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Kinematical Singularities, Crossing Matrix and Kinematical Constraints for Two-Body Helicity Amplitudes

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Helicity amplitudes are expressed via the spinor amplitudes in terms of the Joos invariant amplitudes which have been shown by Williams to be free from kinematical singularities. This procedure allows to analyze the kinematical singularities of helicity amplitudes and separate them out, which results in the definition of regularized helicity amplitudes.

A crossing matrix for helicity amplitudes, is written down, corresponding to the continuation path used to cross spinor amplitudes. We verify explicitly that the corresponding crossing matrix for regularized helicity amplitudes is uniform, as it should be.

Kinematical constraints which generalize, to the case of arbitrary spins and masses, relations which must hold between helicity amplitudes at some values of the energy variable in $\pi N \to \pi N$, $\pi\pi \to N\bar{N}$, and $N\bar{N} \to N\bar{N}$ reactions, appear as a consequence of the existence of poles in the crossing matrix between regularized helicity amplitudes.

I. INTRODUCTION

It is a common practice, in most studies concerning analyticity and crossing properties of scattering amplitudes, to deal with the scattering of spinless, equal mass, particles, and to speak of "the inessential complications due to spin". However spin effects are quite important. It is obvious, for instance, that one must know explicitly the crossing matrix when working on exchange models (bootstrap, peripheralism, Regge poles ...). In Regge-like models also, since the only properties one can conjecture on the Regge residues are their analyticity properties, it is important, in order not to make unreasonable assumptions, to distinguish the kinematical singularities from the dynamical ones. It has also been remarked that, in $\pi N \to \pi N$, $\pi\pi \to N\bar{N}$, $N\bar{N} \to N\bar{N}$ scatterings, the helicity amplitudes must satisfy some relations which prevent them from introducing spurious poles into invariant amplitudes which are supposed to enjoy the Mandelstam analyticity properties. Such constraints put important restrictions on any reasonable assumption made for the purpose of Reggeizing amplitudes in the case of arbitrary spins and masses. [Cf. the "conspiracy or evasion" problem (*1*)].

The purpose of this paper is three-fold:

(i) We find the kinematical singularities of the two-body helicity amplitudes.

(ii) We derive a crossing matrix for helicity amplitudes.

(iii) We generalize, to the case of arbitrary spins and masses, the kinematical constraints already observed in $\pi N \to \pi N$, $\pi\pi \to N\bar{N}$ and $N\bar{N} \to N\bar{N}$ reactions.

Section II is devoted to the search for the kinematical singularities of the helicity amplitudes. The starting point is the work of Williams (*2*) who has shown that the Joos expansion (*3*) of spinor amplitudes leads to invariant amplitudes free from kinematical singularities.[1] This means that these invariant amplitudes are analytic functions of the invariants s and t in a domain which is the image of the analyticity domain of the spinor amplitudes in the four-momentum components restricted to the mass-shell; conversely, the inverse image of the analyticity domain in s and t is the whole analyticity domain of the spinor amplitude. The method we use to find the kinematical singularities of the helicity amplitudes consists in writing them as linear combinations of the Joos invariant amplitudes; the kinematical singularities are then the singularities of the coefficients. Appendix AI is devoted to the cases where some of the external masses are equal and where the general analysis does not apply.

In Section III we show that the crossing path used by Bros, Epstein and Glaser (*5*) to cross the spinor amplitudes is also suitable to perform crossing on helicity amplitudes. We derive explicitly the corresponding crossing matrix (up to an overall sign which is determined in Appendix A-II). Our crossing matrix appears to differ by an overall phase from that of Trueman and Wick* (*6*). In Appendix A-III, we check that the crossing matrix between regularized helicity amplitudes (R.H.A) has no branch points, which provides a good test for the correctness of our crossing matrix. Such a test is possible because, unlike some authors (*7*), (*8*), who have been previously interested in such questions, we use a direct method, independent from the crossing problem, to free helicity amplitudes from their kinematical singularities.

Finally, we show in Section IV, (and in Appendix A-IV for special mass configurations), how the cancellation of poles in the elements of the crossing matrix for R.H.A provides a generalization of the kinematical constraints known in particular cases. Here, we use as a tool the so-called transversity amplitudes introduced by Kotanski (*9*).

All our results are explicitly tested (in Appendix A-V) on cases in which the relations between invariant and helicity amplitudes are known, so that the kinematical singularities, the crossing matrices and the kinematical constraints can be directly derived.

[1] The existence of such an expansion had been previously proved by K. Hepp (*4*).

*[1bis] See Table XI and footnote [8bis].

II. KINEMATICAL SINGULARITIES OF TWO-BODY HELICITY AMPLITUDES

1. Notations and Conventions

A. *Lorentz Transformations and SL(2, C) Matrices*

We briefly recall the correspondence between the Lorentz transformations and the unimodular, 2×2 matrices: with a four-vector p, ($p^2 \doteq p_0{}^2 - \mathbf{p}^2$) we associate the 2×2 matrix $p \cdot \sigma = p_0 \mathbf{1} - \mathbf{p} \cdot \boldsymbol{\sigma}$. It is easy to see that:

(a) $\det p \cdot \sigma = p^2$

(b) if A and B are two unimodular 2×2 matrices and $p' \cdot \sigma = Ap \cdot \sigma B^T$, then $p'^2 = p^2$. Thus $(A, B) \in [SL(2, C) \times SL(2, C)]/Z^2$ is associated with a complex Lorentz transformation $\Lambda : p \to p' = \Lambda p$.

(c) if $B^T = A^\dagger$, the Lorentz transformation Λ associated with (A, B) is real.

(d) $B^T = A^{-1}$ corresponds to a complex rotation, which is real if furthermore $A^{-1} = A^\dagger$.

(e) (A, B) and $(-A, -B)$ correspond to the same Lorentz transformation

(f) $p \cdot \tilde{\sigma} = \tilde{p} \cdot \sigma = p_0 \mathbf{1} + \mathbf{p} \cdot \boldsymbol{\sigma}$ is such that $(p \cdot \sigma)(p \cdot \tilde{\sigma}) = (p \cdot \tilde{\sigma})(p \cdot \sigma) = p^2 \mathbf{1}$

(g) for any unimodular 2×2 matrix A, $A^{T-1} = \epsilon^{-1} A \epsilon$, where $\epsilon = i\sigma_2$.

(h) $p^\mu = \frac{1}{2} \operatorname{Tr}(\sigma^\mu p \cdot \tilde{\sigma})$.

With any 2×2, non singular matrix M, we associate the matrix $D^s(M)^\lambda_\mu$ defined by (*3*)

$$D^s(M)^\lambda_\mu = (\det M)^{s-\mu} \left[\frac{(s+\mu)!\,(s-\mu)!}{(s+\lambda)!\,(s-\lambda)!}\right]^{1/2} (M^2_1)^{\mu-\lambda} (M^1_1)^{\mu+\lambda} P^{(\mu-\lambda),(\mu+\lambda)}_{s-\mu}(Z) \qquad \text{(II-1)}$$

where $Z = (M^1_1 M^2_2 + M^2_1 M^1_2)/\det M$ and $P^{a,b}_n(Z)$ is the Jacobi function of the first kind (*10*). If one takes for M a unimodular matrix A, ($\det M = 1$), one verifies that $D^s(A)$ is a finite-dimensional representation, of dimension $2s + 1$, of the group $SL(2, C)$ of such matrices:

$$D^s(A_1)^\lambda_\nu \, D^s(A_2)^\nu_\mu = D^s(A_1 A_2)^\lambda_\mu .$$

B. *Spinor Amplitudes*

We first recall the definition of spinor states (*3*). Let $(\hat{t}, \hat{n}_1, \hat{n}_2, \hat{n}_3)$ be the standard frame. For a particle at rest, of spin s and mass m, we first define the state $|\, s, \lambda\rangle$, which transforms under a rotation (R, R^{-1}) by

$$U(R)|\, s, \lambda\rangle = D^s(R)^{\lambda'}_\lambda \,|\, s, \lambda'\rangle,$$

where λ is any eigenvalue of the operator $-(1/m)\, W \cdot \mathring{n}_3$. ($W$ is the polarization four-vector of the particle defined from the Poincaré group generators by (*11*) $W_\mu = -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma} J^{\nu\rho} P^\sigma$).

Let now $L(p)$ be a Lorentz transformation[2] obeying

$$L(p)\, \mathring{t} = p/m, \qquad L(p)\, \mathring{n}_1 = n_1(p),$$
$$L(p)\, \mathring{n}_2 = n_2(p), \qquad L(p)\, \mathring{n}_3 = n_3(p),$$

(p is the four momentum of the particle). We define the state $| p, L(p), s, \lambda\rangle$ by

$$| p, L(p), s, \lambda\rangle = U(L(p))\, | s, \lambda\rangle.$$

It is easy to verify that this state transforms under a Lorentz transformation Λ by

$$U(\Lambda) | p, L(p), s, \lambda\rangle = D^s(L^{-1}(\Lambda p)\, \Lambda L(p))_\lambda^{\lambda'}\, | \Lambda p, L(\Lambda p), s, \lambda'\rangle$$

where $L^{-1}(\Lambda p)\, \Lambda L(p)$ is the so-called Wigner rotation (*12*). Now, if $L'(p)$ is an other Lorentz transformation such that $L(p)\, \mathring{t} = p/m$ one finds

$$| p, L'(p), s, \lambda\rangle = D^s(L^{-1}(p)\, L'(p))_\lambda^{\lambda'}\, | p, L(p), s, \lambda'\rangle$$

so that the state $| p, s, A\rangle = D^s(L^{-1}(p))_A^\lambda\, | p, L(p), s, \lambda\rangle$ can be defined independently from $L(p)$. Such a state is called a spinor state (*3*). It transforms simply under a Lorentz transformation Λ by

$$U(\Lambda) | p, s, A\rangle = D^s(\Lambda)_A^{A'}\, | \Lambda p, s, A'\rangle.$$

It is also useful to define three other types of states by

$$\widetilde{| p, s, A\rangle} = D^s(\epsilon)_A^{A'}\, | p, s, A'\rangle,$$

$$| p, s, \dot{A}\rangle = D^s\left(\frac{p \cdot \sigma}{m}\, \epsilon^{-1}\right)_{\dot{A}}^{A'}\, | p, s, A'\rangle,$$

$$\widetilde{| p, s, \dot{A}\rangle} = D^s\left(\frac{p \cdot \sigma}{m}\right)_{\dot{A}}^{A'}\, | p, s, A'\rangle,$$

[2] We shall use all along the phrase "Lorentz transformation L" to refer to the 2×2 unimodular matrix L. The notation Lp, where p is a 4-vector is a shorthand for $(1/2)\, \mathrm{Tr}[\sigma Lp \cdot \sigma L^\dagger]$ [see Sec. II-1-A(h)].

which, under a Lorentz transformation Λ, transform, respectively, according to

$$U(\Lambda)\widetilde{|\, p, s, A\rangle} = D^s(\Lambda^{T-1})_A^{A'} \,\widetilde{|\, \Lambda p, s, A'\rangle},$$

$$U(\Lambda)|\, p, s, \dot{A}\rangle = D^s(\Lambda^*)_{\dot{A}}^{\dot{A}'} \,|\, \Lambda p, s, \dot{A}'\rangle,$$

$$U(\Lambda)\widetilde{|\, p, s, \dot{A}\rangle} = D^s(\Lambda^{\dagger-1})_{\dot{A}}^{\dot{A}'} \,\widetilde{|\, \Lambda p, s, \dot{A}'\rangle}.$$

We also define the conjugate bras by the following scalar products:

$$\langle p, s, A \,|\, p', s', A'\rangle = \delta_3(\mathbf{p} - \mathbf{p}') \,\frac{\omega(p)}{m}\, \delta_{ss'} D^s \left(\frac{p \cdot \tilde{\sigma}}{m}\right)^A_{A'},$$

$$\widetilde{\langle p, s, \dot{A}} \,|\, p', s', \dot{A}'\rangle = \delta_3(\mathbf{p} - \mathbf{p}') \,\frac{\omega(p)}{m}\, \delta_{ss'} \delta^{\dot{A}}_{\dot{A}'} .$$

They transform by the complex conjugate matrices, as they should in such a way that $U(\Lambda)$ might be unitary.

Let us now consider the following two body reaction: $1 + 2 \to 3 + 4$, $p_1 + p_2 = p_3 + p_4$. The spinor amplitude is the following quantity:

$$\mathcal{M}_{\dot{A}_3\dot{A}_4;A_1A_2}(p_3p_4p_1p_2) = \langle p_3\,, s_3\,, \dot{A}_3\,; p_4\,, s_4\,, \dot{A}_4 \,|\, T \,|\, p_1\,, s_1\,, A_1\,; p_2\,, s_2\,, A_2\rangle,$$

where T is defined from the S-matrix by

$$S = \mathbf{1} + i(2\pi)^4\, \delta_4(p_1 + p_2 - p_3 - p_4)\, T.$$

Now, since the two-body spinor states are simply obtained from the one-body spinor states by tensor product, the Lorentz invariance of the interaction is translated into the following covariance formula:

$$\mathcal{M}_{\dot{A}_3\dot{A}_4;A_1A_2}(p_3p_4p_1p_2) = D^{s_1}(\Lambda)_{A_1}^{A_1'}\, D^{s_2}(\Lambda)_{A_2}^{A_2'}\, D^{s_3}(\Lambda)_{\dot{A}_3}^{\dot{A}_3'}\, D^{s_4}(\Lambda)_{\dot{A}_4}^{\dot{A}_4'} \times \mathcal{M}_{\dot{A}_3'\dot{A}_4';A_1'A_2'}(\Lambda p_3 \Lambda p_4 \Lambda p_1 \Lambda p_2), \qquad \text{(II-2)}$$

for any Lorentz transformation Λ.

Apart from this covariance property, the spinor amplitudes enjoy analyticity properties, some of which can be for instance derived from axiomatic field theory, where amplitudes emerge as Fourier transforms of vacuum expectation values of time ordered products of field operators. We recall now the Joos expansion (*3*) of the spinor amplitude in terms of covariant polynomials, the coefficients of which have been proved by Williams (2) to have no kinematical singularity:

$$\mathscr{M}_{A_3A_4;A_1A_2}(p_3p_4p_1p_2) = \sum_{\substack{J,M_1,M_2 \\ \ell_1+\ell_2=\{^J_{J+1}}} (J; \mathscr{J}; s_1, s_2, s_3, s_4)^A_{A_1A_2A_3A_4}\, a_{\ell_1\ell_2}(s,t)$$

$$\times\, (J\ell_1\ell_2)^{M_1M_2}_A\, Y^{\ell_1}_{M_1}(\overrightarrow{e(p_1, p_3)})\, Y^{\ell_2}_{M_2}(\overrightarrow{e(p_2, p_3)}) \qquad \text{(II-3)}$$

where $(J\ell_1\ell_2)^{M_1M_2}_A$ is a Clebsch–Gordan coefficient and $(J; \mathscr{J}; s_1, s_2, s_3, s_4)^A_{A_1A_2A_3A_4}$ is the coupling coefficient for $s_1s_2s_3s_4$ to give J, $\mathscr{J}$ defining the coupling mode.

$Y_M^{\ell}(\mathbf{e})$ is the solid spherical harmonic:

$$Y_M^{\ell}(\mathbf{e}) = \left[\frac{(2\ell+1)(\ell+M)!\,(\ell-M)!}{4\pi}\right]^{1/2} (e_\epsilon)^{|M|}\,[(\mathbf{e}^2)^{1/2}]^{\ell-|M|}\, P^{|M|,|M|}_{\ell-|M|}(\hat{e}_3)$$

$$e_\epsilon = -\epsilon\,\frac{e_1 + i\epsilon e_2}{2}\,; \quad \epsilon = \text{sign}(M); \quad \hat{e}_3 = \frac{e_3}{(\mathbf{e}^2)^{1/2}}\,;$$

$\overrightarrow{e(p_i, p_j)}$ is the semibivector associated with $p_i \wedge p_j$:

$$\overrightarrow{e(p_i, p_j)} = \tfrac{1}{2}[p_i^0\mathbf{p}_j - p_j^0\mathbf{p}_i - i(\mathbf{p}_i \times \mathbf{p}_j)]\quad (\overrightarrow{e(p_i, p_j)})^2 = \tfrac{1}{4}((p_i\cdot p_j)^2 - m_i^2m_j^2)),$$

$$s = (p_1 + p_2)^2; \qquad t = (p_1 - p_3)^2.$$

$a_{\ell_1\ell_2}(s, t)$ is the Joos invariant amplitude.

C. *Helicity States and Amplitudes* (*13*)

(a) *One-particle states*

The definition of an helicity state corresponds to a particular choice for $L(p)$. For a one-particle state one chooses $n_3(p) = L(p)\,\mathring{n}_3$ to be in the 2-plane $\mathring{t}, p$.

(b) *Two-particle states*

One Chooses $n_3(p_1)$ and $n_3(p_2)$ in the 2-plane p_1, p_2:

$$n_3(p_i) = h_{12}(i) = -\frac{m_i^2 P - (p_i \cdot P)\,p_i}{m_i[(p_i\cdot P)^2 - m_i^2P^2]^{1/2}}\,,$$

where $P = p_1 + p_2$. The helicity four-vector $h_{12}(i)$, $(h_{12}(i)^2 = -1)$, is completely defined by the condition that, in the center-of-mass (C.M.) system ($\mathbf{P} = 0$), $\mathbf{h}_{12}(\mathbf{i}) \cdot \mathbf{p}_i$ is positive.

(c) *Helicity amplitudes for two-body reactions*

Up to now, the axes $n_2(p_i)$ were completely arbitrary. In a two-body reaction, since $p_1 + p_2 = p_3 + p_4 = P$ the 4-vector

$$w_\mu = \frac{-2\epsilon_{\mu\nu\rho\sigma} p_1{}^\nu p_2{}^\rho p_3{}^\sigma}{[\Phi(s, t)]^{1/2}} \qquad (w_\mu w^\mu = -1),$$

is orthogonal to p_i and $h_{12}(i)$ $(i = 1, \ldots 4)$.

[$\Phi(s, t) = 0$ is the equation of the boundary of the physical region (see the kinematics for two-body reaction given below). The minus sign, insures that, in the C.M. frame, $\mathbf{W} = \widehat{\mathbf{p}_1 \times \mathbf{p}_3}$ $(\epsilon_{0123} = +1)$, if one chooses for $\Phi^{1/2}$ the positive determination.]

In a two-body reaction, we choose for helicity amplitudes

$$n_2(p_i) = w \qquad (i = 1, \ldots 4),$$

$$M_{\lambda_3\lambda_4;\lambda_1\lambda_2}(s, t, u)$$
$$= \langle p_3, L_{12}(3), s_3, \lambda_3; p_4, L_{12}(4), s_4, \lambda_4 \,|\, T \,|\, p_1, L_{12}(1), s_1, \lambda_1; p_2, L_{12}(2), s_2, \lambda_2\rangle$$

where

$$L_{12}(i) \begin{Bmatrix} \mathring{t} \\ \mathring{n}_2 \\ \mathring{n}_3 \end{Bmatrix} = \begin{Bmatrix} p_i/m_i \\ w \\ h_{12}(i) \end{Bmatrix}.$$

[The Lorentz transformations $L_{12}(i)$ are completely defined by their action on three 4-vectors. In the following, we shall very often omit to precise the action of a Lorentz transformation on the first axis of a frame. Here, for example, we have

$$(L_{12}(i)\, \mathring{n}_1)_\mu = (n_1(p_i))_\mu = \epsilon_{\mu\nu\rho\sigma} \left(\frac{p_i}{m_i}\right)^\nu w^\rho h_{12}(i)^\sigma.]$$

It has been shown by Moussa and Stora (*14*) that it is possible to write this Lorentz transformation in the following form

$$L_{12}(i) = B\left(\frac{P}{s^{1/2}} \to \frac{p_i}{m_i}\right) \Omega_{12}(n_3(P) \to q_{ij})[P]\, \epsilon', \tag{II-4}$$

where $B(p_i/m_i \to p_j/m_j)$ is the following pure Lorentz transformation which takes p_i/m_i onto p_j/m_j:

$$B\left(\frac{p_i}{m_i} \to \frac{p_j}{m_j}\right) = \left(1 + \frac{\sigma \cdot p_j}{m_j} \frac{\sigma \cdot \tilde{p}_i}{m_i}\right) \left[2\left(1 + \frac{p_i \cdot p_j}{m_i m_j}\right)\right]^{-1/2};$$

$[P]$ is an arbitrary Lorentz transformation which takes $\mathring{t}$ onto $P/s^{1/2}$

$$\epsilon' = \begin{cases} -i\sigma_2 & \text{for} \quad i = 2, 4, \\ 1 & \text{for} \quad i = 1, 3. \end{cases}$$

$\Omega_{12}(n_3(P) \to q_{ij})$ is the Lorentz transformation with positive trace which leaves P invariant, takes $n_2(P) = [P]\, \mathring{n}_2$ onto w, and $n_3(P) = [P]\, \mathring{n}_3$ onto

$$q_{ij} = \left(p_i - p_j - \frac{P\,.\,(p_i - p_j)}{P^2}\, P\right)$$
$$\times \left[\frac{[P\,.\,(p_i - p_j)]^2}{P^2} - (p_i - p_j)^2\right]^{-1/2} \qquad \begin{pmatrix} i = 1, 2 \leftrightarrow j = 2, 1 \\ i = 3, 4 \leftrightarrow j = 4, 3 \end{pmatrix}$$

(q_{ij} is in the 2-plane p_i, p_j and is orthogonal to $p_i + p_j$).

We can now express the helicity amplitudes in terms of spinor amplitudes. From

$$|\, p, L(p), s, \lambda\rangle = D^s(L(p))^A_\lambda \,|\, p, s, A\rangle \qquad \text{and}$$
$$\langle p, L(p), s, \lambda\,| = D^s(\epsilon L(p)^{T-1})^A_\lambda \,\langle p, s, \dot{A}\,| = D^s(L(p)\,\epsilon)^A_\lambda \langle ps\dot{A}\,|,$$

with $\epsilon = i\sigma_2$, we get

$$M_{\lambda_3\lambda_4;\lambda_1\lambda_2}(s, t, u) = D^{s_1}(L_{12}(1))^{A_1}_{\lambda_1} D^{s_2}(L_{12}(2))^{A_2}_{\lambda_2} D^{s_3}(L_{12}(3)\,\epsilon)^{A_3}_{\lambda_3}$$
$$\times D^{s_4}(L_{12}(4)\,\epsilon)^{A_4}_{\lambda_4} \mathcal{M}_{A_3A_4;A_1A_2}(p_3p_4p_1p_2). \qquad \text{(II-5)}$$

It is useful to remark that our conventions for helicity amplitudes *differ from those of Jacob and Wick* (*13*) by the fact that we do not multiply in the phase factor $(-1)^{s-\lambda}$ for particle 2 and particle 4. The parity-conservation condition then reads

$$M_{\lambda_3\lambda_4;\lambda_1\lambda_2}(s, t, u) = \eta(-1)^{\Sigma_i(s_i+\lambda_i)} M_{-\lambda_3-\lambda_4;-\lambda_1-\lambda_2}(s, t, u), \qquad \text{(II-6)}$$

where $\eta = \eta_1\eta_2\eta_3\eta_4$ is the product of intrinsic parities.

D. *Kinematics of Two-Body Reactions*

We now define the notations which will be used all along this paper for an arbitrary two-body reaction (Fig. 1).

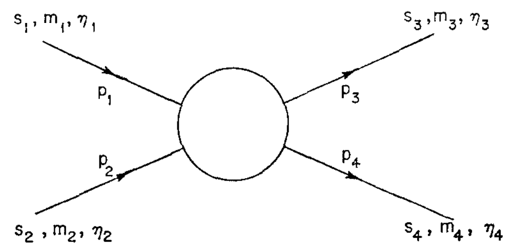

FIG. 1.

$$\begin{aligned} s &= (p_1 + p_2)^2 = (p_3 + p_4)^2, \\ t &= (p_1 - p_3)^2 = (p_2 - p_4)^2, \qquad s + t + u = \sum_i m_i^2, \\ u &= (p_1 - p_4)^2 = (p_2 - p_3)^2, \end{aligned}$$

$$\mathscr{S}_{ij} = \{[s - (m_i + m_j)^2][s - (m_i - m_j)^2]\}^{1/2},$$

$$\varphi_{ij} = [s - (m_i + m_j)^2]^{1/2} : \text{“threshold } i, j\text{”},$$

$$\psi_{ij} = [s - (m_i - m_j)^2]^{1/2} : \text{“pseudo-threshold } i, j\text{”},$$

$$\begin{aligned} \Phi(s, t) = stu &- s(m_2^2 - m_4^2)(m_1^2 - m_3^2) - t(m_1^2 - m_2^2)(m_3^2 - m_4^2) \\ &- (m_1^2 + m_4^2 - m_3^2 - m_2^2)(m_1^2 m_4^2 - m_2^2 m_3^2), \end{aligned}$$

$\Phi(s, t) = 0$ is the equation of the boundary of the physical region. $\Phi(s, t)$ is positive inside this physical region.

$$\left.\begin{aligned} \omega_i &= \frac{s + m_i^2 - m_j^2}{2s^{1/2}}, \\ p_{ij} &= \frac{\mathscr{S}_{ij}}{2s^{1/2}}, \end{aligned}\right\} \qquad \begin{aligned} i &= 1, 2 \leftrightarrow j = 2, 1 \\ i &= 3, 4 \leftrightarrow j = 4, 3 \end{aligned}$$

$p_{12} = p_{21} = k$: C.M. initial momentum,
$p_{34} = p_{43} = p$: C.M. final momentum.

$$\cos \theta_s = \frac{2st + s^2 - s \sum m_i^2 + (m_1^2 - m_2^2)(m_3^2 - m_4^2)}{\mathscr{S}_{12}\mathscr{S}_{34}},$$

$$\sin \theta_s = \frac{(2s^{1/2})[\Phi(s, t)]^{1/2}}{\mathscr{S}_{12}\mathscr{S}_{34}}.$$

2. PRINCIPLE OF THE METHOD

A. *Kinematical Singularities*

We shall say that spin-dependent amplitudes have no kinematical singularities if they have the same analyticity properties as the Joos invariant amplitudes

$a_{\ell_1\ell_2}(s, t)$. Our method to find the kinematical singularities of the helicity amplitudes consists in writing them as linear combinations of the Joos invariant amplitudes with coefficients functions of s and t:

$$M_{\{\lambda\}} = \sum_{(\ell_1,\ell_2)} C_{\{\lambda\}}^{\ell_1\ell_2}(s, t)\, a_{\ell_1\ell_2}(s, t). \tag{II-7}$$

The kinematical singularities will then be the singularities of $C_{\{\lambda\}}^{\ell_1\ell_2}(s, t)$. Let us suppose, for instance, that the C coefficients have a singularity (a square-root branch point, say) at $s = s_0$; let us suppose moreover that $s = s_0$ is also a dynamical singularity (for instance a threshold). Let

$$M_{\{\lambda\}}^{\pm} = \sum_{(\ell_1\ell_2)} C_{\{\lambda\}}^{\pm\ell_1\ell_2}(s, t)\, a_{\ell_1\ell_2}^{\pm}(s, t)$$

be two determinations of $M_{\{\lambda\}}$. If $\sum C_{\{\lambda\}}^{\mp\ell_1\ell_2} a_{\ell_1\ell_2}^{\pm}$ happen to be determinations of another helicity amplitude $M_{\{\lambda'\}}$ corresponding to other helicities $\{\lambda'\}$, then $F_{\{\lambda\}} = M_{\{\lambda\}} + M_{\{\lambda'\}}$ has no kinematical singularity at $s = s_0$. In fact,

$$\begin{aligned} \Delta F = F^+ - F^- &= \sum (C_{\{\lambda\}}^+ + C_{\{\lambda\}}^-)(a_{\ell_1\ell_2}^+ - a_{\ell_1\ell_2}^-) \\ &= \sum \mathscr{C}_{\{\lambda\}} \Delta a_{\ell_1\ell_2} \ : \end{aligned}$$

the discontinuity ΔF of F, is a linear combination of the dynamical dicontinuities $\Delta a_{\ell_1\ell_2}$, with uniform coefficients $\mathscr{C}_{\{\lambda\}} = C_{\{\lambda\}}^+ + C_{\{\lambda\}}^-$.

In order to get expansions of the type (II-7) we use the covariance of the spinor amplitudes Eq. (II-2) and the expression of the helicity amplitudes as functions of spinor amplitudes (Eq. (II-5)), which leads to

$$\begin{aligned} M_{\lambda_3\lambda_4;\lambda_1\lambda_2}(s, t, u) = {} & D^{s_1}(\Lambda L_{12}(1))_{\lambda_1}^{A_1}\, D^{s_2}(\Lambda L_{12}(2))_{\lambda_2}^{A_2}\, D^{s_3}(\Lambda L_{12}(3)\,\epsilon)_{\lambda_3}^{A_3} \\ & \times D^{s_4}(\Lambda L_{12}(4)\,\epsilon)_{\lambda_4}^{A_4}\, \mathscr{M}_{A_3A_4;A_1A_2}(\Lambda p_3 \Lambda p_4 \Lambda p_1 \Lambda p_2) \end{aligned} \tag{II-8}$$

for any Lorentz transformation Λ. We finally get an expansion of the type (II-7) by using the Joos expansion (II-3) of $\mathscr{M}_{A_3A_4;A_1A_2}(\Lambda p_3 \Lambda p_4 \Lambda p_1 \Lambda p_2)$. The choice of Λ will be adapted to the study of each singularity in such a way that this singularity may be factored out as easily as possible.

