These results lead us to the following conclusions concerning the lowest order radiative corrections to muon-decay:

(1) There is no infinite renormalization of the direct term (proportional to D). There is no renormalization of C_1.

(2) The coupling constant C_2 suffers an infinite renormalization.

When we compare the model proposed by Fronsdal and the conventional approach, we see that the situation in Fronsdal's model as far as μ decay is concerned is worse than in the conventional treatment, because of the occurrence of an infinite coupling-constant renormalization.

III. β DECAY OF NEUTRON

We consider the radiative corrections in lowest order to the process

$$n \rightarrow p + e^- + \bar{\nu}_e .$$

We proceed in exactly the same way as in the case of the μ decay. We state the results without going into the details of the calculation.

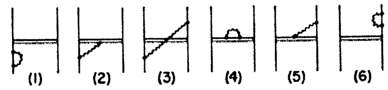

FIG. 5. Diagrams proportional to C_2^2.

When we have only to deal with the direct four-fermion interaction, the lowest order radiative correction gives a divergent renormalization of the coupling constant. Also, after introduction of the charged scalar intermediate boson, we get in addition to an infinite renormalization of $\mathcal{G}$ an infinite renormalization of C_2, contrary to the statement made by Fronsdal. So in this case Fronsdal's model does not give an improvement of the conventional approach either. This result contradicts Fronsdal's statement that the coupling constant renormalization is finite in this case.

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General Method of Constructing Helicity Amplitudes Free from Kinematic Singularities and Zeros

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A simple and straightforward method to identify and remove the kinematic singularities of helicity amplitudes is constructed from the Trueman-Wick crossing relations. A set of amplitudes free of all kinematic singularities and zeros is obtained for two-particle → two-particle reactions of any spins and masses, except that for boson-fermion interactions with general mass assignments there is still a kinematic $s^{1/2}$ singularity left in the amplitude.

I. INTRODUCTION

In dynamical calculations of scattering amplitudes, it is necessary to use kinematic-singularity–free amplitudes, which have only singularities of dynamical origin and satisfy the Mandelstam representation. D. N. Williams[1] has succeeded in constructing a complete set of invariant scalar amplitudes free of kinematic singularities and suitable for dynamical calculations. However, his amplitudes are not suitable for Reggeization. To Reggeize, first we have to remove all the kinematic singularities from the so-called parity-conserving helicity amplitudes and then analytically continue their partial-wave helicity amplitudes with definite parity in the total angular momentum plane.[2] Therefore kinematic-singularity–free helicity amplitudes are not only suitable for dynamical calculation but also suitable for Reggeization. This is our motivation for investigating the kinematic singularities of helicity amplitudes.

Recently, Y. Hara has proposed a method to remove the kinematic singularities of helicity amplitudes by using perturbation field theory, with emphasis on threshold behavior of partial-wave amplitudes and crossing relations.[3] In this paper, we develop a more

[1] D. N. Williams, Construction of Invariant Scalar Amplitudes Without Kinematical Singularities for Arbitrary-Spin Nonzero-Mass Two-Body Scattering Process, Lawrence Radiation Laboratory Report UCRL-1113 (unpublished).

[2] M. Gell-Mann, M. Goldberger, F. Low, E. Marx, and F. Zachariasen, Phys. Rev. 133, B145 (1964).

[3] Y. Hara, Phys. Rev. 136, B507 (1964). For comparison of his results and ours, see Ref. 9.

straightforward method using only the Trueman-Wick crossing relations for helicity amplitudes. Perturbation field theory is not needed. A complete set of amplitudes, which can be shown to be free of all kinematic singularities and zeros, is constructed for interactions of two particles of any spins and masses, except that for boson-fermion interactions of general mass assignments there is still a kinematic $s^{1/2}$ singularity left in the amplitude. Our results are consistent with the usually assumed threshold behavior of partial-wave helicity amplitudes with definite parity.

II. KINEMATIC SINGULARITIES IN THE CROSSED-CHANNEL ENERGY VARIABLE

The partial-wave expansion of a general helicity amplitude for a reaction $a+b\to c+d$, with s being the energy squared and t being the square of the energy in one of the two crossed channels (say, $D+b\to c+A$) is[4]

$$f_{cd;ab}{}^s(s,t)=\sum_J(2J+1)F_{cd;ab}{}^J(s)d_{\lambda\mu}{}^J(\theta_s)\,, \quad \text{(II.1)}$$

where

$$\lambda\equiv a-b \quad \mu\equiv c-d;$$

θ_s is the scattering angle in the s channel, which is taken to be the angle between particles a and c; and $d_{\lambda\mu}{}^J$ is the d function of the rotation matrix element. In the s-channel c.m. system,

$$\cos\theta_s=[2st+s^2-s\sum_i m_i^2+(m_a^2-m_b^2)(m_c^2-m_d^2)]/\mathcal{S}_{ab}\mathcal{S}_{cd}\,, \quad \text{(II.2)}$$

where

$$\begin{aligned}\mathcal{S}_{ab}{}^2&\equiv[s-(m_a-m_b)^2][s-(m_a+m_b)^2]=4sP_{ab}{}^2\,,\\ \mathcal{S}_{cd}{}^2&\equiv[s-(m_c-m_d)^2][s-(m_c+m_d)^2]=4sP_{cd}{}^2\,,\end{aligned} \quad \text{(II.3)}$$

where P_{ab}, P_{cd} are the initial and final momenta in the s-channel c.m. system. We see that $\cos\theta_s$ is an analytic function of t. In general, the d function is related to the Jacobi polynomial by[5]

$$d_{\lambda\mu}{}^J(\theta_s)=\pm\left[\frac{(J+M)!(J-M)!}{(J+N)!(J-N)!}\right]^{1/2}[\cos(\theta_s/2)]^{|\lambda+\mu|}\times[\sin(\theta_s/2)]^{|\lambda-\mu|}P_{(J-M)}{}^{(|\lambda-\mu|,|\lambda+\mu|)}(\cos\theta_s)\,, \quad \text{(II.4)}$$

where

$$M\equiv\text{maximum of }(|\lambda|,|\mu|)\,,$$
$$N\equiv\text{minimum of }(|\lambda|,|\mu|)\,,$$

so that Eq. (II.1) becomes

$$f_{cd;ab}{}^s(s,t)=[\cos(\theta_s/2)]^{|\lambda+\mu|}[\sin(\theta_s/2)]^{|\lambda-\mu|}\sum_J(2J+1)\times F_{cd,ab}{}^J(s)P_{(J-M)}{}^{(|\lambda-\mu|,|\lambda+\mu|)}(\cos\theta_s)\,. \quad \text{(II.5)}$$

We have put other constant factors into $F_{cd;ab}{}^J(s)$. When there is no spin, Eq. (II.5) simply reduces to the familiar Legendre expansion of the amplitude,

$$f_{00;00}{}^s(s,t)=\sum_J(2J+1)F_{00;00}{}^JP_J(\cos\theta_s)\,. \quad \text{(II.6)}$$

We see that the presence of spins has introduced into the helicity amplitudes a definite set of t zeros and singularities through the factors $[\cos(\theta_s/2)]^{|\lambda+\mu|}$ and $[\sin(\theta_s/2)]^{|\lambda-\mu|}$. We argue that these t zeros and singularities are the only kinematic ones. The remaining t singularities are associated with the failure of the Jacobi expansion to converge, a dynamical effect unrelated to particle spins. From the expression of $\sin(\theta_t/2)$ and $\cos(\theta_t/2)$ in s and t in Appendix A, we easily see that the kinematic t singularities of $f_{cd;ab}{}^s$ are all on the boundary of the physical region.

