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Dynamics Based on Rising Regge Trajectories*

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An outline is given of a dynamical scheme based on rising Regge trajectories. The fundamental approximation is that the scattering amplitude can be approximated by the contribution of a finite number of Regge poles. An additional simplifying assumption is that the Regge trajectories are straight lines or, equivalently, that the scattering amplitude is dominated by narrow resonances. Unitarity is introduced by means of the Cheng-Sharp equations, but, in the narrow-resonance approximation, we adopt a very trivial solution of these equations. Crossing is introduced by means of the generalized superconvergence relations due to Igi and to Horn and Schmid. Levinson's theorem is not used; the bootstrap condition is the absence of Kronecker-δ singularities in the J plane. It is hoped that this scheme avoids some of the disadvantages of conventional schemes. In the narrow-resonance approximation one has to solve numerical equations, not integral equations. The scheme is applied to the pseudoscalar, vector, and axial-vector nonets considered as bound states of the $N\bar{N}$ system. As only one channel is being examined, we have to introduce certain parameters from experiment, but we obtain reasonable values for the other parameters.

I. INTRODUCTION

THE current experimental data provide a number of indications that Regge trajectories rise indefinitely with energy, instead of turning over as they do in potential theory or in single-channel unitarity models. It is usually possible to determine two or three points on the trajectory by measurements of the spin of the appropriate particles or of the asymptotic behavior in the crossed channel. If the trajectory is now projected linearly to higher energies, it often turns out that narrow resonances appear at those energies where the trajectory passes through integers or half-integers. The spin of these high-energy resonances has not yet been measured directly, but, owing to the small width and the large Q value, it is presumably high. It is tempting to assume that the resonances are the higher members of the Regge sequence associated with the trajectory in question.

It may be that the strong interactions are characterized by an energy, large compared with the mass of the nucleon, above which the Regge trajectories do turn over. If all particles are built up of real elementary quarks, the quark mass may correspond to such an energy. However, it appears worthwhile to attempt to construct a theory on the assumption either that such an energy does not exist, or that the limit in which it approaches infinity represents a meaningful approximation.

A dynamical scheme based on rising Regge trajectories possesses several attractive features. The most significant is probably that one may be able to work with it in a narrow-resonance approximation. During the last few years many correlations between masses and coupling constants have been obtained by combining group theory, current commutators, or superconvergence relations with the assumption that scattering amplitudes are dominated by a few narrow resonances at low energy. The success of the Gell-Mann–Okubo mass formula, for instance, indicates that one should attempt to incorporate resonances more directly into a dynamical scheme than has been done hitherto. This point of view has been constantly emphasized by Gell-Mann.

It has been remarked by van Hove[1] and by Durand[2] that one can combine the narrow-resonance approximation with Regge asymptotic behavior if and only if the trajectories rise indefinitely. They showed this to be the case by the use of explicit formulas. Their results may be paraphrased by saying that the narrow-resonance approximation is never valid in the region where the trajectories are falling. Thus, for the narrow-resonance approximation to be universally valid, the trajectories must rise indefinitely.

The assumption that Regge trajectories rise indefinitely casts considerable doubt on the Levinson criterion for determining which particles, if any, are elementary. If this criterion is to be valid, it is essential that the phase shifts begin to fall once we have passed above the resonance region in those channels possessing composite resonances. Intuition would suggest that the energy at which the phase shifts begin to fall is of the same order of magnitude as the energy at which the Regge trajectories begin to fall. In a system with indefinitely rising trajectories, therefore, the phase shifts may never fall. This conclusion may be placed on firmer ground by assuming that an infinite number of trajectories rise indefinitely. At each energy where a trajectory passes through a given integer or half-integer, the corresponding phase shift passes through an odd multiple of $\frac{1}{2}\pi$. It will therefore not approach zero, or a multiple of π, asymptotically. One cannot assert this as a rigorous result, as the imaginary part of the lower trajectories will probably be large. If so, there will be no precise connection between the energies at which the real part of the trajectory passes

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[1] L. van Hove, Phys. Letters **24B**, 183 (1967).
[2] L. Durand, Phys. Rev. **161**, 1610 (1967).

through an integer and those at which the phase shift passes through an odd multiple of $\frac{1}{2}\pi$. Nevertheless, it is strongly suggested that the phase shifts will not behave in the manner required by Levinson's criterion.

We have argued elsewhere[3] that the experimental decrease of form factors provides further evidence against Levinson's theorem.

The above arguments, together with the absence of any experimental evidence in favor of Levinson's theorem, suggest that one should attempt to construct a scheme where the theorem is not used. If one is attempting to construct a bootstrap theory, one can use the alternative criterion that there are no non-Regge terms in the asymptotic behavior or, in other words, that there are no Kronecker-δ singularities in the angular momentum plane.[4,5] This assumption can be tested experimentally, and all available evidence points to its being valid.

In previous dynamical schemes, such as the N/D scheme, Levinson's theorem has played the essential role of removing the Castillejo-Dalitz-Dyson (CDD) ambiguity. The phase shifts calculated in such dynamical models had the property of rising through the resonance region and then falling. Furthermore, the region where the phase shift was falling was important in the dynamics. Thus, though one might be able to construct narrow resonances within the framework of such calculations, one could not express the dynamical equations in terms of the resonance parameters alone. In certain special cases one could obtain partial correlations between resonance parameters, the Chew-Low effective-range formula being an example. By dispensing with Levinson's theorem, we again raise the possibility of constructing a dynamics which has a narrow-resonance approximation.

We should emphasize that the presence of indefinitely rising Regge trajectories or the nonvalidity of Levinson's theorem does not necessarily imply that a dynamics based on the N/D method is inapplicable. Approximation schemes have been discussed where the trajectory or phase shift turns over at a value of s which increases indefinitely with the order of the approximation.[3,6] Nevertheless, it is certainly desirable to construct a dynamical scheme based on rising trajectories if it is at all possible.

II. OUTLINE OF THE DYNAMICAL SCHEME

The reasoning of the previous section suggests that we attempt to formulate a dynamical scheme based on equations for the Regge parameters themselves. The infinite rise of the trajectories, as well as the absence of Kronecker-δ singularities in the J plane, can easily be inserted directly into the equations. It has been shown by Cheng and Sharp[7] and by Frautschi, Kaus, and Zachariasen[8] that one can treat a Schrödinger-potential problem by using equations for the Regge parameters. The fundamental approximation is that the amplitude is dominated by a finite number of trajectories in the direct channel. In the simplest approximation only one trajectory is taken. Onc can then write dispersion relations for the Regge parameters. The weight functions in the dispersion relations are determined from unitarity, the subtraction terms from knowledge of the potential.

The scheme proposed in this paper is an application of the Cheng-Sharp scheme to the elementary-particle problem. Naturally, there will be some essential differences between this problem and the potential problem. The infinite rise of the Regge trajectories will be effected by inserting two subtraction terms in the equation for α instead of one. In the narrow-resonance approximation, which we shall use, the imaginary part of α is zero, so that the real part is given simply by the equation $\alpha = as+b$. Experimentally, the Regge trajectories do appear to be roughly linear functions of s. We thus adopt a very trivial solution of the Cheng-Sharp equations to correspond to the narrow-resonance approximation. In fact, the equations are not used explicitly at all. It is well to keep them at the back of one's mind, however, and to regard the linear trajectory as a trivial solution of them; one can then envisage how the scheme will appear when the narrow-resonance approximation is not used and when the imaginary part of α is not neglected. The dynamical scheme which we are suggesting can thus be combined with the narrow-resonance approximation, but it is not tied to this approximation.

Another difference between the potential and relativistic problems lies in the determination of the subtraction terms in the dispersion integrals for the Regge parameters. In the potential problem they are determined from knowledge of the potential while in the relativistic case they will have to be determined from the crossing relation. There is no unique way of applying the crossing conditions, but one attractive possibility is to use the generalized superconvergence relations first proposed by Igi, and discussed more fully by Dolen, Horn, and Schmid, by Logunov, Soloviev, and Tavkhelidze, and by Balázs and Cornwall.[9] These relations

[3] S. Mandelstam, in *Proceedings of the 1966 Tokyo Summer Lectures on Theoretical Physics*, edited by G. Takeda (W. A. Benjamin, Inc., New York, 1966).

[4] G. F. Chew and S. C. Frautschi, Phys. Rev. Letters **7**, 394 (1961).

[5] S. Mandelstam, Phys. Rev. **137**, 949 (1965).

[6] P. Carruthers and M. M. Nieto, Phys. Rev. Letters **18**, 297 (1967); P. Carruthers, Phys. Rev. **154**, 1399 (1967).

[7] H. Cheng and D. Sharp, Ann. Phys. (N. Y.) **22**, 481 (1963); Phys. Rev. **132**, 1854 (1963).

[8] S. C. Frautschi, P. Kaus, and F. Zachariasen, Phys. Rev. **133**, B1607 (1964).