Before doing practical calculations, it is useful to try and guess what singularities we shall meet. For any singularity in s and/or t, it is possible to find a Lorentz transformation Λ such that this singularity does not appear in the Joos expansion of $\mathscr{M}_{A_3A_4;A_1A_2}(\Lambda p_3 \Lambda p_4 \Lambda p_1 \Lambda p_2)$. So, the only singularities one cannot avoid are those which appear in $D(\Lambda L_{12}(i))$. Now, the singularities of a Lorentz transformation which takes the standard frame onto some frame $\{n_i\}$ include those of the basis vectors n_i of this frame. Thus, the kinematical singularities of the helicity amplitudes are at least those of the 4-vectors of the helicity frames, that is [cf. the definition of $h_{12}(i)$ and w in Section II-1-C-(b) and (c)],

$$\Phi(s, t) = 0,$$

$$\left.\begin{array}{l}[(p_1 \cdot P)^2 - m_1{}^2P^2]^{1/2} = 0 \\ [(p_2 \cdot P)^2 - m_2{}^2P^2]^{1/2} = 0\end{array}\right\} \qquad \mathscr{S}_{12} = 0,$$

$$\left.\begin{array}{l}[(p_3 \cdot P)^2 - m_3{}^2P^2]^{1/2} = 0 \\ [(p_4 \cdot P)^2 - m_4{}^2P^2]^{1/2} = 0\end{array}\right\} \qquad \mathscr{S}_{34} = 0.$$

Furthermore, since $h_{12}(i)$ changes sign when one changes the determinations of $\mathscr{S}_{12}$ (if $i = 1, 2$) or $\mathscr{S}_{34}$ (if $i = 3, 4$), one can guess that, in order to free the helicity amplitudes from kinematical singularities at $\mathscr{S}_{12} = 0$ or $\mathscr{S}_{34} = 0$, it will be necessary to associate $M_{\lambda_3\lambda_4;\lambda_1\lambda_2}$ with $M_{\lambda_3\lambda_4;-\lambda_1-\lambda_2}$ or $M_{\lambda_3\lambda_4;\lambda_1\lambda_2}$ with $M_{-\lambda_3-\lambda_4;\lambda_1\lambda_2}$. On the contrary since the helicity 4-vectors do not have the $\Phi(s, t) = 0$ singularity, this singularity will be picked up without associating different amplitudes.

B. *Explicit Choices for Λ.*

Let us suppose that Λ takes some frame R onto the standard frame. First we note that $\Lambda p \cdot \mathring{n}_i = \Lambda p \cdot \Lambda n_i(R) = p \cdot n_i(R)$ so that the components of Λp in the standard frame are equal to those of p in the frame R. Secondly, all the frames which we use are such that the center of mass is at rest ($\mathbf{P} = 0$). So, we choose in Eq. (II-4) $[P] = \Lambda^{-1}$. Let us now, for instance, evaluate $\Lambda L_{12}(1)$:

$$\Lambda L_{12}(1) = \Lambda B\left(\frac{P}{s^{1/2}} \to \frac{p_1}{m_1}\right) \Omega_{12} \left\{\begin{array}{c} P \to P \\ n_3(P) \to q_{12} \\ n_2(P) \to w \end{array}\right\} [P]$$

$$= B\left(\frac{\Lambda P}{s^{1/2}} \to \frac{\Lambda p_1}{m_1}\right) \Lambda\Omega_{12}[P] = B\left(\mathring{t} \to \frac{\Lambda p_1}{m_1}\right) \Omega_{12} \left\{\begin{array}{c} \mathring{t} \to \mathring{t} \\ \mathring{n}_3 \to \Lambda q_{12} \\ \mathring{n}_2 \to \Lambda w \end{array}\right\} .$$

$\Lambda L_{12}(p_1)$ is thus the product of a boost by a rotation. Equivalently it is equal to

$$\Omega_{12} B\,(\mathring{t} \to \Omega_{12}^{-1} \Lambda p_1/m_1).$$

For the general mass case, that is $m_1 - m_2 \neq 0$, $m_3 - m_4 \neq 0$, $m_1 - m_2 \neq m_3 - m_4$, $m_1 + m_2 \neq m_3 + m_4$, we need two different expansions using two frames R. We write down the corresponding expansions in Tables I and II.

TABLE I

EXPLICIT FORM A OF EQ. (II-8) CORRESPONDING TO FRAME R_I, WHICH IS ESPECIALLY CONVENIENT FOR THE STUDY OF KINEMATICAL SINGULARITIES AT $\mathscr{S}_{34} = 0$ IN THE GENERAL MASS CASE.

Frame R_I

$$t(R) = P/s^{1/2}, \qquad n_2(R) = w, \qquad n_3(R) = q_{12}$$

$$\overrightarrow{e(\Lambda p_1, \Lambda p_3)} = \begin{cases} 1/2\omega_1 p \sin\theta_s \\ -i/2kp\sin\theta_s \\ 1/2(\omega_1 p\cos\theta_s - \omega_3 k) \end{cases} \qquad \overrightarrow{e(\Lambda p_2, \Lambda p_3)} = \begin{cases} 1/2\omega_2 p \sin\theta_s \\ i/2kp\sin\theta_s \\ 1/2(\omega_2 p\cos\theta_s + \omega_3 k) \end{cases}$$

$$\Lambda L_{12}(1) = B_1$$
$$\Lambda L_{12}(2) = R_y(\pi)\, B_2$$
$$\Lambda L_{12}(3) = R_y(\theta_s)\, B_3$$
$$\Lambda L_{12}(4) = R_y(\theta_s + \pi)\, B_4$$

where $R_y(\varphi) = \cos(\varphi/2) - i\sigma_2 \sin(\varphi/2)$

$$B_i = \frac{m_i + \omega_i - p_{ij}\sigma_3}{[2m_i(\omega_i + m_i)]^{1/2}} \begin{pmatrix} i = 1, 2 \leftrightarrow j = 2, 1 \\ i = 3, 4 \leftrightarrow j = 4, 3 \end{pmatrix}$$

Expansion A

$$\begin{aligned} M_{\lambda_3\lambda_4;\lambda_1\lambda_2} = \sum_{\Lambda_3,\Lambda_4} & (-)^{s_2-\lambda_2+s_3+\lambda_3} D^{s_1}(B_1)^{\lambda_1}_{\lambda_1} D^{s_2}(B_2)^{\lambda_2}_{\lambda_2} D^{s_3}(B_3)^{-\lambda_3}_{-\lambda_3} D^{s_4}(B_4)^{-\lambda_4}_{-\lambda_4} \\ & \times d^{s_3}(\theta_s)^{\Lambda_3}_{-\lambda_3}\, d^{s_4}(\theta_s)^{\Lambda_4}_{\lambda_4} \sum_{\ell_1,\ell_2,M_1+M_2=\lambda_1-\lambda_2+\Lambda_3+\Lambda_4} \mathscr{C} a_{\ell_1\ell_2}(s,t)\,(p\sin\theta_s)^{|M_1|+|M_2|} \\ & \times (\omega_1 + \epsilon_1 k)^{|M_1|}(\omega_2 - \epsilon_2 k)^{|M_2|}\, P_1[t, (\omega_1 p\cos\theta_s - \omega_3 k)]\, P_2[u, (\omega_2 p\cos\theta_s + \omega_3 k)], \end{aligned} \tag{II-9}$$

where $\mathscr{C}$ is a numerical coefficient,
ϵ_1 and ϵ_2 are the signs of M_1 and M_2.
$P_1[t, z]$ and $P_2[u, z]$ are polynomials in t and z, and in u and z, respectively.

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3. GENERAL-MASS CASE. KINEMATICAL SINGULARITIES AT $\Phi(s, t) = 0$

Since $\Phi(s, t) = 0$ implies $|\cos\theta_s| = 1$, the terms which are singular at $\Phi(s, t) = 0$ are $\cos(\theta_s/2)$ and $\sin(\theta_s/2)$. Using the definition of the D matrices, (Eq. II-1), we evaluate the power of these quantities in expansion A [Eq. (II-9) of Table I; we would get the same results with expansion B]:

$$\begin{aligned} M_{\lambda_3\lambda_4;\lambda_1\lambda_2} = \sum & \sin(\theta_s/2)^{|\Lambda_3+\lambda_3|+|\Lambda_4-\lambda_4|+|M_1|+|M_2|} \\ & \times \cos(\theta_s/2)^{|\Lambda_3-\lambda_3|+|\Lambda_4+\lambda_4|+|M_1|+|M_2|}\, R(s,t), \end{aligned}$$

where $R(s, t)$ is kinematically regular[3] at $\Phi(s, t) = 0$.

[3] From now on we shall use the following terminology: "kinematically uniform at..." for "without a kinematical branch point at...", "kinematically finite at..." for "finite in the absence of a dynamical infinity at" ,"kinematically regular at..." for "without either a kinematical branch point or a pole at...".

TABLE II

EXPLICIT FORM B OF EQ. (II-8) CORRESPONDING TO FRAME R_{II}, WHICH IS ESPECIALLY CONVENIENT FOR THE STUDY OF KINEMATICAL SINGULARITIES AT $\mathscr{S}_{12} = 0$ IN THE GENERAL MASS CASE.

Frame R_{II}

$$t(R) = P/s^{1/2}, \qquad n_2(R) = w, \qquad n_3(R) = q_{34}$$

$$\overrightarrow{e(\Lambda p_1, \Lambda p_3)} = \begin{cases} 1/2\omega_3 k \sin\theta_s \\ -i/2kp \sin\theta_s \\ 1/2(\omega_1 p - \omega_3 k \cos\theta_s) \end{cases} \qquad \overrightarrow{e(\Lambda p_2, \Lambda p_3)} = \begin{cases} -1/2\omega_3 k \sin\theta_s \\ i/2kp \sin\theta_s \\ 1/2(\omega_2 p + \omega_3 k \cos\theta_s), \end{cases}$$

$$\Lambda L_{12}(1) = R_y(-\theta_s)\, B_1$$
$$\Lambda L_{12}(2) = R_y(-\theta_s + \pi)\, B_2$$
$$\Lambda L_{12}(3) = B_3$$
$$\Lambda L_{12}(4) = R_y(\pi)\, B_4$$

Expansion B

$$M_{\lambda_3\lambda_4,\lambda_1\lambda_2} = \sum_{A_1 A_2} (-)^{s_2-\lambda_2+s_3+\lambda_3} D^{s_1}(B_1)^{\lambda_1}_{\lambda_1} D^{s_2}(B_2)^{\lambda_2}_{\lambda_2} D^{s_3}(B_3)^{-\lambda_3}_{-\lambda_3} D^{s_4}(B_4)^{-\lambda_4}_{-\lambda_4}$$
$$\times\, d^{s_1}(-\theta_s)^{A_1}_{\lambda_1} d^{s_2}(-\theta_s)^{A_2}_{-\lambda_2} \sum_{\ell_1,\ell_2,M_1+M_2=A_1+A_2-\lambda_3+\lambda_4} \mathscr{C}' a_{\ell_1\ell_2}(s,t)(k\sin\theta_s)^{|M_1|+|M_2|} \tag{II-10}$$
$$\times\, (\omega_3+\epsilon_1 p)^{|M_1|}(\omega_3+\epsilon_2 p)^{|M_2|}\, P_1'[t, (\omega_1 p - \omega_3 k\cos\theta_s)]\, P_2'[u, (\omega_2 p + \omega_3 k \cos\theta_s)]$$

where $\mathscr{C}'$ is a numerical coefficient,
ϵ_1 and ϵ_2 are the signs of M_1 and M_2.
$P_1'[t, z]$, $P_2'[u, z]$ are polynomials in t and z, and in u and z, respectively.

Let

$$\mathscr{A} = |A_3+\lambda_3| + |A_4-\lambda_4| + |M_1| + |M_2|$$
$$= |A_3+\lambda_3| + |A_4-\lambda_4| + |M_1| + |\lambda_1-\lambda_2+A_3+A_4-M_1|,$$
$$\mathscr{B} = |A_3-\lambda_3| + |A_4+\lambda_4| + |M_1| + |M_2|$$
$$= |A_3-\lambda_3| + |A_4+\lambda_4| + |M_1| + |\lambda_1-\lambda_2+A_3+A_4-M_1|;$$

$\mathscr{A}$ and $\mathscr{B}$ are obviously positive integers.

Any dummy index (A_3, A_4, M_1) appears twice in the four terms of both $\mathscr{A}$ and $\mathscr{B}$, so that, if one varies the value of one index by one unit, $\mathscr{A}$ (resp. $\mathscr{B}$) is changed into $\mathscr{A}+2$, $\mathscr{A}$ or $\mathscr{A}-2$ (resp. $\mathscr{B}+2$, $\mathscr{B}$ or $\mathscr{B}-2$). Now, for $A_3 = -\lambda_3$, $A_4 = \lambda_4$, $M_1 = 0$, $\mathscr{A}$ reduces to $|\lambda_1-\lambda_2-\lambda_3+\lambda_4|$ and cannot decrease, and for $A_3 = \lambda_3$, $A_4 = -\lambda_4$, $M_1 = 0$ $\mathscr{B}$ reduces to $|\lambda_1-\lambda_2+\lambda_3-\lambda_4|$ and cannot decrease.

Thus

$$\mathscr{A}_{\min} = |\lambda - \mu|, \qquad \mathscr{B}_{\min} = |\lambda + \mu|$$
$$(\lambda = \lambda_1 - \lambda_2, \qquad \mu = \lambda_3 - \lambda_4),$$

and other values of $\mathscr{A}$ and $\mathscr{B}$ differ from these by positive even integers. Hence:

$$M_{\lambda_3\lambda_4;\lambda_1\lambda_2} = \sin(\theta_s/2)^{|\lambda-\mu|} \cos(\theta_s/2)^{|\lambda+\mu|} \hat{M}_{\lambda_3\lambda_4;\lambda_1\lambda_2}, \tag{II-11}$$

where $\hat{M}_{\lambda_3\lambda_4;\lambda_1\lambda_2}$ is kinematically regular at $\Phi(s, t) = 0$.

This result was already known (*7*), (*8*), although, to our knowledge, it had not been proved. What we now know, in particular, is that the convergence domain in the $\cos\theta_s$ plane of the partial-wave expansion of the helicity amplitude

$$M_{\lambda_3\lambda_4;\lambda_1\lambda_2} = \sum_J (J + \tfrac{1}{2})\, M^J_{\lambda_3\lambda_4;\lambda_1\lambda_2} d^J(\theta_s)^\mu_\lambda$$
$$= \sin(\theta_s/2)^{|\lambda-\mu|} \cos(\theta_s/2)^{|\lambda+\mu|} \sum_J a^J_{\lambda_3\lambda_4;\lambda_1\lambda_2} P^{|\lambda-\mu|,|\lambda+\mu|}_{J-m}(\cos\theta_s)$$

(where $m = \mathrm{Max}(|\lambda|, |\mu|)$ and $a^J_{\lambda_3\lambda_4;\lambda_1\lambda_2}$ differs from $M^J_{\lambda_3\lambda_4;\lambda_1\lambda_2}$ by numerical factors) is identical to that of the expansion

$$\sum_J a^J_{\lambda_3\lambda_4;\lambda_1\lambda_2} P^{|\lambda-\mu||\lambda+\mu|}_{J-m}(\cos\theta_s).$$

Namely, via a well-known theorem of Szegö [Theorem g.1.1, p. 243 of Ref. (*10*)], on the domain of convergence of expansions in terms of orthogonal polynomials, the convergence domain of $M_{\lambda_3\lambda_4;\lambda_1\lambda_2}$ is completely characterized by dynamical singularities.

4. Kinematical Singularities at Thresholds and Pseudo-Thresholds—General Mass Case

A. *Types of singularities at* $s = (m_1 \pm m_2)^2$ *and* $s = (m_3 \pm m_4)^2$

Using Expansion A or B one finds that the following terms are singular at $\mathscr{S}_{12} = 0$ and $\mathscr{S}_{34} = 0$.

(a) $$D^{s_i}(B_i)^{\lambda_i}_{\lambda_i} = \left(\frac{\omega_i + m_i - p_{ij}}{[2m_i(\omega_i + m_i)]^{1/2}}\right)^{2\lambda_i} = \left(\frac{\omega_i + m_i + p_{ij}}{[2m_i(\omega_i + m_i)]^{1/2}}\right)^{-2\lambda_i},$$

$$\omega_i + m_i = \frac{(s^{1/2} + m_i + m_j)(s^{1/2} + m_i - m_j)}{2s^{1/2}}, \qquad \begin{array}{l} i = 1, 2 \leftrightarrow j = 2, 1, \\ i = 3, 4 \leftrightarrow j = 4, 3. \end{array}$$

If we choose, as we shall always do, the positive determination of $s^{1/2}$ in the neighborhood of $s = (m_i \pm m_j)^2$, one finds that

$$\text{for} \quad s = (m_i + m_j)^2 \qquad \omega_i + m_i = 2m_i\,,$$

$$\text{for} \quad s = (m_i - m_j)^2 \qquad \begin{cases} \omega_i + m_i = 0 & \text{if} \quad m_j > m_i\,, \\ \omega_i + m_i = 2m_i & \text{if} \quad m_i > m_j\,, \end{cases}$$

Summarizing, one sees that, if one turns around $s = (m_i + m_j)^2$ using the path shown in Fig. 2),

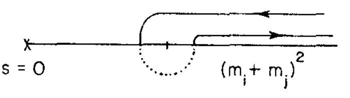

FIG. 2.

$$\left(\frac{\omega_i + m_i - p_{ij}}{[2m_i(\omega_i + m_i)]^{1/2}}\right)^{2\lambda_i}$$

is changed into

$$\left(\frac{\omega_i + m_i - p_{ij}}{[2m_i(\omega_i + m_i)]^{1/2}}\right)^{-2\lambda_i}$$

and that, if one turns around $s = (m_i - m_j)^2$ using the path shown in Fig. 3,

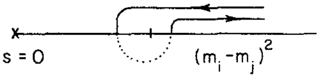

FIG. 3.

$$\left(\frac{\omega_i + m_i - p_{ij}}{[2m_i(\omega_i + m_i)]^{1/2}}\right)^{2\lambda_i}$$

is changed into

$$\left(\frac{\omega_i + m_i - p_{ij}}{[2m_i(\omega_i + m_i)]^{1/2}}\right)^{-2\lambda_i} \quad \text{if} \quad m_i > m_j\,,$$

$$(-1)^{2\lambda_i}\left(\frac{\omega_i + m_i - p_{ij}}{[2m_i(\omega_i + m_i)]^{1/2}}\right)^{-2\lambda_i} \quad \text{if} \quad m_j > m_i\,.$$

(b) $\cos\theta_s$ and $\sin\theta_s$ are singular and behave like $(\mathscr{S}_{12}\mathscr{S}_{34})^{-1}$.

(c) $d^s(\theta_s)$ is singular since $\cos\theta_s$ is.

B. *Kinematical Branch Points at Thresholds and Pseudo-Thresholds.*

(a) *Thresholds*

$s = (m_3 + m_4)^2$. It is easy to see that in expansion A [Eq. (II-9) of Table I], all the terms which appear in the Joos expansion are regular at $s = (m_3 + m_4)^2$.
Now, if one turns around $s = (m_3 + m_4)^2$ using the path shown on Fig. 2, the following changes occur:

$$\theta_s \to \theta_s \pm \pi, \qquad p \to -p;$$
$$\left(\frac{\omega_i + m_i - p}{[2m_i(\omega_i + m_i)]^{1/2}}\right)^{\pm\lambda_i} \to \left(\frac{\omega_i + m_i - p}{[2m_i(\omega_i + m_i)]^{1/2}}\right)^{\mp\lambda_i}, \qquad i = 3, 4.$$

Then, using Expansion A one sees that[4]

$$\hat{M}_{\lambda_3\lambda_4;\lambda_1\lambda_2}(\varphi_{34})$$
$$= (-)^{s_2-\lambda_2+s_3+\lambda_3} \sin(\theta_s/2)^{-|\lambda-\mu|} \cos(\theta_s/2)^{-|\lambda+\mu|}$$
$$\times \left(\frac{\omega_3 + m_3 - p}{[2m_3(\omega_3 + m_3)]^{1/2}}\right)^{-2\lambda_3} \left(\frac{\omega_4 + m_4 - p}{[2m_4(\omega_4 + m_4)]^{1/2}}\right)^{-2\lambda_4} d^{s_3}(\theta_s)^{\Lambda_3}_{-\lambda_3}\, d^{s_4}(\theta_s)^{\Lambda_4}_{\lambda_4} \cdots$$

is changed into:

$$\hat{M}_{\lambda_3\lambda_4;\lambda_1\lambda_2}(-\varphi_{34}) = (-)^{s_2-\lambda_2+s_3+\lambda_3} \sin\left(\frac{\theta_s \pm \pi}{2}\right)^{-|\lambda-\mu|} \cos\left(\frac{\theta_s \pm \pi}{2}\right)^{-|\lambda+\mu|}$$
$$\times \left(\frac{\omega_3 + m_3 - p}{[2m_3(\omega_3 + m_3)]^{1/2}}\right)^{2\lambda_3} \left(\frac{\omega_4 + m_4 - p}{[2m_4(\omega_4 + m_4)]^{1/2}}\right)^{2\lambda_4} d^{s_3}(\theta_s \pm \pi)^{\Lambda_3}_{-\lambda_3}\, d^{s_4}(\theta_s \pm \pi)^{\Lambda_4}_{\lambda_4} \cdots$$

(the dots stand for quantities which are kinematically regular at $s = (m_3 + m_4)^2$).
It is then easy to find

$$\hat{M}_{\lambda_3\lambda_4;\lambda_1\lambda_2}(-\varphi_{34}) = \eta_{34}\hat{M}_{-\lambda_3-\lambda_4;\lambda_1\lambda_2}(\varphi_{34});\ \eta_{34} = (-1)^{s_3-s_4+\lambda}. \qquad \text{(II-12)}$$

$s = (m_1 + m_2)^2$. Turning around $s = (m_1 + m_2)^2$ along the path shown on Fig. 2 and using Expansion B, [Eq. (II-10) of Table II] one finds in the same way

$$\hat{M}_{\lambda_3\lambda_4;\lambda_1\lambda_2}(-\varphi_{12}) = \eta_{12}\hat{M}_{\lambda_3\lambda_4;-\lambda_1-\lambda_2}(\varphi_{12});\ \eta_{12} = (-)^{s_1-s_2-\mu}. \qquad \text{(II-13)}$$

[4] When we study the behavior of a function f in the neighborhood of a square root branch point $z = 0$, we write for convenience $f(z^{1/2})$ and $f(-z^{1/2})$ for the two determinations $f^{\rm I}$ and $f^{\rm II}$ of f at point z.

(b) *Pseudo-thresholds*

The reasoning is quite the same. One finds

$$s = (m_3 - m_4)^2$$
$$\begin{aligned} &\text{if} \quad m_4 > m_3 \qquad \hat{M}_{\lambda_3\lambda_4;\lambda_1\lambda_2}(-\psi_{34}) = \eta_{34}(-)^{2s_3}\, \hat{M}_{-\lambda_3-\lambda_4;\lambda_1\lambda_2}(\psi_{34}) \\ &\text{if} \quad m_3 > m_4 \qquad \hat{M}_{\lambda_3\lambda_4;\lambda_1\lambda_2}(-\psi_{34}) = \eta_{34}(-)^{2s_4}\, \hat{M}_{-\lambda_3-\lambda_4;\lambda_1\lambda_2}(\psi_{34}) \end{aligned} \tag{II-14}$$

$$s = (m_1 - m_2)^2$$
$$\begin{aligned} &\text{if} \quad m_2 > m_1 \qquad \hat{M}_{\lambda_3\lambda_4;\lambda_1\lambda_2}(\psi_{12}) = \eta_{12}(-)^{2s_1}\, \hat{M}_{\lambda_3\lambda_4;-\lambda_1-\lambda_2}(\psi_{12}) \\ &\text{if} \quad m_1 > m_2 \qquad \hat{M}_{\lambda_3\lambda_4;\lambda_1\lambda_2}(-\psi_{12}) = \eta_{12}(-)^{2s_2}\, \hat{M}_{\lambda_3\lambda_4;-\lambda_1-\lambda_2}(\psi_{12}) \end{aligned} \quad . \tag{II-15}$$

C. *Consequences of Parity Conservation*

From formulas (II-12)–(II-15) it is possible to find kinematically "uniform" linear combinations of the $\hat{M}$'s. We define

$$\begin{aligned} A_{\lambda_3\lambda_4\lambda_1\lambda_2} &= \hat{M}_{\lambda_3\lambda_4;\lambda_1\lambda_2} + \eta_{34}\hat{M}_{-\lambda_3-\lambda_4;\lambda_1\lambda_2} \\ &\quad + \eta_{12}[\hat{M}_{\lambda_3\lambda_4;-\lambda_1-\lambda_2} + \eta_{34}(-\lambda_1\,,\, -\lambda_2)\, \hat{M}_{-\lambda_3-\lambda_4;-\lambda_1-\lambda_2}], \\ B_{\lambda_3\lambda_4\lambda_1\lambda_2} &= \hat{M}_{\lambda_3\lambda_4;\lambda_1\lambda_2} - \eta_{34}\hat{M}_{-\lambda_3-\lambda_4;\lambda_1\lambda_2} \\ &\quad + \eta_{12}[\hat{M}_{\lambda_3\lambda_4;-\lambda_1-\lambda_2} - \eta_{34}(-\lambda_1\,,\, -\lambda_2)\, \hat{M}_{-\lambda_3-\lambda_4;-\lambda_1-\lambda_2}], \\ C_{\lambda_3\lambda_4\lambda_1\lambda_2} &= \hat{M}_{\lambda_3\lambda_4;\lambda_1\lambda_2} + \eta_{34}\hat{M}_{-\lambda_3-\lambda_4;\lambda_1\lambda_2} \\ &\quad - \eta_{12}[\hat{M}_{\lambda_3\lambda_4;-\lambda_1-\lambda_2} - \eta_{34}(-\lambda_1\,,\, -\lambda_2)\, \hat{M}_{-\lambda_3-\lambda_4;-\lambda_1-\lambda_2}], \\ D_{\lambda_3\lambda_4\lambda_1\lambda_2} &= \hat{M}_{\lambda_3\lambda_4;\lambda_1\lambda_2} - \eta_{34}\hat{M}_{-\lambda_3-\lambda_4;\lambda_1\lambda_2} \\ &\quad - \eta_{12}[\hat{M}_{\lambda_3\lambda_4;-\lambda_1-\lambda_2} + \eta_{34}(-\lambda_1\,,\, -\lambda_2)\, \hat{M}_{-\lambda_3-\lambda_4;-\lambda_1-\lambda_2}], \end{aligned} \tag{II-16}$$

where

$$\eta_{34}(-\lambda_1\,,\, -\lambda_2) = (-)^{s_3-s_4-\lambda}.$$

Then $A_{\lambda_3\lambda_4\lambda_1\lambda_2}$, $\varphi_{34}^{-1}B_{\lambda_3\lambda_4\lambda_1\lambda_2}$, $\varphi_{12}^{-1}C_{\lambda_3\lambda_4\lambda_1\lambda_2}$, and $\varphi_{12}^{-1}\varphi_{34}^{-1}D_{\lambda_3\lambda_4\lambda_1\lambda_2}$ do not have any kinematical branch point either at $s = (m_3 + m_4)^2$ or at $s = (m_1 + m_2)^2$.

Now, parity invariance [Eq. (II-6)] implies relations between $\hat{M}_{-\lambda_3-\lambda_4-\lambda_1-\lambda_2}$ and $\hat{M}_{\lambda_3\lambda_4\lambda_1\lambda_2}$, namely,

$$\hat{M}_{-\lambda_3-\lambda_4-\lambda_1-\lambda_2} = \eta(-1)^{\Sigma_i(s_i+\lambda_i)}\, \hat{M}_{\lambda_3\lambda_4\lambda_1\lambda_2}\,,$$

where η is the product of intrinsic parities.

Hence

$$\eta_{12}\eta_{34}(-\lambda_1, -\lambda_2)\, \hat{M}_{-\lambda_3-\lambda_4-\lambda_1-\lambda_2} = \eta \hat{M}_{\lambda_3\lambda_4\lambda_1\lambda_2}$$

and

$$\eta_{12}\hat{M}_{\lambda_3\lambda_4-\lambda_1-\lambda_2} = \eta\eta_{34}\hat{M}_{-\lambda_3-\lambda_4\lambda_1\lambda_2}.$$

So, by associating only two different helicity amplitudes, it is possible to write down amplitudes free of kinematical branch points at thresholds.

