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Inelastic Levinson's Theorem, *CDD* Singularities, and Multiple Resonance Poles*

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Some effects of inelastic channels on elastic partial wave amplitudes are discussed. A Levinson's theorem for both the real part of the phase shift and the total phase of the elastic amplitude is derived. The *CDD* singularities required to make the elastic amplitude calculated by single-channel inelastic N/D equations agree with the many-channel calculation without *CDD* singularities are fully characterized in both the η and R methods. Finally, a simple explanation of multiple resonance poles on different Riemann sheets of the amplitude is given in terms of the analyticity properties of η.

I. INTRODUCTION

In this paper we study some of the effects of the presence of inelastic channels on the properties of elastic partial wave amplitudes, and also how the N/D equations are affected by these additional channels. In particular, we consider the following subjects:

1. *Levinson's Theorem.* In the absence of inelastic channels the change of the phase shift between threshold and infinite energy is related to the number of bound states[1] (*1*). With the assumptions that the amplitude is analytic in the angular momentum l and tends to zero for large values of l, we derive a Levinson's theorem for the real part of the phase shift and for the total phase of the amplitude. The Levinson's theorem for the real part of the phase shift δ involves, as usual, the number of bound states, but also depends upon the number of zeros of the *S-matrix* that retreat through the inelastic cut as the angular momentum becomes large. These zeros correspond to the presence of inelastic resonances (*2*). The Levinson's theorem for the total phase φ is found to depend upon the number of bound states and the number of zeros of *the amplitude* that retreat through the inelastic cut as the angular momentum becomes large.

2. *CDD Singularities.* Several methods have been suggested for the inclusion of inelastic effects in the partial wave dispersion relations for the elastic amplitude

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[1] This is strictly correct in potential theory or when the amplitude is computed by the N/D method with no *CDD* poles.

A (3–5). In these methods the input information consists not only of the discontinuity across the left-hand cut but also of another function used to describe the presence of inelastic channels. We discuss here what *CDD* (*6*) singularities must be added to the N/D equations for two of these methods in order to make the single channel calculation agree with a many-channel N/D calculation (*7*).

In the first single-channel method that we discuss, the D-function has the phase $-\delta$; this gives rise to the Frye-Warnock equations (*3*). In the second method, D is required to have the phase $-\varphi$ and the resulting equations are those of Chew and Mandelstam (*4*).

Rather than solve these methods explicitly and compare them with the matrix N/D method, we exploit the fact that at large l both methods agree with the matrix N/D result without *CDD* poles. Whatever *CDD* singularities are required then emerge from the inelastic cuts when the angular momentum is analytically continued to lower values.

3. *Multiple Resonance Poles*. In the presence of inelastic channels, there are generally many poles on different sheets of the amplitude associated with a given resonance as has been discussed by many authors (*8*). These poles become particularly important in the physical manifestation of the resonance if its position is near the threshold of one of the inelastic channels. A discussion of the analytic properties of the function $\eta = e^{-2\delta_I}$, where δ_I is the imaginary part of the phase shift, leads to a simple understanding of the presence of these multiple poles, which does not depend on an expansion of the amplitude near threshold or a restriction to two-particle inelastic channels.

II. THE ELASTIC AMPLITUDE

In the absence of inelasticity only one real function of energy is needed to specify the partial-wave elastic scattering amplitude $A(s)$. This is the phase shift. When inelastic channels are present, however, two real functions will be needed. These two functions can be introduced in several ways. Two ways, which we shall consider in some detail, are:

$$A = \frac{\eta e^{2i\delta} - 1}{2i\rho} = \frac{e^{2i\varphi} - 1}{2i\rho R}, \tag{2.1}$$

where δ, η, φ, and R are functions of the energy and angular momentum and are real above the elastic threshold. The function $\delta(s)$ is the real part of the phase shift, $\varphi(s)$ is the total phase of the amplitude $A(s)$, and $\rho(s)$ is the phase space factor. The functions R and η can be related to the inelastic cross section σ_{in} in a given partial wave by the formulae:

$$\begin{aligned} R &= 1 + \sigma_{in}/\sigma_{el} \\ \eta^2 &= 1 - \frac{\sigma_{in}\, q^2}{(2l+1)\pi}. \end{aligned} \tag{2.2}$$