[9] K. Igi, Phys. Rev. Letters **9**, 76 (1962); R. Dolen, D. Horn, and C. Schmid, *ibid.* **19**, 402 (1967); A. Logunov, L. D. Soloviev, and A. N. Tavkhelidze, Phys. Letters **24B**, 181 (1967); L. A. P. Balázs and J. M. Cornwall, Phys. Rev. **160**, 1313 (1967).

express the integral

$$\int_{s_1}^{N} ds\, \mathrm{Im}A(s,t)$$

in terms of the Regge parameters in the crossed channel. The sum over all the Regge trajectories is not convergent but is asymptotic in N. When the external particles have spin, we can decrease the high-s contribution by dividing the integrand by appropriate functions of s, just as we do with ordinary superconvergence relations. In the narrow-resonance approximation, the relations give us equations between the Regge parameters in the direct and crossed channels. The ordinary superconvergence relations are particular cases of the generalized superconvergence relations, which are valid when the Regge poles in the crossed channel are sufficiently far to the left. The correlations between resonance parameters that have been obtained from superconvergence relations[10] will thus be incorporated automatically in the dynamics.

The dynamical scheme which we shall outline in the following sections should be regarded simply as suggestive, and further work may well reveal the need for substantial modifications or additions. Examination of one channel alone (together with its crossed channels) will not give us enough equations to determine all the resonance parameters. This feature is to be expected and occurs in all bootstrap schemes, since the same parameters occur in the equations for different channels. We shall not attempt to answer the question whether examination of all channels provides us with a uniquely determined, an underdetermined, or an overdetermined system. The question is closely connected with that of the number of trajectories in each channel. The greater the number of trajectories, the larger will be the number of resonance parameters. On the other hand, each trajectory provides a sequence of external particles which give rise to further channels and hence further equations.

One parameter which cannot be determined in the narrow-resonance approximation is the strength of the coupling. The equations will be linear in the coupling constants, and hence one will at most be able to determine ratios of coupling constants. To complete the scheme one will have to go beyond the narrow-resonance approximation, which we shall not do in this paper.

In Sec. III, we shall discuss the dependence of the Regge parameters α and β on the variable s. We shall concentrate mainly on the narrow-resonance approximation, where we shall be able to obtain the dependence explicitly, but we shall also mention the Cheng-Sharp equation which can be used in more general cases. In Sec. IV we shall give arguments which make it plausible that several trajectories with different quantum numbers, and possibly all trajectories, have the same slope. Experimentally, all trajectories so far investigated do appear to have approximately the same slope. In Sec. V we shall discuss the use of the Igi-Horn-Schmid generalized superconvergence relations to obtain crossing formulas, and in Sec. VI, we shall outline an approximation scheme based on the foregoing sections. In Sec. VII we shall apply our scheme to a particular problem, the determination of the pseudoscalar, vector, and axial-vector nonets as bound states of the nucleon-antinucleon system. Since we are only investigating one channel, we shall not be able to calculate all the parameters; we shall take the mass of the nucleon and the relative masses of the mesons from experiment. We shall show that there does exist a solution of our equation. The solution predicts the correct sign for the ratios of the coupling constants and gives a reasonable value for the absolute mass of the mesons.

III. DEPENDENCE OF THE REGGE PARAMETERS ON s

In this section we shall obtain equations for the dependence of the Regge parameters α and β on s, the square of the energy. We shall concentrate on the narrow-resonance approximation, which we shall use in the following sections; the parameters will then have a simple, explicit dependence on s. At the end of the section, we shall indicate how we may go beyond the narrow-resonance approximation. We shall then have to use the unitarity condition to give us nonlinear integral equations for α and β.

We begin with the spinless case. The fundamental approximation which we shall make is that the scattering amplitude can be expressed as a contribution from a finite number of Regge poles

$$a(s,l)=\sum_{r=1}^{n}\frac{\beta_r(s)}{l-\alpha_r(s)}. \tag{3.1}$$

Strictly speaking, we require an expression which converges as the number of poles becomes infinite. Equation (3.1) should therefore be replaced by an expression such as the modified Cheng representation.[11] To our knowledge such a representation has only been worked out for the single-channel problem, but it would be surprising if a generalization did not exist. The modified Cheng representation has been applied to the potential problem.[12] The equations are more complicated than those obtained from (3.1), but do not differ from them in any fundamental way. In the narrow-resonance approximation the modified Cheng representation becomes equivalent to (3.1), and we shall not consider it further in this paper.

The reality conditions are

$$\alpha(s),\ \beta(s)/(4q^2)^{\alpha(s)}\ \text{real analytic}, \tag{3.2}$$

[10] F. Gilman and H. Harari, Phys. Rev. Letters **18**, 1150 (1967); **19**, 723 (1967); and (to be published).

[11] H. Cheng, Phys. Rev. **144**, 1237 (1966).

[12] W. J. Abbe, P. Kaus, P. Nath, and Y. N. Srivastava, Phys. Rev. **141**, 1513 (1966).

where q is the center-of-mass momentum. For simplicity, we have considered the equal-mass case; the results can easily be generalized. In the single-channel problem where only Regge pole is considered, i.e., where the series on the right of (3.1) is approximated by one term, the unitarity condition is

$$\mathrm{Im}\alpha(s)=k(s)\beta(s)\,, \tag{3.3}$$

where k is the usual kinematic factor in the unitarity condition.[7] Note, in particular, that β is real in this approximation. The reality of β is not preserved in higher approximations where more terms are included on the right of (3.1).

In the narrow-resonance approximation, α is real on the real axis and both sides of (3.3) are small. Furthermore, the function $\beta(s)/(4q^2)^{\alpha(s)}$, which is real analytic by (3.2), is also real above threshold, since α and β are both real. This function therefore has no right-hand cut. The analyticity conditions imply that $\alpha(s)$ and $\beta(s)/(4q^2)^{\alpha(s)}$ have no left-hand cut unless two trajectories intersect. In this paper we shall not consider the intersection of trajectories, though our equations could easily be modified to account for this possibility. The Regge parameters $\alpha(s)$ and $\beta(s)/(4q^2)^{\alpha(s)}$ thus have no left-hand or right-hand cuts and they are real, entire functions of s.

When several terms of (3.1) are taken into account, Eq. (3.3) is valid as long as the separation between trajectories is large compared with their imaginary part. It is thus valid in the narrow-resonance approximation. In the multichannel problem, the right-hand side of (3.3) must be replaced by a sum of terms, but it is not difficult to show that each Regge residue is still real. It thus remains true that $\alpha(s)$ and $\beta(s)/(4q^2)^{\alpha(s)}$ are entire functions of s in the narrow-resonance approximation.

We shall assume that Regge trajectories do not rise more than linearly with s. The parameter α will therefore be given by the simple equation

$$\alpha(s)=as+b\,. \tag{3.4}$$

The equation for β is not quite so simple. When α passes through a negative half-integer l_1 other than $-\frac{1}{2}$, β must vanish unless there is a compensating trajectory passing through the positive half-integer $-l_1-1$ at the same value of s. When the leading trajectory passes through a negative half-integer there is no trajectory with positive α, and therefore no compensating trajectory. The Regge residue $\beta(s)$ must therefore vanish whenever $\alpha(s)$ passes through a negative half-integer less than $-\frac{1}{2}$. We can ensure this by inserting a factor $1/\Gamma(\alpha(s)+\frac{3}{2})$ into the expression for β. Since $\alpha(s)=as+b$, this factor is an entire function of s. Thus

$$\frac{\beta(s)}{(4q^2)^{\alpha(s)}}=\frac{E_1(s)}{\Gamma(\alpha(s)+\frac{3}{2})}\,,$$

where E_1 is an entire functions of s. Inserting the equation $\alpha(s)=a(s)+b$, we find that

$$\beta(s)=\frac{E_1(s)(4q^2)^{as+b}}{\Gamma(as+b+\frac{3}{2})}\,,$$

or, by redefinition of the entire function,

$$\beta(s)=\frac{E(s)(4aq^2/e)^{as+b}}{\Gamma(as+b+\frac{3}{2})}\,. \tag{3.5}$$

The factor e^{-as-b} has been inserted in order that the right-hand side of (3.5), without the factor E, should not increase exponentially when s approaches $\pm\infty$. It is easily seen from Stirling's theorem that the expression $(4aq^2/e)^{as+b}/\Gamma(as+b+\frac{3}{2})$ behaves like $1/s$ at infinite s.

There does not appear to be any simple argument for restricting the entire function E to a polynomial of a given degree, or even to a polynomial of arbitrary degree. When s approaches infinity, the narrow-resonance approximation will probably become invalid, and one can therefore not combine Eq. (3.5) with assumptions about the behavior of β at large s. In the approximation scheme which we shall develop by combining (3.5) with the crossing relations, we shall expand E in powers of s; the number of terms kept will depend on the order of the approximation. We shall apparently be able to obtain equations for all the terms, but, as we have pointed out before, questions of this type cannot yet be answered with any degree of certainty.

