Dynamical Theory for Strong Interactions at Low Momentum Transfers but Arbitrary Energies*

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Starting from the Mandelstam representation, it is argued on physical grounds that "strips" along the boundaries of the double spectral regions are likely to control the physical elastic scattering amplitude for arbitrarily high energies at small momentum transfers. Pion-pion scattering is used as an illustration to show how the double spectral functions in the nearest strip regions may be calculated, and an attempt is made to formulate an approximate but "complete" set of dynamical equations. The asymptotic behavior of the solutions of these equations is discussed, and it is shown that if the total cross section is to approach a constant at large energies then at low energy the S-dominant $\pi\pi$ solution is inadmissible. A principle of "maximum strength" for strong interactions is proposed, and it is argued that such a principle will allow large low-energy phase shifts only for $l \lesssim l_{\max}$, where $l_{\max} \sim 1$.

I. INTRODUCTION

THE power of the Mandelstam representation as a basis for a dynamical theory of strong interactions is now widely recognized.[1] Chew and Mandelstam showed how the representation leads to a one-parameter theory of the pion-pion interaction if at low energies only the S-phase shifts are large.[2] The basic approximation used by these authors, as well as by Cini and Fubini,[3] was to represent absorptive parts by the leading terms in a polynomial expansion or, equivalently, to replace double dispersion integrals by single integrals. This procedure can be justified when the only strongly scattered J values are 0 or $\frac{1}{2}$, but when states of higher J interact strongly at low energies, it is not possible to ignore the two-dimensional nature of the dispersion integrals in a consistent dynamical approach. If one does, the asymptotic behavior of the amplitude becomes distorted in a manner conflicting with unitarity.[4] The same difficulty appears in the π-N problem[5] and presumably in all other strongly interacting combinations. It is a difficulty that cannot be ignored, since nature has chosen to give us a $J=\frac{3}{2}$ π-N resonance, a $J=1$ N-N bound state, and very possibly a $J=1$ π-π resonance.[6] In this paper we describe the beginning of an attempt to understand the dynamical role of the double spectral functions with respect to low-energy resonances and bound states. In so doing we find ourselves immediately involved in a consideration of total cross sections and diffraction scattering at very high energies. Indeed, it will be seen that *if* one can make a consistent theory of low-energy phenomena, such a theory automatically covers low-momentum transfer effects at arbitrarily high energies.

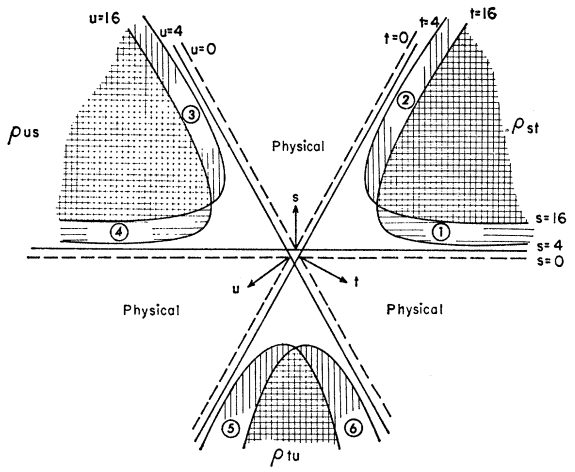

FIG. 1. The Mandelstam diagram for π-π scattering.

We employ the π-π interaction as the basic illustration for our approach, but the essential aspects may be generalized. For a preliminary orientation, consider the Mandelstam diagram[7] in Fig. 1 for one of the three independent π-π amplitudes as a function of s, t, and u.[8] The physical regions for the three different channels are labeled by that variable which is the square of the energy for the channel in question, while the shaded areas are the unphysical regions in which the double spectral functions fail to vanish. The six numbered strip regions are of central importance in our approach. We shall argue (a) that these double-spectral strips dominate those parts of the physical regions which lie

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[1] S. Mandelstam, Phys. Rev. **112**, 1344 (1958); also **115**, 1741, 1752 (1959).

[2] G. F. Chew and S. Mandelstam, Phys. Rev. **119**, 467 (1960); also G. F. Chew, S. Mandelstam, and H. P. Noyes, *ibid.* **119**, 478 (1960).

[3] M. Cini and S. Fubini, Ann. Phys. **3**, 358 (1960).

[4] M. Froissart, Phys. Rev. **123**, 1053 (1961); G. F. Chew and S. Mandelstam, Nuovo cimento **19**, 752 (1961). See also Sec. XI, G. F. Chew, Lawrence Radiation Laboratory Report UCRL-9289, 1960 (unpublished).

[5] S. Frautschi and D. Walecka, Phys. Rev. **120**, 1486 (1960); W. Frazer, *Proceedings of the 1960 Annual International Conference on High-Energy Physics at Rochester* (Interscience Publishers, New York, 1960), p. 282.

[6] W. Frazer and J. Fulco, Phys. Rev. **117**, 1609 (1960).

[7] For a discussion of the general kinematics of processes with two incoming and two outgoing particles, see T. Kibble, Phys. Rev. **117**, 1159 (1960).

[8] The notation and units (e.g., $m_\pi=1$) of reference 2 are used here. It is assumed that the reader is familiar with the concepts introduced in this reference as well as in reference 1.

in strips of comparable width along the boundaries, and (b) that the "strip" double spectral functions can be calculated through relatively tractable elastic unitarity conditions. Thus we are proposing a theory not only for low energies but also for arbitrarily high energies at low momentum transfer. The quantitative reliability of the approximation formulated here is uncertain; it may turn out that one needs to calculate the second strip (between $16m_\pi^2$ and $36m_\pi^2$) more carefully than proposed here in order to achieve an accurate theory. We feel confident, however, that the general approach is sound and that much will be learned by studying the first strip in detail.