TABLE I

RELATIONS BETWEEN R, η, AND THE ANALYTICALLY CONTINUED AMPLITUDE $A(s)$ AND S-MATRIX ($S = 1 + 2i\rho A$)[a]

η	R
$A_+ = \dfrac{\eta e^{2i\delta} - 1}{2i\rho}$	$A_+ = \dfrac{e^{2i\varphi} - 1}{2i\rho R}$
$S_+ = \eta e^{2i\delta}$	$S_+ = 1 + \dfrac{e^{2i\varphi} - 1}{R}$
$\eta^2 = S_+ S_-$	$R = \dfrac{1}{1 - S_+} + \dfrac{1}{1 - S_-}$

[a] A "+" subscript denotes an energy just above the real axis and a "−" subscript denotes an energy just below the axis.

Here σ_{el} is the elastic cross section, q is the center-of-mass momentum, and l is the angular momentum.

We assume that the amplitude $A(s)$ has the familiar analyticity properties in s: it is real analytic with a left-hand cut, and a right-hand cut beginning at s_1. Inelastic thresholds will be denoted by s_i with $s_1 < s_2 < s_3 \cdots$. The only other singularities are the bound-state poles. Some useful relations between η, R and the amplitude above and below the cut are summarized in Table I. We also assume that $A(l, s)$ is an analytic function of the angular momentum l.

III. LEVINSON'S THEOREM FOR THE REAL PART OF THE PHASE SHIFT AND THE TOTAL PHASE OF THE AMPLITUDE

It will be valuable for our consideration of the N/D equations in the next section as well as of interest for its own sake to derive a Levinson's theorem for the phases δ and φ (*1*). We shall assume in this section only the analyticity properties mentioned in Section II and the following:

a. $A(l, s) \to 0$ as $s \to \infty$ so that $S(l, s) \to 1$ as $s \to \infty$.

b. $A(l, s) \to 0$ as $l \to \infty$ for all energies except possibly at the branch points. Therefore, $S(l, s) \to 1$ as $l \to \infty$.

c. The discontinuity across the left-hand cut is finite.

d. For sufficiently large l there are no poles or zeros of $S(l, s)$ for any s.

Since S carries the phase 2δ on the right-hand cut, we obtain an expression for the change in δ between $s = s_1$ and $s = \infty$ by considering the contour integral of the logarithmic derivative of S:

$$I = \int_C ds' \frac{S'(s')}{S(s')} = 2\pi i(N_0 - N_B) \tag{3.1}$$

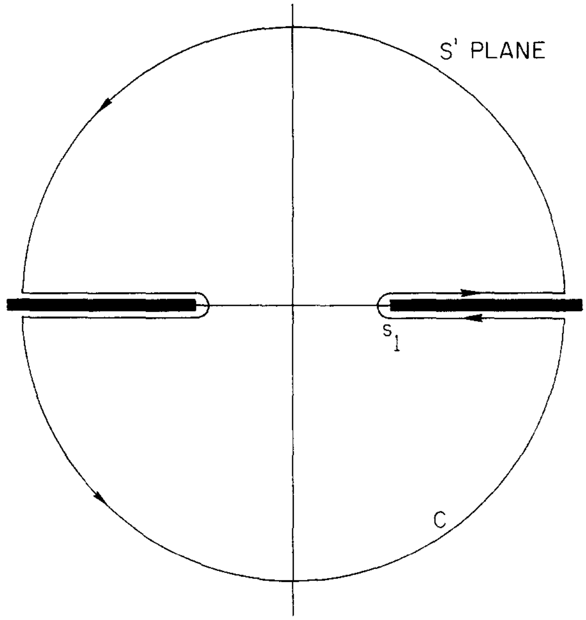

Fig. 1

where the contour C is shown in Fig. 1. The integers N_0 and N_B are the number of zeros and poles of S respectively.

These zeros and poles of S may be classified by their behavior as the angular momentum l is varied. At large l the S matrix approaches 1 for all energies and no zeros or poles are present on the physical sheet. As l is decreased, zeros can emerge from the left-hand cut or the right-hand cut.

On the sheet that is reached by going through the elastic cut there will be a pole corresponding to each zero on the physical sheet (this follows from elastic unitarity). Those poles which retreat through the left-hand cut on this sheet as l becomes large are called elastic resonance poles and those which retreat through the right-hand cut are called inelastic resonance poles (*2*).