Thus the new amplitudes, defined by

$$\begin{aligned}\bar{f}_{cd;ab}{}^s&\equiv f_{cd;ab}{}^s[\cos\tfrac{1}{2}\theta_s]^{-|\lambda+\mu|}[\sin\tfrac{1}{2}\theta_s]^{-|\lambda-\mu|}\\&=\sum_J(2J+1)F_{cd;ab}{}^J(s)\times P_{(J-M)}{}^{(|\lambda-\mu|,|\lambda+\mu|)}(\cos\theta_s)\,,\end{aligned} \quad \text{(II.7)}$$

contain only dynamical t singularities.[6] By assumption of maximal analyticity in S-matrix theory, $\bar{f}_{cd;ab}{}^s$ satisfies a fixed-s dispersion relation in t. In the next section, we shall discuss the kinematic singularities in s. After we remove the kinematic s singularities, the amplitudes will satisfy the Mandelstam representation.

III. KINEMATIC SINGULARITIES IN THE DIRECT-CHANNEL ENERGY VARIABLE

Under crossing, the s-channel and the t-channel helicity amplitudes are related by[7]

[4] M. Jacob and G. C. Wick, Ann. Phys. **7**, 404 (1959). Our normalization convention here is different from theirs. The two are related by

$$f_{cd;ab}(s,t)=2\pi\left(\frac{sp_{ab}}{p_{cd}}\right)^{1/2}f_{cd;ab}{}^{\text{J.W.}}(s,t).$$

$f_{cd;ab}(s,t)$ is related to the S matrix by

$$S_{cd;ab}(s,t)-\delta_{cd;ab}=(2\pi)^4i\delta^4(p_c+p_d-p_a-p_b)(p_a{}^0p_b{}^0p_c{}^0p_d{}^0)^{-1/2}f_{cd;ab}(s,t),$$

while $f_{cd;ab}{}^{\text{J.W.}}(s,t)$ has a simpler relation to the differential cross section, i.e.,

$$\frac{d\sigma}{d\Omega}=|f_{cd;ab}{}^{\text{J.W.}}(s,t)|^2.$$

Also, we have set equal to zero the azimuthal angle, which is independent of the invariant quantities s and t. We use a, b, c, and d as notations for particles as well as the helicity states of the corresponding particle. Which meaning they take can be easily understood from the context. We use A, B, C, and D for the corresponding antiparticles and their helicity states.

[5] Gabor Szego, in *Orthogonal Polynomials* (Edwards Brothers, Inc., Ann Arbor, Michigan, 1948). The author would like to thank David Gross for informing her about the Jacobi polynomial.

[6] Y. Hara has used this condition. See also Ref. 2.

[7] T. L. Trueman and G. C. Wick, Ann. Phys. (N. Y.) **26**, 322 (1964). We make the convention that the phase factor in front of the d function in Eq. (III.1) is unity. This convention corresponds to taking θ_s, the angle between particles a and c in the s c.m. system, and taking θ_t the angle between particles D and c in the t c.m. system.

$$f_{cd;ab}{}^{s}(s,t)=\sum_{c',A',D',b'} d_{A'a}{}^{J_a}(\chi_a)d_{b'b}{}^{J_b}(\chi_b)d_{c'c}{}^{J_c}(\chi_c)d_{D'd}{}^{J_d}(\chi_d)f_{c'A';D'b'}{}^{t}(s,t)\,, \quad \text{(III.1)}$$

where

$$\begin{aligned}
\cos\chi_a&=[-(s+m_a^2-m_b^2)(t+m_a^2-m_c^2)-2m_a^2(m_c^2-m_a^2+m_b^2-m_d^2)]/\mathcal{S}_{ab}\mathcal{T}_{ac}\,,\\
\cos\chi_b&=[(s+m_b^2-m_a^2)(t+m_b^2-m_d^2)-2m_b^2(m_c^2-m_a^2+m_b^2-m_d^2)]/\mathcal{S}_{ab}\mathcal{T}_{bd}\,,\\
\cos\chi_c&=[(s+m_c^2-m_d^2)(t+m_c^2-m_a^2)-2m_c^2(m_c^2-m_a^2+m_b^2-m_d^2)]/\mathcal{S}_{cd}\mathcal{T}_{ac}\,,\\
\cos\chi_d&=[-(s+m_d^2-m_c^2)(t+m_d^2-m_b^2)-2m_d^2(m_c^2-m_a^2+m_b^2-m_d^2)]/\mathcal{S}_{cd}\mathcal{T}_{bd}\,.
\end{aligned} \quad \text{(III.2)}$$

From Eq. (III.2) we can easily obtain the functions $\sin\chi_i$; however, it is more illuminating to write them in the form

$$\begin{aligned}
\sin\chi_a&=2m_a[\phi(s,t)]^{1/2}/\mathcal{S}_{ab}\mathcal{T}_{ac}\,, \quad \sin\chi_c=2m_c[\phi(s,t)]^{1/2}/\mathcal{S}_{cd}\mathcal{T}_{ac}\,,\\
\sin\chi_b&=2m_b[\phi(s,t)]^{1/2}/\mathcal{S}_{ab}\mathcal{T}_{bd}\,, \quad \sin\chi_d=2m_d[\phi(s,t)]^{1/2}/\mathcal{S}_{cd}\mathcal{T}_{bd}\,,
\end{aligned} \quad \text{(III.3)}$$

where

$$\phi(s,t)\equiv st(\textstyle\sum_i m_i^2-s-t)-s(m_b^2-m_d^2)(m_a^2-m_c^2)-t(m_a^2-m_b^2)(m_c^2-m_d^2)$$
$$-(m_a^2m_d^2-m_c^2m_b^2)(m_a^2+m_d^2-m_c^2-m_b^2)\,,$$

$\phi(s,t)=0$ giving the boundary of the physical region. From Eq. (III.1) and the definition of $\bar{f}(s,t)$, we obtain the crossing relations for the $\bar{f}$'s:

$$\begin{aligned}
\bar{f}_{cd;ab}{}^{s}(s,t)&=[\sin\tfrac{1}{2}\theta_s]^{-|\lambda-\mu|}[\cos\tfrac{1}{2}\theta_s]^{-|\lambda+\mu|}\\
&\quad\times\sum_{A',b',c',D'}\{d_{A'a}{}^{J_a}(\chi_a)d_{b'b}{}^{J_b}(\chi_b)d_{c'c}{}^{J_c}(\chi_c)d_{D'd}{}^{J_d}(\chi_a)[\sin\tfrac{1}{2}\theta_t]^{|\lambda'-\mu'|}[\cos\tfrac{1}{2}\theta_t]^{|\lambda'+\mu'|}\bar{f}_{c'A';D'b'}{}^{t}(s,t)\}\\
&\equiv\sum_{A',b',c',D'}\mathfrak{M}_{c'A';D'b'}{}^{cd;ab}(s,t)\bar{f}_{c'A';D'b'}{}^{t}\,,
\end{aligned} \quad \text{(III.4)}$$