Let

$$\begin{aligned} \mathscr{A}_{34} &= \hat{M}_{\lambda_3\lambda_4;\lambda_1\lambda_2} + \eta_{34}\hat{M}_{-\lambda_3-\lambda_4;\lambda_1\lambda_2}, \\ \mathscr{B}_{34} &= \hat{M}_{\lambda_3\lambda_4;\lambda_1\lambda_2} - \eta_{34}\hat{M}_{-\lambda_3-\lambda_4;\lambda_1\lambda_2}, \\ \mathscr{A}_{12} &= \hat{M}_{\lambda_3\lambda_4;\lambda_1\lambda_2} + \eta_{12}\hat{M}_{\lambda_3\lambda_4;-\lambda_1-\lambda_2}, \\ \mathscr{B}_{12} &= \hat{M}_{\lambda_3\lambda_4;\lambda_1\lambda_2} - \eta_{12}\hat{M}_{\lambda_3\lambda_4;-\lambda_1-\lambda_2}; \end{aligned} \tag{II-17}$$

TABLE III

AMPLITUDES FREE OF KINEMATICAL POINTS AT $s = (m_1 \pm m_2)^2$ AND $s = (m_3 \pm m_4)^2$ [a]

Spin Configuration \ Intrinsic parity configuration	$\eta = +1$, $\mathscr{A}_{34} = \mathscr{A}_{12}$, $\mathscr{B}_{34} = \mathscr{B}_{12}$	$\eta = -1$, $\mathscr{A}_{34} = \mathscr{B}_{12}$, $\mathscr{B}_{34} = \mathscr{A}_{12}$
$BB \to BB$ or $BF \to B'F'$ with $m_B < m_F$ and $m_{B'} < m_{F'}$	$\mathscr{A}_{34}$, $\mathscr{S}_{12}^{-1}\mathscr{S}_{34}^{-1}\mathscr{B}_{34}$	$\mathscr{S}_{34}^{-1}\mathscr{B}_{34}$, $\mathscr{S}_{12}^{-1}\mathscr{A}_{34}$
$BB \to F\bar{F}$ or $BF \to B'F'$ with $m_F > m_B$ and $m_{F'} < m_{B'}$	$\psi_{34}^{-1}\mathscr{A}_{34}$, $\mathscr{S}_{12}^{-1}\varphi_{34}^{-1}\mathscr{B}_{34}$	$\varphi_{34}^{-1}\mathscr{B}_{34}$, $\mathscr{S}_{12}^{-1}\psi_{34}^{-1}\mathscr{A}_{34}$
$F\bar{F} \to BB$ or $BF \to B'F'$ with $m_F < m_B$ and $m_{F'} > m_{B'}$	$\psi_{12}^{-1}\mathscr{A}_{34}$, $\varphi_{12}^{-1}\mathscr{S}_{34}^{-1}\mathscr{B}_{34}$	$\psi_{12}^{-1}\mathscr{B}_{34}$, $\varphi_{12}^{-1}\mathscr{S}_{34}^{-1}\mathscr{A}_{34}$
$FF \to FF$ or $BF \to B'F'$ with $m_F < m_B$ and $m_{F'} < m_{B'}$	$\psi_{12}^{-1}\psi_{34}^{-1}\mathscr{A}_{34}$, $\varphi_{12}^{-1}\varphi_{34}^{-1}\mathscr{B}_{34}$	$\psi_{12}^{-1}\varphi_{34}^{-1}\mathscr{B}_{34}$, $\varphi_{12}^{-1}\psi_{34}^{-1}\mathscr{A}_{34}$

[a] $\mathscr{A}$ and $\mathscr{B}$ are defined in Eq. (II-17).

$$\begin{aligned} \text{if} \quad \eta = +1, \quad & \mathscr{A}_{34} = \mathscr{A}_{12}, \\ & \mathscr{B}_{34} = \mathscr{B}_{12}, \\ \text{if} \quad \eta = -1, \quad & \mathscr{A}_{34} = \mathscr{B}_{12}, \\ & \mathscr{B}_{34} = \mathscr{A}_{12}. \end{aligned}$$

$$\begin{aligned} \text{Thus, if} \quad \eta = +1 \quad & \mathscr{A}_{34} \text{ or } \mathscr{A}_{12} \quad \text{and} \quad \varphi_{12}^{-1}\varphi_{34}^{-1}(\mathscr{B}_{34} \text{ or } \mathscr{B}_{12}), \\ \text{if} \quad \eta = -1 \quad & \varphi_{34}^{-1}(\mathscr{B}_{34} \text{ or } \mathscr{A}_{12}) \quad \text{and} \quad \varphi_{12}^{-1}(\mathscr{A}_{34} \text{ or } \mathscr{B}_{12}), \end{aligned} \qquad \text{(II-18)}$$

do not have any kinematical branch point either at $s = (m_3 + m_4)^2$ or at $s = (m_1 + m_2)^2$.

We easily derive results analogous to (II-18) for the case of pseudothresholds, using (II-14) and (II-15). The amplitudes kinematically uniform at $s = (m_1 \pm m_2)^2$ and $s = (m_3 \pm m_4)^2$, are exhibited in Table III.

D. *Kinematical Poles at Thresholds and Pseudo-Thresholds—General Mass Case*

We now look for the possible kinematical poles and zeros at thresholds and pseudo-thresholds, which means that we look for the behavior of the helicity amplitudes near $\mathscr{S}_{12} = 0$ and $\mathscr{S}_{34} = 0$ assuming that the Joos invariant amplitudes are finite and nonzero.

(a) Near $\mathscr{S}_{34} = 0$ using Expansion A (Table I), one sees that

(i) The elements of B_i are finite (their moduli are close to unity),

(ii) The semibivectors are finite.

Moreover, $\cos\theta_s \propto (\mathscr{S}_{34})^{-1}$, $\cos(\theta_s/2)$ and $\sin(\theta_s/2) \propto (\mathscr{S}_{34})^{-1/2}$. $d^s(\theta)$ being a homogeneous polynomial of degree $2s$ in $\sin(\theta/2)$ and $\cos(\theta/2)$, one finds that:

$$\hat{M}_{\lambda_3\lambda_4;\lambda_1\lambda_2} \propto \left(\frac{1}{\mathscr{S}_{34}}\right)^{m_{34}} \qquad m_{34} = s_3 + s_4 - \mathrm{Max}(|\lambda|, |\mu|). \qquad \text{(II-19)}$$

(b) Near $\mathscr{S}_{12} = 0$ using the same reasoning on Expansion B (Table II) one finds

$$\hat{M}_{\lambda_3\lambda_4;\lambda_1\lambda_2} \propto \left(\frac{1}{\mathscr{S}_{12}}\right)^{m_{12}} \qquad m_{12} = s_1 + s_2 - \mathrm{Max}(|\lambda|, |\mu|). \qquad \text{(II-20)}$$

It is quite easy, then, to put together these results and those concerning uniformization at kinematical branch points (Table III). For instance let us consider the case $BB \to BB$, $\eta = +1$. Near $\mathscr{S}_{12} = 0$, $\mathscr{A}_{34}$ can be written

$$\frac{\alpha_{34} + \beta_{34}\mathscr{S}_{12}}{\mathscr{S}_{12}^{m_{12}}},$$

where both α_{34} and β_{34} are kinematically regular at $\mathscr{S}_{12} = 0$. Now, since $\mathscr{A}_{34}$ is kinematically uniform at $\mathscr{S}_{12} = 0$,

$$\beta_{34} \equiv 0 \qquad \text{if} \quad m_{12} \text{ is even,}$$
$$\alpha_{34} \equiv 0 \qquad \text{if} \quad m_{12} \text{ is odd,}$$

so that, in any case,

$\mathscr{S}_{12}^{m_{12}^{-}}\mathscr{A}_{34}$ is kinematically regular at $\mathscr{S}_{12} = 0$. (We have used the notation $N^{\pm}$ to denote N if N is an even integer and $N \pm 1$ if N is an odd integer).

Other cases are treated on the same way. All results are summarized in Table IV, together with those which are related to the behavior at $\Phi = 0$ [Eq. (II-11)] and at $s = 0$ (see below).

5. Kinematical Singularities at $s = 0$

We now study a possible singularity of helicity amplitudes which does not appear in the basis-vectors of the helicity frames, but in Expansions A and B.

A. *Behavior of the Boosts B_i near $s = 0$*

We have

$$\left(\frac{\omega_i + m_i - p_{ij}}{[2m_i(\omega_i + m_i)]^{1/2}}\right)^{2\lambda_i} = \left(\frac{\omega_i - p_{ij}}{m_i}\right)^{\lambda_i} = \left(\frac{m_i^2 - m_j^2 + s - \mathscr{S}_{ij}}{2s^{1/2}m_i}\right)^{\lambda_i}.$$

The behavior of this quantity near $s = 0$ depends on the determination of $\mathscr{S}_{ij}$. For small $|s|$, $\mathscr{S}_{ij} = -\epsilon(|m_i^2 - m_j^2| + O(s))$, where $\epsilon = \pm 1$ depending on the way chosen to define the cuts of $\mathscr{S}_{ij}$ in the s plane; for instance, $\epsilon = +1$ if $\mathscr{S}_{ij}$ is cut as shown in Fig. 4.

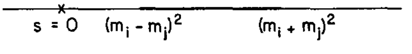

Fig. 4.

Let us now define $\epsilon_{ij} = (m_i - m_j)/|m_i - m_j|$; then

$$[(\omega_i - p_{ij})/m_i] \propto (s^{1/2})^{-\epsilon\epsilon_{ij}}. \tag{II-21}$$

B. *Behavior of* $\cos\theta_s$

$$\cos\theta_s = \frac{s(t-u) + (m_4^2 - m_3^2)(m_2^2 - m_1^2)}{\mathscr{S}_{12}\mathscr{S}_{34}} \tag{II-22}$$

TABLE IV

Notations:

$$\lambda = \lambda_1 - \lambda_2; \qquad \mu = \lambda_3 - \lambda_4; \qquad \eta_{12} = (-)^{s_1 - s_2 - \mu}; \qquad \eta_{34} = (-)^{s_3 - s_4 + \lambda}$$

$\eta = \eta_1\eta_2\eta_3\eta_4$: product of intrinsic parities.

$$m_{12} = s_1 + s_2 - \mathrm{Max}(|\lambda|, |\mu|); \qquad m_{34} = s_3 + s_4 - \mathrm{Max}(|\lambda|, |\mu|)$$

$$N^{\pm} = \begin{cases} N & \text{if } N \text{ is even} \\ N \pm 1 & \text{if } N \text{ is odd} \end{cases}$$

$$\varphi_{ij} = [s - (m_i + m_j)^2]^{1/2} \qquad \psi_{ij} = [s - (m_i - m_j)^2]^{1/2}$$

$$\hat{M}_{\lambda_3\lambda_4;\lambda_1\lambda_2} = \sin\left(\frac{\theta_s}{2}\right)^{-|\lambda-\mu|} \cos\left(\frac{\theta_s}{2}\right)^{-|\lambda+\mu|} M_{\lambda_3\lambda_4;\lambda_1\lambda_2}$$

Amplitudes free of kinematical singularities in the general mass case:

$$F^1_{\lambda_3\lambda_4;\lambda_1\lambda_2} = (s^{1/2})^{|\lambda|+|\mu|} \psi_{12}^a \varphi_{12}^b \psi_{34}^c \varphi_{34}^d \left(\hat{M}_{\lambda_3\lambda_4;\lambda_1\lambda_2} + \begin{array}{c} \eta_{34}\hat{M}_{-\lambda_3-\lambda_4;\lambda_1\lambda_2} \\ \text{or} \\ \eta\eta_{12}\hat{M}_{\lambda_3\lambda_4;-\lambda_1-\lambda_2} \end{array} \right)$$

$$F^2_{\lambda_3\lambda_4;\lambda_1\lambda_2} = (s^{1/2})^{|\lambda|+|\mu|} \psi_{12}^\alpha \varphi_{12}^\beta \psi_{34}^\gamma \varphi_{34}^\delta \left(\hat{M}_{\lambda_3\lambda_4;\lambda_1\lambda_2} - \begin{array}{c} \eta_{34}\hat{M}_{-\lambda_3-\lambda_4;\lambda_1\lambda_2} \\ \text{or} \\ \eta\eta_{12}\hat{M}_{\lambda_3\lambda_4;-\lambda_1-\lambda_2} \end{array} \right)$$

Reactions \ Exponents		a	b	c	d	α	β	γ	δ
$BB \to BB$ or $BF \to B'F'$	$\eta = +1$	m_{12}^-	m_{12}^-	m_{34}^-	m_{34}^-	$m_{12}^+ - 1$	$m_{12}^+ - 1$	$m_{34}^+ - 1$	$m_{34}^+ - 1$
$m_F > m_B$, $m_{F'} > m_{B'}$	$\eta = -1$	$m_{12}^+ - 1$	$m_{12}^+ - 1$	m_{34}^-	m_{34}^-	m_{12}^-	m_{12}^-	$m_{34}^+ - 1$	$m_{34}^+ - 1$
$BB \to F\bar{F}$ or $BF \to B'F'$	$\eta = +1$	m_{12}^-	m_{12}^-	$m_{34}^+ - 1$	m_{34}^-	$m_{12}^+ - 1$	$m_{12}^+ - 1$	m_{34}^-	$m_{34}^+ - 1$
$m_F > m_B$, $m_{F'} < m_{B'}$	$\eta = -1$	$m_{12}^+ - 1$	$m_{12}^+ - 1$	$m_{34}^+ - 1$	m_{34}^-	m_{12}^-	m_{12}^-	m_{34}^-	$m_{34}^+ - 1$
$F\bar{F} \to BB$ or $BF \to B'F'$	$\eta = +1$	$m_{12}^+ - 1$	m_{12}^-	m_{34}^-	m_{34}^-	m_{12}^-	$m_{12}^+ - 1$	$m_{34}^+ - 1$	$m_{34}^+ - 1$
$m_F < m_B$, $m_{F'} > m_{B'}$	$\eta = -1$	m_{12}^-	$m_{12}^+ - 1$	m_{34}^-	m_{34}^-	$m_{12}^+ - 1$	m_{12}^-	$m_{34}^+ - 1$	$m_{34}^+ - 1$
$FF \to FF$ or $BF \to B'F'$	$\eta = +1$	$m_{12}^+ - 1$	m_{12}^-	$m_{34}^+ - 1$	m_{34}^-	m_{12}^-	$m_{12}^+ - 1$	m_{34}^-	$m_{34}^+ - 1$
$m_F < m_B$, $m_{F'} < m_{B'}$	$\eta = -1$	m_{12}^-	$m_{12}^+ - 1$	$m_{34}^+ - 1$	m_{34}^-	$m_{12}^+ - 1$	m_{12}^-	m_{34}^-	$m_{34}^+ - 1$

then at

$$s \sim 0, \cos\theta_s = \epsilon_{12}\epsilon_{34} + O(s) \tag{II-22}$$

(if $\mathscr{S}_{12}$ and $\mathscr{S}_{34}$ are cut in the same way).

C. *Behavior of the Semibivectors*

$$\overrightarrow{e(\Lambda p_1, \Lambda p_3)}^2 = e_1^2 + e_2^2 + e_3^2 = f(t) : \text{a polynomial in } t,$$

$$\overrightarrow{e(\Lambda p_2, \Lambda p_3)}^2 = e_1'^2 + e_2'^2 + e_3'^2 = g(u) : \text{a polynomial in } u.$$

In the frame R_I for instance, $e_1^2 + e_2^2 = \frac{1}{4}m_1^2p^2\sin^2\theta_s$ which is finite, regular, and, in general, nonzero at $s = 0$; thus e_3 is also finite and regular at $s = 0$. So is e_3' for the same reason.

D. *Kinematical Behavior at $s = 0$*

Using expansion A (expansion B would give the same result), one finds that $M_{\lambda_3\lambda_4;\lambda_1\lambda_2}$ behaves near $s = 0$ as

$$-\epsilon\epsilon_{12}\lambda_1 + \epsilon\epsilon_{12}\lambda_2 + \epsilon\epsilon_{34}\lambda_3 - \epsilon\epsilon_{34}\lambda_4 \\ + |A_3 + \epsilon_{12}\epsilon_{34}\lambda_3| + |A_4 - \epsilon_{12}\epsilon_{34}\lambda_4| + \epsilon\epsilon_{12}(M_1 + M_2)(s^{1/2}).$$

Using $M_1 + M_2 = \lambda_1 - \lambda_2 + A_3 + A_4$, we write this exponent in the form

$$|A_3 + \epsilon_{12}\epsilon_{34}\lambda_3| + \epsilon\epsilon_{12}(A_3 + \epsilon_{12}\epsilon_{34}\lambda_3) + |A_4 - \epsilon_{12}\epsilon_{34}\lambda_4| + \epsilon\epsilon_{12}(A_4 - \epsilon_{12}\epsilon_{34}\lambda_4),$$

which is a nonnegative *even* integer which reaches the value zero for $A_3 = -\epsilon_{12}\epsilon_{34}\lambda_3$ and $A_4 = \epsilon_{12}\epsilon_{34}\lambda_4$. Thus, the helicity amplitudes do not have any kinematical singularity at $s = 0$, whatever the spin configuration may be.

However

$$\hat{M}_{\lambda_3\lambda_4;\lambda_1\lambda_2} = \sin(\theta_s/2)^{-|\lambda-\mu|}\cos(\theta_s/2)^{-|\lambda+\mu|} M_{\lambda_3\lambda_4;\lambda_1\lambda_2}$$

(Eq. (II-11)) is singular at $s = 0$ since it behaves like

$$(s^{1/2})^{-|\lambda-\epsilon_{12}\epsilon_{34}\mu|}.$$

Now, since in the case $BF \to B'F'$ (and in this case only) $|\lambda - \mu|$ and $|\lambda + \mu|$ differ by an odd integer, it is impossible in this case to free the helicity amplitudes from their kinematical singularities simultaneously at $\Phi(s, t) = 0$, $\mathscr{S}_{12} = 0$, $\mathscr{S}_{34} = 0$, and $s = 0$.

To summarize, $\mathscr{A}_{34}$ and $\mathscr{B}_{34}$ [Eq. (II-17)] behave near $s = 0$ like

$$(s^{1/2})^{-(|\lambda|+|\mu|)}[\alpha(s, t) + (s^{1/2})^{2\mathrm{Min}(|\lambda|, |\mu|)} \beta(s, t)], \tag{II-23}$$

where $\alpha(s, t)$ and $\beta(s, t)$ have no kinematical singularities at $s = 0$.

6. General Mass Case: Summary and Conclusions

We summarize the results obtained on the kinematical singularities of the helicity amplitudes in the general mass case. We write down in Table II-4 two amplitudes $F^1_{\lambda_3\lambda_4;\lambda_1\lambda_2}$ and $F^2_{\lambda_3\lambda_4;\lambda_1\lambda_2}$ which do not have any kinematical branch points, poles and zeros. However, in the case $BF \to BF$ we must add some remarks. Looking at Table IV, one sees that:

(a) There remains a kinematical square root branch point at $s = 0$.

(b) The regularization at $\mathscr{S}_{12} = 0$ and $\mathscr{S}_{34} = 0$ depends on the choice of the determination of $s^{1/2}$. The results given here are obtained with the determination of $s^{1/2}$ which is positive near $s = (m_i \pm m_j)^2$ (see Fig. 2 and 3). With the other determination we would get slightly different results: since the helicity amplitudes themselves are kinematically regular at $s = 0$, it is easy to relate $F^1_{\lambda_3\lambda_4;\lambda_1\lambda_2}(-s^{1/2})$ to $F^2_{\lambda_3\lambda_4;\lambda_1\lambda_2}(s^{1/2})$; we find:

$$F^1_{\lambda_3\lambda_4;\lambda_1\lambda_2}(-s^{1/2}) = (-)^{|\lambda|+|\mu|+|\lambda-\epsilon_{12}\epsilon_{34}\mu|} F^2_{\lambda_3\lambda_4;\lambda_1\lambda_2}(s^{1/2}) \tag{II-24}$$

These relations can be interpreted [as already done by Y. Hara (*7*)] as a generalization of the Mac Dowell reciprocity relation (*15*). In fact, Eq. (II-23) provides more information than this generalized Mac Dowell relation (II-24) since it gives the actual behavior of R.H.A near $s = 0$.

If parity is not conserved, we have, in general, N linearly independent amplitudes $a_{\ell_1\ell_2}$, where $N = \Pi_i(2s_i + 1)$. Parity conservation implies relations between the a's which are still unknown in the general case, but can be easily expressed (Equation (II-6)) in terms of helicity amplitudes. The corresponding relations between the R.H.A take on the same form, namely

$$F^i_{\lambda_3\lambda_4;\lambda_1\lambda_2} \equiv \eta(-)^{\Sigma_i(s_i+\lambda_i)} F^i_{-\lambda_3-\lambda_4;-\lambda_1-\lambda_2} . \tag{II-25}$$

However, we still have too many amplitudes, since for a given set of λ's, we have defined two combinations of $\hat{M}_{\lambda_f\lambda_i}$ and $\hat{M}_{-\lambda_f\lambda_i}$ or $\hat{M}_{\lambda_f-\lambda_i}$. But it is obvious, from the definition of the F's, that

$$F^i_{\lambda_3\lambda_4;\lambda_1\lambda_2} \equiv \pm\eta_{34}F^i_{-\lambda_3-\lambda_4;\lambda_1\lambda_2} \equiv \pm\eta\eta_{12}F^i_{\lambda_3\lambda_4;-\lambda_1-\lambda_2} . \tag{II-26}$$

The second identity expresses parity conservation. Taking into account these relations, one easily verifies that when parity is conserved, the right number of linearly independent amplitudes can be chosen out of the set of $2N$ regularized helicity amplitudes.

As already pointed out, the same kind of regularization procedure can be applied if parity is not conserved. Uniformization at thresholds and pseudo-thresholds could be explicitly achieved from Eq. (II-16), for four combinations of $\hat{M}_{\lambda_f\lambda_i}$, $\hat{M}_{-\lambda_f\lambda_i}$, $\hat{M}_{\lambda_f-\lambda_i}$, $\hat{M}_{-\lambda_f-\lambda_i}$. Behavior at $s = 0$ would again depend on the spin configuration and on the value of $|\lambda| + |\mu|$. Kinematical poles would then be canceled out by the same type of powers of φ_{12}, ψ_{12}, φ_{34}, ψ_{34} and $s^{1/2}$, since the orders of possible poles are functions of $|\lambda|$ and $|\mu|$ only. We would now arrive at $4N$ R.H.A, $F^i_{\lambda_3\lambda_4;\lambda_1\lambda_2}$, $i = 1, 2, 3, 4$. It is easily verified on (II-16) that, for each i, $F^i_{-\lambda_f\lambda_i}$, $F^i_{\lambda_f-\lambda_i}$, $F^i_{-\lambda_f-\lambda_i}$ are simply related to $F^i_{\lambda_f\lambda_i}$ so that N linearly independent amplitudes $F^i_{\lambda_3\lambda_4;\lambda_1\lambda_2}$ can be selected out.

The results of this section devoted to the regularization of the helicity amplitudes can now be summarized as follows:

(1) New functions $F^i_{\{\lambda\}}$ are defined ($i = 1, 2$ if parity is conserved, $i = 1, 2, 3, 4$ if not). They are related to the Joos amplitudes $a_{\ell_1\ell_2}$ through polynomials in s, t, u.[5]

(2) From the set of functions $F^i_{\{\lambda\}}$, one can pick out a set of linearly independent amplitudes, which we denote $\mathscr{F}_{\{\lambda\}}$. The result of this section can be formally written as

$$\mathscr{F}_{\{\lambda\}} = \sum_{\ell_1\ell_2} C^{\ell_1\ell_2}_{\{\lambda\}} a_{\ell_1\ell_2}, \tag{II-27}$$

where the $C^{\ell_1\ell_2}_{\{\lambda\}}$ coefficients are polynomials in s, t, u.[5]

7. Particular Cases: Equalities between External Masses

Whenever two or more kinematical singularities coincide, the general study does not apply. For instance, one can see that in elastic scattering ($m_1 = m_3$, $m_2 = m_4$, $\mathscr{S}_{12} = \mathscr{S}_{34}$), the general reasoning using expansions A and B fails since the third component of the two involved semibivectors are singular at $\mathscr{S}_{12} = \mathscr{S}_{34} = 0$. It will therefore be necessary to find another expansion. The particular mass configurations and the kinematical singularities which, correspondingly, need a special study are shown in Table (V). Results are summarized in Tables VI–X and details on their derivation are given in Appendix A-I.

[5] (If s corresponds to a BF state, s has to be replaced by $s^{1/2}$ in this statement).

TABLE V

KINEMATIC SINGULARITIES AND MASS CONFIGURATION[a]

Particular mass configuration \ Kinematical singularities needing a special study		$s = 0$	$\mathscr{S}_{12} = 0$	$\mathscr{S}_{34} = 0$
$m_1 = m_3$ $m_2 = m_4$ (see Table VI)	$m_1 \neq m_2$		$\times$	$\times$
$m_1 = m_2$ (see Table VII)	$m_3 \neq m_4$	$\times$	$\times$	
$m_3 = m_4$ (see Table VIII)	$m_1 \neq m_2$	$\times$		$\times$
$m_1 = m_2 = m$ $m_3 = m_4 = m'$ (see Table IX)	$m \neq m'$	$\times$	$\times$	$\times$
$m_1 = m_2 = m_3 = m_4 = m$ (see Table X)		$\times$	$\times$	$\times$

[a] The symbol $\times$ indicates the singularities which need a special study in the corresponding particular mass configurations.

TABLE VI

AMPLITUDE FREE OF KINEMATICAL SINGULARITIES FOR

$m_1 = m_3;\ m_2 = m_4;\ m_1 \neq m_2;\ \eta = +1;\ s_1 = s_3 = s;\ s_2 = s_4 = s'.$

$$(s^{1/2})^{|\lambda-\mu|}\mathscr{S}^{2m}\hat{M}_{\lambda_3\lambda_4;\lambda_1\lambda_2}$$

$$\mathscr{S} = \mathscr{S}_{12} = \mathscr{S}_{34}; \qquad m = s + s' - \mathrm{Max}(|\lambda|, |\mu|)$$

TABLE VII

AMPLITUDES FREE OF KINEMATICAL SINGULARITIES FOR $m_1 = m_2 = m$; $m_3 \neq m_4$; $s_1 = s_2 = s_{\text{in}}$.[a,b]

$$F^1_{\lambda_3\lambda_4;\lambda_1\lambda_2} = (s^{1/2})^{a'}\, k^{b'} \psi^c_{34} \varphi^d_{34} \left(\hat{M}_{\lambda_3\lambda_4;\lambda_1\lambda_2} \begin{array}{c} +\ \eta_{34} \hat{M}_{-\lambda_3-\lambda_4;\lambda_1\lambda_2} \\ \text{or} \\ +\ \eta\eta_{12} \hat{M}_{\lambda_3\lambda_4;-\lambda_1-\lambda_2} \end{array} \right)$$

$$F^2_{\lambda_3\lambda_4;\lambda_1\lambda_2} = (s^{1/2})^{\alpha'}\, k^{\beta'} \psi^\gamma_{34} \varphi^\delta_{34} \left(\hat{M}_{\lambda_3\lambda_4;\lambda_1\lambda_2} \begin{array}{c} -\ \eta_{34} \hat{M}_{-\lambda_3-\lambda_4;\lambda_1\lambda_2} \\ \text{or} \\ -\ \eta\eta_{12} \hat{M}_{\lambda_3\lambda_4;-\lambda_1-\lambda_2} \end{array} \right)$$

Spin and parity configuration \ Values of the exponents		a'	b'	α'	β'
$\lambda_1 + \lambda_2 + \mu$ even	$\eta = +1$	$(2s_{\text{in}})^-$	m^-	$(2s_{\text{in}})^+ - 1$	$m^+_{12} - 1$
	$\eta = -1$	$(2s_{\text{in}})^+ - 1$	$m^+_{12} - 1$	$(2s_{\text{in}})^-$	m^-_{12}
$\lambda_1 + \lambda_2 + \mu$ odd	$\eta = +1$	$(2s_{\text{in}})^+ - 1$	m^-_{12}	$(2s_{\text{in}})^-$	$m^+_{12} - 1$
	$\eta = -1$	$(2s_{\text{in}})^-$	$m^+_{12} - 1$	$(2s_{\text{in}})^+ - 1$	m^-_{12}

[a] c, d, γ, δ are defined as in the general case (See Table IV).

[b] $k = \frac{1}{2}(s - 4m^2)^{1/2}$.