As l is decreased, some of the poles on the second sheet may move onto the first sheet to become bound states (the corresponding zero moves onto the second sheet at the same time). The classification of the poles into inelastic and elastic resonances may thus be extended to a classification of the bound states as elastic bound states or inelastic bound states.

We denote the number of inelastic and elastic bound states by N_{IB} and N_{EB} respectively. Thus

$$N_B = N_{IB} + N_{EB} . \tag{3.2}$$

Inelastic or elastic resonance poles always occur in complex conjugate pairs

(2) so that for each bound state pole there exists a companion resonance pole located on the real energy axis below threshold on the second sheet. We denote by $2N_{\text{I}}$ the number of inelastic resonance poles that are *not* companions for inelastic bound state poles. In a similar way the number of elastic resonance poles that are not companions to elastic bound states will be denoted by $2N_{\text{E}}$. Since every zero of S on the first sheet corresponds to a pole on the second sheet, the total number of zeros is

$$N_0 = 2N_{\text{E}} + 2N_{\text{I}} + N_{\text{IB}} + N_{\text{EB}} . \tag{3.3}$$

We now return to an evaluation of Eq. (3.1). Following closely at this point the development given by Hwa (9), we note that the integrals over the semicircular contours at infinity in Fig. 1 vanish and we can thus write

$$I = \ln S \,|_{C_{\text{R}}} + \ln S \,|_{C_{\text{L}}} , \tag{3.4}$$

where C_{L} and C_{R} are contours around the left-hand and right-hand cuts. The first term on the right side of Eq. (3.4) can be evaluated in terms of the change in the phase shift:

$$\ln S \,|_{C_{\text{R}}} = 4i[\delta(\infty) - \delta(s_1)]. \tag{3.5}$$

In order to evaluate the term $\ln S \,|_{C_{\text{L}}}$ it is convenient to begin at high angular momentum, $l \to \infty$. In this limit, it follows from assumption (b) that $\ln S \,|_{C_{\text{L}}} = 0$.

We now continue S to lower values of l. Since the left-hand cut has a finite discontinuity (assumption (c)), no poles can emerge from the left-hand cut. The integral around C_{L} in Eq. (3.1) thus changes by $2\pi i$ every time a zero crosses the contour and we can write

$$\ln S \,|_{C_{\text{L}}} = 2\pi i N_0^{\text{L}} \tag{3.6}$$

where N_0^{L} is the number of zeros that emerge from the left-hand cut as l is decreased. But N_0^{L} is just the number of elastic resonance poles plus those poles which have become elastic bound states so we have

$$N_0^{\text{L}} = 2(N_{\text{E}} + N_{\text{EB}}). \tag{3.7}$$

Combining Eqs. (3.1)–(3.7), we have the Levinson's theorem for the real part of the phase shift

$$\delta(s_1) - \delta(\infty) = \pi(N_{\text{EB}} - N_{\text{I}}). \tag{3.8}$$

If there are no inelastic channels, the second term is absent and the usual Levinson's theorem holds. As we see from Eq. (3.8), the presence of inelastic channels modifies Levinson's theorem by an amount $-\pi N_{\text{I}}$, where N_{I} is the number of inelastic resonances. (We assume in this discussion that poles on the second sheet that emerge from the right-hand cut as l is decreased do not pass through the left-hand cut on the second sheet and vice versa.)

In an analogous manner, we can derive a Levinson's theorem for the total phase φ of the amplitude A. We define the contour integral $\bar{I}$ by:

$$\bar{I} = \int_C ds' \frac{A'(s')}{A(s')} = 2\pi i(\bar{N}_0 - N_B), \tag{3.9}$$

where N_B represents the number of bound state poles and $\bar{N}_0$ is the number of zeros of A. The contour C is the same as in Fig. 1 and we write, as before,

$$\bar{I} = \ln A \,|_{C_R} + \ln A \,|_{C_L} . \tag{3.10}$$

Here we have

$$\ln A \,|_{C_R} = 2i[\varphi(\infty) - \varphi(s_1)]. \tag{3.11}$$

We shall also assume that under some variation of the angular momentum or coupling strengths, the zeros and poles of A will emerge from the right-hand or left-hand cuts of A. As before, we shall suppose that no poles of A will emerge from the left-hand cut. Thus we may write