The physical motivation for the strip approach lies in two well-established general features of strong interactions: (a) the existence of large phase shifts at low energy in states with $J>\frac{1}{2}$; (b) the occurrence at high energies of forward diffraction peaks, showing a strong concentration of the imaginary part of the amplitude for momentum transfers $\lesssim 4m_\pi$. Even though no direct observations of π-π scattering have been made, we shall assume that these two general features are shared with π-N and N-N scattering.

Dealing first with the low-energy question, we recall that the imaginary part of the amplitude in the s physical region of Fig. 1 is

$$A_s=\frac{1}{\pi}\int du'\,\frac{\rho_{us}(u',s)}{u'-u}+\frac{1}{\pi}\int dt'\,\frac{\rho_{st}(s,t')}{t'-t}, \qquad \text{(I-1)}$$

apart from subtractions (or in other words, single-spectral functions). It appears necessary and permissible[9] to subtract from (I-1) the S-wave imaginary part; however, one may not treat P or higher wave parts as independent of the double spectral functions without developing asymptotic trouble, as shown by Froissart and by Chew and Mandelstam.[4] Thus, if large phase shifts occur for $0<l\lesssim l_m$ at low energy we require that the partial wave projections of (I-1) be large for $l\lesssim l_m$ over an interval of low s, while the projections for $l>l_m$ are small.

Now, what behavior of the double spectral functions at low s could give a sudden decrease in the order of magnitude of the partial-wave projection in going from l_m to l_m+1? A simple guess is that for t large compared with the lower limit of the integrals we have

$$\rho_{st}(s,t)\propto t^{\alpha(s)}, \quad \text{where} \quad [\mathrm{Re}\alpha]_{\max}\approx l_m, \qquad \text{(I-2)}$$

with a similar behavior for ρ_{us}. Then, since

$$\int_{-1}^{+1} d\cos\theta\, P_l(\cos\theta)[t'+2q^2(1-\cos\theta)]^{-1}\underset{t'\to\infty}{\sim} q^{2l}/t'^{l+1},$$

the order of magnitude of the low-energy phase shifts for $l>[\mathrm{Re}\alpha]_{\max}$ will be determined by the lower limits of the dt' and du' integrals, i.e., by the "range" of the interaction. For $l<[\mathrm{Re}\alpha]_{\max}$, however, the lower limit is of secondary importance and the contributions from high values of t' and u' determine the magnitude of the phase shifts. (On the basis of analogy with nonrelativistic potential scattering, one expects, as explained in Sec. V, oscillations arising from $\mathrm{Im}\alpha$ and these permit the integral to have a meaning.) Thus we arrive at the tentative and qualitative conclusion that the existence in the s channel of low-energy resonances (or near resonances) for $J\gtrsim 1$ requires the strip regions Nos. 1 and 4 in Fig. 1 to be important out to large values of t and u, respectively.

If strips No. 1 and No. 4 are important, it follows from the substitution law that strips Nos. 2, 3, 5, and 6 are also important. How can we argue, however, that the interior regions of the double spectral functions are less important? These regions have no direct connection with low-energy resonances, but one would like definite evidence that the double spectral functions become systematically less important as one moves in directions perpendicular to the boundaries. Such evidence is furnished by the forward diffraction peaks at high energies. Referring again to (I-1), we see that the concentration of the imaginary part of the amplitude in the region $-20\lesssim t<0$ when s is large implies that the most important part of the dt' integral is for $t'\lesssim 20$, which is just strip No. 2 of Fig. 1. If the interior regions of ρ_{st} were of major importance, it would be difficult to understand the sparsity of large-angle elastic scattering at high energies.

This line of argument leads one to expect in general a backward elastic peak at high energies due to strip No. 3, although there are reasons why backward peaks may be less prominent than those in the forward direction. We are not arguing here, of course, that a knowledge of the strip double spectral functions is sufficient to describe what happens in the interor of the physical region at high energies (e.g., at angles $\approx 90°$ in the barycentric system). This domain lies outside the present scheme of approximation; it also happens to be a region about which almost nothing is known experimentally.

In the following two sections, formulas originally derived by Mandelstam[1] and obtained through an independent method by Cutkosky[10] are adapted to the calculation of π-π double spectral functions in the strips, and we write down sufficient additional formulas so as

[9] There exists much confusion about the meaning of subtractions in the Mandelstam representation. In this paper we mean by a "permissible" subtraction one that is also "necessary," and vice versa. That is to say, a permissible subtraction term in our sense is independent of the behavior at infinity of the dispersion integral that remains; if it is directly correlated with the remaining integral by asymptotic conditions imposed by unitarity on the full amplitude, then the subtraction is not necessary inasmuch as it is uniquely determined by the double spectral function through analytic continuation (see Froissart, reference 4).

[10] R. Cutkosky, Phys. Rev. Letters 4, 624 (1960) and J. Math. Phys. 1, 429 (1960).

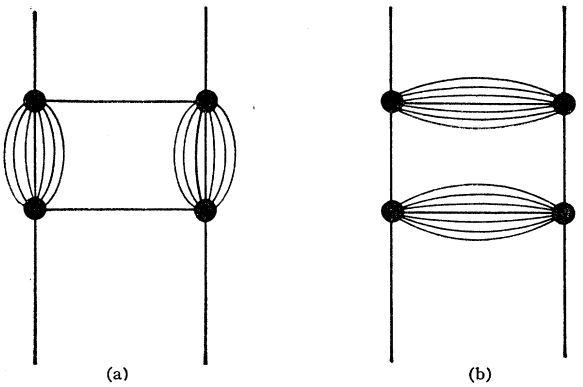

FIG. 2. The two Cutkosky diagrams needed to calculate the double spectral functions in the strip regions.

to achieve a "complete" set of dynamical equations. Section IV then deals with the iterative solution of these equations in the S-dominant case, which is shown to be physically uninteresting because of the total-cross-section behavior at high energies. In Sec. V the interesting case of a constant high-energy limit for the cross section is discussed in a tentative way.