where

$$\lambda'\equiv D-b\,,\quad \mu'\equiv c-A\,,$$

the elements of the crossing matrix $\mathfrak{M}$ being defined by this equation. Now by the result of Sec. II, $\bar{f}^s(s,t)$ is free of kinematic t singularities, and similarly $\bar{f}^t(s,t)$ is free of kinematic s singularities. Therefore all the kinematic s singularities of $\bar{f}^s(s,t)$ are in the crossing-matrix elements in Eq. (III.4) and thus in the functions that are explicitly known. From the kinematics in Appendix A, we see that in addition to the pure s singularities and t singularities at $\mathcal{S}_{ab}=0$, $\mathcal{S}_{cd}=0$, $s=0$, $\mathcal{T}_{ac}=0$, $\mathcal{T}_{bd}=0$, and $t=0$, in the $\mathfrak{M}$'s there are also mixed s and t singularities on the boundary of the physical region, i.e., at $\phi(s,t)=0$. In Appendix B, we show that such apparent mixed s-t singularities of $\mathfrak{M}$ cancel and all the $\mathfrak{M}$'s have only pure s singularities and pure t singularities. This is what one would expect from Eq. (III.4), since $\bar{f}^s(s,t)$ is free of t-kinematic singularities and $\bar{f}^t(s,t)$ is free of s-kinematic singularities, and neither $\bar{f}^s(s,t)$ nor $\bar{f}^t(s,t)$ has dynamical singularities on the physical boundary. Then all the pure s singularities of $\mathfrak{M}$'s are the kinematic s singularities of $\bar{f}^s(s,t)$. If the pure s singularities of each $\mathfrak{M}$ are factorizable and all $\mathfrak{M}$'s in Eq. (III.4) have the same type of pure s singularities, one can easily make $\bar{f}^s$ free of kinematic s singularities by multiplying it by a factor which makes all $\mathfrak{M}$'s in Eq. (III.4) free of s singularities. If this is not the case, one has to seek linear combinations of $\bar{f}^s$ such that the combinations are still free of kinematic t singularities and also suitable for the factorization of the kinematic s singularities. In the following, we shall discuss the factorizability for all cases, with any mass assignments.

In proving the factorizability, we assume that parity is conserved in the interactions. Under parity symmetry,[4]

$$\bar{f}_{c'A';D'b'}{}^{t}(s,t)=\eta_t\bar{f}_{-c'-A';-D'-c}{}^{t}(s,t)\,, \quad \text{(III.5)}$$

where

$$\eta_t\equiv\frac{\eta_A\eta_c}{\eta_D\eta_b}(-)^{J_c+J_a-J_d-J_b}(-)^{\lambda'-\mu'}\,,$$

and η_i is the intrinsic parity of the ith particle. Combining the $\bar{f}^t$'s, which are related by parity symmetry on the left-hand side of Eq. (III.4), we obtain the crossing-matrix elements:

$$\begin{aligned}
M_{c'A';D'b'}{}^{cd;ab}(s,t)&\equiv\mathfrak{M}_{c'A';D'b}{}^{cd;ab}(s,t)+\eta_t\mathfrak{M}_{-c'-A';-D'-b'}{}^{cd;ab}(s,t)\\
&=[\sin\tfrac{1}{2}\theta_s]^{-|\lambda-\mu|}[\cos\tfrac{1}{2}\theta_s]^{-|\lambda+\mu|}[\sin\tfrac{1}{2}\theta_t]^{|\lambda'-\mu'|}[\cos\tfrac{1}{2}\theta_t]^{|\lambda'+\mu'|}\\
&\quad\times\{d_{A'a}{}^{J}(\chi_a)d_{b'b}{}^{J_b}(\chi_b)d_{c'c}{}^{J_c}(\chi_c)d_{D'd}{}^{J_d}(\chi_d)+\eta_t(-)^{(J_a-a)+(J_b-b)+(J_c-c)+(J_d-d)}\\
&\quad\times d_{A'a}{}^{J_a}(\pi-\chi_a)d_{b'b}{}^{J_b}(\pi-\chi_b)d_{c'c}{}^{J_c}(\pi-\chi_c)d_{D'd}{}^{J_d}(\pi-\chi_d)\}\,.
\end{aligned} \quad \text{(III.6)}$$

Equation (III.4) can then be written as

$$\bar{f}_{cd;ab}{}^{s}(s,t)=\sum_{A',b',c',D'\geqslant 0} M_{c'A';D'b'}{}^{cd;ab}(s,t)\bar{f}_{c'A';D'b'}{}^{t}\,. \quad \text{(III.4')}$$

In obtaining Eq. (III.6) from Eq. (III.4), we have used the relation

$$d_{\lambda\mu}{}^{J}(\pi-\theta_s)=(-)^{J-\mu}d_{-\lambda\mu}{}^{J}(\theta_s)\,. \quad \text{(III.7)}$$

From Eq. (III.6), we see that in proving factorizability of kinematic s singularities, mainly we shall play with the d functions. The following relation is useful:

$$d_{\lambda\mu}{}^{J}(\theta)=\pm[\sin\theta]^{-|\lambda-\mu|}[1-\cos\theta]^{|\lambda+\mu|}\times[\cos\tfrac{1}{2}\theta]^{v}\mathcal{P}^{(J-v/2)}(\cos\theta)\,,\qquad\text{(III.8)}$$

where

$$v=1 \quad \text{when } J \text{ is half integer},$$
$$=0 \quad \text{when } J \text{ is integer},$$

$$\mathcal{P}^{(J-v/2)}(\cos\theta)\equiv[\cos\tfrac{1}{2}\theta]^{(2M-v)}\times P_{(J-M)}{}^{(|\lambda-\mu|,|\lambda+\mu|)}(\cos\theta)\,.\qquad\text{(III.9)}$$

The $\mathcal{P}^{(J-v/2)}(\cos\theta)$ is a polynomial of $\cos\theta$ of the order of $J-v/2$.

In the following, we shall study the locations of the kinematic s singularities of $\bar{f}^{s}(s,t)$ for the general mass case (i.e., $m_a\neq m_b$, $m_c\neq m_d$ and not both $m_a\neq m_c$ and $m_b\neq m_d$) and find suitable factors for removing the kinematic s singularities. For special mass assignments, where pairs of masses are equal, the same method applies, but care must be exercised in taking the limits of the above formula. The results for all mass assignments are listed in Sec. IV. To avoid introducing kinematic zeros, we have obtained the kinematic factors for special mass assignments from studying each case individually.

From the kinematics in Appendix A, it is clear that the sines and cosines of the angles and the half-angles of χ_a, χ_b, and θ_s have branch points at $\mathcal{S}_{ab}=0$, i.e., at $s=(m_a+m_b)^2$ and $s=(m_a-m_b)^2$. To investigate the analytic properties of these functions, which are originally defined in the physical region, we first analytically continue them at fixed t outside the physical region of the s channel. Then we vary s around $s=(m_a+m_b)^2$ or $s=(m_a-m_b)^2$ in a counterclockwise sense.[8] We find that the following functions are analytic at both $s=(m_a+m_b)^2$, $s=(m_a-m_b)^2$:

$$\mathcal{S}_{ab}\sin\chi_a\,,\quad \mathcal{S}_{ab}\cos\chi_a\,,\quad \mathcal{S}_{ab}\sin\chi_b\,,\quad \mathcal{S}_{ab}\cos\chi_b\,,$$
$$[\cos\tfrac{1}{2}\chi_a\cos\tfrac{1}{2}\chi_b+\sin\tfrac{1}{2}\chi_a\sin\tfrac{1}{2}\chi_b]\times[s-(m_a+m_b)^2]^{1/2}\,,$$
$$[\cos\tfrac{1}{2}\chi_a\cos\tfrac{1}{2}\chi_b-\sin\tfrac{1}{2}\chi_a\sin\tfrac{1}{2}\chi_b]\times[s-(m_a-m_b)^2]^{1/2}\,,$$
$$[\cos\tfrac{1}{2}\theta_s\cos\tfrac{1}{2}\chi_a+\sin\tfrac{1}{2}\theta_s\sin\tfrac{1}{2}\chi_a]\times\mathcal{S}_{ab}\,,$$
$$\cos\tfrac{1}{2}\theta_s\cos\tfrac{1}{2}\chi_a-\sin\tfrac{1}{2}\theta_s\sin\tfrac{1}{2}\chi_a\,,$$
$$\sin\tfrac{1}{2}\theta_s\cos\tfrac{1}{2}\chi_a+\sin\tfrac{1}{2}\chi_a\cos\tfrac{1}{2}\theta_s\,,$$

and

$$[\sin\tfrac{1}{2}\theta_s\cos\tfrac{1}{2}\chi_a-\cos\tfrac{1}{2}\theta_s\sin\tfrac{1}{2}\chi_a]\times\mathcal{S}_{ab}\,.\qquad\text{(III.10)}$$