We now allow the external particles to have spin, and we shall begin with systems where the particles are both bosons or both fermions. As usual, we shall work with states of fixed helicity λ, μ. Equation (3.5) must be modified in several ways. First, the behavior at $q^2=0$ will depend on the orbital rather than on the total angular momentum. We construct states of fixed incoming and outgoing orbital angular momentum at threshold by taking the usual linear combinations of helicity states. These orbital angular momenta will differ from the total angular momentum by integers which we shall denote by τ and $\tau'(|\tau|<|\lambda|,|\tau'|<|\mu|)$. The states with fixed orbital angular momentum will then have a factor

$$(4q^2)^{as+b+\frac{1}{2}(\tau+\tau')} \tag{3.6}$$

instead of $(4q^2)^{as+b}$.

We also require extra factors when $as+b$ takes on integral values where nonsense states are present. For positive integers or zero, the factors will be as follows:

Factors $(as+b-n)^{1/2}$, $|\lambda|\leq n<|\mu|$ or $|\mu|\leq n<\lambda$ (3.7a)

Factors $(as+b-n)$, $|\lambda|\leq n$, $|\mu|\leq n$ if the trajectory chooses nonsense at $\alpha=n$ (3.7b)

Factors $(as+b-n)$, $|\lambda|>n$, $|\mu|>n$ if the trajectory chooses sense at $\alpha=n$. (3.7c)

There are similar factors when α passes through negative integral values:

Factors $(as+b-n)^{1/2}$,

$$|\lambda|\leq -n-1<|\mu| \text{ or } |\mu|\leq -n-1<|\lambda|. \quad (3.8a)$$

Factors $(as+b-n)$ for one of the inequalities

$$|\lambda|\leq -n-1, \quad |\mu|\leq -n-1 \quad (3.8b)$$

or

$$|\lambda|>-n-1, \quad |\mu|>-n-1. \quad (3.8c)$$

If we are dealing with the leading trajectory, the factors $(as+b-n)$ occur in the case (3.8c). This is because the alternative which occurs at negative half-integral values of α also occurs at negative integral values with $|\lambda|$, $|\mu|>-\alpha-1$. Either β must vanish or there must be a compensating trajectory passing through the point $-\alpha-1$ at the same value of s. For the leading trajectory the second alternative is impossible, and β must vanish. We thus have factors $(as+b-n)$, where n is a negative integer satisfying the inequality (3.8c).

In channels where one of the particles is a fermion, the Γ function in (3.5) becomes $\Gamma(as+b+1)$, since we now require the zeros at negative integers. The extra factors due to spin will involve half-integral instead of integral values of n.

We have emphasized that Eq. (3.5) is really the narrow-resonance approximation to a more general set of equations. Since we shall not go beyond the narrow-resonance approximation in the following sections, we shall confine ourselves to the single-channel, spinless, equal-mass problem, and we shall keep only one term of (3.1). We can then write down the equations by making slight modifications to the nonrelativistic equations of Cheng and Sharp.[7] On doing so, we obtain the following:

$$\alpha(s)=as+b+\frac{1}{\pi}\int ds'\frac{\mathrm{Im}\alpha(s')}{s'-s}, \quad (3.9a)$$

$$\beta(s)=\left[\frac{E(s)(4aq^2/e)^{\alpha(s)}}{\Gamma(as+b+\frac{3}{2})}\right]\times\exp\left(-\frac{1}{\pi}\int ds'\frac{\mathrm{Im}\alpha(s')\ln(4q'^2)}{s'-s}\right)\times\prod_{n=1}^{\infty}\frac{a(s-s_n)}{as+b+n+\frac{1}{2}}, \quad (3.9b)$$

$$\mathrm{Im}\alpha(s)=k\beta(s). \quad (3.9c)$$

We have assumed that $\mathrm{Im}\alpha\to 0$ as $s\to\infty$. If this is not the case, we must modify the dispersion integrals in (3.9a) and (3.9b) in the usual way. The constants s_n in the infinite products of (3.9b) are those values of s for which $\alpha(s)=-n-\frac{1}{2}$. The infinite product, together with the factor $1/\Gamma(as+b+\frac{3}{2})$, ensures that β has zeros at those values of s for which α is a negative half-integer less than $-\frac{1}{2}$.

We could formally have omitted the factor $1/\Gamma(as+b+\frac{3}{2})$ and rewritten infinite product as $\prod[(s_n-s)/s_n]$, but this product does not converge. The infinite product in (3.9b), on the other hand, converges if $\mathrm{Im}\alpha$ approaches zero at infinite s; if $\mathrm{Im}\alpha$ does not approach zero but $\mathrm{Im}\alpha/\alpha$ does, the ratio of the infinite products in (3.9b) taken at two values of s converges. Thus the divergence of the infinite product will be cancelled by a factor independent of s in the entire function $E(s)$, and the final result will converge.

Equations (3.9) provide a system of nonlinear integral equations for α and β, and they have been solved by iteration in the potential model.[7] In the narrow-resonance approximation, $\mathrm{Im}\alpha=0$, and Eqs. (3.9) reduce to the explicit form (3.5). One can easily modify (3.9) to the case where the external particles have spin or to the unequal-mass case. The modifications where more terms of (3.1) are included, or where (3.1) is replaced by a better representation such as the modified Cheng representation, are straightforward in principle, but they complicate the numerical work considerably. We refer the reader to the treatment of the potential model.[8,12]

Whether we are using the narrow-resonance approximation or not, the constants a and b, and the entire function E, remain to be determined. In potential theory, these quantities are obtained from a knowledge of the potential; in the present problem we shall have to use the crossing relations. Before we apply the crossing relations to our problem, however, we shall examine the possibility that the constant a is the same for all trajectories.

IV. SLOPE OF THE REGGE TRAJECTORIES

By considering models where the sequence of resonance on a Regge trajectory corresponds to the sequence of external particles on another Regge trajectory, one can obtain plausible results about the slope of Regge trajectories. The work of Carruthers and Nieto is an example of the type of model we have in mind. Carruthers and Nieto examined the ρN channel and obtained the $D_{3/2}$ resonance as a composite system in that channel. They then combined the ρN and $\rho F_{5/2}$ channels, the $F_{5/2}$ being the next member of the nucleon Regge series, and obtained a $G_{7/2}$ resonance in addition to the $D_{3/2}$. As the $G_{7/2}$ is the next member of the Regge series beginning with the $D_{3/2}$, we can now envisage a more complicated model in which the whole Regge sequence corresponding to the nucleon is included in the external particles, and the whole Regge trajectory corresponding to the $D_{3/2}$ resonance appears as a composite system.

Let us now examine a more general case in which an external particle of mass μ is combined with a series of external particles of spin σ and mass m_σ, where

$$am_\sigma^2+b=\sigma\,, \tag{4.1}$$

i.e.,

$$m_\sigma^2=(\sigma-b)/a\,. \tag{4.2}$$

This system is assumed to produce a Regge sequence of particles of spin σ and mass M_σ, where

$$M_\sigma^2=(\sigma-b')/a'\,. \tag{4.3}$$

We may define the binding energy as the quantity $M_\sigma-m_\sigma-\mu$, and we assume that this quantity remains infinite as σ approaches infinity. We can then easily show that

$$a=a'\,. \tag{4.4}$$

We obtain the same result from the assumption that the expression $M_{\sigma+n}-m_\sigma-\mu$ remains finite. In other words, we assume that a particle in the M sequence is built up from particles of the m sequence of not too different spin, and that the binding energy remains finite as the spin approaches infinity. We can actually make the weaker assumption that the binding energy increases less rapidly then the mass of the particles.

If we do not use the narrow-resonance approximation, so that the function α is given by (3.9a) rather than (3.4), we can still derive the same result, provided that Imα increases less than linearly with s.

The type of approximation which we are using in our paper is fundamentally different from that used in the Carruthers-Nieto model. Nevertheless, it may still be that the definition of the binding energy which we have just given, and the assumption which we have made about it, are physically reasonable. We would then have to restrict the slopes of our trajectories by (4.4). Such a restriction could well be necessary in the sense that, unless it were applied, our equations would not yield convergent results as we increased the number of resonances considered.

If the above assumptions are correct, the Regge trajectories will occur in groups. Each member of the group can be built from another member by means of a Carruthers-Nieto model, and the trajectories in the same group will have the same slope. The simplest system possessing these features is that in which there is only one group, and all trajectories have the same slope. Now, it is a remarkable empirical fact that all trajectories do have the same slope within the accuracy to which the slope can be defined. One may therefore adopt as a working hypothesis the assumption that the constants a are the same for all trajectories. This universal constant a will then fix a scale of mass and only the constants b and the functions E remain to be determined. The arguments for this hypothesis are obviously far from compelling and, since we do not yet know the extent to which our dynamical equations determine our parameters, we must bear in mind the possibility that it may have to be abandoned.