Before we proceed to detailed matters, it is appropriate here to relate our approach to ideas recently expressed by Salzman and Salzman[11] and by Drell[12] as well as by Berestetsky and Pomeranchuk.[13] If we continue to focus attention on the s physical region of Fig. 1, the double spectral function in strip No. 1 corresponds to diagrams in which only two particles are present in intermediate states but any number may be exchanged. In other words, this piece of ρ_{st} is calculated from the Cutkosky diagram[10] shown in Fig. 2(b) and represents purely elastic effects in the s channel. On the other hand, the piece of ρ_{st} in strip No. 2 is calculated from diagram 2(a), in which any number of particles is allowed in intermediate states but only two are exchanged (it is elastic in the t channel). Obviously, then, we are calculating here the diffraction scattering associated with inelastic transitions in which a single pion is exchanged. This is the mechanism of Salzman and Salzman, Drell, and Berestetsky and Pomeranchuk although our method of calculation is quite different. We believe that the Mandelstam-Cutkosky approach through the double-spectral function is more systematic, since it raises no questions it cannot answer about cross sections in unphysical regions. It may lead to very different numerical results.

II. EQUATIONS FOR THE DOUBLE SPECTRAL FUNCTIONS

In his first paper Mandelstam derived formulas for double spectral functions on the basis of the elastic unitarity condition.[1] If for each channel we decompose the absorptive parts into elastic and inelastic components and make a corresponding decomposition of the contributing double spectral functions, then Mandelstam's formulas are exact for the elastic part of the double spectral functions. In other words, the Cutkosky diagram Fig. 2(b) represents the complete $\rho_{st}^{\mathrm{el}(s)}$, and Cutkosky's formula for this diagram[10] is exactly that given by Mandelstam. The basic approximation of our method occurs in the calculation of the inelastic part: What we call the "strip approximation" is the assumption that

$$\rho_{st}^{\mathrm{in}(s)}=\rho_{st}^{\mathrm{el}(t)}, \quad \text{etc.} \tag{II-1}$$

This formula is exact in the strip region, $4<t<16$, but for $t>16$ there will be additional contributions to $\rho_{st}^{\mathrm{in}(s)}$.

Because of the great symmetry of the π-π problem, only three independent elastic double-spectral functions are required to describe the three amplitudes A, B, C of reference 2. We shall call these functions $\rho_{1,2,3}(x,y)$, where by convention the first variable is associated with that channel for which ρ_α is the elastic part of the complete double spectral function. In Fig. 3 the assignment of $\rho_{1,2,3}$ appropriate to the amplitude B is shown. A and C may be obtained by the usual substitutions,[2] which maintain the relative orientation of $\rho_{1,2,3}$ but exchange the channel labels.

The Mandelstam formulas for $\rho_{1,2,3}$ involve three linearly independent absorptive parts which we shall designate by $I_{1,2,3}(x,y)$. Here the first variable corresponds to the channel for which I is the actual imaginary part in the physical region, while the second is a particular choice of one of the remaining two linearly dependent variables. We make the choice so that we have

$$\begin{aligned} A_s(s,t,u)&=I_1(s,t),\\ B_s(s,t,u)&=I_2(s,t),\\ C_s(s,t,u)&=I_3(s,t). \end{aligned} \tag{II-2}$$

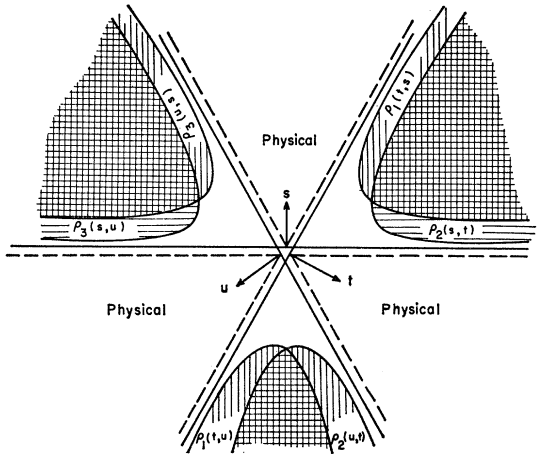

FIG. 3. The assignment of the three independent elastic double-spectral functions to the amplitude B of reference 2. A is obtained by interchanging s and t and C by interchanging t and u.

[11] F. Salzman and G. Salzman, Phys. Rev. Letters **5**, 377 (1960).
[12] S. Drell, Phys. Rev. Letters **5**, 342 (1960).
[13] V. Berestetsky and I. Pomeranchuk, *Proceedings of the 1960 Annual International Conference on High-Energy Physics at Rochester* (Interscience Publishers, New York, 1960), p. 333.

We may note

$$\begin{aligned} I_1(x,y)&=I_1(x,4-x-y),\\ I_2(x,y)&=I_3(x,4-x-y),\\ I_3(x,y)&=I_2(x,4-x-y). \end{aligned} \tag{II-3}$$

In terms of these absorptive parts, the Mandelstam equations[1] for $\rho_{1,2,3}$ turn out to be as follows, when proper allowance is made for the normalization of A, B, C used in reference 2:

$$\rho_1(x,y)=\frac{1}{\pi}[q_x(q_x^2+1)^{\frac{1}{2}}]^{-1}\int\int dy'dy''\frac{I_2^*(y',x)[\frac{3}{2}I_2(y'',x)+I_1(y'',x)+I_3(y'',x)]+\text{c.c.}}{K^{\frac{1}{2}}(x;y,y',y'')}, \tag{II-4}$$

$$\rho_2(x,y)=\frac{1}{\pi}[q_x(q_x^2+1)^{\frac{1}{2}}]^{-1}\int\int dy'dy''[I_1^*(y',x)I_1(y'',x)+I_3^*(y',x)I_3(y'',x)]/K^{\frac{1}{2}}(x;y,y',y''), \tag{II-5}$$

$$\rho_3(x,y)=\frac{1}{\pi}[q_x(q_x^2+1)^{\frac{1}{2}}]^{-1}\int\int dy'dy''[I_1^*(y',x)I_3(y'',x)+\text{c.c.}]/K^{\frac{1}{2}}(x;y,y',y''), \tag{II-6}$$

where

$$\begin{aligned} K(x;y,y',y'')=&y^2+y'^2+y''^2\\ &-2(yy'+yy''+y'y'')-yy'y''/q_x^2, \end{aligned} \tag{II-7}$$

and

$$q_x^2=\tfrac{1}{4}x-1.$$

The upper limits of the integrations over dt' and dt'' are determined by the condition $K=0$. Specifically, one integrates only over that region for which

$$y>y'+y''+\frac{y'y''}{2q_x^2}+2(y'y'')^{\frac{1}{2}}[(1+y'/4q_x^2)(1+y''/4q_x^2)]^{\frac{1}{2}}, \tag{II-8}$$

so that K is positive and vanishes only at the upper limit. It is easy then to verify that $\rho_{1,2,3}(x,y)$ vanishes for $y<16x/(x-4)$.