$$\ln A \,|_{C_L} = 2\pi i \bar{N}_0^{L} \tag{3.12}$$

where $\bar{N}_0^L$ is the number of zeros of A associated with the left-hand cut. If $\bar{N}_0^R$ is the number of zeros of A on the physical sheet which came from the right-hand cut, we have $\bar{N}_0 = \bar{N}_0^L + \bar{N}_0^R$ and deduce that[2]

$$\varphi(s_1) - \varphi(\infty) = \pi(N_B - \bar{N}_0^R). \tag{3.13}$$

The physical interpretation of Eq. (3.13) is less direct than that of Eq. (3.8) since the number of zeros of A is not closely linked with the number of resonances or bound states.

IV. *CDD* SINGULARITIES IN INELASTIC *N/D* METHODS

In this section we consider the problem of solving a set of single channel N/D equations for the elastic amplitude $A(s)$. Our goal is to clarify and to compare the role of *CDD* singularities (singularities of D which are not singularities of the amplitude) for the D functions defined in two ways of formulating this problem. What we discuss here are not the *CDD* singularities associated with elementary particles. Our concern is rather with those *CDD* singularities that must be introduced into a single channel calculation of A in order to make it agree with the more complete method of incorporating inelastic states by means of the matrix N/D technique.[3]

The two methods we consider are distinguished by the way in which D is

[2] If some of the zeros that come from the left-hand cut move off the physical sheet through the right-hand cut, Eq. (3.13) must be modified accordingly.

[3] See in this connection ref. *10*.

defined. In both methods for the cases we study D can be defined such that:

a. $D(s)$ is an analytic function of s with a right-hand cut and possible *CDD* singularities;

b. $D(s) \to 1$ as $s \to \infty$;

c. the zeros of $D(s)$ are in one-to-one correspondence with the bound states of $A(s)$.

In order to complete the definition we give the phase of D on the right-hand cut. In the method we shall call the η method, D_η has the phase $-\delta(s)$ on the right-hand cut (*3*); in the method we shall call the R method, D_R has the phase $-\varphi(s)$ on the right-hand cut (*4*). We shall distinguish the functions employed in the two methods by the subscripts η and R. The relationships between the S-matrix and N and D defined by these requirements are given in Table II. For a derivation and discussion of the integral equations that result from these definitions we refer the reader to the original papers of Frye and Warnock (*3*) (the η method) and Chew and Mandelstam (*4*) (the R method). The equations that relate D to N are given in Table II.

As a basis for our discussion of *CDD* singularities, we may imagine the following procedure:

1. The matrix N/D equations are solved without *CDD* poles;
2. From this solution the functions R and η are computed;
3. The single channel N/D equations employing the R and η methods are then solved.

We then ask when *CDD* singularities must be included in these methods to make the single channel calculations agree with $A(s)$ calculated by the matrix N/D method.

It is more convenient, however, to adopt a procedure that from the beginning assumes and exploits the analyticity of the amplitude in the angular momentum. At large l we shall assume that the amplitude is tending to zero. An examination of Eq. (2.1) reveals

$$\begin{aligned} \delta(s) &\to 0 \\ \eta(s) &\to 1. \end{aligned} \tag{4.1a}$$

If we further assume that the imaginary part of A tends to zero faster than the real part,[4] we have also

$$\varphi(s) \to 0. \tag{4.1b}$$

Only the first assumption will be used in our discussion of the η method while

[4] Thus we exclude here a problem in which the elastic force B_{11} is identically zero. For such a B_{11} the following conclusions about the number of *CDD* poles at large l are not valid since in the absence of *CDD* poles the amplitude A_{11} obtained from the single-channel R method equations would be identically zero.

