Infinitely Rising Regge Trajectories and Crossing Symmetry

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It is found that, in a crossing-symmetric model of a scattering amplitude $A(s,t)$ which is dominated by Regge-pole exchange, the restrictions imposed by analyticity on the Regge trajectory $\alpha(s)$ and by analyticity and polynomial boundedness in s for fixed t on $A(s,t)$, suggest uniquely the asymptotic behavior $\alpha(s)\sim s/\ln s$ as $s\to\infty$, if the large-s and large-t limits of $\ln A(s,t)$ are uniform. The use of a double-dispersion relation for $\ln A(s,t)$ is proposed.

In this paper, the problem of ensuring asymptotic crossing symmetry between two channels of a two-particle scattering amplitude which is dominated by Regge-pole exchange will be considered. In particular, the requirements on the trajectory function $\alpha(s)$ and the residue function $\beta(s)$ will be elucidated when

$$\mathrm{Re}\alpha(s) \underset{s\to\infty}{\longrightarrow} \infty\,. \tag{1}$$

The problem is of relevance, among other things, to crossing-symmetric finite-energy sum-rule bootstraps, which have been treated by the present authors,[1] as well as by others.[2]

Certain subtle restrictions on the permitted form of $\alpha(s)$ are implied by requiring its analyticity and polynomial boundedness. Moreover, although the scattering amplitude $A(s,t)$ does not satisfy a Mandelstam representation with a finite number of subtractions, one expects that $\ln A(s,t)$ does satisfy such a representation, although, in general, there will be extra "dynamical branch points" arising from zeros of $A(s,t)$.

The task is to construct a function $A(s,t)$ which has the following asymptotic behaviors (in this paper, $A(s)\sim_{s\to\infty}B(s)$ means $A(s)/B(s)\to_{s\to\infty}1$):

$$A(s,t) \underset{s\to\infty}{\sim} \beta(t)(-s)^{\alpha(t)}\,, \tag{2}$$

$$A(s,t) \underset{t\to\infty}{\sim} \beta(s)(-t)^{\alpha(s)}\,, \tag{3}$$

where signature and ghost-eliminating factors have been absorbed into $\beta(s)$. Suppose that $\alpha(s)$ does not increase more quickly than $s^2/\ln^2 s$ as $s\to\infty$ for any direction in the complex plane, and that it has only a right-hand cut. Then

$$\alpha(s)=a+bs+\frac{s^2}{\pi}\int^\infty \frac{ds'\,\mathrm{Im}\alpha(s')}{s'^2(s'-s)}\,. \tag{4}$$

If Eq. (1) is satisfied, but if in fact

$$\mathrm{Re}\alpha(s)/s^{1/2} \underset{s\to\infty}{\not\longrightarrow} \infty\,,$$

then $b=0$, and it may be shown from the Boas theorem[3] that

$$\alpha(s) \underset{s\to-\infty}{\longrightarrow} +\infty\,.$$

However, this would lead to a flagrant violation of the Froissart theorem,[4] which requires $\alpha(s)\leqslant 1$ for $s\leqslant 0$. Hence, in fact, one must require

$$\mathrm{Re}\alpha(s)/s^{1/2} \underset{s\to\infty}{\longrightarrow} \infty\,; \tag{5}$$

consequently, the trivial solution $\beta(s)=\text{const}$, $\alpha(s)=C\ln(-s)$, and $C>0$, which is obtained by simply equating the right-hand sides of Eqs. (2) and (3), can be definitely ruled out.