We now need formulas for $I_{1,2,3}$ in terms of $\rho_{1,2,3}$. Making a subtraction of the S-wave imaginary parts, we find

$$\begin{aligned} I_1^{\text{el}}(y,x)=&\tfrac{1}{3}[\text{Im}A^{(0)0}(y)-\text{Im}A^{(0)2}(y)]_{\text{el}}+\frac{1}{\pi}\int dx'\,\rho_1(y,x')\\ &\times\left[\frac{1}{x'-x}+\frac{1}{x'-(4-x-y)}\right.\\ &\left.-\frac{1}{2q_y^2}\ln\left(1+\frac{4q_y^2}{x'}\right)\right], \end{aligned} \tag{II-9}$$

$$\begin{aligned} I_2^{\text{el}}(y,x)=&\tfrac{1}{2}[\text{Im}A^{(0)2}(y)]_{\text{el}}+\frac{1}{\pi}\int dx'\,\rho_2(y,x')\\ &\times\left[\frac{1}{x'-x}-\frac{1}{4q_y^2}\ln\left(1+\frac{4q_y^2}{x'}\right)\right]\\ &+\frac{1}{\pi}\int dx'\,\rho_3(y,x')\\ &\times\left[\frac{1}{x'-(4-x-y)}-\frac{1}{4q_y^2}\ln\left(1+\frac{4q_y^2}{x'}\right)\right], \end{aligned} \tag{II-10}$$

with $I_3^{\text{el}}(y,x)$ given by (II-3).

In order to calculate the inelastic absorptive parts we need the approximation (II-1), which leads us to

$$I_1^{\text{in}}(y,x)=\frac{1}{\pi}\int dx'\,\rho_2(x',y)\left[\frac{1}{x'-x}+\frac{1}{x'-(4-x-y)}\right], \tag{II-11}$$

$$I_2^{\text{in}}(y,x)=\frac{1}{\pi}\int dx'\,\frac{\rho_1(x',y)}{x'-x}+\frac{1}{\pi}\int dx'\,\frac{\rho_3(x',y)}{x'-(4-x-y)}, \tag{II-12}$$

with I_3^{in} given by (II-3). Whether or not the inelastic S wave must be subtracted here is a point that will be considered later. In general, if phenomenology is to be introduced, these last two equations seem a logical place. The double spectral functions occurring therein are correct only outside a boundary calculated by Kolkunov *et al.*[14] which asymptotically approaches the straight lines $x'=16$ and $y=16$. One may therefore wish to add phenomenological contributions to the region inside this boundary.

It should be realized that although unitarity is not completely satisfied in our approximation the inelastic absorptive parts are bounded if a solution can be found. That is to say, for an individual partial wave in the s physical region it follows from our equations that

$$\text{Im}A^{(l)I}(s)=\frac{q_s}{(q_s^2+1)^{\frac{1}{2}}}|A^{(l)I}(s)|^2+q_s(q_s^2+1)^{\frac{1}{2}}\sigma_{\text{in}}{}^{(l)I}(s), \tag{II-13}$$

where $\sigma_{\text{in}}{}^{(l)I}(s)$ is the inelastic partial-wave cross section. Now, (II-13) cannot be satisfied unless we have

$$\sigma_{\text{in}}{}^{(l)I}(s)\leqslant 1/4q_s^2, \tag{II-14}$$

and

$$\sigma_{\text{tot}}{}^{(l)I}(s)=\frac{1}{q_s(q_s^2+1)^{\frac{1}{2}}}\text{Im}A^{(l)I}(s)\leqslant 1/q_s^2, \tag{II-15}$$

[14] V. A. Kolkunov, L. B. Okun, A. P. Rudik and V. V. Sudakov, *Proceedings of the 1960 Annual International Conference on High-Energy Physics at Rochester* (Interscience Publishers, New York, 1960), p. 247.

so *if* we succeed in finding a solution of our dynamical equations we can be sure that our inelastic and total cross sections have the proper upper bounds.

III. THE *S*-WAVE OR SINGLE-SPECTRAL FUNCTION

It is apparent that the problem of calculating the S wave is distinct from that of all waves for $J \gtrsim 1$, the latter being obtainable by quadrature from the double spectral functions once these are known. This distinction was recognized from the beginning by Mandelstam[1] and is the basis for the S-dominant theory of Chew and Mandelstam.[2] We may, in fact, lean almost entirely on the latter work in our handling of the S wave. There is only one change to be made in the N/D technique developed in reference 2: We shall no longer make a partial-wave expansion in evaluating the discontinuity across the left-hand S-wave cuts, but use instead the complete formulas,

$$\mathrm{Im}A^{(0)0}(s) \underset{s<0}{=} \frac{1}{2q_s^2}\int_4^{-4q_s^2} dt \times \mathrm{Re}\{I_1(t,s)+I_3(t,s)+3I_2(t,s)\}, \quad \text{(III-1)}$$

$$\mathrm{Im}A^{(0)2}(s) \underset{s<0}{=} \frac{1}{2q_s^2}\int_4^{-4q_s^2} dt\, \mathrm{Re}\{I_1(t,s)+I_3(t,s)\}.$$

It seems to us probably not important whether any inelastic effects are included on the S-wave right-hand cut. At energies sufficiently high that inelastic scattering becomes substantial, the S wave is unlikely to represent a significant part of the over-all amplitude. It is probably only in the low-energy elastic region that the S wave must be treated accurately.