TABLE II

A Comparison of the R and η Methods for Including Inelasticity in the Single Channel N/D Method

η Method	R Method
$A_+ = N_\eta^+/D_\eta^+$	$A_+ = N_R^+/D_R^+$
$N_\eta^+ = \dfrac{\eta D_\eta^- - D_\eta^+}{2i\rho}$	$N_R^+ = \dfrac{D_R^- - D_R^+}{2i\rho R}$
$S_+ = \eta D_\eta^-/D_\eta^+$	$S_+ = 1 + (D_R^-/D_R^+ - 1)/R$
$D(s) = 1 - \dfrac{1}{\pi}\displaystyle\int_{s_1}^{\infty} \dfrac{2\rho(s')\,\mathrm{Re}\,N(s')}{(s'-s)(1+\eta(s'))}\,ds'$	$D(s) = 1 - \dfrac{1}{\pi}\displaystyle\int_{s_1} ds'\,\dfrac{\rho(s')R(s')N(s')}{s'-s}$

the more restrictive second assumption will be employed when the R method is considered.

As a consequence of Eq. (4.1) the phase differences $\delta(s_1) - \delta(\infty)$ and $\varphi(s_1) - \varphi(\infty)$ vanish at large l and D functions satisfying requirements $a - c$ may be written as

$$D_R(s) = \exp\left[-\frac{1}{\pi}\int_{s_1}^{\infty} ds'\,\frac{\varphi(s')}{s'-s}\right]$$
$$D_\eta(s) = \exp\left[-\frac{1}{\pi}\int_{s_1}^{\infty} ds'\,\frac{\delta(s')}{s'-s}\right] \qquad (4.2)$$

where we put $\varphi(\infty) = \delta(\infty) = 0$. These D functions have no CDD singularities and we conclude that with our assumptions about the force no CDD singularities are required at large l in either method.

We shall now analytically continue these D functions to lower values of l; any necessary CDD singularities will then emerge from the cuts in the problem. This is equivalent to the procedure outlined above because the many channel N/D equations give solutions that are analytic in l if the force is. The many channel N/D equations also do not require CDD poles as l is decreased. We shall see that although a close analogy exists between the η and R methods, the presence of CDD singularities in one method does not imply their occurrence in the other.

A. The R-Method

We shall now examine what happens to D_R as l is decreased. Zeros or poles can emerge from the cut of D_R as l is decreased, but no other form of singularity, since otherwise the amplitude would not have the assumed analyticity in the energy variable. The zeros of D_R which emerge are bound state poles of the amplitude. The emergence of a pole or zero in D_R will correspond to a logarithmic singularity in the phase φ and the contour integral over φ in Eq. (4.2) will have to be distorted. As long as the distorted contour is not dragged to infinity, that is, as long as the emerging pole or zero of D_R remains at a finite point of the energy plane, D_R will maintain its normalization to one as $s \to \infty$. This enables us to exploit the Levinson's theorem for φ, Eq. (3.13). Setting $\varphi(\infty) = 0$, we have

$$\varphi(s_1) = \pi(N_{\mathrm{B}} - N_0^{\mathrm{R}}). \tag{4.3}$$

Near $s \approx s_1$ we have

$$\exp\left[-\frac{1}{\pi}\int_{s_1}^{\infty} ds' \frac{\varphi(s')}{s' - s}\right] \to \text{const } (s - s_1)^{N_{\mathrm{B}} - \bar{N}_0{}^R}. \tag{4.4}$$

Now since D_R is generally finite and nonzero at $s = s_1$, the only form for D_R consistent with the asymptotic property (b) is

$$D_R = \frac{\prod_{i=i}^{N_{\mathrm{B}}} (s - s_{\mathrm{B}i})(s - s_1)^{\bar{N}_0{}^{\mathrm{R}}}}{\prod_{j=1}^{\bar{N}_0{}^{\mathrm{R}}} (s - s_{\mathrm{P}j})(s - s_1)^{N_{\mathrm{B}}}} \exp\left[-\frac{1}{\pi}\int_{s_1}^{\infty} ds' \frac{\varphi(s')}{s' - s}\right] \tag{4.5}$$

where $s_{\mathrm{B}i}$ indicates the location of bound states and $s_{\mathrm{P}i}$ the location of poles of D_R. There are thus $\bar{N}_0{}^{\mathrm{R}}$ *CDD* poles and the emergence of *CDD* poles in D_R is concomitant with the emergence of zeros from the *right-hand*[5] cut of the amplitude A. This, in turn, means that N can have no *CDD* poles at $s_{\mathrm{P}i}$ since then A would be nonzero at these points.