V. CROSSING RELATIONS

We shall apply crossing with the aid of the generalized superconvergence mentioned in Sec. II.[9] These relations are a consequence of Regge asymptotic behavior and the usual analyticity properties. We assume that a scattering amplitude $A(s,t)$ satisfies dispersion relations with only a right-hand cut and has the asymptotic behavior

$$A(s,t)\sim\sum_r\frac{\gamma_r(s)(-t)^{\alpha_r(s)}}{\sin\pi\alpha_r(s)},\quad t\to\infty\,. \tag{5.1}$$

The following relation is then asymptotically true as N becomes large:

$$\int^N dt\,\mathrm{Im}A(s,t)\sim\sum_r\frac{\gamma_r(s)N^{\alpha_r(s)+1}}{\alpha_r(s)+1}. \tag{5.2}$$

By considering the functions $t^nA(s,t)$, we derive the further equations

$$\int^N dt\,t^n\,\mathrm{Im}A(s,t)\sim\sum_r\frac{\gamma_r(s)N^{\alpha_r(s)+n+1}}{\alpha_r(s)+n+1}. \tag{5.3}$$

We can obtain amplitudes $A(s,t)$ with only a right-hand cut by taking the sum of the positive—and negative—signature amplitudes. The right-hand side of (5.1) will receive contributions from the fixed poles in the J plane at nonsense wrong-signature integers[13,14] as well as from the moving poles. One can evaluate the contribution from the fixed poles in terms of the third double-spectral function. At present, our dynamical scheme is far from the stage where contributions from the third double-spectral function should be included.

If the external particles have spin, one divides the amplitude $A(s,t)$ by the factor $(1+z)^{|\lambda+\mu|/2}(1-z)^{|\lambda-\mu|/2}$ before applying (5.1)–(5.3), since the resulting amplitude will be free of kinematic singularities or zeros at $z=\pm1$. Thus

$$\frac{A_{\mu\lambda}(s,t)}{(1+z)^{|\lambda+\mu|/2}(1-z)^{|\lambda-\mu|/2}}\sim\sum_r\frac{\gamma_r(s)(-t)^{\eta_r(s)}}{\sin\pi\eta_r(s)},\quad t\to\infty\,, \tag{5.4a}$$

where

$$\eta_r(s)=\alpha_r(s)-\max(|\lambda|,|\mu|) \tag{5.4b}$$

and

$$z=1+t/2q^2\,. \tag{5.4c}$$

Then

$$\int^N dt\frac{\mathrm{Im}A_{\mu\lambda}(s,t)}{(1+z)^{|\lambda+\mu|/2}(1-z)^{|\lambda-\mu|/2}}\sim\sum_r\frac{\gamma_r(s)N^{\eta_r(s)+1}}{\eta_r(s)+1}, \tag{5.5}$$

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[13] C. E. Jones and V. L. Teplitz, Phys. Rev. **159**, 1271 (1967).
[14] S. Mandelstam and L. L. Wang, Phys. Rev. **160**, 1490 (1967).

or, more generally,

$$\int^{N} dt\, t^{n} \frac{\mathrm{Im}A_{\mu\lambda}(s,t)}{(1+z)^{|\lambda+\mu|/2}(1+z)^{|\lambda-\mu|/2}} \sim \sum_{r} \frac{\gamma_r(s) N^{\eta_r(s)+n+1}}{\eta_r(s)+n+1}. \quad (5.6)$$

In practice, it is usually more convenient to work with the fixed-parity combinations of helicity states than with the helicity states themselves. The powers of t in the denominator will make the integrals in (5.5) and (5.6) less sensitive to the contributions from high t, where the integrand is less accurately known.

The right-hand side of (5.2) depends on the Regge trajectories in the s channel, the left-hand side on those in the t and u channels. Equation (5.2) therefore provides us with a relation between the parameters in the direct and crossed channels.

We now examine in more detail the form of the two sides of (5.5) in the narrow-resonance approximation. The constants $\gamma_r(s)$ will be proportional to the corresponding Regge residues $\beta_r(s)$; we find that

$$\gamma_r(s) = -\mathrm{sgn}(\mu-\lambda) 2^{\max(|\lambda|,|\mu|)+1}(\sqrt{\pi})\Gamma(\alpha(s)+\tfrac{3}{2}) \times\Gamma(\alpha(s)+1)[\Gamma(\alpha(s)+\lambda)\Gamma(\alpha(s)-\lambda) \times\Gamma(\alpha(s)+\mu)\Gamma(\alpha(s)-\mu)]^{-1/2}\beta_r(s)/(q^2)^{\eta_r(s)}. \quad (5.7)$$

The Regge residue $\beta_r(s)$ is expressed in terms of the function E and the constants a and b by using equations (3.5)–(3.8). We note that some of the Γ functions in (3.5) and (5.7) cancel. If we are working to sufficient accuracy, we shall have to include the contributions to the right-hand side of (5.4a) with asymptotic behavior t^{η_r-1}, t^{η_r-2}, etc., from the Regge trajectory α_r. The corresponding functions γ are given by formulas similar to (5.7). Actually we shall use the equations at $s=0$, so that these lower terms should only be included if we also include the daughter and conspirator trajectories.

Turning to the left-hand side of (5.4), we shall confine ourselves to the contributions from the t-channel Regge trajectories; the u-channel trajectories can be handled in a similar way. In the narrow-resonance approximation, a t-channel trajectory with $\alpha(t)=at+b$ will give the following contribution to $\mathrm{Im}B(s,t)$, where B is the amplitude obtained from A by crossing:

$$\mathrm{Im}B_{\mu\lambda}(s,t) = -\frac{1}{a}\sum_{J}(2J+1)\pi\delta\left(t-\frac{b}{a}-\frac{J}{a}\right) \times\beta_{\mu\lambda}(t)d_{\lambda\mu}{}^{J}\left(1+\frac{s}{2q_t^2}\right). \quad (5.8)$$

The sum is over integral or half-integral values of J, depending on the spin. After applying the crossing relation, we obtain the equation

$$\frac{\mathrm{Im}A_{\mu\lambda,\alpha}(s,t)}{(1+z)^{|\lambda+\mu|/2}(1-z)^{|\lambda-\mu|/2}} = \sum_{J}\sum_{\lambda'\mu'\alpha'} C(\lambda,\mu,\alpha,\lambda',\mu',\alpha',s,t) \times(2J+1)\pi\delta\left(t-\frac{b}{a}-\frac{J}{a}\right)\beta_{\mu',\lambda',\alpha'}(t)d_{\lambda'\mu'}{}^{J}\left(1+\frac{s}{2q_t^2}\right) \times\left(2+\frac{t}{2q^2}\right)^{-|\lambda+\mu|/2}\left(-\frac{t}{2q^2}\right)^{-|\lambda-\mu|/2}. \quad (5.9)$$

The index α refers to those quantities other than the helicity which characterize the amplitude, e.g., isotopic spin or $SU(3)$ multiplet. The crossing matrix C will be the product of the helicity crossing matrix and the crossing matrix appropriate to the internal symmetry.

The left-hand side of (5.6) can be handled in a very similar way. By substituting (5.7) and (5.9) in (5.5) or (5.6), and using the formulas for β in terms of the function E and the constants b, one can obtain equations for these quantities. As we have already pointed out, we do not know whether the number of equations is sufficient for a complete calculation. In fact, in the narrow-resonance approximation, these equations are linear in the functions E, and we shall at most be able to obtain the ratios of the coupling constants, not their absolute normalization. To determine the normalization factor, one will have to go beyond the narrow-resonance approximation.

VI. APPROXIMATION SCHEME FOR CALCULATING THE REGGE PARAMETERS

There is obviously no unique way of applying the analyticity and crossing formulas in dynamical calculations. The method which we shall propose in this section should be regarded as one possible suggestion. Further work will almost certainly reveal the need for substantial changes.

We assume that each channel has a leading trajectory, together with subsidiary trajectories the number of which will depend upon the stage in the approximation scheme. Since the Q value of a resonance of given J will increase as we go to lower trajectories, most of the resonances on the lower trajectories may be fairly broad and may not appear experimentally as resonances. Nevertheless, it may still be a reasonable approximation to represent the contribution from these resonances by poles near the real axis, i.e., to assume the narrow-resonance approximation. As we explained in Sec. IV, we shall begin by assuming that all the trajectories have the same slope.

The trajectories are now parametrized according to the formulas given in Sec. III, and the crossing relations (5.5) and (5.6) are applied. Several points remain to be specified, including the values of N and s to be taken in applying the crossing relations, the number of terms of the function E which are kept, and the number of relations (5.6) which are used.