Similarly, for χ_c, χ_d, and θ_s, we find the following functions are analytic at $\mathcal{S}_{ab}=0$:

$$\mathcal{S}_{cd}\sin\chi_c\,,\quad \mathcal{S}_{cd}\cos\chi_c\,,\quad \mathcal{S}_{cd}\sin\chi_d\,,\quad \mathcal{S}_{cd}\cos\chi_d\,,$$
$$[\cos\tfrac{1}{2}\chi_c\cos\tfrac{1}{2}\chi_d+\sin\tfrac{1}{2}\chi_c\sin\tfrac{1}{2}\chi_d]\times[s-(m_c+m_d)^2]^{\frac{1}{2}}\,,$$
$$[\cos\tfrac{1}{2}\chi_c\cos\tfrac{1}{2}\chi_d-\sin\tfrac{1}{2}\chi_c\sin\tfrac{1}{2}\chi_d]\times[s-(m_c-m_d)^2]^{\frac{1}{2}}\,,$$
$$\cos\tfrac{1}{2}\theta_s\cos\tfrac{1}{2}\chi_c+\sin\tfrac{1}{2}\theta_s\sin\tfrac{1}{2}\chi_c\,,$$
$$(\cos\tfrac{1}{2}\theta_s\cos\tfrac{1}{2}\chi_c-\sin\tfrac{1}{2}\theta_s\sin\tfrac{1}{2}\chi_c)\times\mathcal{S}_{cd}\,,$$
$$(\cos\tfrac{1}{2}\theta_s\sin\tfrac{1}{2}\chi_c+\sin\tfrac{1}{2}\theta_s\cos\tfrac{1}{2}\chi_c)\times\mathcal{S}_{cd}\,,$$

and

$$\cos\tfrac{1}{2}\theta_s\sin\tfrac{1}{2}\chi_c-\sin\tfrac{1}{2}\theta_s\cos\tfrac{1}{2}\chi_c\qquad\text{(III.10')}$$

are analytic at $\mathcal{S}_{cd}=0$. The functions $\mathcal{S}_{ab}\mathcal{S}_{cd}\sin\theta_s$ and $\mathcal{S}_{ab}\mathcal{S}_{cd}\cos\theta_s$ are analytic at $\mathcal{S}_{ab}=0$ and $\mathcal{S}_{cd}=0$. The functions $\sin\theta_t/2$ and $\cos\theta_t/2$ are analytic at $\mathcal{S}_{ab}=0$ and $\mathcal{S}_{cd}=0$. From these results, we can easily show that the crossing matrix M in Eq. (III.4′) has singularities at $\mathcal{S}_{ab}=0$ and $\mathcal{S}_{cd}=0$, and these singularities are not factorizable.

Let us look now at the amplitudes

$$\bar{f}_{cd,ab}{}^{s(\pm)}\equiv\bar{f}_{cd,ab}{}^{s}\pm\bar{f}_{-c-d;ab}{}^{s}\,.$$

The crossing-matrix element between $\bar{f}_{cd;ab}{}^{s(\pm)}$ and $\bar{f}_{c'A';D'b'}{}^{t}$ is

$$[\bar{M}_{c'A';D'b'}{}^{cd;ab}(s,t)]^{\pm}\equiv M_{c'A';D'b'}{}^{cd;ab}(s,t)\pm M_{c'A';D'b'}{}^{-c-d;ab}\,.$$

Using Eqs. (III.6) and (III.8), we obtain $\bar{M}^{\pm}$ up to a constant:

$$\begin{aligned}\bar{M}^{\pm}\propto{}&(\sin\tfrac{1}{2}\theta_t)^{|\lambda'-\mu'|}(\cos\tfrac{1}{2}\theta_t)^{|\lambda'+\mu'|}(\sin\theta_s)^{-|\lambda-\mu|}\Bigg\{\Bigg[\left(\frac{1-\cos\chi_a}{\sin\chi_a}\right)^{|A'-a|}\left(\frac{1-\cos\chi_b}{\sin\chi_b}\right)^{|b'-b|}\mathcal{P}^{(J_a-v_a/2)}(\cos\chi_a)\\&\times\mathcal{P}^{(J_b-v_b/2)}(\cos\chi_b)(1+\cos\theta_s)^{m}(\cos\tfrac{1}{2}\theta_s)^{v}(\cos\tfrac{1}{2}\chi_a)^{v_a}(\cos\tfrac{1}{2}\chi_b)^{v_b}\pm\eta_{ab}\left(\frac{1+\cos\chi_a}{-\sin\chi_a}\right)^{|A'-a|}\left(\frac{1+\cos\chi_b}{-\sin\chi_b}\right)^{|b'-b|}\\&\times\mathcal{P}^{(J_a-v_a/2)}(-\cos\chi_a)\mathcal{P}^{(J_b-v_b/2)}(-\cos\chi_b)(1-\cos\theta_s)^{m}(\sin\tfrac{1}{2}\theta_s)^{v}(\sin\tfrac{1}{2}\chi_a)^{v_a}(\sin\tfrac{1}{2}\chi_b)^{v_b}\Bigg]d_{c'c}{}^{J_c}(\chi_c)d_{D'd}{}^{J_d}(\chi_d)\\&+\Bigg[\left(\frac{1-\cos\chi_a}{\sin\chi_a}\right)^{|A'-a|}\left(\frac{1-\cos\chi_b}{\sin\chi_b}\right)^{|b'-b|}\mathcal{P}^{(J_a-v_a/2)}(\cos\chi_a)\mathcal{P}^{(J_b-v_b/2)}(\cos\chi_b)(1-\cos\theta_s)^{m}\\&\times(\sin\tfrac{1}{2}\theta_s)^{v}(\cos\tfrac{1}{2}\chi_a)^{v_a}(\cos\tfrac{1}{2}\chi_b)^{v_b}\pm\eta_{ab}(-)^{2(c+d)}\left(\frac{1+\cos\chi_a}{-\sin\chi_a}\right)^{|A'-a|}\left(\frac{1+\cos\chi_b}{-\sin\chi_b}\right)^{|b'-b|}\mathcal{P}^{(J_a-v_a/2)}(-\cos\chi_a)\\&\times\mathcal{P}^{(J_b-v_b/2)}(-\cos\chi_b)(1+\cos\theta_s)^{m}(\cos\tfrac{1}{2}\theta_s)^{v}(\sin\tfrac{1}{2}\chi_a)^{v_a}(\sin\tfrac{1}{2}\chi_b)^{v_b}\Bigg]d_{c'c}{}^{J_c}(\pi-\chi_c)d_{D'd}{}^{J_d}(\pi-\chi_d)\Bigg\}\,,\qquad\text{(III.11)}\end{aligned}$$

[8] I am indebted to Dr. John Stack for a discussion about this.

with

$$2m+v\equiv|\lambda-\mu|-|\lambda+\mu|\,,\quad \eta_{ab}\equiv\frac{\eta_A\eta_c}{\eta_b\eta_D}(-)^{J_c+J_d+c+d}\,,$$

and

$$\begin{aligned} v_i&=1 \quad \text{when the } i\text{th particle is a fermion}\,,\\ &=0 \quad \text{when the } i\text{th particle is a boson}\,,\end{aligned}$$

where

$v=0$ for $BB\to BB$, $FF\to FF$, $\bar{F}F\to BB$ interactions (B stands for boson and F stands for fermion),

$v=1$ for $BF\to BF$ interaction with the convention $v_a=v_c=1$, $v_b=v_d=0$.