We now demonstrate that the *CDD* poles, which have been shown to correspond to zeros of the amplitude which emerge from the right-hand cut, must emerge from the inelastic part of the cut. First we show that such zeros of the amplitude cannot come from the elastic cut. On the elastic cut we can write

$$S = \frac{D_R^-}{D_R^+} \tag{4.6}$$

where the "+" and "−" refer to above and below the cut. Suppose we are at large l where no *CDD* poles (i.e., zeros of the amplitude) have come onto the physical sheet. If an emerging *CDD* pole is on the sheet reached by going through the elastic cut, then we may continue Eq. (4.6) downward into the complex s-plane until D_R^+ has a pole. But in order for S to be one at this point and the amplitude to vanish, D_R^- must also have a pole at the same position. The function D_R^-, however, is now evaluated in the lower-half s-plane and this contradicts our assumption that there are no *CDD* poles on the physical sheet. Thus no *CDD* poles can come from the elastic cut.

To see that poles can consistently emerge from the inelastic right-hand cut we consider the following expression for S evaluated above the inelastic threshold:

[5] The emergence of zeros of A from its left-hand cut corresponds to zeros of N.

$$S_+ = 1 + \frac{1}{R}\left(\frac{D_R^-}{D_R^+} - 1\right). \tag{4.7}$$

Again we consider large l where no *CDD* poles have yet emerged onto the physical sheet. We now continue downward from the real energy axis through an inelastic cut in order to locate a *CDD* pole. Since

$$R = \frac{1}{1 + S_+} + \frac{1}{1 - S_-}, \tag{4.8}$$

R will also have a pole at the location of the *CDD* pole on this second inelastic sheet. So we see from Eq. (4.7) that $S = 1$ at the *CDD* pole as it should.

We may, in fact, also see the emergence of the *CDD* pole directly from the dispersion equation for D_R :

$$D_R(s) = 1 - \frac{1}{\pi}\int_{s_1}^{\infty} ds' \frac{\rho(s')R(s')N(s')}{s' - s}. \tag{4.9}$$

As we have already noted, the occurrence of a *CDD* pole in the D-function does not result in a corresponding *CDD* pole in the N-function, so from Eq. (4.7) we see that a pole of R crossing the real axis distorts the contour of the integral and gives rise to a pole in D_R. This again demonstrates why the pole must come from the inelastic cut as R is one at all points on the elastic cut.

B. The η Method

In the η method as l is decreased, singularities of D_η as given by Eq. (4.1) may emerge from the right-hand cut. In contrast to the R method, singularities other than poles may emerge from the inelastic part of the cut because in the η method, the N-function, N_η, also carries an inelastic right-hand cut. From the inelastic cut of N_η, cancelling singularities may arise leaving the cut plane analyticity of the full amplitude $A = N/D$ intact. We shall find, in fact, that generally both D_η and N_η have branch-point singularities emerging from the inelastic cut.

As in the R method the poles, zeros or other singularities of D_η will arise from singularities in δ which distort the contour integral over δ in Eq. (4.2). If these distorting singularities do not move off to infinity the normalization of D_η to unity at infinite energies is preserved. The emergence of these singularities of δ generally give rise to a change in the phase shift difference $[\delta(s_1) - \delta(\infty)]$. By Levinson's theorem for δ, Eq. (3.8), this phase shift difference is, in turn, related to the number of bound states and inelastic resonances. The bound states correspond to the emergence of zeros of D_η whereas the inelastic resonances come from zeros of S that migrate onto the physical sheet from the inelastic cut. These zeros of S give rise to cuts in D_η as we shall now demonstrate.

We recall the relation

$$S_+S_- = \eta^2, \tag{4.10}$$

which applies along the inelastic cut where the "+" and "−" subscripts as before refer to above and below the cut. At large l we may continue Eq. (4.10) down into the complex energy-plane to find a zero of S_+ at $s = s_V$. At this point, S_- must be nonzero and finite since it is evaluated on the physical sheet where the boundary condition $S \to 1$, $l \to \infty$ applies. Thus as $s \approx s_V$

$$\begin{aligned} S_+ &\sim (s - s_V) \\ \eta &\sim (s - s_V)^{1/2} \end{aligned} \tag{4.11}$$

We may also employ the relation

$$S_+ = \eta \frac{D_\eta^-}{D_\eta^+} \tag{4.12}$$

to see that

$$D_\eta^+ \sim (s - s_V)^{-1/2} \tag{4.13}$$

since D_η^-, which is evaluated at the corresponding point in the lower-half energy plane of the physical sheet, must be near one in this limit of large l. As l is decreased this inverse-square root singularity of D_η at $s = s_V$ and a corresponding branch point at the mirrored position $s = s_V^*$ may move onto the physical sheet, corresponding to a pair of zeros of S migrating onto the physical sheet from the inelastic cut.