To begin with the value of N, we should obviously make this integration limit as high as possible, since Eqs. (5.5) and (5.6) are only assymptotically true. On the other hand, we cannot take N above the value of t for which we know the function $\mathrm{Im}A(s,t)$ in the integrand. In the narrow-resonance approximation the function $\mathrm{Im}A(s,t)$ will consist of a series of δ functions in t at the positions of the resonances. The integrand in (5.5) and (5.6) will therefore be known up to the lowest resonance on the lowest trajectory which is kept; the first unknown contribution will be at the position of the lowest resonance on the highest trajectory which is omitted. The value of N to be taken should therefore be between these resonances. We may estimate the position of the first omitted trajectory on the assumption, based on the Schrödinger or Bethe-Salpeter equation at high energy, that the trajectories are spaced at integral distances in the J plane. If we have kept a number of trajectories, we may alternatively estimate the position of the first omitted trajectory from the spacing between the trajectories which have been kept.

The energy of the lowest resonance on the lowest trajectory included and that of the lowest resonance on the highest trajectory omitted thus represent lower and upper limits for N. If N is large, the result of the calculations will be insensitive to the precise value of N between the two limits. If N is not large, the results would not be expected to be quantitatively accurate, and their sensitivity to the value of N between the two limits represents a lower limit to their uncertainty. Since we have no *a priori* basis for assigning a precise value to N between the two limits, we shall take it to be midway between the two.

A related uncertainty lies in the form of the asymptotic expansion (5.1). We could equally well have used the series

$$\sum \frac{\gamma_i'(s)(t-t_0)^{\alpha_i(s)}}{\sin\pi\alpha_i(s)}. \tag{6.1}$$

The results will be insensitive to t_0 if a sufficiently large number of terms are taken in the asymptotic series, but they will depend on this parameter if only one or two terms are taken. We shall refer again to this uncertainty in the example of the following section.

With regard to the value of s at which Eqs. (5.5) and (5.6) are to be taken, we must bear in mind that the narrow-resonance approximation neglects all partial waves above a certain angular momentum at any particular value of the energy, so that the approximation would not be expected to be accurate at an unphysical value of the scattering angle where the partial-wave series is badly divergent. The nonresonant contributions from the high angular momenta would probably be large at such angles. We should therefore choose a value of s which avoids unphysical angles in the t and u channels as far as possible. For the equal-mass problem the obvious choice is $s=0$, though other sufficiently small values of s should also be adequate. For the unequal-mass problem, it is impossible to avoid unphysical angles completely; we can avoid one unphysical region only at the expense of going farther into another. The choice $s=0$ still probably provides the best mean. For very unequal masses, we may eventually have to subtract off certain contributions in order to reduce the divergence of the partial-wave expansion; these contributions would be calculated by using the double-dispersion relations. At the moment, however, we shall neglect such complications.

We can obtain further crossing relations by differentiating (5.5) and (5.6) with respect to s. The right-hand sides of these equations will then depend on the derivatives of the Regge residues β with respect to s; they in turn will depend on the higher terms of the entire function $E(s)$ of (5.5). In the lowest approximation, this function is taken to be a constant and the s derivatives of Eqs. (5.5) and (5.6) are not used. In higher approximations, we take the first n terms of the function E, expanded as a power series in s, and we use Eq. (5.6) and its first $(n-1)$ derivatives with respect to s.

Thus, in higher approximations, we would use more values of n in (5.6) corresponding to the larger number of trajectories taken into account; we would also use more derivatives of (5.5) and (5.6) corresponding to the larger number of terms kept in the functions $E(s)$. The question of just how many trajectories to take in each channel, and how to assign the choice of sense or nonsense at the low integers, is not one to which we can give a definite answer. For the characteristics of the leading trajectories we can be guided by experiment; for the lower trajectories we may have to use trial and error. We shall have to examine several trajectories simultaneously in order to obtain a complete set of equations, since the Regge residues β for different trajectories are related to one another. For example, the value of β for the $N\bar{N}$ trajectory at $s=m_\pi^2$ and that for the πN trajectory at $s=m_\pi^2$ are both related to the usual pion-nucleon coupling constant g^2.

We also remark that the number of equations (5.6) which we can use, given any choice of N and of the number of trajectories taken into account, is limited by the accuracy of these equations. The larger the value of n, the larger the difference between the two sides of (5.3), and hence the larger the error in (5.6). Estimates based on potential theory and on the asymptotic distribution of Regge poles calculated by Cheng and Wu[15] indicate that the number of equations which we can take is roughly proportional to N, apart from logarithmic factors. This limitation on the number of equations (5.6) means that we must exhaust the content of any given number of equations, applied to all relevant trajectories,

[15] H. Cheng and T. T. Wu, Phys. Rev. 144, 1232 (1966).

before adding another equation. In deciding the maximum value of n for each helicity state, we must remember that the factors $(1+z)^{|\lambda+\mu|/2}(1-z)^{|\lambda-\mu|/2}$ decrease the integrand at high values of t, in contrast to the factor t^n which increases it. Thus more values of n can be taken for the states of greater helicity; the maximum value of $n-\max(|\lambda|,|\mu|)$ should be the same for all helicity states.

VII. MESONS AS BOUND STATES OF BARYON-ANTIBARYON SYSTEMS

As a simple application of the foregoing scheme, we shall attempt to obtain the mesons (the pseudoscalar, vector, and axial-vector nonets) as bound states of the baryon-antibaryon system. In other words, we examine the baryon-antibaryon system and include the three trajectories of which the pseudoscalar, vector, and axial-vector nonets are the lowest members. The quantum numbers of the mesons do indicate that they should be regarded as bound states of the baryon-antibaryon system rather than of boson systems. The two S-wave bound states of the baryon-antibaryon system have the spin, parity, and charge-conjugation quantum numbers of the pseudoscalar and vector nonets. This is not the case if we attempt to obtain the mesons as bound states of meson systems. For instance, the ρ would be a P-wave bound state of the pion-pion system, and we encounter the well-known difficulty of the absence of a corresponding S-wave bound state.

In the simple approximation which we shall employ, we cannot expect to obtain quantitatively correct results. Our aim in performing the calculation is to show that we obtain a set of equations which yield consistent, reasonable values for the quantities of interest.

The kinematics of the nucleon-antinucleon system have been worked out by Goldberger, Grisaru, MacDowell, and Wong.[16] There are five independent helicity amplitudes. After dividing by factors of $(1-z)^{1/2}$ and $(1+z)^{1/2}$ to make them analytic at $z=\pm 1$, and taking fixed parity combinations, we may define the following amplitudes:

$$f_1=\langle ++|\phi|++\rangle-\langle ++|\phi|--\rangle, \quad (7.1a)$$

$$f_2=\langle ++|\phi|++\rangle+\langle ++|\phi|--\rangle, \quad (7.1b)$$

$$f_3=(1+z)^{-1}\langle +-|\phi|+-\rangle - (1-z)^{-1}\langle +-|\phi|-+\rangle, \quad (7.1c)$$

$$f_4=(1+z)^{-1}\langle +-|\phi|+-\rangle + (1-z)^{-1}\langle +-|\phi|-+\rangle, \quad (7.1d)$$

$$f_5=(4m/\sqrt{s})(1-z^2)^{-1/2}\langle ++|\phi|+-\rangle. \quad (7.1e)$$

We have denoted particles of positive and negative helicity by the symbols + and −. The normalization used will be such that the unitarity condition involves the integral

$$\int \frac{d\Omega}{4\pi}\left(\frac{4q^2}{s}\right)^{1/2}.$$

We may also define fixed partial-wave amplitudes as follows.

Singlet:

$$f_0{}^J=\langle ++|\phi^J|++\rangle-\langle ++|\phi^J|--\rangle, \quad (7.2a)$$

Triplet, $J=l$:

$$f_1{}^J=\langle +-|\phi^J|+-\rangle-\langle +-|\phi^J|-+\rangle, \quad (7.2b)$$

Triplet, $J=l\pm 1$:

$$f_{11}{}^J=\langle ++|\phi^J|++\rangle+\langle ++|\phi^J|--\rangle, \quad (7.2c)$$

$$f_{12}{}^J=2\langle ++|\phi^J|+-\rangle, \quad (7.2d)$$

$$f_{22}{}^J=\langle +-|\phi^J|+-\rangle+\langle +-|\phi^J|-+\rangle. \quad (7.2e)$$

The pseudoscalar trajectory corresponds to the singlet state, the axial-vector trajectory to triplet state with $J=l$, and the vector trajectory to the triplet state with $J=l\pm 1$. In the complete amplitudes (7.1) the pseudoscalar trajectory will only affect f_1; the vector and axial-vector trajectories will affect the other four amplitudes. At $J=1$, the value corresponding to the lowest member, the vector trajectory will affect only f_2, f_4, and f_5, and the axial-vector trajectory will affect only f_3. As $z\to\infty$, the vector trajectory will again affect only f_2, f_4, and f_5, the axial-vector trajectory only f_3.