The expression given by Eq. (III.11) is convenient for observing the analytic structure at $\mathcal{S}_{ab}=0$. To observe the analytic structure of $\bar{M}^\pm$ at $\mathcal{S}_{cd}=0$, we write $\bar{M}^\pm$ in the following form:

$$\begin{aligned}\bar{M}^\pm\propto{}&(\sin\tfrac{1}{2}\theta_t)^{|\lambda'-\mu'|}(\cos\tfrac{1}{2}\theta_t)^{|\lambda'+\mu'|}(\sin\theta_s)^{-|\lambda-\mu|}\Bigg\{\Bigg[\left(\frac{1-\cos\chi_c}{\sin\chi_c}\right)^{|c'-c|}\left(\frac{1-\cos\chi_d}{\sin\chi_d}\right)^{|D'-d|}\mathcal{P}^{(J_c-v_c/2)}(\cos\chi_c)\\ &\times\mathcal{P}^{(J_d-v_d/2)}(\cos\chi_d)(1+\cos\theta_s)^m(\cos\tfrac{1}{2}\theta_s)^v(\cos\tfrac{1}{2}\chi_c)^{v_c}(\cos\tfrac{1}{2}\chi_d)^{v_d}\pm\eta_{cd}\left(\frac{1+\cos\chi_c}{-\sin\chi_c}\right)^{|c'-c|}\left(\frac{1+\cos\chi_d}{-\sin\chi_d}\right)^{|D'-d|}\\ &\times\mathcal{P}^{(J_c-v_c/2)}(-\cos\chi_c)\mathcal{P}^{(J_d-v_d/2)}(-\cos\chi_d)(1-\cos\theta_s)^m(\sin\tfrac{1}{2}\theta_s)^v(\sin\tfrac{1}{2}\chi_c)^{v_c}(\sin\tfrac{1}{2}\chi_d)^{v_d}\Bigg]d_{A'a}^{J_a}(\chi_a)d_{b'b}^{J_b}(\chi_b)\\ &\pm\eta_{ab}(-)^{(A'-a)+(b'-b)}\Bigg[\left(\frac{1-\cos\chi_c}{\sin\chi_c}\right)^{|c'-c|}\left(\frac{1-\cos\chi_d}{\sin\chi_d}\right)^{|D'-d|}\mathcal{P}^{(J_c-v_c/2)}(\cos\chi_c)\mathcal{P}^{(J_d-v_d/2)}(\cos\chi_d)(1-\cos\theta_s)^m\\ &\times(\sin\tfrac{1}{2}\theta_s)^v(\cos\tfrac{1}{2}\chi_c)^{v_c}(\cos\tfrac{1}{2}\chi_d)^{v_d}\pm\eta_{cd}(-)^{2(c+d)}\left(\frac{1+\cos\chi_c}{-\sin\chi_c}\right)^{|c'-c|}\left(\frac{1+\cos\chi_d}{-\sin\chi_d}\right)^{|D'-d|}\mathcal{P}^{(J_c-v_c/2)}(-\cos\chi_c)\\ &\times\mathcal{P}^{(J_d-v_d/2)}(-\cos\chi_d)(1+\cos\theta_s)^m(\cos\tfrac{1}{2}\theta_s)^v(\sin\tfrac{1}{2}\chi_c)^{v_c}(\sin\tfrac{1}{2}\chi_d)^{v_d}\Bigg]d_{A'a}^{J_a}(\pi-\chi_a)d_{b'b}^{J_b}(\pi-\chi_b)\Bigg\}\,,\end{aligned}\qquad\text{(III.11')}$$

where

$$\eta_{cd}=(-)^{J_c+J_d+c+d}.$$

Using the result in Eqs. (III.10) and (III.10'), and after some work, we find the singularities of $\bar{M}^\pm$ at $s=(m_a\pm m_b)^2$ and $s=(m_c\pm m_b)^2$:

$$\begin{aligned}\bar{M}^\pm\propto{}&[s-(m_a+m_b)^2]^{-\frac{1}{2}\alpha_1}[s-(m_a-m_b)^2]^{-\frac{1}{2}\alpha_2}\\ &\times[s-(m_c+m_d)^2]^{-\frac{1}{2}\beta_1}[s-(m_c-m_d)^2]^{-\frac{1}{2}\beta_2},\end{aligned}\qquad\text{(III.12)}$$

where

$\alpha_1=\alpha_2=\alpha_g(\pm)$, $\beta_1=\beta_2=\beta_g(\pm)$ for $BB\to BB$;

$\alpha_1=\alpha_2=\alpha_g(\pm)$, $\beta_1=\beta_g(\pm)$, $\beta_2=\beta_g(\mp)$ for $BB\to F\bar{F}$, (c, d being fermions);

$\alpha_1=\alpha_g(\pm)$, $\alpha_2=\alpha_g(\mp)$, $\beta_1=\beta_2=\beta_g(\pm)$ for $F\bar{F}\to BB$, (a, b being bosons); and

$\alpha_1=\alpha_g(\pm)$, $\alpha_2=\alpha_g(\mp)$, $\beta_1=\beta_g(\pm)$, $\beta_2=\beta_g(\mp)$ for $FF\to FF$.

Here

$$\begin{aligned}\alpha_g(\pm)=-|\lambda-\mu|+\{&\max(\pm\eta_{ab})\text{ of }[J_a+J_b-\tfrac{1}{2}(v_a+v_b)\\ &+\tfrac{1}{2}(|\lambda-\mu|-|\lambda+\mu|)]\}+\tfrac{1}{2}(v_a+v_b),\\ \beta_g(\pm)=-|\lambda-\mu|+\{&\max(\pm\eta_{cd})\text{ of }[J_c+J_d-\tfrac{1}{2}(v_a+v_b)\\ &+\tfrac{1}{2}(|\lambda-\mu|-|\lambda+\mu|)]\}+\tfrac{1}{2}(v_c+v_d),\end{aligned}$$

where "max η of n" means the greatest even number that is equal to or smaller than n when η is $+1$, or the greatest odd number when η is -1. For $FB\to FB$ interactions, i.e., $v=1$, $v_a=v_c=1$, and $v_b=v_d=0$, we have

$$\bar{M}^\pm\propto(\mathcal{S}_{ab})^{-\alpha_g'(\pm)}(\mathcal{S}_{cd})^{-\beta_g'(\pm)}\,,\qquad\text{(III.12')}$$

where

$$\begin{aligned}\alpha_g'(\pm)=-|\lambda-\mu|+\{&\max(\mp\eta_{ab})\\ &\text{of }[J_a+J_b+\tfrac{1}{2}(|\lambda-\mu|-|\lambda+\mu|]\}\,,\\ \beta_g'(\pm)=-|\lambda-\mu|+\{&\max(\pm\eta_{cd})\\ &\text{of }[J_c+J_d+\tfrac{1}{2}(|\lambda-\mu|-|\lambda+\mu|)]\}\,.\end{aligned}$$