It is clear that N_η must cancel the branch point in D_η at $s = s_V$ since S has just a simple zero at this point. This fact can also be seen directly from the relation

$$N_\eta^+ = \frac{\eta D_\eta^- - D_\eta^+}{2ip}. \tag{4.14}$$

A detailed examination of the question of analytically continuing the integral equation for N_η in l is given in ref. *2*.

We emphasize here that the *CDD* requirements in the R and η methods are generally quite different. As we have seen, *CDD* poles in the R method are associated with zeros of the amplitude that emerge onto the physical sheet from the inelastic cut. In the η method, *CDD* singularities become required when zeros of S come onto the physical sheet from the inelastic cut. In the η method, there is a simple physical criterion for *CDD* poles, since the emerging zeros of S correspond to inelastic resonance poles being fed into the second elastic sheet of the amplitude. The requirement of *CDD* poles in the R method, on the other hand, appears to have no simple connection with poles in the amplitude.

V. PROPERTIES OF η AND MULTIPLE RESONANCE POLES

Recently, it has been emphasized by many authors that resonance poles in the scattering amplitude do not occur singly (*8*). A given resonance will generally manifest itself as a pole on many Riemann sheets of the amplitude. Thus if one increases the angular momentum (or decreases the all-over coupling strength), a resonance that progresses through a threshold during this process will have different poles producing the "bump" in the cross section, depending upon whether the resonance energy is above or below the threshold.

We shall show here that these many poles associated with a given resonance have a simple interpretation in terms of the analyticity properties of η. The discussion that we give here frees one from the need to discuss threshold expansions and also is not restricted to two-body inelastic channels. A discussion of multiple resonance poles without using threshold expansions has also been given by Eden and Taylor (*8*).

Although we have not explicitly considered the fact up to now, actually there are n different analytic functions η where n is the number of channels. We shall write η_i to represent the function that is appropriate in Eq. (2.1) between the ith and $(i + 1)$st thresholds. Thus $\eta_1 = 1$. The functions η_i are real between the ith and $(i + 1)$st thresholds but will generally be complex if continued to other regions. Let us suppose a resonance pole at s_V is present on the second elastic sheet, that is, the sheet reached from a point P on the physical sheet by continuing down through the elastic physical region as shown in Fig. 2. The various Riemann sheets that are adjacent to the physical region are labelled in Fig. 2 as $R_1, R_2, \cdots$. The resonance pole we are considering is on sheet R_1. We may now write

$$S_+^i S_-^i = \eta_i^2 \tag{5.1}$$

where S_+^i denotes S evaluated slightly above the real axis between the ith and $(i + 1)$st thresholds. By analytically continuing Eq. (5.1) down to the point $s = s_V$, we find as in Section IV that S_-^i has a simple zero at $s = s_V$:

$$\eta_i^2 = \text{const.}\ (s - s_V) \qquad s \approx s_V . \tag{5.2}$$

Now we imagine the coupling between the channels $2, 3, \cdots i$ and channel 1 being gradually and analytically switched off. During this process the function η_i^2 must approach unity at all points. Thus η_i^2 must have a pole at some point $s = s_P^i$ where $s_P^i \to s_V$ during the decoupling just described in order for η_i^2 to approach unity. That is,