In the lowest approximation we shall keep one trajectory in each of the three channels (pseudoscalar, vector, axial-vector), and shall use Eq. (5.5) but not (5.6). Furthermore, we remarked in the previous section that the generalized superconvergence relations for amplitudes with nonzero helicity should be used before those for amplitudes with zero helicity, since the former relations contain a power of z in the denominator. Thus, the superconvergence relations for f_3, f_4, and f_5, but not those for f_1 and f_2, will be used in our approximation.

We have already pointed out that we cannot obtain equations for all relevant parameters by looking at one channel only. We shall therefore take the following quantities from experiment:

(i) The ratio of the mass of the baryon to the constant $1/a$ (a being the slope of the trajectory) which fixes the dimension of mass. Since we are not examining channels with baryon number equal to 1, we would not expect to obtain the baryon mass in this calculation. We shall take the mass of the baryon to be equal to $1/a$.

(ii) The spacings between the three meson trajectories. We shall take the vector trajectory to be one unit above the pseudoscalar trajectory and half a unit above the axial trajectory. We thus do not attempt to calculate ratios of the masses of the pseudoscalar, vector, and axial-vector mesons. In particular, we

[16] M. L. Goldberger, M. T. Grisaru, S. W. MacDowell, and D. Y. Wong, Phys. Rev. 120, 2250 (1960).

assume the mass of the pseudoscalar and vector mesons to be equal, as is required by $SU(6)$.

(iii) We shall also make an assumption regarding the ratio of the Regge residues for the three amplitudes f_{11}^J, f_{12}^J, and f_{22}^J which depend on the vector trajectory. Owing to the factorization theorem, there are only two independent amplitudes in the approximation where one trajectory is kept. The ratio in question is that between the magnetic and electric coupling of the vector meson. We shall discuss the estimation of this ratio below. The electric coupling is, in fact, considerably smaller than the magnetic coupling, and our final results would not be very different if we neglected the electric coupling completely.

In our lowest approximation we shall take the functions E in (3.5) to be constants, so that there is only one known parameter associated with each Regge residue β. Three quantities remain to be determined, the position of one of the trajectories (the constant b) and the two ratios between the Regge residues for the three trajectories. Since we have three generalized superconvergence relations, we should be able to determine these quantities. We remark again that the absolute value of any of the Regge residues cannot be determined in the narrow-resonance approximation.

We next treat the internal symmetry of our problem. We shall assume exact $SU(3)$, and shall further assume that each of our mesons consists of a degenerate octet and singlet. The baryons will be assumed to be octets. The crossing matrix in (5.9) will contain the $SU(3)$ crossing matrix as a factor. Thus, if (5.9) and (5.7) are substituted in (5.4), the equations will have the form

$$x=CMx\,,$$

where x is a vector in helicity and $SU(3)$ space, C the $SU(3)$ crossing matrix, and M a helicity matrix. Each solution of this equation will be proportional to an eigenfunction of the $SU(3)$ crossing matrix. We emphasize that this is only true in the approximation where we do not allow more than one trajectory for each value of the parity and charge conjugation.

The $SU(3)$ crossing matrix for an octet-octet channel has been given by de Swart.[17] There are two eigenfunctions which involve only singlets (S) and octets (DD,DF,FD,FF):

(i) $DD=5,\ DF=0,\ FD=0,\ FF=9,$
$$S=16,\ \text{eigenvalue } 1; \quad (7.3\text{a})$$

(ii) $DD=0,\ DF=1,\ FD=1,\ FF=0,$
$$S=0,\ \text{eigenvalue } -1. \quad (7.3\text{b})$$

Unfortunately, neither of these eigensolutions is factorizable between the D and F states, as it should be in our approximation where we have only one trajectory.

[17] J. J. de Swart, Nuovo Cimento **31**, 420 (1964).

We can obtain a factorizable eigenfunction which resembles (7.3a) by writing

$$DD=5,\quad DF=FD=3\sqrt{5},\quad FF=9,\quad S=16. \quad (7.4)$$

This is not far from the result which we would obtain for $N\bar{N}$ scattering through the vector meson by magnetic coupling according to the estimates of Sugawara and von Hippel[18]:

$$DD=5,\quad DF=FD=2\sqrt{5},\quad FF=4,\quad S=16. \quad (7.5)$$

We can now resolve (7.4) into components along the directions of the two eigenfunctions (7.3a) and (7.3b) If we weight each amplitude according to the multiplicity of the states involved, the eigenfunctions of the crossing matrix will be orthogonal, so that we can take components in the usual way. We shall demand that the component in the direction (7.3a) satisfy the equations of Sec. V. The component in the direction (7.3b) will then not satisfy our equations. We prefer to demand that our equations be true for the component along the direction (7.3a) than for that along the direction (7.3b), since not every resonance gives a positive-definite contribution to a given partial wave when resolved in the latter direction. We are thus more likely to obtain cancelling contributions from higher resonances when resolving in the direction (7.3b). The D/F ratio for these higher resonances would have to be of the opposite sign to that of the original resonance. Another way of viewing the problem is to regard the single resonance which we have taken as representing the contribution of two or more resonances with D/F ratios of both signs. The resultant resonance could then have the decomposition (7.3a) in $SU(3)$ space.

We shall also use the paper of Sugawara and von Hippel to estimate the ratio of the electric to the magnetic coupling of the vector meson. The magnetic-magnetic transition has been given in (7.5); the electric-electric and electric-magnetic transitions are pure FF and are equal to 1 and 2 on the same scale. If we denote the components in the direction (7.3a) in $SU(3)$ space by $(g_M)^2$, $(g_E)^2$, and $(g_{EM})^2$, we easily find that

$$g_E^2=(3/31)g_M^2,\quad (g_{EM})^2=(6/31)g_M^2. \quad (7.6)$$

Throughout our work we shall assume that the only important crossed channel to the nucleon-antinucleon channel is the other nucleon-antinucleon channel. The effect of the nucleon-nucleon channel will be neglected owing to the absence of any low-mass resonances. The existence of only one important crossed channel implies that signature may be neglected in the nucleon-antinucleon channel. In other words, we have exchange degeneracy, and the trajectories of odd and even signature coincide. It has been noted empirically that exchange degeneracy is approximately valid for meson trajectories. For instance, the ρ and A_2 trajectories

[18] H. Sugawara and F. von Hippel, Phys. Rev. **145**, 1331 (1966).

appear to coincide within the limits of experimental error.

We can now apply the formulas of Secs. IV and V to set up our equations. We shall work in terms of the coupling constants $g_P{}^2$ for the pseudoscalar meson, $g_A{}^2$ for the axial-vector meson, and $g_M{}^2$, $g_E{}^2$, and $(g_{EM})^2$ for the vector meson. The residues of the functions f have the following values in terms of the coupling constants:

$$\begin{aligned}\mathrm{Im}f_1(s,t)&=\tfrac{1}{2}(\mu_P{}^2/4)g_P{}^2\delta(s-\mu_P{}^2)\,,\\ \mathrm{Im}f_2(s,t)&=\tfrac{1}{2}M^2g_E{}^2(1+t/2q^2)\delta(s-\mu_V{}^2)\,,\\ \mathrm{Im}f_3(s,t)&=-\tfrac{1}{2}(M^2-\tfrac{1}{4}\mu_A{}^2)g_A{}^2\delta(s-\mu_A{}^2)\,,\\ \mathrm{Im}f_4(s,t)&=\tfrac{1}{2}(\mu_V{}^2/4)g_M{}^2\delta(s-\mu_V{}^2)\,,\\ \mathrm{Im}f_5(s,t)&=-\tfrac{1}{2}M^2(g_{EM})^2\delta(s-\mu_V{}^2)\,.\end{aligned}\tag{7.7}$$

From Eqs. (5.8) and (7.7), we can express the Regge residues $\beta(\mu^2)$, where μ is the mass of the appropriate particle, in terms of the g's. Thus

$$\begin{aligned}\beta_0(\mu_P{}^2)&=-(1/8\pi)\mu_P{}^2g_P{}^2\,,\\ 3\beta_{11}(\mu_V{}^2)&=-(1/2\pi)M^2g_E{}^2\,,\\ 3\beta_1(\mu_A{}^2)&=(1/\pi)(M^2-\tfrac{1}{4}\mu_A{}^2)g_A{}^2\,,\\ 3\beta_{22}(\mu_V{}^2)&=-(1/4\pi)\mu_V{}^2g_M{}^2\,,\\ 3\beta_{12}(\mu_V{}^2)&=-(1/\sqrt{2}\pi)M^2(g_{EM})^2\,.\end{aligned}$$

We have defined our unit of mass so that $a=1$. The subscripts on the β's indicate that they are the Regge resonances of the functions $f_0{}^J$, $f_{11}{}^J$, $f_1{}^J$, $f_{22}{}^J$, and $f_{12}{}^J$.