With no prejudice at this stage as to whether the constant λ defined in reference 2 can be assigned an arbitrary value in the present approach, we give here for future reference the relations between λ and $a_{0,2}$, the S-wave amplitudes at the symmetry point, $s=\frac{4}{3}$:

$$a_{0,2} = -\binom{5}{2}\lambda - \frac{2}{\pi}\int dt'\left[\binom{3}{0}I_2(t',\tfrac{4}{3})+2I_1(t',\tfrac{4}{3})\right] \times\left[\frac{1}{t'-\frac{4}{3}}+\frac{3}{8}\ln\left(1-\frac{8}{3t'}\right)\right]. \quad \text{(III-2)}$$

It goes without saying that if all the equations set down in this section and the last are satisfied we shall have a solution with complete crossing symmetry.

IV. DISCUSSION OF THE DYNAMICAL EQUATIONS—*S*-DOMINANT SOLUTIONS

In Secs. II and III we have written down coupled equations for the double spectral functions. We now turn to a tentative discussion, making no pretense of rigor, of how one might try to solve these equations and what features we expect to arise in the solution.

A first consideration is how many parameters the theory contains. From conventional field theoretical arguments, one would guess that the minimal set of parameters includes the particle masses and one real dimensionless constant such as λ. Unstable zero-spin "elementary" particles of the Castillejo-Dalitz-Dyson type[15] could also occur, but we shall assume that none exist. It has been previously emphasized that no further free parameters are permitted in a consistent S-matrix theory of pion-pion scattering,[4] but in an incomplete approach such as ours it may become necessary to introduce additional, less fundamental, parameters to represent the effect of regions where formulas (II-11) and (II-12) are not exact. On the other side of the coin, we must keep in mind the possibility that physically interesting solutions do not allow a continuous range of λ and that the value of this parameter may be uniquely determined by requirements of consistency.

For reasons of simplicity isotopic spin and exchange terms are ignored in the discussion of this section, since these complications are inessential to the arguments we shall make. The elastic double spectral function is then given by

$$\rho(x,y)=\frac{1}{\pi q_x(q_x^2+1)^{\frac{1}{2}}}\int\int dy'dy''\,\frac{I^*(y',x)I(y'',x)}{K^{\frac{1}{2}}(x;\,y,y',y'')}. \quad \text{(IV-1)}$$

The absorptive part I is $I^{\mathrm{el}}+I^{\mathrm{in}}$, where

$$I^{\mathrm{in}}(y,x)=\frac{1}{\pi}\int dx'\,\frac{\rho(x',y)}{x'-x}, \quad \text{(IV-2)}$$

and

$$I^{\mathrm{el}}(y,x)=I^{\mathrm{el}(0)}(y)+\frac{1}{\pi}\int dx'\,\rho(y,x') \times\left[\frac{1}{x'-x}-\frac{1}{4q_y^2}\ln\left(1+\frac{4q_y^2}{x'}\right)\right]. \quad \text{(IV-3)}$$

In I^{el} the S wave has been subtracted out of the double spectral function, and appears as a separate term $I^{\mathrm{el}(0)}$, which is to be determined by the procedure discussed in Sec. III above. The absence of an S-wave subtraction in I^{in} will be explained shortly.

We now wish to argue that I^{el} plays the role of a "potential". If we consider scattering by a superposition of Yukawa potentials,[16]

$$V(r)=-\int dy\,\rho_0(y)(e^{-yr}/r),$$

[15] L. Castillejo, R. H. Dalitz, and F. J. Dyson, Phys. Rev. **101**, 453 (1956).

[16] R. Blankenbecler, M. L. Goldberger, N. N. Khurie, and S. B. Treiman, Ann. Phys. **10**, 62 (1960).

the nonrelativistic equations turn out to be identical with (IV-1) and (IV-2) if we omit the factor $(q_x^2+1)^{-\frac{1}{2}}$ from (IV-1) and replace $I^{\mathrm{el}}(y,x)$ by the real function $\rho_0(y)$. Our problem differs, then, in that it has relativistic kinematics and contains a complex energy-dependent "potential," allowing inelastic scattering. Furthermore, our potential is not given at the beginning but must be calculated in a self-consistent way from the scattering amplitude.

It is known[16] that if the potential is considered as given one may solve Eqs. (IV-1) and (IV-2) in terms of a series of functions, each of which has its threshold displaced from the preceding threshold and each of which can be calculated by quadrature from functions with lower thresholds. The possibility of this decomposition follows from the property (II-8) of the integration region in (IV-1). Precisely, if we write

$$I(y,x)=\sum_{n=1}^{\infty} I_n(y,x), \tag{IV-4}$$

where

$$I_1(y,x)=I^{\mathrm{el}}(y,x), \tag{IV-5}$$

$$I_n(y,x)_{n>1}=\frac{1}{\pi}\int dx' \frac{\rho_n(x',y)}{x'-x}, \tag{IV-6}$$

then for

$$\rho_n(x,y)=\frac{1}{\pi q_x(q_x^2+1)^{\frac{1}{4}}}\int\int dy'dy'' \times\frac{\sum_{n'=1}^{n-1} I_{n'}{}^*(y',x)I_{n-n'}(y'',x)}{K^{\frac{1}{2}}(x;\,y,y',y'')}, \tag{IV-7}$$

it is easy to verify that we have satisfied (IV-1) and (IV-2) with

$$\rho(x,y)=\sum_{n=2}^{\infty}\rho_n(x,y), \quad \rho_n(x,y)=0 \quad \text{for} \quad y<4n^2, \tag{IV-8}$$

and

$$I^{\mathrm{in}}(y,x)=\sum_{n=2}^{\infty} I_n(y,x), \quad I_n(y,x)=0 \quad \text{for} \quad y<4n^2. \tag{IV-9}$$

Thus the solution builds up, step by step, by a "bootstrap" mechanism starting with the "potential," $I^{\mathrm{el}}=I_1$.

