compatible with the model under study, we prefer to start with the general form (2) and attempt an independent fit.

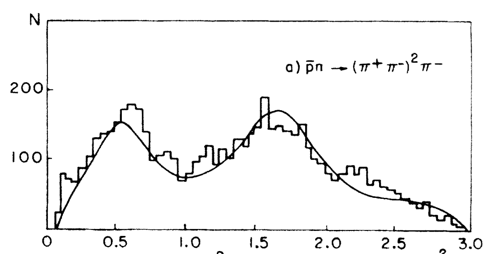
We then come back to the general form (2) with $n \geq 1$. In order to restrict the possible values of n and m, we utilize a very striking experimental feature, that is, the spectacular hole found in the Dalitz plot for $\bar{p}n \to \pi^+\pi^-\pi^-$ at values of s and t such that $\alpha_s \simeq \alpha_t \simeq 1.5$.[9] We then keep only those terms in (2) that vanish at $\alpha_s + \alpha_t = 3$. This is the case for $n+m \leq 3$. We are then led to the form

$$A(s,t) = c_{10}\frac{\Gamma(1-\alpha_s)\Gamma(1-\alpha_t)}{\Gamma(1-\alpha_s-\alpha_t)} + c_{11}\frac{\Gamma(1-\alpha_s)\Gamma(1-\alpha_t)}{\Gamma(2-\alpha_s-\alpha_t)} + c_{20}\frac{\Gamma(2-\alpha_s)\Gamma(2-\alpha_t)}{\Gamma(2-\alpha_s-\alpha_t)} + c_{21}\frac{\Gamma(2-\alpha_s)\Gamma(2-\alpha_t)}{\Gamma(3-\alpha_s-\alpha_t)} + c_{30}\frac{\Gamma(3-\alpha_s)\Gamma(3-\alpha_t)}{\Gamma(3-\alpha_s-\alpha_t)}. \tag{8}$$

Note that in this model one has, in principle, the freedom to decouple or not to decouple the ρ and the f^0 independently. Therefore, it is no longer true in our case, as it was in Lovelace's, that if the ρ is uncoupled, the f^0 must also be uncoupled.

We assume the same trajectory function as given by (7). The imaginary part is larger than that required to reproduce the ρ and f^0 width, since in this theory the bump at the ρ mass is also due to the ϵ, which is much broader; similarly, the bump at the f^0 is a multiple one. Therefore, it seems reasonable to attribute to a given bump the width of its broadest component.

Our qualitative results are as follows: The coefficients c_{10} and c_{11} are the most important ones, and the angular distributions for $\bar{p}n \to \pi^+\pi^-\pi^-$ are very sensitive to the ratio c_{11}/c_{10}, which must be about 2. This is mainly because the "hole" in the Dalitz plot at $\alpha_s \simeq \alpha_t \simeq 1.5$ is too deep to be the result of any one term, and the conditions $c_{11} = 2c_{10}$ and $c_{20} = c_{21}$ ensure a destructive interference in that region. The dependence on the other parameters is less critical. c_{20} and c_{21} turn out to



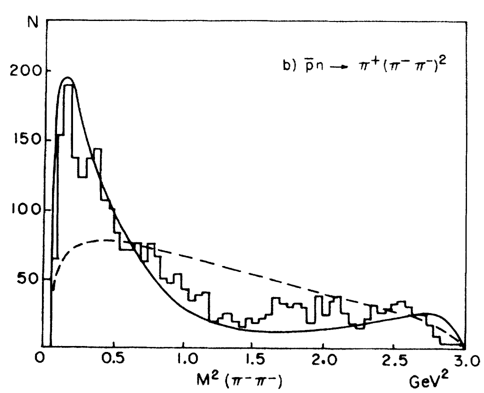


FIG. 1. (a) $\pi^+\pi^-$ and (b) $\pi^-\pi^-$ invariant-mass distributions for the decay $\bar{p}n \to \pi^+\pi^-\pi^-$. The continuous curves correspond to the values of c_{nm} stated in the text. The dashed curve in (b) is the phase space.

[8] A. Pais, Ann. Phys. (N. Y.) **9**, 548 (1960); C. Zemach, Phys. Rev. **133**, 1201 (1964).

[9] P. Anninos, L. Gray, P. Hagerty, T. Kalogeropoulos, S. Zenone, R. Bizzarri, G. Ciapetti, M. Gaspero, I. Laakso, S. Lichtman, and G. C. Moneti, Phys. Rev. Letters **20**, 402 (1968).

[10] M. Foster, Ph. Gavillet, G. Labrosse, L. Montanet, R. A. Salmeron, P. Villamoes, C. Ghesquiere, E. Lillestol, Nucl. Phys. (to be published).

be in the range $-0.3 \leqslant c_{20} \simeq c_{21} \leqslant 0.2$, and are in fact compatible with zero.

Fits with one single term are certainly ruled out. Our best fit is displayed in Figs. 1 and 2. It corresponds to the following set of parameters: $c_{10}=1$, $c_{11}=1.89$, $c_{20}=c_{21}=0$, and $c_{30}=0.57$. In Fig. 3 are shown the predictions that follow for mass distributions in the other charge modes.