We see that the singularities of $\bar{f}^{s(\pm)}$ at $\mathcal{S}_{ab}=0$ can be factored out. In addition to the singularities at $\mathcal{S}_{ab}=0$, $\mathcal{S}_{cd}=0$, there is a singularity at $s=0$. In this case of $m_a\neq m_b$, $m_c\neq m_d$, the entire $s^{1/2}$ singularity is introduced by the factor $\sin(\theta_s/2)$. For reactions of the type $BB\to BB$, $FF\to FF$, $F\bar{F}\to BB$, the difference of $|\lambda+\mu|$ and $|\lambda-\mu|$ is even, so we can remove the $s^{1/2}$ singularity from $\bar{f}^{s(\pm)}$ simply by multiplying it by $(s)^{\frac{1}{2}[\max(|\lambda-\mu|,|\lambda+\mu|)]}$. For reactions of the type $BF\to BF$, the difference of $|\lambda-\mu|$ and $|\lambda+\mu|$ is odd, so we can not remove the $s^{1/2}$ singularity from $\bar{f}^{s(\pm)}$. Multiplying by a factor of $(s)^{\frac{1}{2}[\max(|\lambda-\mu|,|\lambda+\mu|)]}$, we can remove its possible pole only at $s=0$. It still has a branch point at $s=0$. However, $\bar{f}^{s(\pm)}$ is closely related to the partial-wave helicity amplitude of definite parity, and by the McDowell reciprocity relation[3] the partial-wave helicity amplitude of definite parity has a simple reflection property in $(s)^{1/2}$. Thus we can conveniently work with the $s^{1/2}$ plane; in fact, in the calculation of the partial-wave amplitude of $BF \to BF$ interaction, we are forced to work in this plane. Therefore, for $BB \to BB$, $FF \to FF$, and $F\bar{F} \to BB$ interactions, the functions

$$[\bar{f}_{cd;ab}{}^s \pm \bar{f}_{-c-d;ab}{}^s] \times [s-(m_a+m_b)^2]^{\frac{1}{2}\alpha_1}[s-(m_a-m_b)^2]^{\frac{1}{2}\alpha_2} \times [s-(m_c+m_d)^2]^{\frac{1}{2}\beta_1}[s-(m_c-m_d)^2]^{\frac{1}{2}\beta_2} \times (s)^{\frac{1}{2}[\text{max of }(|\lambda-\mu|,|\lambda+\mu|)]} \quad \text{(III.13)}$$

are analytic in s and t. For $FB \to FB$ interactions, the functions

$$[\bar{f}_{cd;ab}{}^s \pm \bar{f}_{-c-d;ab}{}^s](\mathcal{S}_{ab})^{\alpha_g'(\pm)}(\mathcal{S}_{cd})^{\beta_g'(\pm)} \times (s)^{\frac{1}{2}[\text{max of }(|\lambda-\mu|,|\lambda+\mu|)]} \quad \text{(III.13')}$$

are analytic in $(s)^{1/2}$ and t and finite at $s=0$.

IV. SUMMARY

In the following, we list all the amplitudes that are free of kinematic singularities.[9] We can easily show that these results are consistent with the usually assumed threshold behavior of partial-wave helicity amplitudes with definite parity.[2,3]

The definition of $\bar{f}^s$ by

$$\bar{f}_{cd;ab}{}^s(s,t) = [\sin\theta_s/2]^{-|\lambda-\mu|}[\cos\theta_s/2]^{-|\lambda+\mu|} f_{cd;ab}{}^s(s,t)$$

is always used. For $FB \to FB$ interactions, we take the convention that particles a and c are fermions.

A. Equal Masses, $m_a = m_b = m_c = m_d \equiv m$

The desired amplitude here is

$$(s-4m^2)^{\frac{1}{2}\alpha_e}(s)^{-\frac{1}{2}\beta_e}\bar{f}_{cd;ab}{}^s(s,t), \quad \text{(IV.1)}$$

where

$$\lambda \equiv a-b, \quad \mu \equiv c-d,$$

$$\alpha_e \equiv \tfrac{1}{2}(v_a+v_b+v_c+v_d) + \{\text{the max } \eta_e \text{ of } [J_a+J_b+J_c+J_d-\tfrac{1}{2}(v_a+v_b+v_c+v_d)]\} - |\lambda-\mu| - |\lambda+\mu|,$$

$$\beta_e \equiv \text{the max } (\eta_s) \text{ of } [|(|J_d-J_b|-|J_a-J_c|)|+1],$$

with

$$v_i = 1, \quad \text{if } J_i \text{ is half integer}$$
$$= 0, \quad \text{if } J_i \text{ is integer}$$

$$\eta_e \equiv \frac{\eta_A \eta_c}{\eta_b \eta_D}(-)^{2(J_a+J_b)},$$

$$\eta_s \equiv \eta_e(-)^{|\lambda-\mu|}.$$

"The max η of n" means the greatest even number smaller than or equal to n when η is $+1$ or the greatest odd number with η is -1.

Using this result, we can easily find the kinematic singularity-free helicity amplitudes for the nucleon-nucleon scattering $NN \to NN$. In this case,

$$J_a = J_b = J_c = J_d = \tfrac{1}{2},$$
$$v_a = v_b = v_c = v_d = 1,$$
$$\eta_N = +1, \quad \eta_{\bar{N}} = -1,$$
$$\eta_e = +1.$$

We find that

$$(s-4m^2)f_{++;++}{}^s, \quad (s-4m^2)f_{++;--}{}^s,$$
$$(\cos\tfrac{1}{2}\theta_s)^{-2}f_{+-;+-}{}^s, \quad (\sin\tfrac{1}{2}\theta_s)^{-2}f_{+-;-+}{}^s,$$

and

$$(s)^{-1/2}(\sin\tfrac{1}{2}\theta_s \cos\tfrac{1}{2}\theta_s)^{-1}f_{++;--}{}^s$$

are free of all kinematic singularities. These results are in agreement with those obtained by M. L. Goldberger *et al.*[10]

B. $m_a = m_c \equiv m_1$, $m_b = m_d \equiv m_2$

The following amplitudes are free of kinematic s and t singularities for $BB \to BB$, $FB \to FB$, $FF \to FF$ interactions:

$$(\mathcal{S})^{\alpha_e}(s)^{\frac{1}{2}\beta_l}\bar{f}_{cd;ab}{}^s(s,t),$$

where α_e is the same as that of Eq. (IV.1),

$$\beta_l \equiv |\lambda-\mu|,$$
$$\mathcal{S} \equiv \{[s-(m_1+m_2)^2][s-(m_1-m_2)^2]\}^{1/2}.$$

For $F\bar{F} \to BB$ interactions, one can easily find the kinematic singularity-free amplitudes by the method for the general mass case.

Applying this general result to the $\pi N \to \pi N$ interaction, we find that

$$[\cos(\theta_s/2)]^{-1}f_{+0;+0}{}^s \quad \text{and} \quad s^{1/2}[\sin(\theta_s/2)]^{-1}f_{+0;-0}{}^s$$

are free of kinematic singularities. These results agree with those obtained first by G. Chew *et al.*[11]

C. $m_a = m_b \equiv m_1$, $m_c = m_d \equiv m_2$

The following functions are analytic in s and t for $BB \to BB$, $\bar{F}F \to BB$, $FF \to FF$ interactions:

$$[\bar{f}_{cd;ab}{}^s \pm \bar{f}_{-c-d;ab}{}^s](s-4m_1{}^2)^{\frac{1}{2}\alpha_g(\pm)} \times (s-4m_2{}^2)^{\frac{1}{2}\beta_g(\pm)}(s)^{-\frac{1}{2}\gamma}, \quad \text{(IV.2)}$$

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[9] Our results are consistent with those of Hara's; however, our approaches are different. To find the kinematic singularities of $\bar{f}_{cd;ab}{}^s \pm \bar{f}_{-c-d;ab}$ at $\mathcal{S}_{ab}=0$ and $\mathcal{S}_{cd}=0$, Hara used the assumption of the threshold condition of the partial-wave helicity amplitudes with definite parity. He did not pin down the exact value of the power of the singularities. We first find the exact power of kinematic singularities at $\mathcal{S}_{ab}=0$ and $\mathcal{S}_{cd}=0$ of $\bar{f}_{cd;ab}{}^s \pm \bar{f}_{-c-d;ab}{}^s$ and then we do find that our result is consistent with the threshold condition of the partial-wave helicity amplitudes with definite parity, but the power is not equal to that obtained from the threshold condition. Hara's results may have kinematic zeros.