Thus, the term in some imagined expansion of $A(s,t)$ which gives rise to the large-s behavior [Eq. (2)] and the one which gives rise to the large-t behavior cannot be identical. The most straightforward possibility is that envisaged in the interference model,[5] where one simply adds the two terms, obtaining

$$A(s,t)\approx\beta(t)(-s)^{\alpha(t)}+\beta(s)(-t)^{\alpha(s)}\,. \tag{6}$$

For large s, it is required that the second term does not interfere with the first, and this can be arranged by asking that $\beta(s)[K(s)]^{\alpha(s)}$ remain bounded as $s\to\infty$, where

$$K(s) \underset{s\to\infty}{\longrightarrow} \infty\,.$$

However, since $\alpha(s)$ satisfies Eq. (5), it is to be expected that $\beta(s)$, and therefore $A(s,t)$, would diverge faster than any power of $(-s)$ as $s\to-\infty$. However, $A(s,t)$ should be polynomially bounded in s, for fixed t (although the order of the polynomial increases as t increases). This assumption, for example, ensures the

[1] K. Dietz, J. Honerkamp, and J. Kupsch, Nucl. Phys. **B6**, 639 (1968); D. Atkinson, K. Dietz, and J. Honerkamp, Z. Physik **216**, 281 (1968).

[2] M. Ademollo, H. R. Rubinstein, G. Veneziano, and M. A. Virasoro, Phys. Rev. Letters **19**, 1402 (1967); C. Schmid, *ibid.* **20**, 628 (1968); D. J. Gross, Phys. Letters **19**, 1303 (1967).

[3] R. P. Boas, *Entire Functions* (Academic Press Inc., New York, 1954); N. N. Khuri, Phys. Rev. Letters **18**, 1094 (1967).

[4] M. Froissart, *Theoretical Physics* (International Atomic Energy Agency, Vienna, 1963), p. 379.

[5] V. Barger and D. Cline, Phys. Rev. **155**, 1792 (1967); N. N. Khuri, Phys. Rev. Letters **16**, 913 (1966).

existence of the Froissart-Gribov transform. Thus the interference model, at least in the crude form in which it is given here, cannot be satisfactory either. The truth must lie somewhere between the two extremes of having one term that reproduces both the s- and t-channel asymptotics, and having the sum of two terms, one for each channel.

As in Ref. 1, the following expression will be used for $A(s,t)$:

$$A(s,t)=\beta(s)[\alpha(s)+g(t,s)]^{\alpha(s)}, \qquad (7)$$

where $g(t,s)$ will be chosen so that Eqs. (2) and (3) are satisfied. As has already been explained, $A(s,t)$ does not satisfy a Mandelstam representation with a finite number of subtractions, but $\ln A(s,t)$ will be polynomially bounded, and so it, as well as $g(t,s)$, will, in general, satisfy a double-dispersion representation, with extra branch points, arising from the zeros and poles of $A(s,t)$ and $\beta(s)$ [see Eq. (17) below]. Consider first, however, what would happen if $g(s,t)$ had no extra branch points, and suppose that the order of the s and t integrals can be freely interchanged.

To satisfy Eq. (3), one must have

$$g(t,s)/t \underset{t\to\infty}{\longrightarrow} -1. \qquad (8)$$

For large s, the form (7) reduces to

$$A(s,t) \underset{s\to\infty}{\sim} \beta(s)[\alpha(s)]^{\alpha(s)} \exp[g(t,s)]; \qquad (9)$$

and, for this to agree with Eq. (2), one must have

$$g(t,s) \underset{s\to\infty}{\sim} \alpha(t)\ln(-s)+\ln\beta(t)-\ln[\beta(s)\alpha(s)^{\alpha(s)}]. \quad (10)$$

The simplest situation arises if

$$g(t,s)/\alpha(s) \underset{s\to\infty}{\longrightarrow} 0; \qquad (11)$$

and then, from Eq. (10), one must have

$$\ln\beta(s) \underset{s\to\infty}{\sim} -\alpha(s)\ln\alpha(s), \qquad (12)$$

a behavior that has been suggested by Jones and Teplitz.[6]