We find a substantial contribution from the ρ. The argument for decoupling the ρ, as explained above, is certainly not compelling, and we think that the final

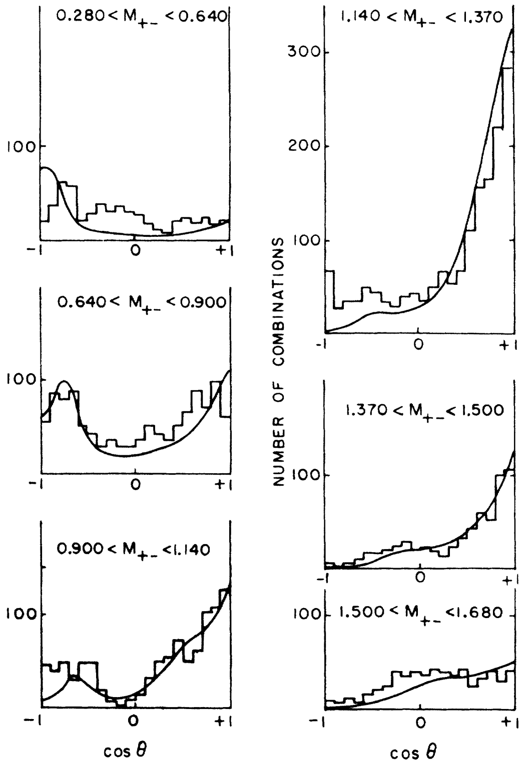


FIG. 2. Fit to the angular distributions of the π^- relative to the $(\pi^+\pi^-)$ dipion line of flight, in the dipion center of mass, for $\bar{p}n \to \pi^+\pi^-\pi^-$. The curves are from the model as specified in the text $[M_{+-} \equiv M(\pi^+,\pi^-)]$.

answer lies in the ability of a model to fit all data in the various channels.

For the total rates we obtain the ratios

$$R(\bar{p}p \to 3\pi^0) : R(\bar{p}n \to \pi^+\pi^-\pi^-) : R(\bar{p}p_{T=1} \to \pi^+\pi^-\pi^0) = 1 : 2.44 : 2.88. \quad (9)$$

From experiment the following frequencies are known:

$$f(\bar{p}n \to \pi^+\pi^-\pi^-) = (2.4 \pm 0.4) \times 10^{-2} \quad \text{(Ref. 9)},$$
$$\tfrac{1}{2} f(\bar{p}p_{T=1} \to \pi^+\pi^-\pi^0) = (1.1 \pm 0.4) \times 10^{-2} \quad \text{(Ref. 10)}.$$

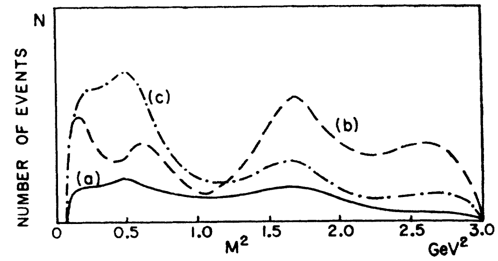


FIG. 3. Predictions of the model for mass distributions in (a) $\bar{p}p \to (\pi^0\pi^0)^2\pi^0$; (b) $\bar{p}p_{T=1} \to (\pi^+\pi^-)^2\pi^0$; (c) $\bar{p}p_{T=1} \to (\pi^+\pi^0)^2\pi^-$. [By $\bar{p}p \to (\pi^+\pi^-)^2\pi^0$ we mean that the invariant-mass-squared distribution for the $(\pi^+\pi^-)$ system is plotted.] The relative scale is correct, while the absolute scale is arbitrary.

To convert these frequencies into rates, we also need the conversion factor $(\bar{p}p \to \text{all})/(\bar{p}n \to \text{all})$, which is estimated to be about 1.7.[11] Using these data, we get

$$R(\bar{p}p_{T=1} \to \pi^+\pi^-\pi^0)/R(\bar{p}n \to \pi^+\pi^-\pi^-) \simeq 1.6^{+1.1}_{-0.8},$$

while, as shown in Eq. (9), we predict 1.16. However, we do not attach much significance to this agreement because of the large errors involved, especially in the separation between the triplet and the singlet part of $\bar{p}p \to \pi^+\pi^-\pi^0$.

The neglect of Pomeranchuk contributions is justified *a posteriori*. This is probably because, as Harari has stated, forgetting about the Pomeranchuk contributions amounts (in this case) to neglecting a nonresonating background.

Finally, one wonders if the c_{nm} can be computed from first principles. In these processes, if the $\pi\pi$ physical scattering amplitude is simple, as Lovelace proposes, one may hope to perform a mass continuation. This is not easily done, even using five-point functions,[13] because of the ambiguities stated at the beginning.[4] However, even the physical $\pi\pi$ amplitude may not be that simple.

We have also studied the E-meson decay with the same method, and reasonable agreement is found, using a simple expression for the decay amplitude, although the Dalitz plot has a different structure from the $\bar{N}N$ one.[14]

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[11] R. Bizzarri, Nuovo Cimento **53A**, 951 (1968).
[12] The model of Lovelace predicts 0.68 for this ratio.
[13] M. A. Virasoro, Phys. Rev. Letters **22**, 37 (1969); K. Bardakci and H. Ruegg, Phys. Rev. **181**, 1884 (1969).
[14] These results have been reported H. R. Rubinstein, in Proceedings of the 1969 Coral Gables Conference on Fundamental Interactions at High Energy (to be published).