[10] M. L. Goldberger, M. T. Grisaru, S. W. McDowell, and D. Y. Wong, Phys. Rev. **120**, 2250 (1960).

[11] G. F. Chew, M. L. Goldberger, F. E. Low, and Y. Nambu, Phys. Rev. **106**, 1337 (1957).

where

$$\alpha_g(\pm)\equiv-|\lambda-\mu|+\{\max(\pm\eta_{ab})\text{ of }[J_a+J_b-\tfrac{1}{2}(v_a+v_b)+\tfrac{1}{2}(|\lambda-\mu|-|\lambda+\mu|)]\}+\tfrac{1}{2}(v_a+v_b),$$

$$\beta_g(\pm)\equiv-|\lambda-\mu|+\{\max(\pm\eta_{cd})\text{ of }[J_c+J_d-\tfrac{1}{2}(v_c+v_d)+\tfrac{1}{2}(|\lambda-\mu|-|\lambda+\mu|)]\}+\tfrac{1}{2}(v_c+v_d),$$

$$\gamma\equiv\text{the max }(\eta_s)\text{ of }[|(|J_d-J_b|-|J_c-J_a|)|+1],$$

$$\eta_s\equiv\frac{\eta_A\eta_c}{\eta_b\eta_D}(-)^{2(J_a+J_b)}(-)^{\lambda-\mu},$$

$$\eta_{ab}\equiv\frac{\eta_A\eta_c}{\eta_b\eta_D}(-)^{J_d+J_c+d+c},\quad \eta_{cd}\equiv(-)^{J_d+J_c+d+c}.$$

For $BF\to BF$ interactions, the result for the general mass case applies except that the $s^{1/2}$ singularity needs to be re-evaluated.

Applying Eq. (IV.2) to $N\bar{N}$, with

$$\eta_{ab}=+1,\quad \eta_{cd}=+1,\quad \eta_s=(-)^{\lambda-\mu}$$

and taking only the plus sign of Eq. (IV.2), we easily obtain the kinematic singularity-free amplitudes:

$$(s-4m_n^2)^{1/2}f_{00,++}{}^s$$

and

$$(\sin\theta_s)^{-1}[s(s-4m_\pi^2)]^{-\frac{1}{2}}f_{00;+-}{}^s.$$

D. General Mass Case ($m_a\neq m_b$, $m_c\neq m_d$ and Not Both $m_a=m_c$, $m_b=m_d$)

For $BB\to BB$, $F\bar{F}\to BB$, $FF\to FF$ interactions, the following amplitudes are free from kinematic s and t singularities:

$$[\bar{f}_{cd;ab}{}^s\pm\bar{f}_{-c-d;ab}{}^s]\times[s-(m_a+m_b)^2]^{\frac{1}{2}\alpha_1}[s-(m_a-m_b)^2]^{\frac{1}{2}\alpha_2}\times[s-(m_c+m_d)^2]^{\frac{1}{2}\beta_1}[s-(m_c-m_d)^2]^{\frac{1}{2}\beta_2}(s)^{\frac{1}{2}\gamma_g},\quad\text{(IV.3)}$$

where α_1, α_2, β_1, and β_2 are the same as those of Eq. (III.12) and

$$\gamma_g\equiv\max\text{ of }(|\lambda-\mu|,|\lambda+\mu|).$$

For $BF\to BF$ interactions, the amplitudes

$$[\bar{f}_{cd;ab}{}^s\pm\bar{f}_{-c-d;ab}{}^s]\times(\mathcal{S}_{ab})^{\alpha_g'(\pm)}\times(\mathcal{S}_{cd})^{\beta_g'(\pm)}(s)^{\frac{1}{2}\gamma_g}$$

with

$$\alpha_g'(\pm)\equiv-|\lambda-\mu|+\{\max(\mp\eta_{ab})\text{ of }[J_a+J_b+\tfrac{1}{2}(|\lambda-\mu|-|\lambda+\mu|)]\},$$

$$\beta_g'(\pm)\equiv-|\lambda-\mu|+\{\max(\pm\eta_{cd})\text{ of }[J_c+J_d+\tfrac{1}{2}(|\lambda-\mu|-|\lambda+\mu|)]\},$$

are free of kinematic singularities except for the $s^{1/2}$ singularity.

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APPENDIX A: THE KINEMATICS

The boundary of the physical region[12] is

$$\phi(s,t)=0\equiv st(\textstyle\sum_i m_i^2-s-t)-s(m_b^2-m_d^2)(m_a^2-m_c^2)-t(m_a^2-m_b^2)(m_c^2-m_d^2)-(m_a^2m_d^2-m_c^2m_b^2)\times(m_a^2+m_d^2-m_c^2-m_b^2).\quad\text{(A1)}$$

We define θ_s as the scattering angle in the s channel (i.e., $a+b\to c+d$), which is taken to be the angle between particles a and c. In the s c.m. system, we have

$$\cos\theta_s=[2st+s^2-s\textstyle\sum_i m_i^2+(m_a^2-m_b^2)(m_c^2-m_d^2)]/\mathcal{S}_{ab}\mathcal{S}_{cd},$$

where

$$\mathcal{S}_{ab}{}^2\equiv[s-(m_a-m_b)^2][s-(m_a+m_b)^2]=4sp_{ab}{}^2,$$
$$\mathcal{S}_{cd}{}^2\equiv[s-(m_c-m_d)^2[s-(m_c+m_d)^2]=4sp_{cd}{}^2,\quad\text{(A2)}$$
$$\sin\theta_s=2[s\phi(s,t)]^{1/2}/\mathcal{S}_{ab}\mathcal{S}_{cd}\quad\text{for}\quad 0\leqslant\theta_s\leqslant\pi.$$

θ_t is the scattering angle between particles D and c in the t channel (i.e., $D+b\to c+A$). In the t c.m. system,

$$\cos\theta_t=[2st+t^2-t\textstyle\sum_i m_i^2+(m_d^2-m_b^2)(m_c^2-m_a^2)]/\mathcal{T}_{ac}\mathcal{T}_{bd},\quad\text{(A3)}$$

$$\sin\theta_t=2[t\phi(s,t)]^{1/2}/\mathcal{T}_{ac}\mathcal{T}_{bd},$$

where

$$\mathcal{T}_{ac}{}^2\equiv[t-(m_a+m_c)^2][t-(m_a-m_c)^2]=4tp_{ac}{}^2,$$
$$\mathcal{T}_{bd}{}^2\equiv[t-(m_b+m_d)^2][t-(m_b-m_d)^2]=4tp_{bd}{}^2$$