Consider the double-dispersion relation for $g(t,s)$:

$$g(t,s)=P(t,s)+\frac{s^{m'}}{\pi}\int ds'\frac{\rho_{(s)}(s')}{s'^{m'}(s'-s)}+\frac{t^{n'}}{\pi}\int dt\,\frac{\rho_t(t')}{t'^{n'}(t'-t)}$$
$$+\frac{s^m t^n}{\pi^2}\int ds'\int dt'\frac{\rho_{st}(s',t')}{s'^m t'^n(s'-s)(t'-t)}, \qquad (13)$$

where $P(t,s)$ is a polynomial in s and t. In order to reproduce the behavior (10), it must be possible to express those parts of Eq. (13) that depend on s (i.e., the double integral, the s single integral, and the part of the subtraction polynomial not independent of s) in terms of the following large-s expansion:

$$\alpha(t)\ln(-s)+f(t)+O(1), \qquad (14)$$

where $f(t)$ is some function of t only. On the other hand, for fixed s and increasing t, $g(t,s)$ must tend to $-t$, independently of s. This behavior cannot come from the double integral in Eq. (13), which *must* give an s-dependent leading term in the large-t expansion (there is no function that has a constant Hilbert transform). Hence, the $-t$ behavior must come from the rest of Eq. (13) (i.e., the s-independent part of the subtraction polynomial and the single integral over t'). It follows, then, that $f(t)$ in Eq. (14) must tend to infinity less quickly than t as $t\to\infty$, for otherwise there would be an s-dependent term which interferes in the large-t limit. This means that the large-t behavior of $\ln\beta(t)$ in Eq. (10) will be given by the s-dependent parts of Eq. (13), and hence must be

$$\ln\beta(t) \underset{t\to\infty}{\sim} -t; \qquad (15)$$

and then Eq. (12) gives

$$\alpha(s) \underset{s\to\infty}{\sim} s/\ln s. \qquad (16)$$

This rather remarkable result, namely, that $\alpha(s)$ and $\beta(s)$ are asymptotically determined, rests crucially on the assumption that the large-t limit is uniform in s, which would follow, for example, if the double spectral function in Eq. (13) had only a finite number of oscillations as a function of s.[7]

No attention has been paid so far to the extra "dynamical branch points" that $g(t,s)$ may have because of the zeros of $A(s,t)$:

$$g(t,s)=[A(s,t)/\beta(s)]^{\alpha^{-1}(s)}-\alpha(s)$$
$$\underset{s\to\infty}{\sim} \ln A(s,t). \qquad (17)$$

Suppose, in fact, that $A(s,t)$ vanishes on the surfaces

$$S=f_i(t), \quad i=1,2,\cdots,n. \qquad (18)$$

We define

$$\tilde{A}(s,t)\equiv A(s,t)/\prod_{i=1}^{n}[s-f_i(t)], \qquad (19)$$

[6] C. E. Jones and V. L. Teplitz, Phys. Rev. Letters 19, 135 (1967).

[7] There is an apparent contradiction between the result (16) and the model of Veneziano [G. Veneziano, Nuovo Cimento 57, 190 (1968)], which was brought to the authors' notice after the completion of the present paper. Veneziano satisfies crossing symmetry with linear trajectories. Strictly speaking, however, his form does not have Regge asymptotics [Eq. (2)] because the resonance poles necessarily lie on the real axis. The present paper indicates that when $\alpha(s)$ is allowed to have an imaginary part, then crossing symmetry forces the asymptotic form (16). Thus the Veneziano model cannot be quite right; but, since the behavior (16) is almost linear, the model may not be badly wrong.