TABLE VIII

AMPLITUDES FREE OF KINEMATICAL SINGULARITIES FOR:

$m_3 = m_4 = m'$; $m_1 \neq m_2$; $s_3 = s_4 = s_f$.[a,b]

$$F^1_{\lambda_3\lambda_4;\lambda_1\lambda_2} = (s^{1/2})^{c'}\,\psi^a_{12}\varphi^b_{12}p^{d'}\begin{pmatrix} & +\eta_{34}\hat{M}_{-\lambda_3-\lambda_4;\lambda_1\lambda_2} \\ \hat{M}_{\lambda_3\lambda_4;\lambda_1\lambda_2} & \text{or} \\ & +\eta\eta_{12}\hat{M}_{\lambda_3\lambda_4;-\lambda_1-\lambda_2}\end{pmatrix}$$

$$F^2_{\lambda_3\lambda_4;\lambda_1\lambda_2} = (s^{1/2})^{\gamma'}\,\psi^\alpha_{12}\varphi^\beta_{12}p^{\delta'}\begin{pmatrix} & -\eta_{34}\hat{M}_{-\lambda_3;-\lambda_4\lambda_1\lambda_2} \\ \hat{M}_{\lambda_3\lambda_4;\lambda_1\lambda_2} & \text{or} \\ & -\eta\eta_{12}\hat{M}_{\lambda_3\lambda_4;-\lambda_1-\lambda_2}\end{pmatrix}$$

Spin and parity configurations \ Values of the exponents		c'	d'	γ'	δ'
$\lambda_3 + \lambda_4 + \lambda$ even	$\eta = +1$	$(2s_f)^-$	m^-_{34}	$(2s_f)^+ - 1$	$m^+_{34} - 1$
	$\eta = -1$	$(2s_f)^-$	m^-_{34}	$(2s_f)^+ - 1$	$m^+_{34} - 1$
$\lambda_3 + \lambda_4 + \lambda$ odd	$\eta = +1$	$(2s_f)^+ - 1$	m^-_{34}	$(2s_f)^-$	$m^+_{34} - 1$
	$\eta = -1$	$(2s_f)^+ - 1$	m^-_{34}	$(2s_f)^-$	$m^+_{34} - 1$

[a] a, b, α and β are defined as in the general case (see Table IV).

[b] $p = \frac{1}{2}(s - 4m'^2)^{1/2}$.

TABLE IX

AMPLITUDES FREE OF KINEMATICAL SINGULARITIES FOR
$m_1 = m_2 = m,\ m_3 = m_4 = m',\ m \neq m';\ s_1 = s_2 = s_{\text{in}};\ s_3 = s_4 = s_f$.[a]

$$F^1_{\lambda_3\lambda_4;\lambda_1\lambda_2} = (s^{1/2})^a k^b p^d \left(\hat{M}_{\lambda_3\lambda_4;\lambda_1\lambda_2} + \begin{matrix}\eta_{34}\hat{M}_{-\lambda_3-\lambda_4;\lambda_1\lambda_2}\\ \text{or}\\ \eta\eta_{12}\hat{M}_{\lambda_3\lambda_4;-\lambda_1-\lambda_2}\end{matrix}\right)$$

$$F^2_{\lambda_3\lambda_4;\lambda_1\lambda_2} = (s^{1/2})^\alpha k^\beta p^\delta \left(\hat{M}_{\lambda_3\lambda_4;\lambda_1\lambda_2} - \begin{matrix}\eta_{34}\hat{M}_{-\lambda_3-\lambda_4;\lambda_1\lambda_2}\\ \text{or}\\ \eta\eta_{12}\hat{M}_{\lambda_3\lambda_4;-\lambda_1-\lambda_2}\end{matrix}\right)$$

Spin and parity configurations \ Value of the exponents		a	b	d	α	β	δ
$\Sigma\lambda_i$ even	$\eta = +1$	0	m^-_{12}	m^-_{34}	0	$m^+_{12} - 1$	$m^+_{34} - 1$
	$\eta = -1$	-1	$m^+_{12} - 1$	m^-_{34}	-1	m^-_{12}	$m^+_{34} - 1$
$\Sigma\lambda_i$ odd	$\eta = +1$	-1	m^-_{12}	m^-_{34}	-1	$m^+_{12} - 1$	$m^+_{34} - 1$
	$\eta = -1$	0	$m^+_{12} - 1$	m^-_{34}	0	m^-_{12}	$m^+_{34} - 1$

[a] $k = \frac{1}{2}(s - 4m^2)^{1/2}$, $p = \frac{1}{2}(s - 4m'^2)^{1/2}$; $\eta = +1$ for $BB \to BB$, $FF \to FF$ and $F\bar{F} \to F\bar{F}$; $\eta = -1$ for $BB \to F\bar{F}$ and $F\bar{F} \to BB$.

TABLE X

AMPLITUDE FREE OF KINEMATICAL SINGULARITIES FOR
$m_1 = m_2 = m_3 = m_4 = m,\ \eta = +1;\ s_1 = s_2 = s_3 = s_4 = S$

$\Sigma\lambda_i$ even:	$\Sigma\lambda_i$ odd:
$p^{2n}\hat{M}_{\lambda_3\lambda_4\lambda_1\lambda_2}$	$(p^{2n}/s^{1/2})\,\hat{M}_{\lambda_3\lambda_4\lambda_1\lambda_2}$

$$p = \tfrac{1}{2}(s - 4m^2)^{1/2}$$
$$n = 2S - \text{Max}(|\lambda|, |\mu|)$$

III. CROSSING MATRIX FOR HELICITY AMPLITUDES

Let $M^{(s)}_{\lambda_3\lambda_4;\lambda_1\lambda_2}(s, t, u)$ and $M^{(t)}_{\lambda_3\lambda_1;\lambda_4\lambda_2}(s, t, u)$ be, respectively, the helicity amplitudes for the two reactions:

$$1 + 2 \rightarrow 3 + 4 \text{ with momenta } p_i \text{, helicities } \lambda_i$$

and

$$\bar{4} + 2 \rightarrow 3 + \bar{1} \text{ with momenta } q_i \text{, helicities } \lambda_i \text{,}$$

as defined in Eq. (II-5) from the corresponding spinor amplitudes $\mathcal{M}^{(s)}_{A_3A_4;A_1A_2}(p_3p_4\,;\,p_1p_2)$ and $\mathcal{M}^{(t)}_{A_3A_1;A_4A_2}(q_3q_1\,;\,q_4q_2)$. For suitable analytic properties of the spinor amplitudes, these helicity amplitudes are initially defined as analytic functions of s, t, u in complex neighborhoods S^+ and T^+ of the corresponding physical regions. The index $+$ means that the s (resp. t) physical region (s and t real), are reached within S^+ (resp. T^+) from the upper half s (resp. t) plane. The purpose of this section is:

(1) to analytically continue $M^{(t)}$ from T^+ to S^+,

(2) to find the relation between $M^{(s)}$ and the analytic continuation of $M^{(t)}$, (crossing matrix).

1. Crossing Properties of the Spinor Amplitudes

The crossing properties of the spinor amplitudes on the mass shell have been proved by Bros, Epstein, and Glaser (*5*) in the framework of quantum field theory (*L.S.Z.* formulation). Their proof holds for four particles with arbitrary nonvanishing masses, whenever there exists a strictly positive minimum mass for all states different from the vacuum. We shall first briefly recall their results:

(a) $\mathcal{M}^{(s)}_{A_3A_4;A_1A_2}(p_3p_4\,;\,p_1p_2)$ and $\mathcal{M}^{(t)}_{A_3A_4;A_1A_2}(q_3q_4;\,q_1q_2)$[6] being boundary values of the same function $\mathcal{M}_{A_3A_4;A_1A_2}(k_3k_4\,;\,k_1k_2)$ with initial analyticity domain Δ in complex k_i space, $k_1 + k_2 = k_3 + k_4$, the holomorphy envelope of Δ always contains a connected open set of the mass shell manifold ($k_i^2 = m_i^2$) which connects the physical regions of the two reactions. In other words, $\mathcal{M}^{(t)}_{A_3A_1;A_4A_2}(q_3q_1\,;\,q_4q_2)$ can be continued from the physical region of the t reaction to a point $q_4 = -p_4$, $q_2 = p_2$, $q_3 = p_3$, $q_1 = -p_1$, where $\{p_1, p_2, p_3, p_4\}$ belongs to the s-physical region, and $q_i^2 = m_i^2$ all along the continuation path.

[6] Note the order of the indices and the momenta in spinor amplitudes. It refers to the order in which fields are written down in the definition of spinor amplitudes as Fourier transforms of vacuum expectation values of time-ordered products.

Furthermore, one knows that for any set $\{A_i\}$ of spinor indices,

$$\mathscr{M}^{(s)}_{A_3A_4;A_1A_2}(p_3p_4\,;p_1p_2) = (-)^{\sigma(P)}\,\mathscr{M}^{(t)}_{A_3A_1;A_4A_2}(p_3 - p_1\,; -p_4p_2) \qquad \text{(III-1)}$$

where $\sigma(P) = 1$ if 1 and 4 are fermions, and $\sigma(P) = 0$ in all other cases.

(b) A continuation path can be explicitly defined by its image in the space of invariants $\{s, t, u; s + t + u = \sum_i m_i^2\}$: one first connects the region T^+ to the region U^- (u-physical region reached from below) by a domain $\Omega_s^+(s_1) : \{s_1 < 0, \operatorname{Im} t > 0;\ |\,t\,| > R(s_1);\ |\,s - s_1\,| < \epsilon(s_1, t)\}$, and then U^- to S^+ by $\Omega_t^-(t_1) : \{t_1 < 0, \operatorname{Im} s > 0;\ |\,s\,| > R(t_1);\ |\,t - t_1\,| < \epsilon(t_1, s)\}$. Finally, one chooses a path C in the union of

$$T^+,\ \Omega_s^+(s_1),\ U^-,\ \Omega_t^-(t_1),\ S^+.$$

Any path Γ in $\{q_i\}$ space, $q_i^2 = m_i^2$, the image of which is C can then be taken to analytically continue $\mathscr{M}^{(t)}$ from a t-physical point $\{q_i\}$ to a point $\{-p_4, p_2, p_3, -p_1\}$, $\{p_i\}$ in the s-physical region.

2. Analytic Continuation of the Helicity Amplitudes

We know from Section II that the helicity amplitudes $M^{(t)}_{\{\lambda\}}(s, t, u)$ are analytic functions of s, t, and u, $(s + t + u = \sum_i m_i^2)$, in the image of the analyticity domain of the spinor amplitude $\mathscr{M}^{(t)}_{\{A_i\}}(q_i)$ deprived from the sets $\Phi(s, t) = 0$, $\mathscr{T}_{31} = 0$, $\mathscr{T}_{42} = 0$. $\mathscr{T}_{31}$ and $\mathscr{T}_{42}$ are defined in the t-channel as $\mathscr{S}_{12}$ and $\mathscr{S}_{34}$ are in the s-channel and correspond to the threshold and pseudo-threshold singularities in the t-channel.

$M^{(t)}_{\{\lambda\}}(s, t, u)$ is formally defined [Eq. (II-5)] by

$$M^{(t)}_{\{\lambda\}}(s, t, u) = \mathscr{D}(L_{31})^{\{A\}}_{\{\lambda\}}\,\mathscr{M}^{(t)}_{\{A\}}(q_i), \qquad \text{(III-2)}$$

where $\mathscr{D}(L_{31})$ involves the four helicity Lorentz transformations corresponding to the t-channel.[7]

In order to continue $M^{(t)}$ from T^+ to S^+, we have to show that the analyticity domain of $\mathscr{M}^{(t)}$ which connects the two physical regions contains an analyticity domain for $\mathscr{D}(L_{31})$.

[7] When we analytically continue outside the physical region, the helicity frame becomes complex, so that the full Lorentz transformation from the standard frame to the helicity frame is no longer determined (Sec. II-1-A) by $(L(p), L^{\dagger}(p))$, but by some pair of 2×2 matrices (L, L'), $L' = L^{\dagger}(p^*)$. We shall be interested only in the matrix L and its analytic continuation, which we shall still call a Lorentz transformation, although it is only part of it.

For the sake of definiteness, let us consider what happens for particle 1. In the t-physical region, the part of $\mathscr{D}(L_{31})$ relative to this particle is the matrix $D^{s_1}(L_{31}(1))$, where $L_{31}(1)$ is the helicity Lorentz transformation:

$$L_{31}(1)\begin{Bmatrix}\mathring{t}\\ \mathring{n}_2\\ \mathring{n}_3\end{Bmatrix} = \begin{Bmatrix}q_1/m_1\\ W_t\\ h_{31}(1)\end{Bmatrix},$$

where W_t is defined by:

$$W_{t_\mu} = -\frac{2\epsilon_{\mu\nu\rho\sigma}q_4{}^\nu q_2{}^\rho q_3{}^\sigma}{[\Phi(s,t,u)]^{1/2}}$$

$$h_{31}(1) = -2\,\frac{m_1{}^2P_{31} - (q_1\cdot P_{31})\,q_1}{m_1\mathscr{T}_{31}}$$

$$P_{31} = P_{42} = P_t = q_1 + q_3 = q_2 + q_4\,.$$

All these definitions are t-channel analogus of Eq. (II-4).

The Lorentz transformation $L_{31}(1)$ is singular wherever the basis vectors of the helicity frame associated with particle 1 are. It is singular for $\Phi = 0$ (cf. the denominator of W_t) and $\mathscr{T}_{31} = 0$ [cf. the denominator of $h_{31}(1)$]. Then, $L_{31}(1)$ can be analytically continued along any path which avoids these singular points, and the result of this continuation is the same (up to a sign, as it will be seen below) for all paths which lead to the same determination of $\mathscr{T}_{31}$ and $\Phi^{1/2}$.

Of course the same argument holds for the other particles. For bosons, a sign ambiguity is of no consequence since the matrix elements of $D^{s_i}(L_{31}(i))$ are homogeneous functions of degree $2s_i$, of the matrix elements of $L_{31}(i)$. For fermions, $D^{s_i}(L_{31}(i))$ may depend on the path followed in q_i space, just as $L_{31}(i)$ does. $L_{31}(i)$ can be written as $\sigma\cdot Q(i)/[Q^2(i)]^{1/2}$ where the object $Q(i)$ has components which have singularities of the type $\Phi^{1/2}$ and $\mathscr{T}_{31}$ or $\mathscr{T}_{42}$; $L_{31}(i)$ has one additional possible singularity, which is not a singularity of the basis vectors of the helicity frames, namely $Q^2(i) = 0$, which leads to the above mentioned sign ambiguity. This last equation is a relation between the *components* of the q_i's which cannot be expressed as a condition on the *invariants*. As a consequence, 2 paths Γ_1 and Γ_2 in q_i space, on the mass-shell, which have the same image C in the invariant space but which lead to opposite values of $[Q^2(i)]^{1/2}$ are not equivalent for $D^{s_i}(L_{31}(i))$. However, we shall prove the following.

LEMMA. *The analytic continuation along a path Γ of the tensor product $L_{31}(i)\otimes L_{31}(j)$ only depends on the image C of Γ, and not on Γ itself.*

Consequence. Since there are always an even number of fermions involved in a reaction, different paths Γ with the same image C lead to the same determination of $\prod_i^{\otimes} D^{s_i}(L_{31}(i))$.

Proof of the lemma. We write, as in Eq. (II-4),

$$L_{31}(i) = B\left(\frac{P_t}{t^{1/2}} \to \frac{q_i}{m_i}\right) \Omega_{31}(n_3(P_t) \to q(i))[P_t]\, \epsilon', \tag{III-3}$$

where

$$q(i) = q_{42} \quad \text{for} \quad i = 4 \text{ or } 2 \quad \text{and} \quad q(i) = q_{31} \quad \text{for} \quad i = 3 \text{ or } 1$$
$$\epsilon' = 1 \quad \text{for} \quad i = 4 \text{ or } 3 \quad \text{and} \quad \epsilon' = -i\sigma_2 \quad \text{for} \quad i = 2 \text{ or } 1.$$

We consider $X_{ij} = L_{31}(i) \otimes L_{31}(j)$,

(a) X_{ij} is bilinear in $[P_t]$ so that the sign ambiguity of $[P_t]$ is irrelevant;

(b) $$B\left(\frac{P_t}{t^{1/2}} \to \frac{q_i}{m_i}\right) = \frac{m_i t^{1/2} + \sigma \cdot q_i \tilde{\sigma} \cdot P_t}{[2m_i t^{1/2}(m_i t^{1/2} + q_i \cdot P_t)]^{1/2}}.$$

The singularities of B in q-space can be expressed through functions of the invariants only, so that B is the same for different Γ's which have the same image C.

(c) We can always choose for $[P_t]$ a Lorentz transformation which takes $\mathring{n}_2$ onto W_t, so that Ω_{31} is a Lorentz transformation, in the 2-plane orthogonal to P_t and W_t, which takes $n_3(P_t)$ onto $q(i)$. Then $\Omega_{31}(j) = K\Omega_{31}(i)$, where K represents a pure Lorentz transformation which takes $q(i)$ onto $q(j)$ ($K = +\mathbf{1}$ if i and j belong to the same 2-body state). The singularities of K are again expressed through invariant functions of the q_i's, and X_{ij} is linear in K and bilinear in $\Omega_{31}(i)$ so that the sign ambiguity due to $\Omega_{31}(i)$ is also irrelevant.

Conclusion. Taking in the definition of the analyticity domain of Bros, Epstein, and Glaser (5) (see Section III-1), large enough values of $R(s_1)$ and $R(t_1)$ in such a way that any path C also avoids all kinematical singularities, any path Γ in q_i-space with image C is in the analyticity domain of the matrix $\mathscr{D}(L_{31})_{\{\lambda\}}^{\{A\}}$ of Eq. (III-2). An explicit choice for C will be exhibited in Section III-3-A.

3. Crossing Matrix for Helicity Amplitudes

Labeling with the index c the "crossed" quantities, that is to say quantities which have been continued from the t-physical region to the s-physical region along a path Γ, we formally have

$$M_{\{\lambda\}}^{(t)^c}(s, t, u) = \mathscr{D}^c(L_{31})_{\{\lambda\}}^{\{A\}} \mathscr{M}_{\{A\}}^{(t)^c}(q_i).$$

The crossing property (III-1) of the spinor amplitudes then leads to the following crossing relation:

$$M_{\{\lambda\}}^{\{s\}}(s, t, u) = (-)^{\sigma(P)} \sum_{\{A\}} \mathscr{D}^{-1^c}(L_{31})_{\{A\}}^{\{\lambda'\}} \mathscr{D}(L_{12})_{\{\lambda\}}^{\{A\}} M_{\{\lambda'\}}^{(t)^c}(s, t, u). \qquad \text{(III-4)}$$

More explicitly, the matrices $\mathscr{D}(L_{31})$ and $\mathscr{D}(L_{12})$ are, from Eq. (II-5),

$$\mathscr{D}(L_{31})_{\{\lambda'\}}^{\{A\}} = (-)^{s_3+\lambda_3'+s_1+\lambda_1'} D^{s_1}(L_{31}(1))_{-\lambda_1'}^{A_1} D^{s_2}(L_{31}(2))_{\lambda_2'}^{A_2} D^{s_3}(L_{31}(3))_{-\lambda_3'}^{A_3} D^{s_4}(L_{31}(4))_{\lambda_4'}^{A_4},$$

$$\mathscr{D}(L_{12})_{\{\lambda\}}^{\{A\}} = (-)^{s_4+\lambda_4+s_3+\lambda_3} D^{s_1}(L_{12}(1))_{\lambda_1}^{A_1} D^{s_2}(L_{12}(2))_{\lambda_2}^{A_2} D^{s_3}(L_{12}(3))_{-\lambda_3}^{A_3} D^{s_4}(L_{12}(4))_{-\lambda_4}^{A_4}.$$

The crossing matrix in Eq. (III-4) is then

$$Z_{\{\lambda\}}^{\{\lambda'\}} = (-)^{\sigma(P)} (-)^{\lambda_3-\lambda_3'+s_4+\lambda_4-s_1-\lambda_1'} D^{s_1}(L_{31}^{-1^c}(1)\, L_{12}(1))_{\lambda_1}^{-\lambda_1'}$$

$$\times D^{s_2}(L_{31}^{-1^c}(2)\, L_{12}(2))_{\lambda_2}^{\lambda_2'} D^{s_3}(L_{31}^{-1^c}(3)\, L_{12}(3))_{-\lambda_3}^{-\lambda_3'} D^{s_4}(L_{31}^{-1^c}(4)\, L_{12}(4))_{-\lambda_4}^{\lambda_4'}. \qquad \text{(III-5)}$$

The explicit calculation of Z therefore reduces to that of the tensor product

$$\prod_i^{\otimes} D^{s_i}(L_{31}^{-1^c}(i)\, L_{12}(i)).$$

We now show that $L_{31}^{-1^c}(i)\, L_{12}(i)$ can be determined, up to a sign, by considering the basis vectors of the associated helicity frames. We shall then write the crossing matrix Z, the overall sign being determined directly in Appendix A-II.

A. *Determination of* $L_{31}^{-1^c}(i)\, L_{12}(i)$, *up to a Sign*

To complete the definition of helicity amplitudes outside their respective physical regions, we define $\mathscr{T}_{31}$, $\mathscr{T}_{42}$, $\mathscr{S}_{12}$, $\mathscr{S}_{34}$ as analytic functions of s and t in the following cut planes:

— the t-plane is cut from

$$\text{Min}((m_3 - m_1)^2, (m_2 - m_4)^2)$$

to

$$\text{Max}((m_3 + m_1)^2, (m_4 + m_2)^2)$$

and $\mathscr{T}_{31}$ and $\mathscr{T}_{42}$ are positive for large real t;

— the s-plane is cut from

$$\mathrm{Min}((m_1 - m_2)^2, (m_3 - m_4)^2)$$

to

$$\mathrm{Max}((m_1 + m_2)^2, (m_3 + m_4)^2)$$

and $\mathscr{S}_{12}$ and $\mathscr{S}_{34}$ are positive for large real s.

— The Φ-plane is cut along the positive real axis and $\Phi^{1/2}$ is taken with a positive determination in the physical regions.

— Finally, in order to get the usual determination of the scattering angle, $0 \leqslant \theta \leqslant \pi$, we are led to define $s^{1/2}$ (resp. $t^{1/2}$, $u^{1/2}$) with a cut along the positive real axis and a positive determination in S^+ (resp. T^+, U^+). Then $\sin\theta_s = 2s^{1/2}\Phi^{1/2}/\mathscr{S}_{12}\mathscr{S}_{34}$ is $\geqslant 0$ in S^+, as well as $\sin\theta_t$ and $\sin\theta_u$ in T^+ and U^+, respectively.

For the sake of definiteness, we shall consider the planes of the variables s, t, u, and Φ with the cuts and determinations of the kinematical functions which have just been defined. Let us exhibit a path C in the Bros–Epstein–Glaser domain which allows the analytic continuation of the helicity amplitudes.

Let ρ be a large real parameter. We start from $M_0 : t = 4\rho$, $s = -2\rho$, $u = \sum_i m_i^2 - 2\rho$ in T^+.

We have a first arc:

$$t = \rho(1 + 3e^{i\varphi}), \qquad s = -2\rho$$
$$u = \sum_i m_i^2 + \rho(1 - 3e^{i\varphi}), \qquad 0 \leqslant \varphi \leqslant \pi$$

ending in $M_1 : t = -2\rho$, $s = -2\rho$, $u = \sum_i m_i^2 + 4\rho$ in U^- and a second arc

$$\begin{aligned} t &= -2\rho, \\ s &= \rho(1 - 3e^{-i\varphi}), \qquad\qquad 0 \leqslant \varphi \leqslant \pi, \\ u &= \sum_i m_i^2 + \rho(1 + 3e^{-i\varphi}), \end{aligned}$$

ending in $M_2 : t = -2\rho$, $s = 4\rho$, $u = \sum_i m_i^2 - 2\rho$ in S^+. Φ is a third-degree polynomial in s, t, u, which for large ρ may be approximated by

$$\Phi \simeq 2\rho^3(9e^{2i\varphi} - 1), \qquad 0 \leqslant \varphi \leqslant \pi \text{ on the first arc,}$$
$$\Phi \simeq 2\rho^3(9e^{-2i\varphi} - 1), \qquad 0 \leqslant \varphi \leqslant \pi \text{ on the second arc,}$$

so that $\Phi^{1/2}$ goes back to a positive value in S^+ as it should. Also, since we have never crossed the real s or t axis, $\mathscr{S}_{ij}$ and $\mathscr{T}_{ij}$ have not left the cut planes where they have been defined.

The image of C in the s, t, u, and Φ complex planes are shown in Figs. 5(a)–(d).

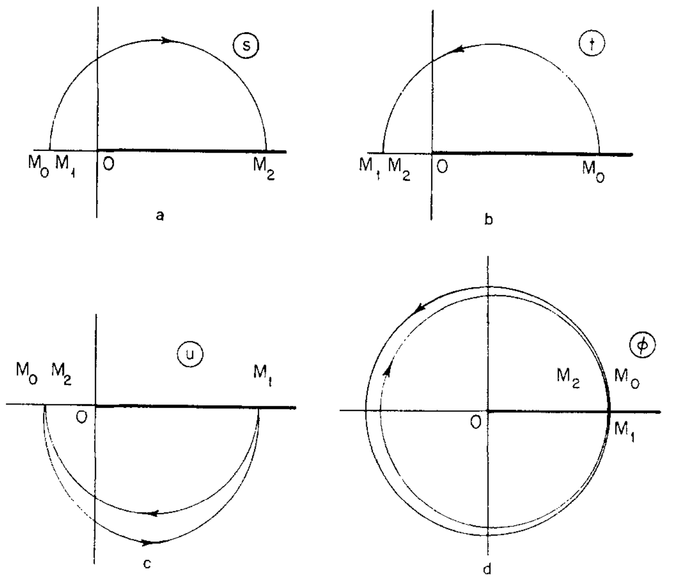

FIG. 5.

All functions being now well defined all along the continuation path, we turn to the analytic continuations of the t-helicity frames. In S^+, these continuations are

Basis-vector 0. p_i/m_i if $i = 2, 3$; $-p_i/m_i$ if $i = 4, 1$.

Basis-vector 2. $-W_s$ for $i = 1, 2, 3, 4$ because we have verified that we ended with a positive determination of $\Phi^{1/2}$, and furthermore,

$$(\epsilon_{\mu\nu\rho\sigma} q_4{}^\nu q_2{}^\rho q_3{}^\sigma)^c = -\epsilon_{\mu\nu\rho\sigma} p_1{}^\nu p_2{}^\rho p_3{}^\sigma.$$

Basis-vector 3.

$$h_{31}^c(i) = -2\,\frac{m_i{}^2(p_3 - p_1) - (p_i \cdot (p_3 - p_1))\, p_i}{m_i T(i)},$$

where $T(i)$ is the analytic continuation of $\mathscr{T}_{34}$ for $i = 1, 3$ and $\mathscr{T}_{12}$ for $i = 4, 2$.

This defines, up to a sign, the analytic continuation of $L_{31}(i)$. Finally, we recall that

$$L_{12}(i) \begin{Bmatrix} \mathring{t} \\ \mathring{n}_2 \\ \mathring{n}_3 \end{Bmatrix} = \begin{Bmatrix} p_i/m_i \\ W_s \\ h_{12}(i) \end{Bmatrix}.$$

(a) $i = 4$ *or* 1 (*crossed particles*)

Let us consider the Lorentz transformation $\mathscr{L}(i) = L_{31}^{c^{-1}}(i)\, L_{12}(i)$. From the preceding study, we have

$$\mathscr{L}(i) \begin{Bmatrix} \mathring{t} \\ \mathring{n}_1 \\ \mathring{n}_2 \\ \mathring{n}_3 \end{Bmatrix} = \begin{Bmatrix} -\mathring{t} \\ n_1'(i) \\ -\mathring{n}_2 \\ n_3'(i) \end{Bmatrix} .$$

$n_3'(i)$ [resp. $n_1'(i)$] is the 3-axis [resp. 1-axis] of the transformed frame. Note that the time axis, (and the 2-axis), of the standard frame are reversed, so that one needs a pair of unimodular matrices to specify $\mathscr{L}(i)$. Let $\mathscr{L}'(i)$ be the 2×2 matrix which accompanies $\mathscr{L}(i)$ in the definition of the complex Lorentz transformation $(\mathscr{L}(i), \mathscr{L}'(i))$.[8]

It is easily seen that $(\mathscr{L}(i), -\mathscr{L}'(i))$ transforms the standard frame according to

$$(\mathscr{L}(i), -\mathscr{L}'(i)) \begin{Bmatrix} \mathring{t} \\ \mathring{n}_1 \\ \mathring{n}_2 \\ \mathring{n}_3 \end{Bmatrix} = \begin{Bmatrix} \mathring{t} \\ -n_1'(i) \\ \mathring{n}_2 \\ -n_3'(i) \end{Bmatrix} .$$

It follows that this is a rotation around the second axis, which turns out to be real, so that

$$\mathscr{L}(i) = -[\mathscr{L}'(i)]^{-1} \equiv R(i) = \pm(\cos \tfrac{1}{2}\chi_i' - i\sigma_2 \sin \tfrac{1}{2}\chi_i') = \pm\alpha_i . \quad \text{(III-6a)}$$

The rotation angle χ_i' will be calculated in the following.