The β's at other values of s can now be found from (3.5)–(3.8). For the pseudoscalar trajectory, corresponding to the function $f_0{}^J$, both helicities are zero and we can use (3.5) directly (with the entire function E set equal to a constant):

$$\beta_0(s)=\frac{\mathrm{const}(4q^2/e)^{s+b_P}}{\Gamma(s+b_P+\frac{3}{2})}\,.\tag{7.8}$$

The constant can be expressed in terms of $g_P{}^2$ by normalizing at $s=-b_P=\mu_P{}^2$ and using (7.8). Thus

$$\beta_0(s)=-\frac{(1/16\sqrt{\pi})g_P{}^2\mu_P{}^2(4q^2/e)^{s+b_P}}{\Gamma(s+b_P+\frac{3}{2})}\,.\tag{7.9a}$$

The formula for the vector trajectory is somewhat more complicated, as three helicity states are involved and the appropriate linear combinations have threshold dependences of $(4q^2)^{s+b_V-1}$, $(4q^2)^{s+b_V}$, and $(4q^2)^{s+b_V+1}$. As the nucleon-nucleon threshold is some distance above the energy region of the mesons, we shall not attempt to obtain the correct threshold behavior but shall assume a uniform threshold dependence of $(4q^2)^{s+b_V-1}$. We also require factors of the form (3.7) and (3.8), since nonsense states are present at $J=0$. We shall assume that the trajectory chooses nonsense at $J=0$, since the contrary assumption would introduce the well-known ghost problem.[19] The helicity matrix element in which we shall be interested is that with $\lambda=1$, $\mu=0$, corresponding to the function $f_{12}{}^J$. According to (3.7) and (3.8), this matrix element will contain a factor $\{(s+b_V)(s+b_V+1)\}^{1/2}$. Hence

$$\beta_{12}(s)=\frac{\mathrm{const}(4q^2/e)^{s+b_V-1}[(s+b_V)(s+b_V+1)]^{1/2}}{\Gamma(s+b_V+\frac{3}{2})}\,.$$

Normalizing at $s=-b_V+1=\mu_V{}^2$ and using (7.7), we find that

$$\beta_{12}(s)=-\frac{(1/8\sqrt{\pi})(g_{EM})^2M^2(4q^2/e)^{s+b_V-1}[(s+b_V)(s+\mathrm{b}_V+1)]^{1/2}}{\Gamma(s+b_V+\frac{3}{2})}\,.\tag{7.9b}$$

For the axial-vector trajectory, $J=l$, and the threshold factor is $(4q^2)^{s+b_A}$. We assume that the trajectory chooses nonsense at $J=0$, as no appropriate particle has been seen. The factors from (3.7) and (3.8) in β_1 are then $(s+b_V+1)$. Thus, applying (3.5) and normalizing according to (7.8), we find that

$$\beta_1(s)=\frac{(1/8\sqrt{\pi})\frac{1}{4}eg_A{}^2(4q^2/e)^{s+b_A}(s+b_A+1)}{\Gamma(s+b_A+\frac{3}{2})}\,.\tag{7.9c}$$

We must next combine our formulas (7.9) with (5.4) and (5.7) to find the asymptotic behavior of our amplitudes at $s=0$ and large t. Looking first at f_1, we find from (7.9a), (5.4), and (5.7) that

$$f_1(0,t)\sim\frac{\frac{1}{8}g_P{}^2\mu_P{}^2(4/e)^{b_P}(-t)^{b_P}}{\Gamma(b_P+1)\,\sin\pi b_P}\,.\tag{7.10}$$

We are not directly interested in the asymptotic behavior of f_1, as we are only using the superconvergence relations for f_3, f_4, and f_5. However, at $s=0$, there will be a conspiracy associated with the condition

$$f_1-f_3-zf_4=0\,.\tag{7.11}$$

We shall assume that the conspiracy is of the type III of Freedman and Wang,[20] since any other type would require the existence of hitherto undiscovered particles. The function f_3 will then not contribute to the leading term of (7.11) as $z\to\infty$, and

$$f_4(0,t)\sim-(2M^2/t)f_1(0,t)\,,$$

[19] As we are assuming exchange degeneracy, the ρ and A_2 trajectories are the same.

[20] D. Z. Freedman and J. M. Wang, Phys. Rev. **160**, 1560 (1967).

or, from (7.10),

$$f_4(0,t)\sim -\frac{\frac{1}{4}M^2 g_P{}^2\mu_P{}^2(4/e)^{b_P}(-t)^{b_P-1}}{\Gamma(b_P+1)\,\sin\pi(b_P-1)}. \tag{7.12a}$$

Note that the vector trajectory does not contribute to the asymptotic behavior of $f_4(0,t)$, as it contains kinematic factors which vanish in that helicity state at $s=0$. The asymptotic behavior of $f_5(0,t)$ can be found directly from (7.9b), (5.4), and (5.7):

$$f_5(0,t)\sim -\frac{\frac{1}{2}(g_{EM})^2 M^2(4/e)^{b_V-1}(-t)^{b_V-1}}{\Gamma(b_V)\,\sin\pi(b_V-1)}. \tag{7.12b}$$

Similarly, the asymptotic behavior of $f_3(0,t)$ can be found from (7.9c), (5.4), and (5.7):

$$f_3(0,t)\sim -\frac{\frac{1}{2}g_A{}^2 M^2(4/e)^{b_A-1}(-t)^{b_A-1}}{\Gamma(b_A)\,\sin\pi(b_A-1)}. \tag{7.12c}$$

Having obtained the asymptotic behavior (7.12), we can easily write down the generalized superconvergence relations for our functions. They are as follows:

$$\int^N dt\,\mathrm{Im}f_4(0,t)\sim -\frac{\frac{1}{4}M^2 g_P{}^2\mu_P{}^2(4/e)^{b_P}N^{b_P}}{b_P\Gamma(b_P+1)}, \tag{7.13a}$$

$$\int^N dt\,\mathrm{Im}f_5(0,t)\sim -\frac{\frac{1}{2}(g_{EM})^2 M^2(4/e)^{b_V-1}N^{b_V}}{\Gamma(b_V+1)}, \tag{7.13b}$$

$$\int^N dt\,\mathrm{Im}f_3(0,t)\sim -\frac{\frac{1}{2}g_A{}^2 M^2(4/e)^{b_A-1}N^{b_A}}{\Gamma(b_A+1)}. \tag{7.13c}$$

In applying the criteria given in the last section to find a suitable value for N, we observe that our assumptions regarding the Regge trajectories imply that $\mu_P{}^2=\mu_V{}^2$, while $\mu_A{}^2=\mu_V{}^2+\frac{1}{2}$ (in units for which $a=1$). If the trajectories in the same channel are spaced at unit distance apart, the next pseudoscalar and vector particles will have the squares of their masses equal to $\mu_P{}^2+1$. Hence, if we take the limit of integration to be midway between the position of the first two resonances in the pseudoscalar and vector channels, we obtain $N=\mu_P{}^2+\frac{1}{2}$.

We observe that the limit of integration N is at the position of the lowest resonance in the axial-vector channel. Now, according to our criterion for choosing N, a value of N midway between the nth and the $(n+1)$th resonance corresponds to taking the first n resonances on the left-hand side of the superconvergence relations (5.5). Hence a value of N at the position of the nth resonance should correspond to taking the first $n-1$ resonances together with half the contribution of the nth resonance. In evaluating the left-hand side of (7.13), we shall therefore take only half the contribution of the axial-vector meson. Since the masses of the resonances in this channel are greater than those in the pseudoscalar and vector channels, this would appear to be more reasonable than to take the full contribution of the lowest resonance in all three channels.

The left-hand side of (7.13) can now be evaluated by using (7.7) at $t=0$, together with the helicity crossing matrix. This crossing matrix has been evaluated in Ref. 16[21]; we could also use the general formulas of Trueman and Wick and of Cohen-Tannoudji, Morel, and Navelet.[22] For the particular case $s=0$ and for the elements f_3, f_4, and f_5, the crossing matrix is as follows:

$$\begin{pmatrix} f_4(0,t)\\ f_5(0,t)\\ f_3(0,t)\end{pmatrix}=\begin{pmatrix} -\dfrac{M^2}{t} & -\dfrac{M^2}{4M^2-t} & \dfrac{4M^4}{t(4M^2-t)} & \dfrac{4M^4}{t(4M^2-t)} & 0\\ 0 & \dfrac{2M^2}{4M^2-t} & \dfrac{2M^2}{4M^2-t} & \dfrac{2M^2}{4M^2-t} & \dfrac{4M^2}{4M^2-t}\\ \dfrac{M^2}{t} & -\dfrac{M^2}{4M^2-t} & -\dfrac{2M^2(2M^2-t)}{t(4M^2-t)} & -\dfrac{2M^2(2M^2-t)}{t(4M^2-t)} & 0\end{pmatrix}\begin{pmatrix} f_1(t,0)\\ f_2(t,0)\\ f_3(t,0)\\ f_4(t,0)\\ f_5(t,0)\end{pmatrix}. \tag{7.14}$$