One may in fact attach a direct physical significance to the individual terms I_n. These are the contributions to the absorptive part from intermediate states containing $2n$ pions,[17] so $4\pi q_s^{-1}(q_s^2+1)^{-\frac{1}{2}}I_n(s,0)$ is the cross section for a process leading to $2n$ pions. It is perhaps puzzling that in our theory *only* pions are produced, since the basic approximation seems to allow any inelastic process that can be achieved through single pion exchange. Because of the bootstrap nature of our equations, however, if we are to achieve production of, say, $N\bar{N}$ pairs, we must start with the matrix element for $\pi+\pi\rightarrow N+\bar{N}$. This latter process, however, cannot be achieved through single pion exchange, so $N\bar{N}$ production never gets started. The same statement may obviously be made for other baryon pairs as well as for $K\bar{K}$; thus *if* single pion exchange actually plays the dominant role we have assigned it, then production of particles other than pions should be small no matter how high the energy.

It is possible to augment our double spectral function with terms corresponding to single-baryon or single-kaon exchange so as to generate production of baryons and K mesons, and undoubtedly such a modification will eventually be tried. Such terms occur only in the interior region of the double-spectral function, however, so it is not immediately apparent that at the same time consideration need not be given to multipion exchange.

Even though our "potential" $I_1=I^{\mathrm{el}}$ is only determined *a posteriori* by formula (IV-3) we expect that for small values of λ an iteration procedure will converge rapidly because $I^{\mathrm{el}(0)}\sim\lambda^2$ while $\rho\sim\lambda^4$. Thus the "potential" will be dominated by the S-wave part of I^{el}, with the higher waves constituting only a perturbation. In fact the S-wave dominates $I^{\mathrm{el}}(t,s)$ at all t, and this will be taken as the defining property of an S-dominant solution in our treatment of arbitrary energies, since it leads to the characteristic features of the low-energy S-dominant solutions studied by Chew *et al.*[2] In the numerical calculation of Chew *et al.* the $l=1$ contribution alone was kept from the higher waves and with an adiabatic increase of λ turned out to be negligible in comparison to the S wave *throughout* the range of λ that was physically interesting. This situation may change for large λ when all higher waves are included in our present treatment, or it may change even for small λ if the iteration is begun with a "potential" that includes contributions beyond the elastic S-wave; these possibilities are discussed in the following section. Here we wish to make a negative argument that, from the point of view which considers high-energy diffraction scattering at the same time as low-energy elastic scattering, S-dominant solutions are physically inadmissable.

Our argument is based on the asymptotic behavior of $I(t,s)$ for $s=0$ as t tends to infinity. If total and elastic cross sections asymptotically tend to constants, then since in the t channel

$$I^{\mathrm{in}}(t,0)=\frac{1}{16\pi}(t-4)^{\frac{1}{2}}t^{\frac{1}{2}}\sigma^{\mathrm{in}}(t),$$

$$I^{\mathrm{el}}(t,0)=\frac{1}{16\pi}(t-4)^{\frac{1}{2}}t^{\frac{1}{2}}\sigma^{\mathrm{el}}(t),$$

it follows that both $I^{\mathrm{in}}(t,0)$ and $I^{\mathrm{el}}(t,0)$ should increase

[17] In problems involving different quantum numbers, the intermediate states are not necessarily restricted to $2n$ particles. For example, in simple Yukawa potential theory I_1 would represent single-particle exchange and ρ_n would represent n-particle exchange.

linearly for large t. If the first diffraction peak approaches a constant width, this statement may be extended to $-20\ m_\pi^2 \lesssim s < 0$, a circumstance noticed independently by Gribov[18] and the present authors. Gribov goes on to argue that a strict linear behavior is difficult to reconcile with the unitarity condition in the s channel and that the cross section probably decreases asymptotically faster than $\propto (\ln t)^{-1}$. We are inclined to doubt such a circumstance (see below), but logarithmic factors do not in any case affect our argument concerning S-dominant solutions.

The argument is simply that if the S-wave dominates $I^{el}(t,s)$ for all t then $I^{el}(t,0)$ cannot increase at infinity because $\mathrm{Im}A_{el}{}^{(0)}(t)$ is bounded by unity. The S dominant case corresponds, then, to $\sigma^{el}(t)$ falling off as t^{-1}. Now, it might be argued that there could be S dominance at low energies but not at high. However, if $I^{in}(t,s)$ is to go linearly with t for $-20m_\pi^2 \lesssim s < 0$, then according to (IV-2) and our assumption of strip dominance there must be at least some values of x in the range $4m_\pi^2 < x \lesssim 20m_\pi^2$ for which the average behavior (there may be oscillations) of $\rho(x,y)$ is at least linear in y. Then in (IV-3), according to the argument in the introduction, the P wave would be expected to be important already at low energies.

We cannot be sure, of course, that there must be a P resonance. However, the low-energy situation with a double-spectral function that gives the required high-energy limit in the crossed channel is certainly nothing like that in the solutions found by Chew *et al.*[2] To be more specific, when λ is small and there are no other parameters, our starting term in I^{el} is $\mathrm{Im}A_{el}{}^{(0)}(t)$, which behaves $\propto (\ln t)^{-2}$ as $t \to \infty$ when calculated by the N/D method in the purely elastic approximation.[2,19] It has been shown by Gribov[18] that the successive iterations $\rho_2, \rho_3, \cdots$ calculated from this starting term will also behave $\propto (\ln t)^{-2}$. According to (IV-6) and (IV-9), $I^{in}(t,s)$ will have a similar behavior, provided the sum over n does not go more strongly than an individual term, so the inelastic S wave which is present in our new framework cannot be so large as to change the asymptotic situation. Furthermore, it is easy to show from (IV-7) that in any finite order, $\rho_n(x,y) \underset{x\to\infty}{\propto} x^{-1}$. There are two immediate consequences: the S-wave term dominates $I^{el}(t,s)$ [Eq. (IV-3)], and I^{in} converges without subtraction [Eq. (IV-2)].