(b) $i = 2, 3$ (*uncrossed particles*)

$$\mathscr{L}(i) \begin{Bmatrix} \mathring{t} \\ \mathring{n}_1 \\ \mathring{n}_2 \\ \mathring{n}_3 \end{Bmatrix} = \begin{Bmatrix} \mathring{t} \\ n_1'(i) \\ -\mathring{n}_2 \\ n_3'(i) \end{Bmatrix} ; \qquad \text{hence } \mathscr{L}'(i)^c = [\mathscr{L}(i)]^\dagger \text{ in this case.}$$

$\mathscr{L}(i)$ is again a rotation, and it reverses the second axis. We write:

$$\mathscr{L}(i) = R_3(\pi)\, R(i),$$

[8] $\mathscr{L}'(i)$ is not the Hermitian conjugate $\mathscr{L}(i)^\dagger$. In fact, $\mathscr{L}'(i) = L_{12}^\dagger(i)\, [L_{31}^{\dagger-1}(i)]^c$ and it is quite possible that $[L_{31}^{-1\dagger}]^c \neq [L_{31}^{-1c}(i)]^\dagger$, because the analytic continuation of $L_{31}^\dagger(i)$ along the path C is the Hermitian conjugate of that of $L^3{}_1(i)$ along the path C^*, complex conjugate to C.

where $R_3(\pi)$ is a rotation through an angle π around the 3-axis and where $R(i)$ is a real rotation through an angle χ_i' around the second axis,

$$R(i) \begin{Bmatrix} \mathring{t} \\ \mathring{n}_1 \\ \mathring{n}_2 \\ \mathring{n}_3 \end{Bmatrix} = \begin{Bmatrix} \mathring{t} \\ -n_1'(i) \\ \mathring{n}_2 \\ n_3'(i) \end{Bmatrix} .$$

We write

$$\mathscr{L}(i) = \pm i\sigma_3(\cos \tfrac{1}{2}\chi_i' - i\sigma_2 \sin \tfrac{1}{2}\chi_i') = \mp\alpha_i \tag{III-6b}$$

B. *Determination of the Rotation Angles* χ_i': *The Crossing Matrix.*

We calculate χ_i', $-\pi \leqslant \chi_i' \leqslant \pi$, by

$$\cos \chi_i' = -\mathring{n}_3 \cdot (R(i)\, \mathring{n}_3)$$

$$\sin \chi_i' = -\mathring{n}_1 \cdot (R(i)\, \mathring{n}_3) = \mathring{n}_3 \cdot (R(i)\, \mathring{n}_1),$$

since χ_i' is the angle through which $\mathring{\mathbf{n}}_3$ has been rotated around the 2-axis.

Calculation of $\cos \chi_i'$

$$\cos \chi_i' = \epsilon_i \mathring{n}_3 \cdot (L_{31}^{-1^{\mathrm{c}}}(i)\, L_{12}(i)\, \mathring{n}_3),$$

with $\epsilon_i = +1$ for $i = 1, 4$ and $\epsilon_i = -1$ for $i = 2, 3$, according to the definition of $R(i)$.

By definition of $h_{12}(i)$ and $h_{31}(i)$,

$$L_{12}(i)\, \mathring{n}_3 = h_{12}(i), \qquad L_{31}^{\mathrm{c}}(i)\, \mathring{n}_3 = h_{31}^{\mathrm{c}}(i);$$

then

$$\cos \chi_i' = \epsilon_i h_{31}^{\mathrm{c}}(i)\, h_{12}(i),$$

$$h_{12}(i) = -2\, \frac{m_i^2 P_{12} - (p_i \cdot P_{12})\, p_i}{m_i S(i)},$$

$$h_{31}^{\mathrm{c}}(i) = -2\, \frac{m_i^2 (p_3 - p_1) - (p_i \cdot (p_3 - p_1))\, p_i}{m_i T(i)},$$

$$S(i) = \begin{pmatrix} \mathscr{S}_{12} \\ \mathscr{S}_{34} \end{pmatrix} \begin{matrix} \text{for } i = 1, 2 \\ \text{for } i = 3, 4 \end{matrix} \qquad T(i) = \begin{pmatrix} \mathscr{T}_{31} \\ \mathscr{T}_{42} \end{pmatrix} \begin{matrix} \text{for } i = 3, 1 \\ \text{for } i = 4, 2, \end{matrix}$$

which yields

$$\mathcal{S}_{12}\mathcal{T}_{31}\cos\chi_1' = (s + m_1^2 - m_2^2)(t + m_1^2 - m_3^2) + 2m_1^2\Delta,$$
$$\mathcal{S}_{12}\mathcal{T}_{42}\cos\chi_2' = (s + m_2^2 - m_1^2)(t + m_2^2 - m_4^2) - 2m_2^2\Delta,$$
$$\mathcal{S}_{34}\mathcal{T}_{31}\cos\chi_3' = (s + m_3^2 - m_4^2)(t + m_3^2 - m_1^2) - 2m_3^2\Delta,$$
$$\mathcal{S}_{34}\mathcal{T}_{42}\cos\chi_4' = (s + m_4^2 - m_3^2)(t + m_4^2 - m_2^2) + 2m_4^2\Delta,$$

with

$$\Delta = m_2^2 + m_3^2 - m_4^2 - m_1^2.$$

Calculation of $\sin\chi_i'$

Similarly,

$$\sin\chi_i' = \mathring{n}_1(L_{31}^{c^{-1}}(i)\, L_{12}(i)\, \mathring{n}_3);$$

i.e.,

$$\sin\chi_i' = n_1^{c}(i)\, h_{12}(i).$$

$h_{12}(i)$ has already been defined, $n_1^c(i)$ is the analytic continuation of the first basis-vector of the helicity frame of particle i in the t-channel. One gets

$$n_{1_\mu}(i) = \epsilon_{\mu\nu\rho\sigma} W_t^{\nu} h_{31}^{\rho}(i)(q_i^{\sigma}/m_i).$$

For instance, for $i = 1$, one has $n_{1_\mu}(1) = -(2/\mathcal{T}_{31})\,\epsilon_{\mu\nu\rho\sigma} W_t^{\nu} q_3^{\rho} q_1^{\sigma}$, whose analytic continuation is

$$n_{1_\mu}^{c}(1) = -(2/\mathcal{T}_{31})\,\epsilon_{\mu\nu\rho\sigma} W_s^{\nu} p_3^{\rho} p_1^{\sigma}.$$

Repeating this argument for $i = 2, 3, 4$ yields

$$\sin\chi_1' = -\frac{2m_1\Phi^{1/2}}{\mathcal{S}_{12}\mathcal{T}_{31}}, \qquad \sin\chi_3' = \frac{2m_3\Phi^{1/2}}{\mathcal{S}_{34}\mathcal{T}_{31}},$$
$$\sin\chi_2' = -\frac{2m_2\Phi^{1/2}}{\mathcal{S}_{12}\mathcal{T}_{42}}, \qquad \sin\chi_4' = \frac{2m_4\Phi^{1/2}}{\mathcal{S}_{34}\mathcal{T}_{42}}.$$

These angles χ_i' determine the crossing matrix, *up to an overall sign* $\eta^{(s)}$, *independent from the helicities.* Taking into account the rotations through π around the 3-axis which occur for $i = 2, 3$, the crossing matrix (III-5) now reads

$$M^{(s)}_{\lambda_3\lambda_4;\lambda_1\lambda_2}(s, t, u) = \eta^{(s)}\, (-)^{\sigma(P)}\, e^{-i\pi(\lambda_2-\lambda_3)} \sum_{\lambda_i'} (-)^{\lambda_3-\lambda_3'+s_4+\lambda_4-s_1-\lambda_1'}\, d^{s_1}(\chi_1')_{\lambda_1}^{-\lambda_1'}$$
$$\times\, d^{s_2}(-\chi_2')_{\lambda_2}^{\lambda_2'}\, d^{s_3}(-\chi_3')_{-\lambda_3}^{-\lambda_3'}\, d^{s_4}(\chi_4')_{-\lambda_4}^{\lambda_4'}\, M^{(t)}_{\lambda_3'\lambda_1';\lambda_4'\lambda_2'}(s, t, u). \qquad \text{(III-7)}$$

We finally clean out this equation by redefining angles χ_i' in order to write the crossing matrix as $\prod_i^{\otimes} d^{s_i}(\chi_i)_{+\lambda_i}^{+\lambda_i'}$, using identities such as:

$$d^s(\chi' \pm \pi)_\lambda^{\lambda'} = (-)^{s \pm \lambda'} d^s(\chi')_{\bar{\lambda}}^{-\lambda'}$$

and

$$d^s(\chi')_\lambda^{\lambda'} = (-)^{\lambda' - \lambda} d^s(-\chi')_\lambda^{\lambda'} = (-)^{\lambda' - \lambda} d^s(\chi')_{-\lambda}^{-\lambda'}.$$

Furthermore, we show in Appendix A-II that $\eta^{(s)} = (-)^{2s_2 + 2s_4}$, so that we get the final result, which is given in Table XI, and tested in Appendix A-V on the case of πN elastic scattering.

4. Crossing Matrix for the Regularized Helicity Amplitudes

In the crossing matrix of Table XI, we can express $M^{(s)}$ and $M^{(t)}$ as functions of the R.H.A $\mathscr{F}^{(s)}$ and $\mathscr{F}^{(t)}$, which have been defined in Section II. The crossing

TABLE XI*

The Crossing Matrix for Helicity Amplitudes

$$M^{(s)}_{\lambda_3\lambda_4;\lambda_1\lambda_2}(s, t, u) = (-)^{\sigma(P)+2s_2+2s_4}\, e^{i\pi(\lambda_2-\lambda_3)} \sum_{\lambda_i'} d^{s_1}(\chi_1)_{\lambda_1}^{\lambda_1'} d^{s_2}(\chi_2)_{\lambda_2}^{\lambda_2'}$$

$$\times\, d^{s_3}(\chi_3)_{\lambda_3}^{\lambda_3'} d^{s_4}(\chi_4)_{\lambda_4}^{\lambda_4'} M^{(t)}_{\lambda_3'\lambda_1';\lambda_4'\lambda_2'}(s, t, u)$$

$\sigma(P) = 1$ if 1 and 4 are fermions and 0 in all other cases;

$$\cos\chi_1 = -\frac{(s + m_1^2 - m_2^2)(t + m_1^2 - m_3^2) + 2m_1^2 \Delta}{\mathscr{S}_{12}\mathscr{T}_{31}} \qquad \sin\chi_1 = \frac{2m_1\sqrt{\Phi}}{\mathscr{S}_{12}\mathscr{T}_{31}}$$

$$\cos\chi_2 = \frac{(s + m_2^2 - m_1^2)(t + m_2^2 - m_4^2) - 2m_2^2 \Delta}{\mathscr{S}_{12}\mathscr{T}_{42}} \qquad \sin\chi_2 = \frac{2m_2\sqrt{\Phi}}{\mathscr{S}_{12}\mathscr{T}_{42}}$$

$$\cos\chi_3 = \frac{(s + m_3^2 - m_4^2)(t + m_3^2 - m_1^2) - 2m_3^2 \Delta}{\mathscr{S}_{34}\mathscr{T}_{31}} \qquad \sin\chi_3 = -\frac{2m_3\sqrt{\Phi}}{\mathscr{S}_{34}\mathscr{T}_{31}}$$

$$\cos\chi_4 = -\frac{(s + m_4^2 - m_3^2)(t + m_4^2 - m_2^2) + 2m_4^2 \Delta}{\mathscr{S}_{34}\mathscr{T}_{42}} \qquad \sin\chi_4 = -\frac{2m_4\sqrt{\Phi}}{\mathscr{S}_{34}\mathscr{T}_{42}}$$

$$\Delta = m_2^2 + m_3^2 - m_4^2 - m_1^2$$

*8bis We thank Dr. Trueman for a clarifying correspondence, according to which formula of Table XI of this paper can be put into coincidence with formula (43) of Ref. (*6*) by taking into account the particle "2" convention together with a reversal of the normal to the reaction plane in the t channel. Our choice of conventions is the same for all reactions once the order of particles is fixed.

relation then becomes

$$\mathscr{F}^{(s)}_{\{\lambda\}} = X^{\{\lambda'\}}_{\{\lambda\}} \mathscr{F}^{(t)}_{\{\lambda'\}}, \tag{III-8}$$

where the crossing matrix X enjoys the following properties:

(1) X is uniform since $\mathscr{F}^{(s)}$ and $\mathscr{F}^{(t)}$ do not have any kinematical singularity (except at $s = 0$ and/or $t = 0$ if channels s and/or t correspond to BF states). This is checked in Appendix A-III.

(2) X is furthermore a rational function of s and t since it is uniform and can be expressed in terms of algebraic functions. We write $X^{\{\lambda'\}}_{\{\lambda\}} = P/Q$, where P and Q are polynomials.

(3) Since $\mathscr{F}^{(s)}_{\{\lambda\}}$ is kinematically finite, zeros of Q have to be cancelled out by zeros of some combinations of the functions $\mathscr{F}^{(t)}_{\{\lambda'\}}$. The study of this problem is the purpose of the following section.

IV. KINEMATICAL CONSTRAINTS

1. Principle of the Method

The aim of this section is to generalize to the case of arbitrary spins and masses certain relations which must hold between R.H.A. at some values of the energy variable as it has been already remarked in $\pi N \to \pi N$ (*16*), $\pi\pi \to N\bar{N}$ (*16*), and $N\bar{N} \to N\bar{N}$ (*17*) reactions and which we call kinematical constraints.

We recall an example ($\pi\pi \to N\bar{N}$ scattering) of how such relations are derived. If one expresses the invariant amplitudes A and B (see appendix A-V) as functions of the R.H.A. F^1_{++} and F^2_{+-}, one finds

$$\begin{cases} A = \dfrac{m_N}{p^2}\left[F^1_{++} + \dfrac{m_N(t-u)}{2} F^2_{+-}\right], \\ B = -2m_N F^2_{+-}, \end{cases} \tag{IV-1}$$

which exhibits a purely kinematical pole for A at $p^2 = 0$, i.e., $s = 4m_N{}^2$, unless the linear combination of R.H.A., which is the residue of this pole, vanishes. The kinematical constraint states that

$$F^1_{++}(s, t) + \frac{m_N(t-u)}{2} F^2_{+-}(s, t) \quad \text{must vanish at} \quad s = 4m_N{}^2 \text{ as } s - 4m_N{}^2.$$

If one does not want to make any assumption on the behavior of A at $s = 4m_N{}^2$, one can say that the above linear combination of R.H.A. can be divided by $s - 4m_N{}^2$ without loosing its analyticity properties. Similar results are obtained in πN and $N\bar{N}$ elastic scatterings by applying the same method, i.e., expressing invariant amplitudes in terms of R.H.A.

It is hopeless to try and generalize such a method for the case of arbitrary spins and masses since we do not know, up to now, how to invert expansions of the type (II-9) or (II-10), that is to say, to express the Joos invariant amplitudes $a_{\ell_1\ell_2}(s, t)$ in terms of helicity amplitudes. Furthermore, there is no simple way known at the moment to express parity conservation in terms of the set of $a_{\ell_1\ell_2}(s, t)$. The method which we propose is based on the analyticity properties of the crossing matrix elements. In fact, we concluded Section III by noting that the crossing matrix elements between R.H.A. are meromorphic functions. Our method consists in looking for and canceling their poles. We find that crossing matrix elements giving t-channel R.H.A. in terms of s-channel R.H.A. behave near $\mathscr{S}_{12} = 0$ and $\mathscr{S}_{34} = 0$ as

$$\mathscr{S}_{12}^{-(2s_1+2s_2)^{\pm}} \quad \text{and} \quad \mathscr{S}_{34}^{-(2s_3+2s_4)^{\pm}}.$$

Apart from these poles the crossing-matrix elements have poles at $s = 0$ (branch point if the s-channel is a $BF \to BF$ one). This does not give more information than is contained in Eq. (II-23) and (II-24), which express that the helicity amplitudes themselves are kinematically regular at $s = 0$. On the other hand, we verify that, despite the presence of $\sin(\theta_t/2)^{|\lambda-\mu|} \cos(\theta_t/2)^{|\lambda+\mu|}$ in their denominators, all the crossing-matrix elements are finite at $\Phi(s, t) = 0$, except in the case $m_1 = m_2$, $m_3 = m_4$ where $\psi_{12} = \psi_{34} = s^{1/2}$ and $\Phi(s, t)$ simultaneously vanish at $s = 0$ (see Appendix A-IV).

Since the R.H.A. in the t-channel have no kinematical singularity at $\mathscr{S}_{12}=0$ and $\mathscr{S}_{34}=0$, one could be tempted to say that the vanishing of the s-channel R.H.A. at $\mathscr{S}_{12} = 0$ and $\mathscr{S}_{34} = 0$ provides a generalization of the kinematical constraints. However such a statement does not take into account the fact that, for instance, the determinant of $d(\chi)$ is unity and thus has no pole even though all the matrix elements may have one. Such an argument would imply, in $\pi\pi \to N\bar{N}$ scattering for instance, that both F^1_{++} and F^2_{+-} have to vanish at $s = 4m_N^2$ as $s - 4m_N^2$, which is wrong.

These difficulties can obviously be avoided if it is possible to diagonalize the crossing matrix for R.H.A. or at least the crossing matrix for helicity amplitudes. Now this has been done by Kotański (*9*), who remarked that, since the reaction plane is the same for both channels, one can define amplitudes for which the crossing matrix is "diagonal" (in fact, the crossing matrix has only one nonzero element in each row and column) by choosing the spin quantization axis along the normal to this plane, that is to say along w. Such amplitudes have been called by Kotański "transversity amplitudes". Our method to derive the kinematical constraints then consists in writing the crossing matrix for transversity amplitudes and looking for the singularities of its elements.

2. Kinematical Constraints in the General-Mass Case

A. *Transversity Amplitudes*

We briefly recall here the properties of transversity amplitudes. We call transversity of particle i the eigenvalue of the operator

$$-(1/m_i)\, w_\mu W^\mu(p_i)$$

where again

$$w_\mu = \frac{-2\epsilon_{\mu\nu\rho\sigma} p_1{}^\nu p_2{}^\rho p_3{}^\sigma}{[\Phi(s,t)]^{1/2}},$$

and $W(p_i)$ is the polarization four-vector of particle i. Whereas, for particle i the helicity frame is defined by

$$p_i,\ n_1(p_i),\ n_2(p_i) = w, \qquad n_3(p_i) = h_{12}(i),$$

a "transversity frame" can be defined by

$$p_i,\ n_1(p_i),\ n_2{}^{\mathrm{T}}(p_i) = -h_{12}(i),\ n_3{}^{\mathrm{T}}(p_i) = w.$$

So, transversity states are related to helicity states by

$$| p_i, L_t(p_i), s_i, \tau_i\rangle = D^{s_i}(R)^{\lambda_i}_{\tau_i} \,| p_i, L_h(p_i), s_i, \lambda_i\rangle.$$

Here R is a rotation through $-\frac{1}{2}\pi$ around the first axis, i.e., specified by Euler angles $\frac{1}{2}\pi$, $\frac{1}{2}\pi$, $-\frac{1}{2}\pi$, and the subscripts t and h refer respectively to transversity and helicity frames. In Reference (*9*), Kotański, has given the properties of $D^s(R)$. We recall here the most useful one, namely, that it diagonalizes the matrices $d^s(\chi)$:

$$[D^s(R)\, d^s(\chi)\, D^s(R^*)]^\lambda_\tau = e^{i\chi\tau}\delta_\tau{}^\lambda,$$

or

$$[D^s(R^*)\, d^s(\chi)\, D^s(R)]^\lambda_\tau = e^{-i\chi\tau}\delta_\tau{}^\lambda. \tag{IV-2}$$

Transversity amplitudes are related to helicity amplitudes through:

$$T_{\tau_3\tau_4;\tau_1\tau_2} = D^{s_1}(R)^{\lambda_1}_{\tau_1} D^{s_2}(R)^{\lambda_2}_{\tau_2} D^{s_3}(R^*)^{\lambda_3}_{\tau_3} D^{s_4}(R^*)^{\lambda_4}_{\tau_4} M_{\lambda_3\lambda_4;\lambda_1\lambda_2}. \tag{IV-3}$$

Among all the properties of transversity amplitudes given by Kotański we recall

(a) *Parity-conservation condition.* From Eqs. (IV-3) and (II-6) (parity conservation condition for helicity amplitudes) and from

$$(-)^{s+\lambda}\, D^s(R)^\lambda_\tau = e^{i\pi\tau} D^s(R)^{-\lambda}_\tau, \tag{IV-4}$$

we deduce that

$$T_{\tau_3\tau_4;\tau_1\tau_2} = \eta(-1)^{\tau_1+\tau_2-\tau_3-\tau_4}\, T_{\tau_3\tau_4;\tau_1\tau_2};$$

that is,

$$T_{\tau_3\tau_4;\tau_1\tau_2} = 0 \quad \text{if} \quad \eta(-)^{\tau_1+\tau_2-\tau_3-\tau_4} = -1. \tag{IV-5}$$

(b) *Crossing matrix.* From Eqs. (IV-2), (IV-3), and from the crossing matrix for helicity amplitudes (Table XI), we derive the crossing matrix for transversity amplitudes,

$$T^t_{-\tau_3-\tau_1;-\tau_4-\tau_2} = (-)^{\sigma(P)}\,(-)^{s_1+s_2+s_3+s_4}\, e^{-i\pi(\tau_2-\tau_3)} \times e^{+i(\tau_1\chi_1-\tau_2\chi_2+\tau_3\chi_3-\tau_4\chi_4)}\, T^s_{\tau_3\tau_4;\tau_1\tau_2}, \tag{IV-6}$$

where the χ angles are defined by their cosines and sines in Table XI.

B. *Relations between Transversity Amplitudes and* R.H.A.

We shall show that crossing relation (IV-5) implies a specific behaviour of the transversity amplitudes which yields the kinematical constraints between regularized helicity amplitudes. Before doing that, it is necessary to study how the transversity amplitudes are related to R.H.A. near all the singularities.

(a) $s = 0$. The general study of Section II has shown that helicity amplitudes are kinematically regular at $s = 0$. Since the transversity amplitudes are linear combinations of helicity amplitudes with numerical coefficients, they do not have a kinematical singularity at $s = 0$.

(b) $\mathscr{S}_{12} = 0$ *and* $\mathscr{S}_{34} = 0$. Applying Eq. (II-12)–(II-15) to $M_{\lambda_3\lambda_4;\lambda_1\lambda_2}$, which appears in the expansion (IV-3), and using Eq. (IV-4), we find

$$T^s_{\tau_3\tau_4;\tau_1\tau_2}(-\varphi_{12}) = e^{\pm i\pi(\tau_1+\tau_2)}\, T^s_{\tau_3\tau_4;\tau_1\tau_2}(\varphi_{12})$$

according as $\theta_s(-\varphi_{12}) = \theta_s(\varphi_{12}) \pm \pi$;

$$T^s_{\tau_3\tau_4;\tau_1\tau_2}(-\psi_{12}) = e^{\pm i\pi(\tau_1-\tau_2)\epsilon_{12}}\, T^s_{\tau_3\tau_4;\tau_1\tau_2}(\psi_{12})$$

according as $\theta_s(-\psi_{12}) = \theta_s(\psi_{12}) \pm \pi$;

$$T^s_{\tau_3\tau_4;\tau_1\tau_2}(-\varphi_{34}) = e^{\pm i\pi(\tau_3+\tau_4)}\, T^s_{\tau_3\tau_4;\tau_1\tau_2}(\varphi_{34})$$

according as

$$\theta_s(-\varphi_{34}) = \theta_s(\varphi_{34}) \pm \pi; \qquad \text{(IV-7)}$$

and

$$T^s_{\tau_3\tau_4;\tau_1\tau_2}(-\psi_{34}) = e^{\pm i\pi(\tau_3-\tau_4)\epsilon_{34}}\, T^s_{\tau_3\tau_4;\tau_1\tau_2}(\psi_{34})$$

according as

$$\theta_s(-\psi_{34}) = \theta_s(\psi_{34}) \pm \pi. \qquad \text{(IV-7)}$$

Furthermore, recalling Eqs. (II-19) and (II-20), which give the behavior of helicity amplitudes near $\mathscr{S}_{12} = 0$ and $\mathscr{S}_{34} = 0$, we see that

$$\hat{T}^s_{\tau_3\tau_4;\tau_1\tau_2} = \mathscr{S}_{12}^{s_1+s_2}\, \mathscr{S}_{34}^{s_3+s_4}\, T^s_{\tau_3\tau_4;\tau_1\tau_2} \qquad \text{(IV-8)}$$

is kinematically *finite* at $\mathscr{S}_{12} = 0$ and $\mathscr{S}_{34} = 0$. Furthermore, the coefficients of the expansion of $\hat{T}_{\tau_3\tau_4;\tau_1\tau_2}$ in terms of R.H.A. are not all equal to zero at $\mathscr{S}_{12}=0$ and $\mathscr{S}_{34} = 0$.

(c) $\Phi(s, t) = 0$. Replacing the $M_{\lambda_3\lambda_4;\lambda_1\lambda_2}$ appearing in Eq. (IV-3) by

$$\sin(\theta_s/2)^{|\lambda-\mu|} \cos(\theta_s/2)^{|\lambda+\mu|}\, \hat{M}_{\lambda_3\lambda_4;\lambda_1\lambda_2}$$

(where $\hat{M}_{\lambda_3\lambda_4;\lambda_1\lambda_2}$ is kinematically regular at $\Phi(s, t) = 0$) and using

$$D^s(R)^\lambda_\tau = e^{i\pi\tau}\, (-)^{s+\lambda}\, D^s(R)^\lambda_{-\tau}$$

and

$$D^s(R^*)^\lambda_\tau = e^{-i\pi\tau}\, (-)^{s+\lambda}\, D^s(R^*)^\lambda_{-\tau},$$

we find:

$$T^s_{\tau_3\tau_4;\tau_1\tau_2}(-\Phi^{1/2}) = (-)^{s_1-s_2+s_4-s_3}\, (-)^{\tau_1+\tau_2-\tau_3-\tau_4}\, T^s_{-\tau_3-\tau_4;-\tau_1-\tau_2}(\Phi^{1/2}) \qquad \text{(IV-9)}$$

if $\theta_s(-\Phi^{1/2}) = -\theta_s(\Phi^{1/2})$ and

$$T^s_{\tau_3\tau_4;\tau_1\tau_2}(-\Phi^{1/2}) = (-)^{s_1-s_2-s_4+s_3}\, (-)^{\tau_1+\tau_2-\tau_3-\tau_4}\, T^s_{-\tau_3-\tau_4;-\tau_1-\tau_2}(\Phi^{1/2}), \qquad \text{(IV-10)}$$

$$\text{if} \quad \theta_s(-\Phi^{1/2}) = 2\pi - \theta_s(\Phi^{1/2}).$$

It would have been possible to guess such results qualitatively: since the transversity four-vector w has a $\Phi^{1/2}$ branch point, $T^s_{\tau_3\tau_4;\tau_1\tau_2}(-\Phi^{1/2})$ is related to $T^s_{-\tau_3-\tau_4;-\tau_1-\tau_2}(\Phi^{1/2})$. On the contrary, the singularities at $\mathscr{S}_{12} = 0$ and $\mathscr{S}_{34} = 0$

appear only in the second basis vector, $-h_{12}(i)$, of the transversity frames. So, it is not necessary to combine two or more transversity amplitudes to get amplitudes which are kinematically uniform at $\mathscr{S}_{12} = 0$ and $\mathscr{S}_{34} = 0$.