which has no zeros. Then

$$\ln\tilde{A}(s,t)=\ln A(s,t)-\sum_{i=1}^{n}\ln[s-f_i(t)]. \qquad (20)$$

So long as n is finite and $f_i(t)$ is polynomially bounded in t, the sum in Eq. (20) behaves only like $\ln t$ for large t, and none of the previous arguments are affected. On the other hand, if these conditions are violated, a nonuniformity in the large-t limit is possible. For example, the zeros of $A(s,t)$ could produce a term

$$-\ln[1+h(s)e^{f(t)}] \qquad (21)$$

in the expression (17) for $g(t,s)$, where

$$h(s) \underset{s\to\infty}{\longrightarrow} 0, \qquad (22)$$

$$f(t) \underset{t\to\infty}{\sim} t. \qquad (23)$$

Clearly, this example goes beyond the class of functions that have been considered so far, since (21) tends to zero for large s, and to $-t$ for large t, but in a nonuniform way.[8] Thus it is not claimed that the behaviors (14) and (15) have been proved in general; but they would seem to be the simplest ways of satisfying Eqs. (2) and (3). Within this class of solution, it is to be expected that a Mandelstam representation for $\ln A(s,t)$ might be of some practical or theoretical use, whereas, for more general possibilities, this would seem more doubtful.

It can be shown that, if Eq. (11) is *not* true, then $g(t,s)$ is not polynomially bounded in s and t, and so falls outside the class of amplitudes for which one can write a double-dispersion relation for $\ln A(s,t)$. For in this case, Eq. (9) is replaced by

$$A(s,t) \underset{s\to\infty}{\sim} \beta(s)g(t,s)^{\alpha(s)}\exp[\alpha^2(s)/g(t,s)]. \qquad (24)$$

If this is to be consistent with Eq. (2), one must have

$$g(t,s) \underset{s\to\infty}{\sim} \exp\left[\frac{1}{\alpha(s)}\left(\alpha(t)\ln(-s) + \ln\beta(t)-\ln\beta(s)-\frac{\alpha^3(s)}{g(t,s)}\right)\right]. \qquad (25)$$

Because $\beta(s)$ *must* decrease more quickly than $\alpha(s)^{-\alpha(s)}$ [since Eq. (11) is not true in this case], this means that $g(t,s)$ cannot be polynomially bounded in all complex t directions for fixed s because of the $\ln\beta(t)$ term in Eq. (25).

Experimental evidence has been growing to support the idea of linear trajectories[9]; and this behavior would, of course, be experimentally indistinguishable from the form

$$\alpha(s) \underset{s\to\infty}{\sim} s/\ln s$$

that has been uniquely picked out, by the foregoing conclusions, for the asymptotic region. In all the above work, s has been taken dimensionless, the scaling factor being in fact arbitrary. However, a most important point is that the slope of the trajectory is determined to be the same as this scaling factor. Hence, there is only one parameter left to be determined by experiment. If the slope is taken to be 1 GeV^{-2}, as suggested by experiment, then the scaling factor should be 1 GeV^2.

The result of the paper can be interpreted as an asymptotic bootstrap, in which the power law of the trajectory has been determined from very general considerations, and in which the slope has been determined in terms of the energy scale.

The next step, in a detailed bootstrap, would be the inclusion of unitarity, perhaps through the Cheng-Sharp equations,[10] as in a paper by Mandelstam.[11] However, one eventually envisages a treatment going substantially beyond the narrow-resonance model, which can at best be only a crude approximation.[1]

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[8] Examples of crossing-symmetric amplitudes with almost arbitrary trajectories have recently been constructed. However, these examples fall into the class of amplitudes for which the large-s limits are not uniform with respect to t. See, N. N. Khuri. Phys. Rev. **176**, 2026 (1968); C. Caser, J. M. Kaplan, and R. Omnes (unpublished); J. Kupsch, Bonn University (unpublished).

[9] A. H. Rosenfeld *et al.*, Rev. Mod. Phys. **40**, 77 (1968).

[10] H. Cheng and D. Sharp, Ann. Phys. (N. Y.) **22**, 481 (1963); Phys. Rev. **132**, 1854 (1963); S. Frautschi, P. E. Kaus, and F. Zachariasen, *ibid.* **B133**, 1607 (1964).

[11] S. Mandelstam, Phys. Rev. **166**, 1539 (1968).