In the following section we investigate the extremely important possibility that the infinite sum over n may in some cases behave asymptotically with a higher power than any individual term. If iterations are made starting with small λ, such behavior is conceivable as λ becomes large if the double-spectral function ever grows to such a magnitude that its effect becomes comparable to that of the S wave. We believe that approximations made in the earlier work of Chew, Mandelstam, and Noyes[2] did not give the higher waves a real chance to grow and that the possibility of success for the adiabatic approach must still be considered open. We now turn to a possible mechanism by which the infinite sum over n may lead to asymptotic behavior of a truly strong-interaction type.

[18] V. N. Gribov, Nuclear Phys. **22**, 249 (1961).

[19] This special characteristic of the one-parameter S-dominant case was pointed out to us by K. Wilson. He has independently formulated the same coupled elastic unitarity relations as ours, [(Harvard University Physics Department, 1960) (to be published)] and has undertaken a numerical calculation of the one-parameter S-dominant case.

V. THE STRONG-INTERACTION LIMIT

One may adopt at least two different points of view about the strong-interaction requirement that $I(t,s)$ should increase linearly (or almost linearly) for large t if s is fixed in the region $-20m_\pi^2 \lesssim s < 0$. One may start with this requirement as a boundary condition and attempt to study its implications; this is the approach of Gribov.[18] Alternatively one may seek a mechanism that circumvents the asymptotic inhibitions discussed in Sec. IV and allows an increasing behavior for our amplitudes at infinity; this is the approach to be discussed here. We shall give reasons for believing that the asymptotic power will continuously increase with the "strength" of interaction up to a maximum power determined by unitarity.

Our basic motivation arises in the work of Regge on nonrelativistic potential scattering.[20] Regge shows for a wide class of potentials that

$$I(t,s) \underset{t\to\infty}{\propto} t^{\alpha(s)}, \tag{V-1}$$

where $\alpha(s)$ is real for $s < 4m_\pi^2$ and for attractive potentials increases with the strength of attraction. We believe that we have found within the iteration scheme a mechanism for such asymptotic behavior that applies as well to the relativistic case. The mechanism is illustrated by the following example: Suppose that the asymptotic behavior of the "potential" is *given* to be

$$I_1(t,s) \underset{t\to\infty}{\longrightarrow} t^{\alpha_0} I_1(s), \tag{V-2}$$

where α_0 is real and greater than $-\frac{1}{2}$. As discussed below, such a factorability of s and t dependence is not likely to be realistic, but it is easy to analyze. Carrying out the kind of asymptotic calculations with (IV-1) performed by Gribov[18] for the case where logarithmic decrease is absent, we then find

$$\rho_n(t,s) \underset{t\to\infty}{\longrightarrow} t^{\alpha_0} \frac{(\ln t)^{n-1}}{(n-1)!} \rho_n(s), \tag{V-3}$$

where

$$\rho_n(s) = G(s) \sum_{n'=1}^{n-1} I_{n'}{}^*(s) I_{n-n'}(s), \tag{V-4}$$

with

$$I_n(t,s) \longrightarrow t^{\alpha_0} \frac{(\ln t)^{n-1}}{(n-1)!} I_n(s). \tag{V-5}$$

[20] T. Regge, Nuovo cimento **14**, 951 (1959); **18**, 947 (1960).

In the relativistic case,

$$G(s)=\frac{F(\alpha_0)}{\pi}\frac{q_s^{2\alpha_0+1}}{(q_s^2+1)^{\frac{1}{2}}}, \tag{V-6}$$

where

$$F(\alpha_0)=\int_0^1 dz\,\frac{z^{\alpha_0}}{(1-z)^{\frac{1}{2}}}, \tag{V-7}$$

while in the nonrelativistic problem one omits the factor $(q_s^2+1)^{-\frac{1}{2}}$ in $G(s)$. It of course follows from (IV-2) that for $n\geqslant 2$,

$$I_n(s)=\frac{1}{\pi}\int ds'\,\frac{\rho_n(s')}{s'-s}. \tag{V-8}$$

We have not been able to deduce the limiting behavior of $I_n(s)$ for large n, but note that the sequence of numbers defined by

$$\sigma_n=C\sum_{n'=1}^{n-1}\sigma_{n'}\sigma_{n-n'} \tag{V-9}$$

has the property

$$\lim_{n\to\infty}\frac{\sigma_{n+1}}{\sigma_n}=4c\sigma_1. \tag{V-10}$$

Thus it is not implausible to conjecture on the basis of (V-4) and (V-8) that at least in some average sense

$$\lim_{n\to\infty}\frac{I_{n+1}(s)}{I_n(s)}=\Delta(s), \tag{V-11}$$

independent of n. If such were the case, then

$$\lim_{t\to\infty}I(t,s)=t^{\alpha_0}\lim_{t\to\infty}\sum_{n=1}^{\infty}\frac{(\ln t)^{n-1}}{(n-1)!}I_n(s) \tag{V-12}$$

may lead to an asymptotic t dependence

$$\sim t^{\alpha_0}e^{\Delta(s)\ln t}=t^{\alpha(s)}, \tag{V-13}$$

where $\alpha(s)=\alpha_0+\Delta(s)$.

Observe also that if the "potential" in our example $I_1=I^{e1}$ is changed by a scale factor, then $\Delta(s)$ changes by the same factor. Thus the asymptotic power varies with the strength of interaction, as implied by Regge's analysis[20] which associates $\mathrm{Re}\alpha(s)$ with the maximum angular momentum for which there is a resonance in the s channel. If, therefore, we can find solutions of our equations which have the above qualitative behavior, and arbitrary parameters occur, we expect by varying these parameters to change the asymptotic power $\alpha(s)$. The condition to be satisfied, for strong interactions is

$$\alpha(s)=1 \quad \text{for} \quad -20m_\pi^2\lesssim s<0. \tag{V-14}$$

Froissart[4] has been able to show that a higher value than unity for α in this region is inconsistent with the Mandelstam representation; in some sense, therefore, we shall be choosing our parameters to give the maximum possible strength of interaction.