C. *Kinematical Constraints in the General Mass Case*

Since $T^t_{-\tau_3-\tau_1;-\tau_4-\tau_2}$ is kinematically regular at $\mathscr{S}_{12} = 0$, $\mathscr{S}_{34} = 0$, and $s = 0$, $T^s_{\tau_3\tau_4;\tau_1\tau_2}$ will have to cancel all singularities which may appear in $\exp[i(\tau_1\chi_1 - \tau_2\chi_2 + \tau_3\chi_3 - \tau_4\chi_4)]$ [Eq. (IV-6)] at $\mathscr{S}_{12} = 0$ and $\mathscr{S}_{34} = 0$. We then say that, $T^s_{\tau_3\tau_4;\tau_1\tau_2}$ has to behave near $\mathscr{S}_{12} = 0$, $\mathscr{S}_{34} = 0$ like $\exp[-i(\tau_1\chi_1 - \tau_2\chi_2 + \tau_3\chi_3 - \tau_4\chi_4)]$, which we rewrite as

$$\exp\left(-i\left[(\chi_1 + \chi_2)\frac{\tau_1 - \tau_2}{2} + (\chi_1 - \chi_2)\frac{\tau_1 + \tau_2}{2} + (\chi_3 + \chi_4)\frac{\tau_3 - \tau_4}{2} + (\chi_3 - \chi_4)\frac{\tau_3 + \tau_4}{2}\right]\right).$$

We now study the behavior of $\exp[i(\chi_1 \pm \chi_2)]$ (resp. $\exp[i(\chi_3 \pm \chi_4)]$) near $\mathscr{S}_{12} = 0$ (resp. $\mathscr{S}_{34} = 0$). Since near $\mathscr{S}_{12} = 0$ (resp. $\mathscr{S}_{34} = 0$), $\cos(\chi_1 \pm \chi_2)$ and $\sin(\chi_1 \pm \chi_2)$ [resp. $\cos(\chi_3 \pm \chi_4)$ and $\sin(\chi_3 \pm \chi_4)$] behave like $[s - (m_1 \mp m_2)^2]^{-1}$ (resp. $[s - (m_3 \mp m_4)^2]^{-1}$ [see Appendix A-III, Sec. I-1, and Eq. (A-III-1)], $\exp[i(\chi_1 \pm \chi_2)]$ (resp. $\exp[i(\chi_3 \pm \chi_4])$) will have either a pole or a zero at $s = (m_1 \pm m_2)^2$ [resp. $s = (m_3 \pm m_4)^2$]. ($\exp[i(\chi_i + \chi_j)]$ has a zero if $\exp[-i(\chi_i + \chi_j)]$ has a pole and vice versa). From the expression of $\sin \chi_i$ one can see that the behavior of $\exp[i(\chi_1 \pm \chi_2)]$ and $\exp[i(\chi_3 \pm \chi_4)]$ depends on the determination of $[\Phi(s, t)]^{1/2}$. We will show below that, in order to write explicitly the kinematical constraints, it is sufficient to know the correlations between the behaviors of $\exp[i(\chi_1 \pm \chi_2)]$, $\exp[i(\chi_3 \pm \chi_4)]$, and for instance that of $e^{i\theta_s}$. This can be done directly using relations given in Eq. (A-III-1). Results are given in Table XII.

TABLE XII

near $s = (m_1 + m_2)^2 : e^{i\theta_s} \simeq \varphi_{12}^{\pm 1} \Leftrightarrow e^{-i(\chi_1 - \chi_2)} \simeq \varphi_{12}^{\pm 2}$
near $s = (m_1 - m_2)^2 : e^{i\theta_s} \simeq \psi_{12}^{\pm 1} \Leftrightarrow e^{-i(\chi_1 + \chi_2)} \simeq \psi_{12}^{\pm 2\epsilon_{12}}$
near $s = (m_3 + m_4)^2 : e^{i\theta_s} \simeq \varphi_{34}^{\pm 1} \Leftrightarrow e^{-i(\chi_3 - \chi_4)} \simeq \varphi_{34}^{\pm 2}$
near $s = (m_3 - m_4)^2 : e^{i\theta_s} \simeq \psi_{34}^{\pm 1} \Leftrightarrow e^{-i(\chi_3 + \chi_4)} \simeq \psi_{34}^{\pm 2\epsilon_{34}}$
Where $\epsilon_{ij} = \dfrac{m_i - m_j}{\lvert m_i - m_j \rvert}$

Table XII allows us to write the kinematical constraints in terms of the amplitudes $\tilde{T}^{s}_{\tau_3\tau_4;\tau_1\tau_2}$ defined in Eq. (IV-8):

Near $\varphi_{12} = 0$

$$\tilde{T}^{s}_{\tau_3\tau_4;\tau_1\tau_2} \simeq \varphi_{12}^{s_1+s_2\pm(\tau_1+\tau_2)} \qquad \text{if} \quad e^{i\theta_s} \simeq \varphi_{12}^{\pm 1}$$

Near $\psi_{12} = 0$

$$\tilde{T}^{s}_{\tau_3\tau_4;\tau_1\tau_2} \simeq \psi_{12}^{s_1+s_2\pm\epsilon_{12}(\tau_1-\tau_2)} \qquad \text{if} \quad e^{i\theta_s} \simeq \psi_{12}^{\pm}$$

Near $\varphi_{34} = 0$

$$\tilde{T}^{s}_{\tau_3\tau_4;\tau_1\tau_2} \simeq \varphi_{34}^{s_3+s_4\pm(\tau_3+\tau_4)} \qquad \text{if} \quad e^{i\theta_s} \simeq \varphi_{34}^{\pm 1}$$

Near $\psi_{34} = 0$

$$\tilde{T}^{s}_{\tau_3\tau_4;\tau_1\tau_2} \simeq \psi_{34}^{s_3+s_4\pm\epsilon_{34}(\tau_3-\tau_4)} \qquad \text{if} \quad e^{i\theta_s} \simeq \psi_{34}^{\pm 1}. \tag{IV-11}$$

In order to make the meaning of Eq. (IV-11) clear, we add the following important remarks.

(a) In terms of regularized helicity amplitudes, Eq. (IV-11) are actually *constraints*: this means that near $s = S_i$, ($S_1 = (m_1 + m_2)^2$, $S_2 = (m_1 - m_2)^2$, $S_3 = (m_3 + m_4)^2$, $S_4 = (m_3 - m_4)^2$), some linear combinations of R.H.A., with coefficients which do not vanish at $s = S_i$, must behave near $s = S_i$ like $[(s - S_i)^{1/2}]^{N_i}$, where N_i is a nonnegative integer.

(b) Since $e^{i\theta_s/2} \simeq (s - S_i)^{\pm 1/4}$ if $\theta_s[-(s - S_i)^{1/2}] = \theta[(s - S_i)^{1/2}] \pm \pi$, we see that Eq. (IV-11) which make use of crossing are compatible with Eq. (IV-7) which only use regularization of helicity amplitudes. This, by the way, provides a consistency check on the evaluation of the crossing angles χ_i. Furthermore, this proves that, if N_i is odd, then $\tilde{T}^{s}_{\tau_3\tau_4;\tau_1\tau_2}[(s - S_i)^{1/2}]^{-1}$ is kinematically uniform at $s = S_i$.

(c) If one changes the determination of $\Phi^{1/2}$:

(i) the behavior of $e^{i\theta_s}$ changes into its inverse at $s = S_i$;

(ii) $T^{s}_{\tau_3\tau_4;\tau_1\tau_2}$ changes into $T^{s}_{-\tau_3-\tau_4;-\tau_1-\tau_2}$ up to a phase as shown in Eqs. (IV-9) and (IV-10). So, Eq. (IV-11) with either the plus or the minus sign corresponds to the same set of kinematical constraints for R.H.A.

(d) — For practical applications, Eq. (IV-11) can be understood in the following way:

$$\frac{\partial^p}{\partial s^p}\left(\hat{T}^{i}_{\tau_3\tau_4;\tau_1\tau_2}(s,t)\right) = 0 \qquad \text{for} \quad \left\{\begin{matrix} s = S_i \\ i\mathscr{S}_{12}\mathscr{S}_{34}\cos\theta_s = \mathscr{S}_{12}\mathscr{S}_{34}\sin\theta_s \end{matrix}\right\}$$

$$\text{with} \qquad p = 0, 1, \ldots, \left(\frac{N_i^-}{2} - 1\right), \qquad i = 1, \ldots, 4; \qquad \text{(IV-12)}$$

$$S_1 = (m_1 + m_2)^2; \quad S_2 = (m_1 - m_2)^2; \quad S_3 = (m_3 + m_4)^2; \quad S_4 = (m_3 - m_4)^2;$$

$$N_1 = s_1 + s_2 + \tau_1 + \tau_2\,, \qquad N_2 = s_1 + s_2 + \epsilon_{12}(\tau_1 - \tau_2),$$

$$N_3 = s_3 + s_4 + \tau_3 + \tau_4\,, \qquad N_4 = s_3 + s_4 + \epsilon_{34}(\tau_3 - \tau_4),$$

$$N_i^- = \begin{cases} N_i & \text{if } N_i \text{ is even,} \\ N_i - 1 & \text{if } N_i \text{ is odd,} \end{cases}$$

$$\epsilon_{ij} = \frac{m_i - m_j}{|\, m_i - m_j \,|}\,;$$

$$\hat{T}^{i}_{\tau_3\tau_4;\tau_1\tau_2}(s,t) = \frac{\hat{T}^{s}_{\tau_3\tau_4;\tau_1\tau_2}}{[(s - S_i)^{1/2}]^{N_i - N_i^-}}\,,$$

$\hat{T}_{\tau_3\tau_4;\tau_1\tau_2}$ being expressed in terms of R.H.A. through (IV-8), (IV-3) and Table IV. Quite equivalently, constraints read:

$$\frac{\partial^p}{\partial s^p}\left(\hat{T}'^{i}_{\tau_3\tau_4;\tau_1\tau_2}(s,t)\right) = 0 \qquad \text{for} \quad \left\{\begin{matrix} s = S_i \\ i\mathscr{S}_{12}\mathscr{S}_{34}\cos\theta_s = -\mathscr{S}_{12}\mathscr{S}_{34}\sin\theta_s \end{matrix}\right\}$$

$$\text{with} \qquad p = 0, 1, \ldots, \left(\frac{N_i'^-}{2} - 1\right); \qquad i = 1, \ldots, 4; \qquad \text{(IV-13)}$$

$$N_1' = s_1 + s_2 - \tau_1 - \tau_2\,, \qquad N_2' = s_1 + s_2 - \epsilon_{12}(\tau_1 - \tau_2),$$

$$N_3' = s_3 + s_4 - \tau_3 - \tau_4\,, \qquad N_4' = s_3 + s_4 - \epsilon_{34}(\tau_3 - \tau_4);$$

$$\hat{T}'^{i}_{\tau_3\tau_4;\tau_1\tau_2}(s,t) = \frac{\hat{T}^{s}_{\tau_3\tau_4;\tau_1\tau_2}}{[(s - S_i)^{1/2}]^{N_i' - N_i'^-}}\,.$$

In this way the kinematical constraints appear as the vanishing of linear combinations of the R.H.A. and their derivatives at $s = S_i$ with coefficients polynomial in t.

3. Kinematical Constraints in Particular Mass Cases

The general analysis applies in the case $\mathscr{S}_{12} = \mathscr{S}_{34} = \mathscr{S}$, $\epsilon_{12} = \epsilon_{34}$. Eq. (IV-12) becomes

$$\frac{\partial^p}{\partial s^p}(\hat{T}^i_{\tau_3\tau_4;\tau_1\tau_2}(s,t)) = 0 \qquad \text{for} \quad \begin{Bmatrix} s = S_i \\ i\mathscr{S}^2 \cos\theta_s = \mathscr{S}^2 \sin\theta_s \end{Bmatrix}$$

$$\text{with} \qquad p = 0, 1, \ldots, \left(\frac{N_i^-}{2} - 1\right); \qquad i = 1, 2, \tag{IV-14}$$

$$S_1 = (m_1 + m_2)^2, \qquad S_2 = (m_1 - m_2)^2,$$
$$N_1 = s_1 + s_2 + s_3 + s_4 + \tau_1 + \tau_2 + \tau_3 + \tau_4;$$
$$N_2 = s_1 + s_2 + s_3 + s_4 + \epsilon_{12}(\tau_1 - \tau_2 + \tau_3 - \tau_4);$$
$$\hat{T}^i_{\tau_3\tau_4;\tau_1\tau_2}(s,t) = \frac{\hat{T}^s_{\tau_3\tau_4;\tau_1\tau_2}}{[(s - S_i)^{1/2}]^{N_i - N_i^-}},$$

or equivalently Eq. (IV-13) becomes

$$\frac{\partial^p}{\partial s^p}(\hat{T}'^i_{\tau_3\tau_4;\tau_1\tau_2}(s,t)) = 0 \qquad \text{for} \quad \begin{Bmatrix} s = S_i \\ i\mathscr{S}^2 \cos\theta_s = -\mathscr{S}^2 \sin\theta_s \end{Bmatrix},$$

$$\text{with} \qquad p = 0, 1, \ldots, \left(\frac{N_i'^-}{2} - 1\right); \qquad i = 1, 2, \tag{IV-15}$$

$$N_1' = s_1 + s_2 + s_3 + s_4 - \tau_1 - \tau_2 - \tau_3 - \tau_4;$$
$$N_2' = s_1 + s_2 + s_3 + s_4 - \epsilon_{12}(\tau_1 - \tau_2 + \tau_3 - \tau_4);$$
$$\hat{T}'^i_{\tau_3\tau_4;\tau_1\tau_2}(s,t) = \frac{\hat{T}^s_{\tau_3\tau_4;\tau_1\tau_2}}{[(s - S_i)^{1/2}]^{N_i' - N_i'^-}}.$$

New problems arise when ψ_{12} and/or ψ_{34} coincide with $s^{1/2}$. A particular study of this case is given in Appendix A-IV.

Finally we show in appendix A-V how all the kinematical constraints which have been already observed in some particular cases can be derived in the framework developped in the present section.

V. SUMMARY AND CONCLUSIONS

This study of the helicity amplitudes with respect to their analyticity properties as functions of the invariants leads to the definition of new amplitudes which are analytic in the image of the analyticity domain of the spinor amplitudes in the space of the four-momenta, on the mass-shell. Our results hold for a large set of mass cases, and whatever spins are involved, with the exception of a $BF \to BF$

reaction, when we are left with a square-root type singularity on the c.m. energy variable squared, which is related to the so-called "Mac Dowell reciprocity relation." We would like to emphasize that the two basic tools in the work are

(1) The analyticity of the Joos expansion for spinor amplitudes *(2)*, *(3)*

(2) The existence of an analyticity domain for the spinor amplitudes, which connects the various physical regions associated to a given two-body process (crossing property) *(5)*.

We have also made use of the interesting properties of the transversity amplitudes *(9)*, in connection with the question of kinematical constraints.

Our results are of three types.

(1) The helicity amplitudes can be made free of kinematical singularities. More precisely, whether parity is conserved or not, the relevant number of linearly independent new amplitudes $\mathscr{F}_{\{\lambda\}}$ can be defined, which are related to the Joos functions a_ℓ [Eq. (II-27)] through polynomial coefficients in s, t, and u.[9]

(2) A crossing matrix is derived for the helicity amplitudes using an analytic continuation path from one physical region to the other. For the regularized helicity amplitudes $\mathscr{F}$, the corresponding crossing matrix has elements of the form P/Q where P and Q are polynomials in s, t and u*.

(3) By using the so-called transversity amplitudes, for which crossing yields a relation between only one amplitude in the s-channel and an other one in the t-channel, the problem raised by the existence of zeros in Q is more easily solved than for the $\mathscr{F}$ functions. The solution is expressed by means of relations between the $\mathscr{F}$ functions and some number of their derivatives with respect to the c.m. squared energy, which hold at certain values of this variable, namely at thresholds and pseudo-thresholds.

We think that such considerations on the analytic properties of helicity amplitudes can lead to two kinds of applications:

(1) For any phenomenological model which makes use of crossing properties, especially for Regge type models, the $\mathscr{F}$ functions are good candidates to satisfy a Mandelstam representation and to be approximated by Regge type functions. Furthermore, taking into account the kinematical constraints yields some information either about possible relations between the residues of a given Regge pole or about the existence of families of poles, depending on the ideas one may have about the question of "evasion or conspiracy" *(1)*, *(17)*. These properties, together

[9] When the considered channel corresponds to a BF state, one should replace the relevant c.m. squared energy (s, t or u) by its square root.

*[9bis] When the considered channel corresponds to a BF state, one should replace the relevant c.m. squared energy (s, t or u) by its square root.

with crossing relations, allow to write phenomenological formulae which are at least "kinematically correct".

A last remark concerning the kinematical constraints is that no constraint can be found for vanishing c.m.-squared energy in the general mass case. The relations which have been discovered by Gribov and Volkov (*17*) in the reaction $N\bar{N} \to N\bar{N}$ at $t = 0$ (t is the c.m.-squared energy), can be generalized to any spin, but only in *the equal mass case*. The true generalization of these relations is given by the kinematical constraints at pseudothresholds. The existence of these constraints is then a very general property of any helicity amplitudes involving high enough spins, thus although such relations can hold at zero c.m.-squared energy in some mass cases, they seem to be of a nature very different from that of constraints which are implied, inside the Regge pole model, between the so-called "daughter" trajectories (*18*). Indeed, in the $NN \to NN$ reaction, for example, there is no need for daughter trajectories (equal mass case), whereas, just because of the equality of the masses, a kinematical constraint exists at zero c.m.-squared energy.

(2) Another application of the present study could be, on a more theoretical ground, to allow the generalization to the caseof nonzero spins of the results which have been obtained for the analyticity domain of the amplitude from axiomatic field theory (*19*), in the spinless case. Here we must notice that the enlargement of the analyticity domain which has been obtained makes use of a positiveness property of the absorptive part, which is a consequence of unitarity. However, the content of unitarity seems difficult to express so simply by some positivity condition in the case where there are several two-body coupled channels, which occurs for an elastic reaction between particles with nonzero spins (*20*).

Finally, it would be interesting to build out of the set of R.H.A. with the corresponding kinematical constraints, new amplitudes with the same analyticity properties, but free of any constraint. This would yield a new basis for an analytic expansion of spinor amplitudes. Parity would be easy to express and amplitudes would be labelled by meaningful indices, simply related to the individual spin components.

APPENDIX A-I. Kinematical Singularities in Particular Mass Cases

We give here the details of the calculations corresponding to the particular mass configurations shown in Table V.

We have not studied very pathological cases such as, for instance, $m_1 + m_2 = m_2 + m_4$, $m_1 - m_2 \neq m_3 - m_4$. Furthermore, we assume that when two particles have the same mass they are of the same species; in particular, they have the same spin. We shall always assume that parity is conserved. In all cases, the study of the kinematical singularity at $\Phi(s, t) = 0$ goes as in the general case.

I. "Elastic Scattering": $m_1 = m_3$, $m_2 = m_4$, $m_1 \neq m_2$

We define new notations:

$$\mathscr{S}_{12} = \mathscr{S}_{34} = \mathscr{S} = ([s - (m_1 + m_2)^2][s - (m_1 - m_2)^2])^{1/2},$$
$$p = k = \mathscr{S}/2s^{1/2},$$
$$\omega_1 = \omega_3 = E, \qquad \omega_2 = \omega_4 = \omega,$$
$$\cos\theta_s = 1 + t/2p^2.$$

A new situation arises from the confluence of $\mathscr{S}_{12} = 0$ and $\mathscr{S}_{34} = 0$ to $\mathscr{S} = 0$ (Table II-5). Expansions A and B are no longer suitable. Now $p\cos(\theta_s/2)$, $p\sin(\theta_s/2)$ are regular at $\mathscr{S} = 0$. So, we introduce a new frame, which we call RIII, characterized by $n_3(R) = (q_{12} + q_{34})[-(q_{12} + q_{34})^2]^{-1/2}$, which leads to the Expansion C shown in Table XIII.

TABLE XIII

Explicit Form C of Eq. (II-8) Corresponding to Frame RIII, Which is Convenient to Study Kinematical Singularities at $\mathscr{S} = 0$ when $m_1 = m_3$ and $m_2 = m_4$.

Frame R III

$$t(R) = \frac{P}{s^{1/2}}, \qquad n_2(R) = w, \qquad n_3(R) = \frac{q_{12} + q_{34}}{[-(q_{12} + q_{34})^2]^{1/2}}$$

$$\overrightarrow{e(\Lambda p_1, \Lambda p_3)} = \begin{cases} Ep\sin(\theta_s/2) \\ -i\dfrac{p^2}{2}\sin\theta_s \\ 0 \end{cases} \qquad \overrightarrow{e(\Lambda p_2, \Lambda p_3)} = \begin{cases} \frac{1}{2}p\sin(\theta_s/2)(\omega - E) \\ \dfrac{i}{2}p^2\sin\theta_s \\ \frac{1}{2}p\cos(\theta_s/2)(\omega + E) \end{cases}$$

$$\Lambda L_{12}(1) = R_y(-\theta_s/2)\,B_1$$
$$\Lambda L_{12}(2) = R_y(-\theta_s/2 + \pi)\,B_2$$
$$\Lambda L_{12}(3) = R_y(\theta_s/2)\,B_3$$
$$\Lambda L_{12}(4) = R_y(\theta_s/2 + \pi)\,B_4$$

Expansion C

$$M_{\lambda_3\lambda_4;\lambda_1\lambda_2} = \sum (-1)^{s_2-\lambda_2+s_3+\lambda_3} D^{s_1}(B_1)^{\lambda_1}_{\lambda_1} D^{s_2}(B_2)^{\lambda_2}_{\lambda_2} D^{s_3}(B_3)^{-\lambda_3}_{-\lambda_3}$$
$$D^{s_4}(B_4)^{-\lambda_4}_{-\lambda_4} d^{s_1}\left(-\frac{\theta_s}{2}\right)^{\Lambda_1}_{\lambda_1} d^{s_2}\left(-\frac{\theta_s}{2}\right)^{\Lambda_2}_{-\lambda_2} d^{s_3}\left(\frac{\theta_s}{2}\right)^{\Lambda_3}_{-\lambda_3} d^{s_4}\left(\frac{\theta_s}{2}\right)^{\Lambda_4}_{\lambda_4} \sum \cdots \tag{A-I.1}$$

The dots stand for quantities kinematically regular at $\mathscr{S} = 0$ ($\mathscr{S} = \mathscr{S}_{12} = \mathscr{S}_{34}$)

1. *Kinematical Singularities at* $\mathscr{S} = 0$

$s = (m_1 + m_2)^2$. Using Expansion C [Eq. (A-I.1) of Table XIII] and turning around $s = (m_1 + m_2)^2$ along the path shown on Fig. 2, we find the following changes:

$$\begin{cases} \frac{1}{2}\theta_s \to \frac{1}{2}\theta_s \pm \pi, \\ p \to -p, \\ \left(\frac{\omega_i + m_i - p}{[2m_i(\omega_i + m_i)]^{1/2}}\right)^{\pm 2\lambda_i} \to \left(\frac{\omega_i + m_i - p}{[2m_i(\omega_i + m_i)]^{1/2}}\right)^{\mp 2\lambda_i}, \end{cases}$$

which lead to

$$\hat{M}_{\lambda_3\lambda_4;\lambda_1\lambda_2}(-\varphi_{12}) = (-)^{\Sigma_i(s_i+\lambda_i)}\,\hat{M}_{-\lambda_3-\lambda_4;-\lambda_1-\lambda_2}(\varphi_{12}) = \eta\hat{M}_{\lambda_3\lambda_4;\lambda_1\lambda_2}(\varphi_{12})$$
$$= \hat{M}_{\lambda_3\lambda_4;\lambda_1\lambda_2}(\varphi_{12}),$$

since with our assumption on the species of equal mass particles, $\eta = +1$ if $m_1 = m_3$, $m_2 = m_4$. Thus, $\hat{M}_{\lambda_3\lambda_4\lambda_1\lambda_2}$ has no kinematical branch point at $s = (m_1 + m_2)^2$.

$s = (m_1 - m_2)^2$. At pseudo-threshold we find

$$\hat{M}_{\lambda_3\lambda_4;\lambda_1\lambda_2}(-\psi_{12}) = \eta(-)^{2s_1+2s_3}\,\hat{M}_{\lambda_3\lambda_4;\lambda_1\lambda_2}(\psi_{12}) = \hat{M}_{\lambda_3\lambda_4;\lambda_1\lambda_2}(\psi_{12})$$

since $\eta = +1$ and $s_1 = s_3$. Thus $\hat{M}_{\lambda_3\lambda_4;\lambda_1\lambda_2}$ has no kinematical branch point at $s = (m_1 - m_2)^2$ either.

Kinematical poles. Since $d^s(\pm\theta_s/2)$ is a homogeneous polynomial of degree $2s$ in $\sin(\theta_s/4)$ and $\cos(\theta_s/4)$ and since $\sin(\theta_s/4) \propto (\mathscr{S})^{-1/2}$ and $\cos(\theta_s/4) \propto (\mathscr{S})^{-1/2}$, one has

$$\hat{M}_{\lambda_3\lambda_4;\lambda_1\lambda_2} \propto (\mathscr{S})^{-2[s_1+s_2-\mathrm{Max}(|\lambda|,|\mu|)]}.$$

2. *Kinematical Singularity at* $s = 0$

In principle, no special study is needed at $s = 0$. However, we get a result slightly different from that obtained in the general case: since it is not necessary to associate two different helicity amplitudes in order to get an amplitude without kinematical singularity at $\mathscr{S} = 0$, it is possible to pick up the whole kinematical singularity at $s = 0$ even in the $BF \to BF$ case. Recalling that

$$\hat{M}_{\lambda_3\lambda_4;\lambda_1\lambda_2} \propto (s^{1/2})^{-|\lambda-\epsilon_{12}\epsilon_{34}\mu|}$$

where now $\epsilon_{12}\epsilon_{34} = +1$,

$$(s^{1/2})^{|\lambda-\mu|}\,\mathscr{S}^{2(s_1+s_2-\mathrm{Max}(|\lambda|,|\mu|)}\,\hat{M}_{\lambda_3\lambda_4;\lambda_1\lambda_2} \qquad \text{(A-I.2)}$$

has no kinematical singularity. (Results are given in Table VI).

II. $m_1 = m_2 = m$; $m_3 \neq m_4$; $s_1 = s_2 = s_{\text{In}}$

The general study applies to $\mathscr{S}_{34}$ and φ_{12}. ψ_{12} now coincides with $s^{1/2}$.

1. *Kinematical Branch Point at* $s = 0$

$$k = \frac{(s - 4m^2)^{1/2}}{2}, \qquad \omega = \omega_1 = \omega_2 = \frac{s^{1/2}}{2},$$
$$\cos\theta_s \propto s^{1/2}.$$

We use expansion B [Eq. (II-10)] for which the third components of the semi-bivectors are regular at $s = 0$. One turn around $s = 0$ induces the following changes:

$$\omega_3 \pm p \to -(\omega_3 \pm p),$$
$$\omega_4 \pm p \to -(\omega_4 \pm p),$$
$$(\omega - k)/m \to -m/(\omega - k),$$
$$-\theta_s \to -\pi + \theta_s$$

From

$$d^{s_1}(-\pi + \theta_s)^{A_1}_{\lambda_1} = (-)^{s_1 - A_1}\, d^{s_1}(-\theta_s)^{A_1}_{-\lambda_1},$$

$$d^{s_2}(-\pi + \theta_s)^{A_2}_{-\lambda_2} = (-)^{s_2 - A_2}\, d^{s_2}(-\theta_s)^{A_2}_{\lambda_2},$$

one gets

$$\hat{M}_{\lambda_3\lambda_4;\lambda_1\lambda_2}(-s^{1/2}) = (-1)^{\lambda_1+\lambda_2+\mu}\, \eta_{12} \hat{M}_{\lambda_3\lambda_4;-\lambda_1-\lambda_2}(s^{1/2}).$$

Thus, with the notations defined in Eq. (II-17):

for $\lambda_1 + \lambda_2 + \mu$ even $\begin{cases} \mathscr{A}_{12} \\ ks^{1/2}\mathscr{B}_{12} \end{cases}$ have no kinematical branch points at $s = 0$ and $s = 4m^2$;

for $\lambda_1 + \lambda_2 + \mu$ odd $\begin{cases} s^{1/2}\mathscr{A}_{12} \\ k\mathscr{B}_{12} \end{cases}$ have no kinematical branch points at $s = 0$ and $s = 4m^2$.

2. *Kinematical Pole at* $s = 0$

Applying the general analysis, one finds that each term of Expansion B [Eq. (II-10) of Table II] behaves like

$$(s^{1/2})^{\epsilon\epsilon_{34}(A_1 + A_2)};$$

and since the sum over A_1 and A_2 runs from $-s_{\text{In}}$ to $+s_{\text{In}}$ the worst behavior will be

$$(s^{1/2})^{-2s_{\text{In}}}$$

so that:

$$(s^{1/2})^{(2s_{\text{in}})^-} \mathscr{A}_{12} \quad \text{and} \quad (s^{1/2})^{(2s_{\text{in}})^+-1} \mathscr{B}_{12} \quad \text{for} \quad \lambda_1 + \lambda_2 + \mu \quad \text{even}$$

and

$$(s^{1/2})^{(2s_{\text{in}})^+-1} \mathscr{A}_{12} \quad \text{and} \quad (s^{1/2})^{(2s_{\text{in}})^-} \mathscr{B}_{12} \quad \text{for} \quad \lambda_1 + \lambda_2 + \mu \quad \text{odd} \quad \text{(A-I.3)}$$

have no kinematical singularity at $s = 0$. (Results are given in Table VII).

III. $m_3 = m_4 = m'$; $m_1 \neq m_2$; $s_3 = s_4 = s_f$

This case is quite analogous to the preceding one. One finds easily that

$$(s^{1/2})^{(2s_f)^-} \mathscr{A}_{34} \quad \text{and} \quad (s^{1/2})^{(2s_f)^+-1} \mathscr{B}_{34} \quad \text{for} \quad \lambda_3 + \lambda_4 + \lambda \quad \text{even}$$

and

$$(s^{1/2})^{(2s_f)^+-1} \mathscr{A}_{34} \quad \text{and} \quad (s^{1/2})^{(2s_f)^-} \mathscr{B}_{34} \quad \text{for} \quad \lambda_3 + \lambda_4 + \lambda \quad \text{odd} \quad \text{(A-I.4)}$$

have no kinematical singularity at $s = 0$. (Results are given in Table VIII).