APPENDIX B

We want to show that the elements of the crossing matrix relating $\bar{f}^s$ and $\bar{f}^t$ in Eq. (III.4) do not have mixed s and t singularity. As we see from the kinematics, all the mixed s and t singularity of the sines and cosines of the angles θ_s, θ_t, and angles χ_i are on the boundary of the physical region $\phi(s,t)=0$. In some special mass cases, $\phi(s,t)=0$ gives $s=0$ and $t=0$, but that is not our concern here. We consider only the part of $\phi(s,t)=0$ due to the vanishing of its factor of mixed s and t. Then at $\phi(s,t)=0$, the cosines of all the angles are either $+1$ or -1 and are analytic there. The sines of all θ_s, θ_t, and χ_i's have $(\phi)^{1/2}$ singularity. Whether the sine or cosine of the half angle has $(\phi)^{1/2}$ singularity will depend upon whether the cosine of the angle is $+1$ or -1 at $\phi=0$. One can show that at $\phi(s,t)=0$, except at $s=0$, $t=0$ given by $\phi(s,t)=0$,

$$\cos\chi_a\cos\chi_c=\cos\chi_b\cos\chi_d=\cos\theta_s=\pm1,\quad\text{(B1)}$$

$$\cos\chi_a\cos\chi_b=\cos\chi_c\cos\chi_d=\cos\theta_t=\pm1.\quad\text{(B2)}$$

[12] T. Kibble, Phys. Rev. 117, 1159 (1959).

Equation (B1) implies that at $\phi(s,t)=0$, if

$$\sin\tfrac{1}{2}\theta_s \approx \phi^{1/2}, \quad \text{i.e.,} \quad \cos\theta_s = +1,$$

either

$$\sin\tfrac{1}{2}\chi_a \approx \phi^{1/2} \quad \text{and} \quad \sin\tfrac{1}{2}\chi_c \approx \phi^{1/2}, \quad \text{i.e.,} \quad \cos\chi_a = \cos\chi_c = +1,$$

or

$$\cos\tfrac{1}{2}\chi_a \approx \phi^{1/2} \quad \text{and} \quad \cos\tfrac{1}{2}\chi_c \approx \phi^{1/2}, \quad \text{i.e.,} \quad \cos\chi_a = \cos\chi_c = -1,$$

and either

$$\sin\tfrac{1}{2}\chi_b \approx \phi^{1/2} \quad \text{and} \quad \sin\tfrac{1}{2}\chi_d \approx \phi^{1/2}, \quad \text{i.e.,} \quad \cos\chi_b = \cos\chi_d = +1,$$

or

$$\cos\tfrac{1}{2}\chi_b \approx \phi^{1/2} \quad \text{and} \quad \cos\tfrac{1}{2}\chi_d \approx \phi^{1/2}, \quad \text{i.e.,} \quad \cos\chi_b = \cos\chi_d = -1;$$

if

$$\cos\tfrac{1}{2}\theta_s \approx \phi^{1/2}, \quad \text{i.e.,} \quad \cos\theta_s = -1,$$

either

$$\sin\tfrac{1}{2}\chi_a \approx \phi^{1/2} \quad \text{and} \quad \cos\tfrac{1}{2}\chi_b \approx \phi^{1/2}, \quad \text{i.e.,} \quad \cos\chi_a = -\cos\chi_b = +1,$$

or

$$\cos\tfrac{1}{2}\chi_a \approx \phi^{1/2} \quad \text{and} \quad \sin\tfrac{1}{2}\chi_b \approx \phi^{1/2}, \quad \text{i.e.,} \quad \cos\chi_a = -\cos\chi_b = -1, \tag{B3}$$

etc., and a similar argument for $\cos\theta_t$.

From Eq. (III.4), the crossing-matrix element is

$$\mathfrak{M}_{c'A';D'b'}{}^{cd;ab} = [\sin\tfrac{1}{2}\theta_s]^{-|\lambda-\mu|}[\cos\tfrac{1}{2}\theta_s]^{|\lambda+\mu|} \times d_{A'a}{}^{J_a}(\chi_a) d_{b'b}{}^{J_b}(\chi_b) d_{c'c}{}^{J_c}(\chi_c) d_{D'd}{}^{J_d}(\chi_d) [\sin\tfrac{1}{2}\theta_t]^{|\lambda'-\mu'|}[\cos\tfrac{1}{2}\theta_t]^{|\lambda'+\mu'|}. \tag{B4}$$

Up to a constant, we have

$$d_{\lambda\mu}{}^J(\theta) \propto [\cos\tfrac{1}{2}\theta]^{|\lambda+\mu|}[\sin\tfrac{1}{2}\theta]^{|\lambda-\mu|} P_{(J-M)}{}^{(|\lambda-\mu|,|\lambda+\mu|)}(\cos\theta).$$

When $\cos\theta = \pm 1$, the Jacobi polynomial is a constant,

$$d_{\lambda\mu}{}^J(\theta) \propto [\cos\tfrac{1}{2}\theta]^{|\lambda+\mu|}[\sin\tfrac{1}{2}\theta]^{|\lambda-\mu|}. \tag{B5}$$

Therefore at $\phi(s,t)=0$,

$$\mathfrak{M}_{c'A';D'b'}{}^{cd;ab} \propto [\sin\tfrac{1}{2}\theta_s]^{-|\lambda-\mu|}[\cos\tfrac{1}{2}\theta_s]^{-|\lambda+\mu|}[\sin\tfrac{1}{2}\chi_a]^{|A'-a|}[\cos\tfrac{1}{2}\chi_a]^{|A'+a|}[\sin\tfrac{1}{2}\chi_b]^{|b'-b|}[\cos\tfrac{1}{2}\chi_b]^{|b'+b|} \times [\sin\tfrac{1}{2}\chi_c]^{|c'-c|}[\cos\tfrac{1}{2}\chi_c]^{|c'+c|}[\sin\tfrac{1}{2}\chi_d]^{|D'-d|}[\cos\tfrac{1}{2}\chi_d]^{|D'+d|}[\sin\tfrac{1}{2}\theta_t]^{|\lambda'-\mu'|}[\cos\tfrac{1}{2}\theta_t]^{|\lambda'+\mu'|}. \tag{B6}$$

If $\cos\theta_s = +1$ and $\cos\theta_t = +1$, then $\cos\chi_a = \cos\chi_b = \cos\chi_c = \cos\chi_d = +1$ or -1. Using Eq. (B3) and Eq. (B6), we find

$$\mathfrak{M}_{c'A';D'b'}{}^{cd;ab} \propto (\phi)^{\frac{1}{2}n}, \tag{B7}$$

where

$$n = -|\lambda-\mu| + |\lambda'-\mu'| + |A'\mp a| + |b'\mp b| + |c'\mp c| + |D'\mp d|. \tag{B8}$$

The top signs are for $\cos\chi_i = +1$ and the bottom signs are for $\cos\chi_i = -1$. It is easy to see that n is even integer. To show that n is always positive, we use the inequality

$$|a| + |b| \geqslant |a \pm b| \geqslant |a| - |b|,$$

where a, b are any numbers. Therefore

$$|A'-a| + |b'-b| + |c'-c| + |D'-d| \geqslant |\pm(A'-a) \pm (b'-b) \pm (c'-c) \pm (D'-d)| = |(a-b)-(c-d)-[(D'-b')-(c'-A')]| = |(\lambda-\mu)-(\lambda'-\mu')| \geqslant |\lambda-\mu| - |\lambda'-\mu'|.$$

Therefore $n>0$ for $\cos\chi_i = +1$, similarly for $\cos\chi_i = -1$. We can do the same thing for other cases of $\cos\theta_s = +1$, $\cos\theta_t = -1$; $\cos\theta_s = -1$, $\cos\theta_t = +1$; and $\cos\theta_s = -1$, $\cos\theta_t = -1$. Therefore, none of the $\mathfrak{M}$'s has mixed s and t singularities.