It is necessary to emphasize that a more detailed study of potential scattering, to be published by one of us (S.F.) independently, shows that the mechanism for starting the logarithmic buildup to an increment of power is probably not directly related to the asymptotic behavior of the "potential". It seems likely instead that the finite momentum components of the potential after a few interations lead to a "starting" power $\alpha_0=-\frac{1}{2}$ and that the mechanism even beyond this point is a good deal more complicated because the s and t dependence is not separable.[21] Of course, Regge[20] assures us that for nonrelativistic potential scattering the buildup of a power increment is actually achieved.

The reader may wonder why we are not convinced by Gribov's argument about the necessity of a logarithmic decrease in the total cross section.[18] The answer is that since $I_n(s)$ as given by (V-8) is complex for $s>4m_\pi^2$ there is every reason to expect $\Delta(s)$ as given by (V-11) to be complex in this region, a point emphasized by Mandelstam.[22] (In Regge's analysis[20] of nonrelativistic potential scattering the same complexity of the asymptotic power occurs.) The imaginary part of $\alpha(s)$ then causes oscillations in the complete amplitude which invalidate Gribov's arguments.

It should be stated here that the condition (V-14) actually applies only to *one* of the three independent absorptive parts in the $\pi\pi$ problem, that corresponding to non-charge-exchange scattering in the forward direction. In the notation of Sec. II, this is $I_2(t,s)$. It may well develop that I_1 and I_3 do not increase so rapidly at infinity, corresponding to the oft-conjectured circumstance that backward and charge-exchange scattering become asymptotically negligible in comparison to the forward coherent diffraction peak.

In the event that the adiabatic approach fails, we have thought of several different possible ways of starting the iteration procedure so as to achieve (V-14). The most immediate idea is to include both P and S elastic absorptive parts in the starting guess for the "potential". Some kind of smooth cutoff would have to be inserted in order to represent the suppressive influence of the oscillations which appear only at a later stage when all waves are included. It would be very helpful if some *a priori* knowledge of the detailed nature of the oscillations were available and could be put into the "starting potential." In the absence of such knowledge, one must hope that the high momentum components of the "potential" are relatively unimportant.

The starting guess for the S and P waves would be characterized by a finite number of parameters—probably through effective range formulas—and these would be varied so as to give the best match with the result of the first complete iteration. One would then

[21] K. Wilson (private communication) has reached similar conclusions.

[22] S. Mandelstam, Discussion at the Conference on Strong Interactions, Berkeley, 1960 [Revs. Modern Phys. (to be published)].

proceed to second, third, etc. iterations to see if convergence could be achieved. The previous experience of Chew and Mandelstam[1] suggests that there may perhaps be no free parameters at all in the final result, if this corresponds to the kind of P-dominant solution discussed by these authors. Previously, that is, there were two parameters, λ and λ_1. The latter, however, corresponded to a cutoff (which should no longer be needed) and we have a new condition (V-14) to add to previously recognized requirements. From past experience with strong-interaction theory, however, we know not to count chickens before they are hatched.

VI. CONCLUSION

In conclusion, we refer to our Letter[23] on the same topic as this paper in order to revise and amplify certain remarks made there. On the question of the maximum angular momentum for which the low-energy phase shift may be large, we are still confident that this is determined by $[\mathrm{Re}\alpha]_{\max}$, as explained in our introduction above. When the Letter was written we believed that α must be real and equal to unity. Now we must add two qualifications: (1) Even if $\alpha(s)=1$ for $-20m_\pi^2 \lesssim s<0$, if there are oscillations it need not necessarily be true that $\mathrm{Re}\alpha(s)=1$ for $s>4m_\pi^2$. On dimensional grounds it would nevertheless be surprising if there were a substantial deviation of $\mathrm{Re}\alpha$ from unity at low energies. (2) It is really only for the $I=0$ state of the low-energy $\pi\pi$ system that coherent diffraction in the crossed channel at high energy has immediate relevance. In other words, the statement we should have made in our Letter is that constant limits for total cross sections imply a low-energy $\pi\pi$ force in the $I=0$ state of a strength that may produce large S and P phase shifts but probably nothing higher. Of course Bose statistics happen to exclude the $l=1$, $I=0$ state, but there is reason to expect forces of the same order of magnitude in all three isotopic spin states.

In our Letter we made reference to the Pomeranchuk theorem regarding equality of high-energy particle and antiparticle cross sections as a guarantee that no arbitrary constant (such as a cutoff or λ_1) could be associated with the P wave. This point is implicitly contained in Froissart's paper,[4] where it is also pointed out that any condition limiting "backward" direction scattering amplitudes to an asymptotic power less than unity will equally well provide such a guarantee. In view of the now-recognized nonseparability of s and t dependence, we, of course, no longer would attempt to express the Pomeranchuk theorem through a formula such as Eq. (4) of our Letter.

In any event we still believe in the essential points of the Letter, which were that: (1) Unitarity at high energies in one channel puts on the interaction strength a limit that is carried over by analytic continuation to the low energy region of a crossed channel, and (2) perhaps, nature approaches this limit as closely as possible. It remains to be seen whether such a notion of maximal strength for strong interactions allows a determination of the low-energy coupling constants heretofore regarded as arbitrary.

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Note added in proof. After completion of this manuscript we received a paper by K. A. Ter-Martirosyan, J. Exptl. Theoret. Phys. (U.S.S.R.) **39**, 827 (1960) [translation: Soviet Phys.—JETP **12**, 575 (1961)], in which the same equations obtained here are derived.

[23] G. F. Chew and S. C. Frautschi, Phys. Rev. Letters **5**, 580 (1960).