IV. $m_1 = m_2 = m$; $m_3 = m_4 = m'$; $m \neq m'$; $s_1 = s_2 = s_{\text{in}}$, $s_3 = s_4 = s_f$

The general study applies to the thresholds, but now ψ_{12} and ψ_{34} coincide with $s^{1/2}$.

$$\begin{cases} \omega_1 = \omega_2 = \omega_3 = \omega_4 = \frac{1}{2}(s^{1/2}) \\ k = \frac{1}{2}(s - 4m^2)^{1/2} \qquad p = \frac{1}{2}(s - 4m'^2)^{1/2} \\ \cos\theta_s \text{ is regular at } s = 0. \end{cases}$$

Expansions A, B and C are not suitable since the third components of the semibivectors are singular at $s = 0$ in the corresponding frames. However, one can see that in these three frames the second component of the semibivectors, equal to $\pm ipk \sin\theta_s$, is regular at $s = 0$. So, we define a new frame (frame *R*IV) in which the third axis is the second axis of frames *R*I, *R*II and *R*III, namely, w. This leads to Expansion D shown in Table XIV.

1. *Kinematical Branch Point at* $s = 0$

One turn around $s = 0$ transforms

$$\left(\frac{\omega_i - p_{ij}}{m_i}\right) \quad \text{into} \quad -\left(\frac{\omega_i - p_{ij}}{m_i}\right)^{-1}.$$

TABLE XIV

EXPLICIT FORM D OF EQ. (II-8) CORRESPONDING TO FRAME RIV, WHICH IS CONVENIENT TO STUDY THE KINEMATICAL SINGULARITY AT $s = 0$ WHEN $m_1 = m_2$, $m_3 = m_4$.

Frame R IV

$$t(R) = \frac{P}{s^{1/2}}, \qquad n_2(R) = -\frac{q_{12} + q_{34}}{[-(q_{12} + q_{34})^2]^{1/2}}, \qquad n_3(R) = w$$

$$\overrightarrow{e(\Lambda p_1, \Lambda p_3)} = \begin{pmatrix} \frac{s^{1/2}}{4} \sin(\theta_s/2)(k + p) \\ -\frac{s^{1/2}}{4} \cos(\theta_s/2)(p - k) \\ -\frac{i}{2} pk \sin \theta_s \end{pmatrix} \qquad \overrightarrow{e(\Lambda p_2, \Lambda p_3)} = \begin{pmatrix} \frac{s^{1/2}}{4} \sin(\theta_s/2)(p - k) \\ -\frac{s^{1/2}}{4} \sin(\theta_s/2)(p + k) \\ \frac{i}{2} pk \sin \theta_s \end{pmatrix}$$

$$\Lambda L_{12}(1) = R_x(\pi/2)\, R_y(-\theta_s/2)\, B_1$$
$$\Lambda L_{12}(2) = R_x(\pi/2)\, R_y(-\theta_s/2 + \pi)\, B_2$$
$$\Lambda L_{12}(3) = R_x(\pi/2)\, R_y(\theta_s/2)\, B_3$$
$$\Lambda L_{12}(4) = R_x(\pi/2)\, R_y(\theta_s/2 + \pi)\, B_4$$

$R_x(\pi/2)$ is a rotation through $\pi/2$ around the 1-axis

$$R_x(\pi/2) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \qquad \text{(Euler angles } -\pi/2,\ \pi/2,\ \pi/2).$$

Expansion D

$$M_{\lambda_3\lambda_4;\lambda_1\lambda_2} = \sum (-)^{s_2-\lambda_2+s_3+\lambda_3} D^{s_1}(B_1)^{\lambda_1}_{\lambda_1} D^{s_2}(B_2)^{\lambda_2}_{\lambda_2} D^{s_3}(B_3)^{-\lambda_3}_{-\lambda_3} D^{s_4}(B_4)^{-\lambda_4}_{-\lambda_4}$$
$$D^{s_1}\left(-\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}\right)^{A_1}_{C_1} d^{s_1}\left(-\frac{\theta_s}{2}\right)^{C_1}_{\lambda_1} D^{s_2}\left(-\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}\right)^{A_2}_{C_2} d^{s_2}\left(-\frac{\theta_s}{2}\right)^{C_2}_{-\lambda_2} D^{s_3}\left(-\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}\right)^{A_3}_{C_3}$$
$$d^{s_3}\left(\frac{\theta_s}{2}\right)^{C_3}_{-\lambda_3} D^{s_4}\left(-\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}\right)^{A_4}_{C_4} d^{s_4}\left(\frac{\theta_s}{2}\right)^{C_4}_{\lambda_4} (s^{1/2})^{|M_1|+|M_2|} \ldots \tag{A-I.4}$$

$$M_1 + M_2 = A_1 + A_2 + A_3 + A_4$$

The dots stand for quantities kinematically regular at $s = 0$.

From

$$D^s(-\pi/2, \pi/2, \pi/2)^A_C\, d^s(\theta)^C_\lambda = (-)^{s+A}\, e^{-i\pi\lambda}\, D^s(-\pi/2, \pi/2, \pi/2)^A_C\, d^s(\theta)^C_{-\lambda}$$

we find, with Expansion D, $\hat{M}_{\lambda_3\lambda_4;\lambda_1\lambda_2}(-s^{1/2}) = \hat{M}_{-\lambda_3-\lambda_4;-\lambda_1-\lambda_2}(s^{1/2})(-)^{\Sigma_i s_i}$, or taking parity conservation (II-6) into account,

$$\hat{M}_{\lambda_3\lambda_4;\lambda_1\lambda_2}(-s^{1/2}) = \eta(-)^{\Sigma_i \lambda_i}\, \hat{M}_{\lambda_3\lambda_4;\lambda_1\lambda_2}(s^{1/2}).$$

2. *Kinematical Pole at* $s = 0$

Each term of the sum in Expansion D behaves like $(s^{1/2})^{|M_1|+|M_2|}$ and since $|M_1| + |M_2| \geqslant 0$, there is no kinematical pole at $s = 0$.

To summarize:

if $\eta = +1$ (i.e., $BB \to BB$, $FF \to FF$, $F\bar{F} \to F\bar{F}$),

$$\hat{M}_{\lambda_3\lambda_4;\lambda_1\lambda_2} \quad \text{for} \quad \sum_i \lambda_i \quad \text{even},$$

$$(s^{-1/2})\,\hat{M}_{\lambda_3\lambda_4;\lambda_1\lambda_2} \quad \text{for} \quad \sum_i \lambda_i \quad \text{odd},$$

and if $\eta = -1$ (i.e., $BB \to F\bar{F}$, $F\bar{F} \to BB$),

$$(s^{-1/2})\,\hat{M}_{\lambda_3\lambda_4;\lambda_1\lambda_2} \quad \text{for} \quad \sum_i \lambda_i \quad \text{even},$$

$$\hat{M}_{\lambda_3\lambda_4;\lambda_1\lambda_2} \quad \text{for} \quad \sum_i \lambda_i \quad \text{odd} \tag{A-I.5}$$

have no kinematical singularity at $s = 0$. (Results are given in Table IX).

V. $m_1 = m_2 = m_3 = m_4 = m$; $s_1 = s_2 = s_3 = s_4 = S$; $\eta = +1$

Near $s = 4m^2$ we apply the study of case I and near $s = 0$ we apply the study of case IV and find directly that

$$p^{2(2S-\mathrm{Max}(|\lambda|,|\mu|))}\,\hat{M}_{\lambda_3\lambda_4;\lambda_1\lambda_2} \quad \text{for} \quad \sum_i \lambda_i \quad \text{even},$$

$$p^{2(2S-\mathrm{Max}(|\lambda|,|\mu|))}\,\frac{\hat{M}_{\lambda_3\lambda_4;\lambda_1\lambda_2}}{s^{1/2}} \quad \text{for} \quad \sum_i \lambda_i \quad \text{odd} \tag{A-I.6}$$

have no kinematical singularity. (Results are given in Table X).

APPENDIX A-II.

Determination of the Overall Sign $\eta^{(s)}$ in the Crossing Matrix

The method we used in Section III-3 cannot yield the overall sign $\eta^{(s)}$ since we only follow the helicity frames all along the continuation path instead of the relevant 2×2 matrices. We first prove that $\eta^{(s)}$ can only be a product of $(-)^{2s_i}$ factors.

As shown in Section III-2, fermions can be associated by pairs so that the tensor product of the corresponding 2×2 matrices $L_{31}(i)$ can be unambiguously continued. This means that the *relative sign* only of $L_{31}^{c^{-1}}(i)\, L_{12}(i)$ and $L_{31}^{c^{-1}}(j) L_{12}(j)$ can be determined independently of the chosen continuation path Γ in q_i-space. Let us define ϵ_Γ by:

$$\mathscr{L}(1) = L_{31}^{c^{-1}}(1)\, L_{12}(1) = \epsilon_\Gamma \alpha_1$$

where α_1 has been defined in Eq. (III-6a); $L_{31}^{c}(1)$ is the analytic continuation of $L_{31}(1)$ along a path Γ. Now $\mathscr{L}(i)$, $(i = 2, 3, 4)$, as obtained through analytic continuation along the same path, can be written as

$$\mathscr{L}(i) = \epsilon_\Gamma \epsilon_i' \alpha_i \qquad \text{where} \qquad \epsilon_i' = \pm 1.$$

This proves that $\eta^{(s)} = \epsilon_\Gamma^{\sum_i 2s_i} \epsilon_2'^{2s_2} \epsilon_3'^{2s_3} \epsilon_4'^{2s_4}$ and, owing to the fact that $\sum_i 2s_i$ is even, $\eta^{(s)}$ is independent of Γ and furthermore it is a product of $(-)^{2s_i}$ factors. In particular, $\eta^{(s)} = +1$ for a reaction involving four bosons.

Since $(-)^{\sum_i 2s_i} \equiv 1$, the only possible values of $\eta^{(s)}$ are

$$(-)^{2s_1},\ (-)^{2s_2},\ (-)^{2s_3},\ (-)^{2s_4},\ (-)^{2s_1+2s_2},\ (-)^{2s_1+2s_3},\ (-)^{2s_1+2s_4}.$$

A first simplification is related to the involutary character of the crossing operation.

Substituting angles χ_i, as defined in Table XI, instead of angles χ_i' involved in Eq. (III-7) yields

$$M_{\{\lambda\}}^{(s)} = (-)^{\sigma(P)}\, \eta^{(s)}\, e^{i\pi(\lambda_2-\lambda_3)} \prod_i^{\otimes} d^{s_i}(\chi_i)_{\lambda_i}^{\lambda_i'}\, M_{\{\lambda'\}}^{(t)} \tag{A-II.1}$$

and permuting s and t in the above relation, we get

$$M_{\{\lambda'\}}^{(t)} = (-)^{\sigma(P)}\, \eta^{(t)}\, e^{i\pi(\lambda_2'-\lambda_3')} \prod_i^{\otimes} d^{s_i}(\chi_i^{(t)})_{\lambda_i'}^{\lambda_i''}\, M_{\{\lambda''\}}^{(s)}. \tag{A-II.2}$$

Since the analytic continuation is made along the same path, the angles χ_i and $\chi_i^{(t)}$ can be compared at the same point. One finds that

$$\begin{aligned} \chi_i &= -\chi_i^{(t)} && \text{for crossed particles,} \\ \chi_i &= \chi_i^{(t)} && \text{for uncrossed particles.} \end{aligned} \tag{A-II.3}$$

If one now inverts (A-II.1) and substitutes (A-II.3) into (A-II.2), comparing the two expressions obtained for $M_{\{\lambda'\}}^{(t)}$ yields

$$\eta^{(s)} = (-)^{2s_2+2s_3}\, \eta^{(t)}. \tag{A-II.4}$$

The four possibilities left for $\eta^{(s)}$ are then

$$(-)^{2s_1},\ (-)^{2s_4},\ (-)^{2s_1+2s_2},\ (-)^{2s_1+2s_3}.$$

As a consequence of our preceding discussion about the sign ambiguity, we can choose a definite path Γ, and write for this path

$$\epsilon_i \alpha_i = L_{31}^{c^{-1}}(i)\, L_{12}(i) \qquad \text{where} \qquad \epsilon_i = \pm 1,$$
$$\eta^{(s)} = \epsilon_1^{2s_1} \epsilon_2^{2s_2} \epsilon_3^{2s_3} \epsilon_4^{2s_4} = (\epsilon_1 \epsilon_2)^{2s_2} (\epsilon_1 \epsilon_3)^{2s_3} (\epsilon_1 \epsilon_4)^{2s_4}.$$

By inspection, we see that the determination of only two relative signs, $\epsilon_1 \epsilon_3$ and $\epsilon_2 \epsilon_4$, say, allows us to discriminate between the four possibilities. Exhibiting explicitly the matrix $\epsilon = i\sigma_2$ which occurs in the definition of the 2–body helicity states, we define matrices β_i through

$$\epsilon_1 \alpha_1 = \epsilon \beta_1, \qquad \epsilon_2 \alpha_2 = \epsilon \beta_2 \epsilon^{-1},$$
$$\epsilon_3 \alpha_3 = \beta_3, \qquad \epsilon_4 \alpha_4 = \beta_4 \epsilon^{-1}.$$

$\epsilon_1 \epsilon_3$ and $\epsilon_2 \epsilon_4$ are thus determined by a direct calculation of

$$\gamma_{31} = \epsilon_1 \epsilon_3 \alpha_3^{-1} \epsilon^{-1} \alpha_1 = \beta_3^{-1} \beta_1,$$
$$\gamma_{42} = \epsilon_4 \epsilon_2 \epsilon^{-1} \alpha_4^{-1} \epsilon^{-1} \alpha_2 \epsilon = \beta_4^{-1} \beta_2. \tag{A-II.5}$$

Taking ρ large enough in the definition of path C in Section III-3-A, we get a continuation path which stays constantly far away from the singularities, the precise location of which depends on the masses. Then $\eta^{(s)}$ does not depend on the mass case, so that it can be evaluated when masses are equal.

In this case, Eq. (A-II.5) reduces to

$$\gamma_{31} = -i \epsilon_1 \epsilon_3 \sigma_1 = \beta_3^{-1} \beta_1,$$
$$\gamma_{42} = i \epsilon_2 \epsilon_4 \sigma_1 = \beta_4^{-1} \beta_2. \tag{A-II.6}$$

DIRECT CALCULATION OF $\gamma_{31} = \beta_3^{-1} \beta_1$

Going back to the definition of $L_{31}(i)$ and $L_{12}(i)$ as given by Eqs. (III-3) and (II-4), respectively, and setting there $L'_{31}(i) = L_{31}(i)\, \epsilon'^{-1}$ and $L'_{12}(i) = L_{12}(i)\, \epsilon'^{-1}$, we get:

$$\gamma_{31} = L'^{-1}_{12}(3)\, L'^{c}_{31}(3)\, L'^{-1^c}_{31}(1)\, L'_{12}(1).$$

Particles 3 and 1 belong to the same (final) state in the t-channel. Then Eq. (III-3) yields:

$$L'_{31}(3)\, L'^{-1}_{31}(1) = B\left(\frac{P_t}{t^{1/2}} \to \frac{q_3}{m}\right) B^{-1}\left(\frac{P_t}{t^{1/2}} \to \frac{q_1}{m}\right).$$

This matrix, with a positive trace (as can be seen by taking $q_3 = q_1$), represents a pure Lorentz transformation in the 2-plane q_1, q_3, which takes q_1/m onto q_3/m. Then

$$L'_{31}(3)\, L'^{-1}_{31}(1) = \frac{m^2 + \sigma q_3 \sigma \tilde{q}_1}{[2m^2(m^2 + q_3 \cdot q_1)]^{1/2}},$$

the analytic continuation of which is:

$$L'^{c}_{31}(3)\, L'^{-1^c}_{31}(1) = \frac{m^2 - \sigma p_3 \sigma \tilde{p}_1}{(m^2 t)^{1/2}}.$$

Now Eq. (A-II-6) shows that γ_{31} (and γ_{42}) do not depend on the standard frame. So, *once the analytic continuation is performed*, we can take as a new frame the c.m. frame in the s-channel with 2-axis along w and 3-axis along q_{12}. Let θ_s be the scattering angle, and let $E = s^{1/2}/2$; then

$$L'_{12}(1) = \frac{m + \sigma \cdot p_1}{[2m(E+m)]^{1/2}}, \qquad L'_{12}(3) = \frac{m + \sigma \cdot p_3}{[2m(E+m)]^{1/2}} R(\theta_s)$$

where $R(\theta_s) = \cos(\theta_s/2) - i\sigma_2 \sin(\theta_s/2)$. We then easily find

$$\gamma_{31} = \frac{R^{-1}(\theta_s)(\boldsymbol{\sigma} \cdot \mathbf{p}_3 - \boldsymbol{\sigma} \cdot \mathbf{p}_1)}{t^{1/2}}.$$

Evaluating $\frac{1}{2}\,\mathrm{Tr}(i\sigma_1 \gamma_{31})$ from (Eq. A-II.6), we find

$$\epsilon_1 \epsilon_3 = \frac{2ip \sin \theta_s/2}{t^{1/2}} \qquad \text{with} \qquad 2p = (s - 4m^2)^{1/2} > 0$$

above the s-cut.

In the s physical region, t is negative and $t^{1/2} = i\,|\,t\,|^{1/2}$, θ_s is positive, so that $\sin(\theta_s/2) = |\,t\,|^{1/2}/2p$. Thus:

$$\epsilon_1 \epsilon_3 = +1.$$

Direct Calculation of $\gamma_{42} = \beta_4^{-1}\beta_2$

The preceding calculation can be exactly reproduced, because 4 and 2 also belong to the same (initial) state in the t-channel. One finds

$$\gamma_{42} = \frac{R^{-1}(\theta_s)(\boldsymbol{\sigma} \cdot \mathbf{p}_4 - \boldsymbol{\sigma} \cdot \mathbf{p}_2)}{t^{1/2}}$$

in the s-channel c.m. frame above-defined, so that $\gamma_{42} = -\gamma_{31}$. Comparison with Eq. (A-II.6) yields

$$\epsilon_2 \epsilon_4 = \epsilon_1 \epsilon_3 .$$

Looking at the possible values of $\eta^{(s)}$, we conclude that

$$\eta^{(s)} = (-)^{2s_1+2s_3} = (-)^{2s_2+2s_4}.$$

APPENDIX A-III. Consistency between Crossing and Regularization

We have defined kinematically regular helicity amplitudes $\mathscr{F}$ in both s- and t-channels. A consistency check of their being kinematically uniform at thresholds and pseudo-thresholds can be performed by looking at the behavior of the corresponding matrix at such points. As shown in section IV, the occurence of poles in this matrix does not mean that regularization is wrong or that individual amplitudes have to vanish, but yields the kinematical constraints. So, we shall just make sure that crossing matrix between R.H.A is uniform at $\mathscr{S}_{12} = 0, \mathscr{S}_{34} = 0$, $\mathscr{T}_{31} = 0$, $\mathscr{T}_{42} = 0$. We first study the crossing angles χ_i at such points and then the corresponding properties of the crossing matrix.

I. Singularities of the Trigonometric Functions of the Crossing Angles

After turning around a threshold or a pseudo-threshold branch point of the s (Resp. t) channel, we see that $\cos\chi_i$, $\sin\chi_i$, $\cos\theta_s$ and $\sin\theta_s$ (Resp. $\cos\theta_t$ and $\sin\theta_t$) change sign, which means that the corresponding angles are increased or decreased by π. We first show that these alterations are not independent.

1. *Angle Relations*

A. *Branch point at* $\mathscr{S}_{12} = 0$

Let us calculate $\sin(\chi_1 + \epsilon\chi_2)$ where $\epsilon = \pm 1$, using Table XI. We find

$$\mathscr{S}_{12}^2\mathscr{T}_{31}\mathscr{T}_{42}\sin(\chi_1 + \epsilon\chi_2) = 2\Phi^{1/2}[s - (m_1 + \epsilon m_2)^2][t(m_1 - \epsilon m_2) + (m_2^2 - m_4^2)\, m_1 - \epsilon m_2(m_1^2 - m_3^2)].$$

Recalling that $\mathscr{S}_{12}^2 = [s - (m_1 + m_2)^2][s - (m_1 - m_2)^2]$, we conclude:

(1) $\sin(\chi_1 + \chi_2)$ does not have either a pole or a zero at $s = (m_1 + m_2)^2$. Then $\sin[(\chi_1 + \chi_2)/2]$ and $\cos[(\chi_1 + \chi_2)/2]$ have no branch point at $\varphi_{12} = 0$, so that, upon turning around $\varphi_{12} = 0$, $\chi_1 \to \chi_1 + \epsilon_1\pi$ whereas

$$\chi_2 \to \chi_2 - \epsilon_1\pi \ (\epsilon_1 = \pm 1).$$

(2) Similarly $\sin(\chi_1 - \chi_2)$ does not have either a pole or a zero at $s = (m_1 - m_2)^2$, and, upon turning around $\psi_{12} = 0$, we see that χ_1 and χ_2 now behave identically.

To relate the behavior of θ_s to that of the χ's, we conclude $\sin(\theta_s + \epsilon\chi_2)$, $\epsilon = \pm 1$. We find

$$\begin{aligned}\mathscr{S}_{12}^2\mathscr{S}_{34}\mathscr{T}_{42}\sin(\theta_s + \epsilon\chi_2) &= 2\Phi^{1/2}(s^{1/2} + m_1 + \epsilon m_2)(s^{1/2} - m_1 + \epsilon m_2)(s^{1/2}(t - m_2^2 - m_4^2)\\ &\quad + \epsilon m_2(s - m_3^2 + m_4^2)).\end{aligned}$$

With the positive determination of $s^{1/2}$ (above the cut), we conclude that

$$\begin{aligned} & \sin(\theta_s - \chi_2) \text{ does not have a pole behavior at } \varphi_{12} = 0;\\ \text{if } m_1 > m_2, \quad & \sin(\theta_s + \chi_2) \text{ does not have a pole behavior at } \psi_{12} = 0;\\ \text{if } m_1 < m_2, \quad & \sin(\theta_s - \chi_2) \text{ does not have a pole behavior at } \psi_{12} = 0.\end{aligned}$$

B. *Branch points at* $\mathscr{S}_{34} = 0$, $\mathscr{T}_{31} = 0$, $\mathscr{T}_{42} = 0$

Similar methods are used. Symbols φ_{42}, ψ_{42}, φ_{31}, ψ_{31} are defined in the t-channel, as φ_{12}, ψ_{12}, φ_{34}, ψ_{34} are in the s-channel.

All results are summarized in the following table, which gives the correlated behaviors of various angles when one turns around a given branch point.

$$\begin{aligned}
&\varphi_{12} \quad \text{If } \theta_s \to \theta_s + \epsilon\pi \text{ then } \chi_1 \to \chi_1 - \epsilon\pi, \quad \chi_2 \to \chi_2 + \epsilon\pi;\\
&\psi_{12} \quad \text{if } \theta_s \to \theta_s + \epsilon\pi \text{ then } \chi_1 \to \chi_1 + \epsilon\epsilon_{12}\pi, \chi_2 \to \chi_2 + \epsilon\epsilon_{12}\pi;\\
&\varphi_{34} \quad \text{if } \theta_s \to \theta_s + \epsilon\pi \text{ then } \chi_3 \to \chi_3 - \epsilon\pi, \quad \chi_4 \to \chi_4 + \epsilon\pi;\\
&\psi_{34} \quad \text{if } \theta_s \to \theta_s + \epsilon\pi \text{ then } \chi_3 \to \chi_3 - \epsilon\epsilon_{34}\pi, \chi_4 \to \chi_4 - \epsilon\epsilon_{34}\pi.
\end{aligned} \qquad \text{(A-III.1)}$$

$$\begin{aligned}
&\varphi_{42} \quad \text{If } \theta_t \to \theta_t + \epsilon\pi \text{ then } \chi_4 \to \chi_4 + \epsilon\pi, \quad \chi_2 \to \chi_2 + \epsilon\pi;\\
&\psi_{42} \quad \text{if } \theta_t \to \theta_t + \epsilon\pi \text{ then } \chi_4 \to \chi_4 - \epsilon\epsilon_{42}\pi, \chi_2 \to \chi_2 + \epsilon\epsilon_{42}\pi;\\
&\varphi_{31} \quad \text{if } \theta_t \to \theta_t + \epsilon\pi \text{ then } \chi_3 \to \chi_3 - \epsilon\pi, \quad \chi_1 \to \chi_1 - \epsilon\pi;\\
&\psi_{31} \quad \text{if } \theta_t \to \theta_t + \epsilon\pi \text{ then } \chi_3 \to \chi_3 + \epsilon\epsilon_{31}\pi, \chi_1 \to \chi_1 - \epsilon\epsilon_{31}\pi.
\end{aligned} \qquad \text{(A-III.2)}$$

We recall that $\epsilon_{ij} = \text{sign}(m_i - m_j)$,

$$\epsilon = \pm 1.$$

II. Consistency between Crossing and Regularization

Although we have checked consistency for all types of reactions, we shall only, for the sake of brevity, reproduce here one such check explicitly, namely, in the case of four fermions, with relative intrinsic parity $\eta = +1$.

Let $F^{i}_{\{\lambda\}}$ (Resp. $G^{i}_{\{\lambda'\}}$) be the R.H.A. in the s (Resp. t) channel, where $i = 1, 2$. Putting together the regularization formulae (Table IV) and crossing relations (Table XI), we obtain

$$F^{1}_{\lambda_3\lambda_4;\lambda_1\lambda_2} = \sum_{\{\lambda'\}} (C^{\lambda_3'\lambda_1'\lambda_4'\lambda_2'}_{\lambda_3\lambda_4\lambda_1\lambda_2} + \eta_{34} C^{\lambda_3'\lambda_1'\lambda_4'\lambda_2'}_{-\lambda_3-\lambda_4\lambda_1\lambda_2})[\varphi_{42}^{\Delta_{42}}\psi_{42}\varphi_{31}^{\Delta_{31}}\psi_{31}G^{1}_{\lambda_3'\lambda_1';\lambda_4'\lambda_2'}$$

$$+ \varphi_{42}\psi_{42}^{\Delta_{42}}\varphi_{31}\psi_{31}^{\Delta_{31}}G^{2}_{\lambda_3'\lambda_1';\lambda_4'\lambda_2'}], \qquad \text{(A-III.3)}$$

where

$$C^{\lambda_3'\lambda_1'\lambda_4'\lambda_2'}_{\lambda_3\lambda_4\lambda_1\lambda_2} = e^{i\pi(\lambda_2-\lambda_3)}\, \eta^{(s)}\, (-)^{\sigma(p)}\, \frac{\sin(\theta_t/2)^{|\lambda'-\mu'|}\cos(\theta_t/2)^{|\lambda'+\mu'|}}{\sin(\theta_s/2)^{|\lambda-\mu|}\cos(\theta_s/2)^{|\lambda+\mu|}}\, \frac{(s^{1/2})^{|\lambda|+|\mu|}}{(t^{1/2})^{|\lambda'|+|\mu'|}}$$

$$\times \frac{\mathscr{S}_{12}^{m_{12}^-}\mathscr{S}_{34}^{m_{34}^-}\psi_{12}^{\Delta_{12}-1}\psi_{34}^{\Delta_{34}-1}}{\mathscr{T}_{42}^{m_{42}^+}\mathscr{T}_{31}^{m_{31}^+}} \prod_i d^{s_i}(\chi_i)^{\lambda_i'}_{\lambda_i}, \qquad \text{(A-III.4)}$$

$$\lambda' = \lambda_4' - \lambda_2'; \qquad \mu' = \lambda_3' - \lambda_1',$$

$$m_{42} = s_4 + s_2 - \text{Max}(|\lambda'|, |\mu'|), \qquad \text{and} \qquad \Delta_{ij} = m_{ij}^+ - m_{ij}^- .$$
$$m_{31} = s_3 + s_1 - \text{Max}(|\lambda'|, |\mu'|),$$

1. *Branch Points* $\mathscr{S}_{34} = 0$

The only quantity in (A-III.3) which is singular at $\mathscr{S}_{34} = 0$ is $C^{\{\lambda'\}}_{\{\lambda\}}$. So, we have to show that the combination $C^{\{\lambda'\}}_{\lambda_f\lambda_i} + \eta_{34}C^{\{\lambda'\}}_{-\lambda_f\lambda_i}$ is uniform both at $\varphi_{34} = 0$ and $\psi_{34} = 0$.