$$\eta_i^2 = \frac{s - s_V}{s - s_P^i}\, \bar{\eta}_i^2 \tag{5.3}$$

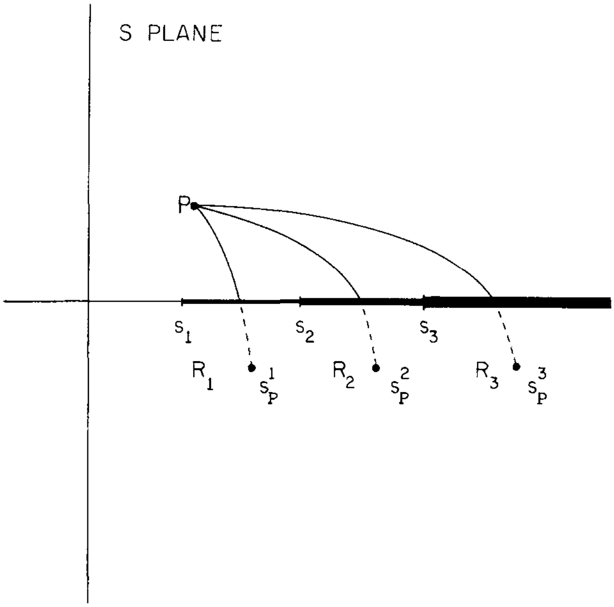

FIG. 2

where $\bar{\eta}_i^2$ is neither singular nor zero at s_V and s_P^i.

Continuing Eq. (5.1) down to the point $s = s_P^i$, we find that S_+^i will have a pole there since S_-^i, which is now evaluated on the physical sheet, should be nonsingular at this point. Thus the existence of a pole in the amplitude on sheet R_1 of Fig. 2 leads to the presence of poles at $s = s_P^i$ on the sheets R_i. There are in addition, of course, mirror image poles to all of these just discussed on Riemann sheets reached by analytically continuing upward from below the physical cut.

We can, of course, invert the above argument. That is, a pole on the sheet R_i for $i > 1$ gives rise to a pole of η_i^2. From this we conclude that there must be a zero of η_i^2 at $s = s_V$. Since $S_+^i S_-^i = \eta_i^2$, the zero at $s = s_V$ may either occur in S_+ or S_-. In either event it can be shown that n poles occur on n different sheets of the amplitude, not including the mirror image poles.[6]

These poles associated with a given resonance are just those discussed recently by a number of authors (*8*). By making use of threshold expansions they have been able to show that if s_P^i is close to the threshold $(i + 1)$ then s_P^{i+1} is also near this threshold. The existence of these poles, as we have seen, follows simply from the analyticity properties of η.

[6] The poles considered here are on the sheets adjacent to the physical region and are, therefore, the most important for the physical manifestation of a resonance. There are, in addition, other correlated poles on more distant sheets of the S-matrix. These poles are discussed by Eden and Taylor (*8*).

VI. CONCLUSION

Much information is contained in the assumption that the scattering amplitude is an analytic function of the angular momentum l. By making use of this analyticity, we have been able to connect by analytic continuation the region of high l, where properties of the amplitude are generally simple, to regions of lower l, where resonances and bound states may occur. By making use of this connection between high and low l, we have been able to deduce Levinson's theorems for the real part of the phase shift and for the total phase of the scattering amplitude. The number of multiples of π by which the real part of the phase shift changes between threshold and infinite energy is equal to the number of elastic bound states minus the number of inelastic resonance poles which are not companions of inelastic bound states. The number of multiples of π by which the total phase of the scattering amplitude changes between threshold and infinite energy is equal to the number of bound states minus the number of zeros of the amplitude that emerge through its right-hand cut.

With reasonable assumptions about the forces, we have been able to give a complete analysis of *CDD* singularities that are required in the R and the η N/D methods in order to make their solutions agree with those of the matrix N/D method. In the R method *CDD* poles are required whenever there is a zero of the amplitude on its physical sheet that retreats through the right-hand cut at large l. In the η method *CDD* singularities are required whenever there is a zero of the S-matrix on its physical sheet that retreats through the right-hand cut at large l. In the R method the *CDD* singularities are poles, but the criteria for these poles are difficult to state in physical terms. In the η method the *CDD* singularities are usually branch points; however, the locations of these singularities have a simple interpretation as the positions of inelastic resonances (*2*).

Finally, by assuming analyticity in the interchannel coupling strengths, we have demonstrated in a simple way the existence of many poles on different Riemann sheets of the amplitude all corresponding to one resonance.

Note Added: After this work was completed our attention was drawn to a paper by J. Finkelstein (*Phys. Rev* **140,** B111 (1965)) where criteria similar to ours for *CDD* poles in the R method are obtained.

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