If we now insert the expressions (7.7) on the right of (7.14), with a factor $\frac{1}{2}$ for the axial-vector function f_3, and integrate, we find that

$$\begin{pmatrix} \displaystyle\int^N dt\, f_4(0,t)\\ \displaystyle\int^N dt\, f_5(0,t)\\ \displaystyle\int^N dt\, f_3(0,t)\end{pmatrix}=\tfrac{1}{2}M^2\begin{pmatrix} -\dfrac{1}{4} & \dfrac{M^2}{(4M^2-\mu_V{}^2)} & -\dfrac{M^2}{4M^2-\mu_V{}^2} & 0 & -\dfrac{M^2}{2\mu_A{}^2}\\ 0 & \dfrac{\mu_V{}^2}{2(4M^2-\mu_V{}^2)} & \dfrac{2M^2}{4M^2-\mu_V{}^2} & \dfrac{-4M^2}{4M^2-\mu_V{}^2} & -\dfrac{1}{4}\\ \dfrac{1}{4} & -\dfrac{(2M^2-\mu_V{}^2)}{2(4M^2-\mu_V{}^2)} & -\dfrac{M^2}{4M^2-\mu_V{}^2} & 0 & \dfrac{2M^2-\mu_A{}^2}{4\mu_A{}^2}\end{pmatrix}\begin{pmatrix} g_P{}^2\\ g_M{}^2\\ g_E{}^2\\ (g_{EM})^2\\ g_A{}^2\end{pmatrix} \tag{7.15}$$

[21] We are interested in the crossing matrix between the s and the t channels, whereas Goldberger, Grisaru, MacDowell, and Wong write down the crossing matrix between the s and the u channels. The elements of our crossing matrix between f_1, f_2, and f_3 on the one hand, and f_4 and f_5 on the other, will therefore have the opposite sign to that of GGMW. We thank Dr. D. Wong for pointing out some misprints in their crossing matrix.

[22] T. L. Trueman and G. C. Wick, Ann. Phys. (N. Y.) **26**, 322 (1964); G. Cohen-Tannoudji, A. Morel, and H. Navelet (to be published). The former paper contains some phase ambiguities.

$$
=\tfrac{1}{2}M^2\begin{bmatrix} -\dfrac{1}{4} & \dfrac{M^2}{4M^2-\mu_V^2}\left(1-\dfrac{3}{31}\right) & -\dfrac{M^2}{2\mu_A^2} \\ 0 & \dfrac{M^2}{4M^2-\mu_V^2}\left(\dfrac{\mu_V^2}{2M^2}-\dfrac{18}{31}\right) & -\dfrac{1}{4} \\ \dfrac{1}{4} & \dfrac{M^2}{4M^2-\mu_V^2}\left(-\dfrac{2M^2-\mu_V^2}{2M^2}-\dfrac{3}{31}\right) & \dfrac{2M^2-\mu_A^2}{4\mu_A^2} \end{bmatrix}\begin{bmatrix} g_P^2 \\ g_M^2 \\ g_A^2 \end{bmatrix} \tag{7.16}
$$

from (7.6). Equations (7.13) and (7.16) provide us with an eigenvalue equation. From our assumptions about the Regge trajectories, we can put

$$\mu_P^2=\mu_V^2=-b_P=-b_V+1\,, \tag{7.17a}$$

$$\mu_A^2=-b_A+1=\mu^2+\tfrac{1}{2}\,. \tag{7.17b}$$

Also, since we have assumed the nucleon to have unit mass,

$$M^2=1\,, \tag{7.17c}$$

and, finally,

$$N=\mu_V^2+\tfrac{1}{2}\,. \tag{7.17d}$$

We thus have an eigenvalue equation for μ_V^2 and, having solved it, we can find the ratio of the coupling constants. There exists a solution with

$$\mu_V^2=0.29,\quad g_P^2=0.21g_V^2,\quad g_A^2=0.18g_V^2. \tag{7.18}$$

VIII. CONCLUDING REMARKS

The solution to the equations of the model of the last section does possess the correct qualitative properties. We were able to solve the eigenvalue equation with a value for the mass of the pseudoscalar and vector mesons between 0 and 1, and with the ratios of the coupling constants both positive. The coupling constant g_P^2 appears to be somewhat small, but this is not surprising in view of the very simplified model taken. The error may possibly be due to our assumption that the entire function $E(s)$, which occurs as a factor in β, is constant. Experimentally, the data for backward charge-exchange NN scattering indicate that the residue for the pion trajectory or its conspirator passes through zero near $s=0$. If this is so, the effect of the trajectory at $s=0$, for a given value of g_P^2, would be reduced.

The qualitative nature of our result is not very sensitive to the precise value chosen for N [Eq. (7.17d)] or to the choice $t_0=0$ in (6.1). We have seen that these choices represented uncertainties in the lower stages of our approximation scheme.

One important qualitative feature of any dynamical problem is the sign of the force in a particular channel due to the exchange of a particular particle. In this respect the present scheme is similar to more conventional schemes as may be seen by using Eq. (5.9). If the dynamical equations are to possess a solution, the sign of the crossing matrix must be such that the left-hand side is positive. When working with the conventional dynamical schemes, one has to calculate the left-hand discontinuity from an equation similar to (5.9), and, again, the result must be positive in a channel where a resonance is present. When applied to the particular case of the crossing matrix (7.16), the sign requirement is that the elements in the first row must be positive, those in the other two rows must be negative. By examining the matrix we can find the signs of the forces in the three channels due to the exchange of the three mesons, and a sufficiently large number of them are attractive to give a consistent solution. Needless to say, one should not press the analogy between the present scheme and the conventional schemes too far; the quantitative features are completely different.

The calculations performed in the previous section are considerably simpler than conventional calculations of masses and coupling constants. We had to solve a three-by-three eigenvalue problem, whereas even the simplest conventional calculations require the solution of an integral equation. The results which we obtained are much less detailed than results of conventional calculations. We only obtained the position and strength of resonances, whereas conventional calculations give the complete scattering amplitude as a function of the energy. However, this extra detail is probably not worth while as long as we do not consider exchange of higher resonances. In the present scheme it may well be feasible to include the exchange of a fairly large number of resonances.

Even in the lowest approximation our calculations were not a complete solution of the bootstrap equations, as we only treated one channel, and we therefore had to take certain quantities from experiment. The next step would be to consider NP scattering, NV scattering, and NA scattering, where P, V, and A represent the pseudoscalar, vector, and axial-vector mesons. The parameters occurring in the lowest approximation would be the same parameters which occurred in our problem, and we would obtain further equations which could be used to determine some, and possibly all, of the parameters which we had to take from experiment. The crossing matrix would not involve the octet and singlet mesons in the same way, and we would no longer have to assume the existence of degenerate nonets. When considering the meson-baryon channels, we would have to include the Δ trajectory and probably the $D_{3/2}$ trajectory in addition to the baryon trajectory. We would in turn have to consider the meson-Δ and meson-$D_{3/2}$ channels, and also channels such as the $\Delta\bar{N}$ and $\Delta\bar{\Delta}$ channels, as these channels would involve the same parameters as the meson-nucleon channels. We might then have a complete set of equations which could be solved without introducing any experimental parameters.

In higher approximations, one would include more than one trajectory with a given set of quantum numbers. One would then have further parameters, but one would also have more equations from which they might be determined. These extra equations would arise, firstly by taking values of n other than zero in (5.6) and secondly by considering channels consisting of the resonances on the new trajectories in combination with the original resonances or with one another. One would also take the higher-spin resonances on the original trajectory as external particles in further possible channels. As long as one had external particles with spin, one would have to decide whether the trajectories chose sense or nonsense at the lower integers or half-integers. One might have to include trajectories with all possibilities. Another question to be decided would be what parity, charge conjugation, and $SU(3)$ states to induce at each stage of our calculation. We need hardly add that the problem of symmetry breaking occurs here as in all other approaches.

We shall not attempt to answer such questions in this paper. One may be able to obtain some guidance from experiment, but it is unlikely that such guidance will be unambiguous, as the resonances associated with the lower trajectories will probably be broad and therefore hard to detect. The quark model may possibly provide further guidance, even if real quarks do not exist, but at the moment, it is not evident whether or how the quark model emerges from our scheme.

In this paper we shall also not attempt to include partially conserved axial-vector current or current commutators within our scheme. It may be that the current-commutation rules can be deduced from the other dynamical equations when the scattering amplitudes are continued off the mass shell; alternatively, it may be necessary to postulate them in order to obtain sufficient equations to complete the scheme. If the latter alternative is true, we would not be able to obtain a complete and rigorous dynamical scheme without going off the mass shell, but we could incorporate the commutation relations in a preliminary approach by making use of the low-energy theorems associated with them.

In view of all the unanswered questions, we must regard the present work as suggesting a method of approach rather than as a complete scheme. It does appear to open the possibility of a treatment free from some of the drawbacks of more conventional methods.

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