A. *Branch point at* $\varphi_{34} = 0$

From (A-III.1), the singular terms of $C^{\{\lambda'\}}_{\{\lambda\}}$ in (A-III.4) have the following behavior upon turning around $\varphi_{34} = 0$:

$$\sin(\theta_s/2)^{|\lambda-\mu|}\cos(\theta_s/2)^{|\lambda+\mu|} \to \sin(\theta_s/2)^{|\lambda+\mu|}\cos(\theta_s/2)^{|\lambda-\mu|}\,(-)^{\lambda+\epsilon\mu},$$
$$d^{s_3}(\chi_3)^{\lambda_3'}_{\lambda_3}\, d^{s_4}(\chi_4)^{\lambda_4'}_{\lambda_4} \to d^{s_3}(\chi_3)^{\lambda_3'}_{-\lambda_3}\, d^{s_4}(\chi_4)^{\lambda_4'}_{-\lambda_4}\,(-)^{s_3+s_4-\epsilon\mu};$$

on the other hand, $e^{i\pi(\lambda_2-\lambda_3)} = (-)^{2s_3}\, e^{i\pi(\lambda_2+\lambda_3)}$, so that we find

$$C^{\{\lambda'\}}_{\lambda_3\lambda_4\lambda_1\lambda_2}(-\varphi_{34}) = (-)^{s_3-s_4-\lambda}\, C^{\{\lambda'\}}_{-\lambda_3-\lambda_4\lambda_1\lambda_2}(\varphi_{34}) = \eta_{34}C^{\{\lambda'\}}_{-\lambda_3-\lambda_4\lambda_1\lambda_2}(\varphi_{34})$$

since $(-)^{s_3-s_4} = (-)^{s_4-s_3}$ in the case under consideration ($FF \to FF$). We then verify that the relevant combination of coefficients $C^{\{\lambda'\}}_{\{\lambda\}}$ is uniform as it should be.

B. *Branch point at* $\psi_{34} = 0$

In the same way, turning around $\psi_{34} = 0$ yields

$$\sin(\theta_s/2)^{|\lambda-\mu|} \cos(\theta_s/2)^{|\lambda+\mu|} \to \sin(\theta_s/2)^{|\lambda+\mu|} \cos(\theta_s/2)^{|\lambda-\mu|} (-)^{\lambda+\epsilon\mu},$$

$$d^{s_3}(\chi_3)_{\lambda_3}^{\lambda_3'} d^{s_4}(\chi_4)_{\lambda_4}^{\lambda_4'} \to d^{s_3}(\chi_3)_{-\lambda_3}^{\lambda_3'} d^{s_4}(\chi_4)_{-\lambda_4}^{\lambda_4'} (-)^{s_3+s_4-\epsilon\epsilon_{34}(\lambda_3+\lambda_4)},$$

$$\psi_{34}^{\Delta_{34}-1} \to -\psi_{34}^{\Delta_{34}-1}.$$

Then

$$C_{\lambda_3\lambda_4\lambda_1\lambda_2}^{\{\lambda'\}}(-\psi_{34}) = -(-)^{3s_3+s_4+\lambda} (-)^{\lambda_3(1+\epsilon_{34})+\lambda_4(1-\epsilon_{34})} C_{-\lambda_3-\lambda_4\lambda_1\lambda_2}^{\{\lambda\}}(\psi_{34}).$$

Since in the present case $(-)^{2\lambda_3} = (-)^{2\lambda_4} = -1$, the phase factor in the above relation is in fact independent of ϵ_{34} and it is again equal to η_{34}.

2. *Branch Points at* $\mathscr{S}_{12} = 0$

In Eq. (A-III.3), we replace $\eta_{34} C_{-\lambda_3-\lambda_4\lambda_1\lambda_2}^{\{\lambda'\}}$ by $\eta_{12} C_{\lambda_3\lambda_4-\lambda_1-\lambda_2}^{\{\lambda'\}}$, and the above argument holds replacing indices 3 and 4 by 1 and 2.

3. *Branch Points at* $\mathscr{T}_{31} = 0$

From (A-III.3), replacing

$$\sum_{\lambda_3'\lambda_1'\lambda_4'\lambda_2'} \quad \text{by} \quad \frac{1}{2}\left[\sum_{\lambda_3'\lambda_1'\lambda_4'\lambda_2'} + \sum_{-\lambda_3'-\lambda_1'\lambda_4'\lambda_2'}\right]$$

and using Eq. (II-26) for $G_{\{\lambda'\}}^i$; namely,

$$G_{\lambda_3'\lambda_1';\lambda_4'\lambda_2'}^1 = \eta_{31} G_{-\lambda_3'-\lambda_1';\lambda_4'\lambda_2'}^1,$$

and

$$G_{\lambda_3'\lambda_1';\lambda_4'\lambda_2'}^2 = -\eta_{31} G_{-\lambda_3'-\lambda_1';\lambda_4'\lambda_2'}^2,$$

with

$$\eta_{31} = (-)^{s_3-s_1+\lambda'},$$

we obtain (using the shorthand $C_{\lambda_f\lambda_i}^{\lambda_f'\lambda_i'}$ instead of $C_{\lambda_3\lambda_4\lambda_1\lambda_2}^{\lambda_3'\lambda_1'\lambda_4'\lambda_2'}$)

$$F_{\lambda_f\lambda_i}^1 = \frac{1}{2} \sum_{\lambda_3'\lambda_1'\lambda_4'\lambda_2'} \{C_{\lambda_f\lambda_i}^{\lambda_f'\lambda_i'} + \eta_{31} C_{\lambda_f\lambda_i}^{-\lambda_f'\lambda_i'} + \eta_{34}[C_{-\lambda_f\lambda_i}^{\lambda_f'\lambda_i'} + \eta_{31} C_{-\lambda_f\lambda_i}^{-\lambda_f'\lambda_i'}]\} \varphi_{42}^{\Delta_{42}} \psi_{42} \varphi_{31}^{\Delta_{31}} \psi_{31} G_{\lambda_f'\lambda_i'}^1$$

$$+ \frac{1}{2} \sum_{\lambda_3'\lambda_1'\lambda_4'\lambda_2'} \{C_{\lambda_f\lambda_i}^{\lambda_f'\lambda_i'} - \eta_{31} C_{\lambda_f\lambda_i}^{-\lambda_f'\lambda_i'} + \eta_{34}[C_{-\lambda_f\lambda_i}^{\lambda_f'\lambda_i'} - \eta_{31} C_{-\lambda_f\lambda_i}^{-\lambda_f'\lambda_i'}]\} \varphi_{42} \psi_{42}^{\Delta_{42}} \varphi_{31} \psi_{31}^{\Delta_{31}} G_{\lambda_f'\lambda_i'}^2.$$

A. *Branch point at* $\varphi_{31} = 0$

Let us study the behavior of $C^{\lambda_f'\lambda_i'}_{\lambda_f\lambda_i}$ after one turn around $\varphi_{31} = 0$. According to (A-III.2),

$$\sin(\theta_s/2)^{|\lambda'-\mu'|} \cos(\theta_s/2)^{|\lambda'+\mu'|} \to (-)^{\lambda'+\epsilon\mu'} \sin(\theta_s/2)^{|\lambda'+\mu'|} \cos(\theta_s/2)^{|\lambda'-\mu'|},$$

$$d^{s_1}(\chi_1)^{\lambda_1'}_{\lambda_1} d^{s_3}(\chi_3)^{\lambda_3'}_{\lambda_3} \to (-)^{s_1+s_3-\epsilon(\lambda_1'+\lambda_3')} d^{s_1}(\chi_1)^{-\lambda_1'}_{\lambda_1} d^{s_3}(\chi_3)^{-\lambda_3'}_{\lambda_3},$$

so that

$$C^{\lambda_f'\lambda_i'}_{\lambda_f\lambda_i}(-\varphi_{31}) = \eta_{31} C^{-\lambda_f'\lambda_i'}_{\lambda_f\lambda_i}(\varphi_{31}) \qquad \text{(in the spin case considered).}$$

Then

$$X^{\pm}_{\lambda_f\lambda_i}(\varphi_{31}) = C^{\lambda_f'\lambda_i'}_{\lambda_f\lambda_i}(\varphi_{31}) \pm \eta_{31} C^{-\lambda_f'\lambda_i'}_{\lambda_f\lambda_i}(\varphi_{31}) = \pm X^{\pm}_{\lambda_f\lambda_i}(-\varphi_{31})$$

and the coefficients of G^1 and G^2 are uniform, as they should be.

B. *Branch point at* $\psi_{31} = 0$

Following the same procedure, one verifies that the behavior of $X^{\pm}$ is independent of ϵ_{31} and one finds

$$X^{\pm}_{\lambda_f\lambda_i}(\psi_{31}) = \pm X^{\pm}_{\lambda_f\lambda_i}(-\psi_{31}),$$

so that again the crossing matrix element is regular at $\psi_{31} = 0$.

4. *Branch Points at* $\mathcal{T}_{42} = 0$

In (A-III.3), we now replace

$$\sum_{\lambda_3'\lambda_1'\lambda_4'\lambda_2'} \quad \text{by} \quad \frac{1}{2}\left[\sum_{\lambda_3'\lambda_1'\lambda_4'\lambda_2'} + \sum_{\lambda_3'\lambda_1'-\lambda_4'-\lambda_2'}\right]$$

and use (II-26),

$$G^1_{\lambda_3'\lambda_1';\lambda_4'\lambda_2'} = \eta_{42} G^1_{\lambda_3'\lambda_1';-\lambda_4'-\lambda_2'},$$

$$G^2_{\lambda_3'\lambda_1';\lambda_4'\lambda_2'} = -\eta_{42} G^2_{\lambda_3'\lambda_1';-\lambda_4'-\lambda_2'}.$$

We turn the crank once more, and now change λ_i' into $-\lambda_i'$ instead of λ_f' into $-\lambda_f'$ in the above argument.

III. Conclusion

Of course the same line can be followed to verify that the crossing-matrix elements between F^2 and G^1, G^2 are also uniform. This has been done together with other spin configurations. We shall not reproduce here these rather tedious calculations.

APPENDIX A-IV. Particular Cases: Constraints at $s = 0$

(a) $m_1 = m_2$; $m_3 \neq m_4$, $s_1 = s_2 = s_{\text{in}}$

The conclusions about the behaviors of $T^s_{\tau_3\tau_4;\tau_1\tau_2}$ in the neighborhood of $\varphi_{12} = 0$, $\varphi_{34} = 0$ and $\psi_{34} = 0$ are the same as in the general case. But now there is a kinematical constraint at $s = 0$ because the pseudothreshold ψ_{12} coincides with $s^{1/2}$.

$$\cos(\chi_1 + \chi_2) \quad \text{and} \quad \sin(\chi_1 + \chi_2) \quad \text{behave like } 1/s.$$

Furthermore, from the regularization (Eq. A-I.3), $(s^{1/2})^{2s_{\text{in}}} \hat{M}^s_{\lambda_3\lambda_4;\lambda_1\lambda_2} = a + bs^{1/2}$, where a and b are kinematically finite at $s = 0$.

Then $(s^{1/2})^{2s_{\text{in}}} M^s_{\lambda_3\lambda_4;\lambda_1\lambda_2} = a' + b's^{1/2}$ where a' and b' are kinematically finite at $s = 0$.

Thus $(s^{1/2})^{2s_{\text{in}}} T^s_{\tau_3\tau_4;\tau_1\tau_2}$ is kinematically finite at $s = 0$. According to the crossing formula (IV-6), $T^s_{\tau_3\tau_4;\tau_1\tau_2} \simeq \exp[-i(\chi_1 + \chi_2)^{(1/2)(\tau_1-\tau_2)}]$ near $s = 0$ with $e^{i(\chi_1+\chi_2)} \simeq (s)^{\mp 1}$ if $[\Phi(s, t)]^{1/2} = \pm im_1(m_3^2 - m_4^2)$ at $s = 0$.

Conclusion. Near $s = 0$,

$$(s^{1/2})^{2s_{\text{in}}} T^s_{\tau_3\tau_4;\tau_1\tau_2} \simeq (s^{1/2})^{2s_{\text{in}} \pm (\tau_1-\tau_2)}$$
$$\text{if} \quad [\Phi(s, t)]^{1/2} = \pm im_1(m_3^2 - m_4^2) \quad \text{at} \quad s = 0. \qquad \text{(A-IV.1)}$$

(b) $m_3 = m_4$; $m_1 \neq m_2$; $s_3 = s_4 = s_f$

The same kind of results holds

(i) ψ_{34} coincides with $s = 0$,

(ii) $(s^{1/2})^{2s_f} T^s_{\tau_3\tau_4;\tau_1\tau_2}$ is finite at $s = 0$,

(iii) $e^{i(\chi_3+\chi_4)} \simeq (s)^{\mp 1}$ if $[\Phi(s, t)]^{1/2}|_{s=0} = \pm im_3(m_1^2 - m_2^2)$.

Conclusion. Near $s = 0$,

$$(s^{1/2})^{2s_f} T^s_{\tau_3\tau_4;\tau_1\tau_2} \simeq (s^{1/2})^{2s_f \pm (\tau_3-\tau_4)}$$
$$\text{if} \quad [\Phi(s, t)]^{1/2} = \pm im_3(m_1^2 - m_2^2) \quad \text{at} \quad s = 0. \qquad \text{(A-IV.2)}$$

(c) $m_1 = m_2\,;\, m_3 = m_4$

In this case both ψ_{12} and ψ_{34} coincide with $s^{1/2}$. Furthermore, the $s = 0$ singularity appears simultaneously in $\cos \chi_i$, $T^s_{\tau_3\tau_4;\tau_1\tau_2}$ and $T^t_{-\tau_3-\tau_1;-\tau_4-\tau_2}$. For this reason it is simpler to return to helicity amplitudes. In the present case the functions of table XI reduce to

$$\cos \chi_1 = -\cos \chi_2 = -\frac{(t + m_1^2 - m_3^2)(-s)^{1/2}}{\mathscr{T}_{13}(4m_1^2 - s)^{1/2}}$$

$$\sin \chi_1 = \sin \chi_2 = \frac{2m_1(\mathscr{T}^2_{13} + st)^{1/2}}{\mathscr{T}_{13}(4m_1^2 - s)^{1/2}},$$

$$\cos \chi_3 = -\cos \chi_4 = +\frac{(t + m_3^2 - m_1^2)(-s)^{1/2}}{\mathscr{T}_{13}(4m_3^2 - s)^{1/2}}$$

$$\sin \chi_3 = \sin \chi_4 = -\frac{2m_3(\mathscr{T}^2_{13} + st)^{1/2}}{\mathscr{T}_{13}(4m_3^2 - s)^{1/2}},$$

$$\sin \frac{\theta_t}{2} = \frac{t^{1/2}(-s)^{1/2}}{\mathscr{T}_{13}}.$$

We deduce, in the neighborhood of $s = 0$,

$$M^t_{\lambda_3'\lambda_1';\lambda_4'\lambda_2'} = \sin(\theta_t/2)^{|\lambda'-\mu'|} \cos(\theta_t/2)^{|\lambda'+\mu'|}\, \hat{M}^t_{\lambda_3'\lambda_1';\lambda_4'\lambda_2'} \simeq (-s)^{|\lambda'-\mu'|/2}. \tag{A-IV.3}$$

Furthermore, from the regularization (A-I.5),

$$M^s_{\lambda_3\lambda_4;\lambda_1\lambda_2} \simeq (s^{1/2})^\epsilon \qquad \text{where} \qquad \begin{cases} \epsilon = +1 & \text{if} \quad \eta(-)^{\Sigma_i\lambda_i} = -1, \\ \epsilon = 0 & \text{if} \quad \eta(-)^{\Sigma_i\lambda_i} = +1. \end{cases} \tag{A-IV.4}$$

The inverse crossing relation reads

$$\begin{aligned} M^t_{\lambda_3'\lambda_1';\lambda_4'\lambda_2'} = {} & (-1)^{\sigma(P)} (-1)^{2s_2+2s_4} e^{-i\pi(\lambda_2'-\lambda_3')} \sum_{\{\lambda\}} d^{s_1}(-\chi_1)^{\lambda_1}_{\lambda_1'} d^{s_2}(\chi_2)^{\lambda_2}_{\lambda_2'} \\ & \times d^{s_3}(\chi_3)^{\lambda_3}_{\lambda_3'} d^{s_4}(-\chi_4)^{\lambda_4}_{\lambda_4'} M^s_{\lambda_3\lambda_4;\lambda_1\lambda_2}. \end{aligned} \tag{A-IV.5}$$

Using (A-IV.3) and (A-IV.4), we deduce the kinematical constraints at $s = 0$,

$$\sum_{\{\lambda\}} d^{s_1}(-\chi_1)^{\lambda_1}_{\lambda_1'} d^{s_2}(\chi_2)^{\lambda_2}_{\lambda_2'} d^{s_3}(\chi_3)^{\lambda_3}_{\lambda_3'} d^{s_4}(-\chi_4)^{\lambda_4}_{\lambda_4'} M^s_{\lambda_3\lambda_4;\lambda_1\lambda_2} \simeq (s^{1/2})^{|\lambda'-\mu'|}$$

$$\text{where} \quad \lambda' = \lambda_4' - \lambda_2' \qquad \text{and} \qquad \mu' = \lambda_3' - \lambda_1'. \tag{A-IV.6}$$

APPENDIX A-V. EXAMPLES

In this appendix, we test all the results obtained in this paper on examples for which one knows directly the relations between helicity amplitudes and invariant amplitudes supposed to enjoy Mandelstam analyticity properties.

I. TEST OF THE REGULARIZATION OF HELICITY AMPLITUDES

A. *General Mass Case*

(1) $\underset{1}{\tfrac{1}{2}^+} + \underset{2}{0^-} \to \underset{3}{\tfrac{1}{2}^+} + \underset{4}{0^-} \qquad \eta = +1.$

In terms of the usual invariant amplitudes A and B, supposed to be kinematically regular, helicity amplitudes are expressed by:

$$M_{\lambda_3 0;\lambda_1 0} = \bar{u}(p_3)^{\lambda_3}\,[A + (B/2)\gamma\cdot(p_2 + p_4)]\,u_{\lambda_1}(p_1)$$

where

$$u_\lambda(p_i) = \frac{1}{\sqrt{2}}\begin{pmatrix} D^{1/2}(L_{12}(i))\cdot{}_\lambda \\ D^{1/2}(L_{12}^{+-1}(i))\cdot{}_\lambda \end{pmatrix}$$

and

$$\bar{u}^\lambda(p_i) = \frac{1}{\sqrt{2}}\,(D^{1/2}(L_{12}^{-1}(i))^\lambda.\ D^{1/2}(L^\dagger(i))^\lambda.)$$

We get, after some algebraic calculations:

$$\begin{aligned}
M_{\frac{1}{2}0;\frac{1}{2}0} &= \frac{\cos\frac{\theta_s}{2}\left\{[(\omega_1+m_1)(\omega_3+m_3)-pk]\left(A - B\,\frac{m_1+m_3}{2}\right) + B(s^{1/2})[(\omega_1+m_1)(\omega_3+m_3)+pk]\right\}}{[2m_1 2m_3(\omega_1+m_1)(\omega_3+m_3)]^{1/2}} \\
&= M_{-\frac{1}{2}0;-\frac{1}{2}0} \qquad \text{(parity invariance)}, \\
M_{\frac{1}{2}0;-\frac{1}{2}0} &= \frac{\sin\frac{\theta_s}{2}\left\{[(\omega_1+m_1)(\omega_3+m_3)+pk]\left(A - B\,\frac{m_1+m_3}{2}\right) + B(s^{1/2})[(\omega_1+m_1)(\omega_3+m_3)-pk]\right\}}{[2m_1 2m_3(\omega_1+m_1)(\omega_3+m_3)]^{1/2}} \\
&= -M_{-\frac{1}{2}0;\frac{1}{2}0} \qquad \text{(parity invariance)}.
\end{aligned} \tag{A-V.1}$$

For $M_{\frac{1}{2}0;\frac{1}{2}0}$, one has

$$\lambda = \tfrac{1}{2}, \qquad \mu = \tfrac{1}{2}, \qquad \eta_{12} = +1, \qquad \eta_{34} = -1, \qquad m_{12} = 0, \qquad m_{34} = 0.$$

From Eq. (II-17) one finds:

$$\mathscr{A}_{34} = \mathscr{A}_{12} = \frac{2(\omega_1 + m_1)(\omega_3 + m_3)\left[A + B\left(s^{1/2} - \frac{m_1 + m_3}{2}\right)\right]}{[2m_1 2m_3(\omega_1 + m_1)(\omega_3 + m_3)]^{1/2}},$$

$$\mathscr{B}_{34} = \mathscr{B}_{12} = \frac{2pk\left[-A + B\left(s^{1/2} + \frac{m_1 + m_3}{2}\right)\right]}{[2m_1 2m_3(\omega_1 + m_1)(\omega_3 + m_3)]^{1/2}}.$$

One verifies directly that, for $m_1 > m_2$, $m_3 > m_4$,

$$F^1_{\frac{1}{2}0,\frac{1}{2}0} = s^{1/2}\mathscr{A}_{34} \quad \text{and} \quad F^2_{\frac{1}{2}0;\frac{1}{2}0} = s^{1/2}\mathscr{S}^{-1}_{12}\mathscr{S}^{-1}_{34}\mathscr{B}_{34}$$

have no kinematical singularity, except for the $s^{1/2}$ factor which occurs in all $BF \to BF$ reactions. Using the fact that $(\omega_i + m_i)^{1/2}$ is singular at the pseudo-threshold when $m_i < m_j$, one treats easily the other mass configurations of table (IV),

(2) $\underset{1}{0^-} + \underset{2}{0^-} \to \underset{3}{\tfrac{1}{2}^+} + \underset{4}{\bar{\tfrac{1}{2}}^-} \qquad \eta = -1.$

From

$$M_{\lambda_3\lambda_4;00} = \bar{u}^{\lambda_3}(p_3)[A + (B/2)\gamma\cdot(p_2 - p_1)]\, v_{\lambda_4}(p_4)$$

where

$$v_\lambda(p_i) = \begin{pmatrix} D^{1/2}(L_{12}(i)\,\epsilon)\cdot{}_\lambda \\ D^{1/2}(L^{+-1}_{12}(i)\,\epsilon^{-1})\cdot{}_\lambda \end{pmatrix}(2)^{-1/2}$$

we get

$$\begin{aligned} M_{\frac{1}{2}\frac{1}{2};00} = &\Big\{Ap(\omega_3 + m_3 + \omega_4 + m_4) - \frac{B}{2}(\omega_1 - \omega_2)(\omega_3 + m_3 - \omega_4 - m_4) \\ &- Bk\cos\theta_s[(\omega_3 + m_3)(\omega_4 + m_4) - p^2]\Big\} \\ &\times [2m_3 2m_4(\omega_3 + m_3)(\omega_4 + m_4)]^{-1/2} \\ = &-M_{-\frac{1}{2}-\frac{1}{2};00}\,; \end{aligned} \qquad \text{(A-V.2)}$$

$$M_{\frac{1}{2}-\frac{1}{2};00} = \frac{Bk\sin\theta_s[(\omega_3 + m_3)(\omega_4 + m_4) + p^2]}{[2m_3 2m_4(\omega_3 + m_3)(\omega_4 + m_4)]^{1/2}} = M_{-\frac{1}{2}\frac{1}{2};00}.$$

Noting the identities

$$(\omega_3 + m_3)^{1/2}(\omega_4 + m_4)^{1/2} + (\omega_3 - m_3)^{1/2}(\omega_4 - m_4)^{1/2} \equiv \psi_{34},$$

$$(\omega_3 + m_3)^{1/2}(\omega_4 + m_4)^{1/2} - (\omega_3 - m_3)^{1/2}(\omega_4 - m_4)^{1/2} \equiv \psi_{34}\frac{m_3 + m_4}{s^{1/2}},$$

$$(\omega_3 + m_3)^{1/2}(\omega_4 - m_4)^{1/2} + (\omega_3 - m_3)^{1/2}(\omega_4 + m_4)^{1/2} \equiv \varphi_{34},$$

$$(\omega_3 + m_3)^{1/2}(\omega_4 - m_4)^{1/2} - (\omega_3 - m_3)^{1/2}(\omega_4 + m_4)^{1/2} \equiv \varphi_{34}\frac{m_3 - m_4}{s^{1/2}},$$

we simplify (A-V.2) into

$$M_{\frac{1}{2}\ \frac{1}{2};00} = \frac{1}{(4m_3m_4)^{1/2}}\Big[A\varphi_{34} - \frac{B}{2}(m_1{}^2 - m_2{}^2)(m_3 - m_4)\frac{\varphi_{34}}{s} - B\frac{\psi_{34}\mathscr{S}_{12}\cos\theta_s}{2s}(m_3 + m_4)\Big],$$

$$M_{\frac{1}{2}-\frac{1}{2};00} = \frac{B\mathscr{S}_{12}\sin\theta_s\psi_{34}}{2s^{1/2}(4m_3m_4)^{1/2}}.$$

Now, with $\lambda_3 = \frac{1}{2}\ \lambda_4 = \frac{1}{2}$,

$$\lambda = 0, \qquad \mu = 0, \qquad \eta_{12} = +1, \qquad \eta_{34} = +1,$$
$$m_{12} = 0, \qquad m_{34} = +1,$$
$$\mathscr{A}_{34} = \mathscr{B}_{12} = 0, \qquad \text{and} \qquad \mathscr{B}_{34} = \mathscr{A}_{12} = 2\hat{M}_{\frac{1}{2}\frac{1}{2};00},$$

$$F^1_{\frac{1}{2}\ \frac{1}{2};00} = \varphi_{34}\mathscr{B}_{34} = \frac{1}{(m_3m_4)^{1/2}}\Big[A\varphi_{34}^2 - \frac{B}{2}\,[(t-u)(m_3+m_4) + (m_1{}^2 - m_2{}^2)(m_3 - m_4)]\Big]$$

has no kinematical singularity.

With, $\lambda_3 = \frac{1}{2}$, $\lambda_4 = -\frac{1}{2}$,

$$\lambda = 0, \qquad \mu = 1, \qquad \eta_{12} = -1, \qquad \eta_{34} = +1,$$
$$m_{12} = -1, \qquad m_{34} = 0,$$
$$\mathscr{A}_{34} = \mathscr{B}_{12} = 2\hat{M}_{\frac{1}{2}-\frac{1}{2};00}, \qquad \text{and} \qquad \mathscr{B}_{34} = \mathscr{A}_{12} = 0,$$
$$F^2_{\frac{1}{2}-\frac{1}{2};00} = s^{1/2}\mathscr{S}_{12}^{-1}\psi_{34}^{-1}\mathscr{A}_{34} = \frac{B}{(m_3m_4)^{1/2}}$$

has no kinematical singularity (compare with table IV); $BB \to F\bar{F}$, $\eta = -1$).

B. *Particular Mass Cases*

(1) $m_1 = m_3$, $m_2 = m_4$, $m_1 \neq m_2$ (*πN elastic scattering*)

With $m_1 = m_3$, $m_2 = m_4$ Eq. (A-V.1) become

$$M_{\frac{1}{2}0;\frac{1}{2}0} = \cos\left(\frac{\theta_s}{2}\right)\Big[A + B\frac{s - m_1{}^2 - m_2{}^2}{2}\Big], \qquad E = \omega_1 = \omega_3, \quad \omega = \omega_2 = \omega_4,$$

$$M_{\frac{1}{2}0;-\frac{1}{2}0} = \sin\left(\frac{\theta_s}{2}\right)\Big[A\frac{E}{m_1} + B\omega\Big],$$

$$\hat{M}_{\frac{1}{2}0;\frac{1}{2}0} = A + B\frac{s - m_1{}^2 - m_2{}^2}{2},$$

and

$$s^{1/2}\hat{M}_{\frac{1}{2}0;-\frac{1}{2}0} = s^{1/2}\left[A\frac{E}{m_1} + B\omega\right]$$

are kinematically regular (Table VI).

(2) $m_1 = m_2 = m,\ m_3 \neq m_4$

With $m_1 = m_2 = m$, Eq. (A-V.2) become

$$M_{\frac{1}{2}\frac{1}{2};00} = \frac{1}{(4m_3m_4)^{1/2}}\left[A\varphi_{34} - B\psi_{34}(s-4m^2)^{1/2}\frac{\cos\theta_s}{s^{1/2}}(m_3+m_4)\right],$$
$$M_{\frac{1}{2}\frac{1}{2};00} = \frac{B(s-4m^2)^{1/2}\sin\theta_s\psi_{34}}{4m_3m_4}.$$

(i) With $\lambda_3 = \frac{1}{2}, \lambda_4 = \frac{1}{2}, \lambda_1 + \lambda_2 + \mu$ is even, $\mathscr{B}_{12} = 0$, and $\mathscr{A}_{12} = 2\hat{M}_{\frac{1}{2}\frac{1}{2};00} = 2M_{\frac{1}{2}\frac{1}{2};00}$ is regular at $s = 0$ since $\cos\theta_s \propto s^{1/2}$.

(ii) With $\lambda_3 = \frac{1}{2}, \lambda_4 = -\frac{1}{2}, \lambda_1 + \lambda_2 + \mu$ is odd, $\mathscr{A}_{12} = 0$, and

$$\mathscr{B}_{12} = 2\hat{M}_{\frac{1}{2}-\frac{1}{2};00} = [-B(s-4m^2)^{1/2}\,\psi_{34}]/(m_3m_4)^{1/2}$$

is regular at $s = 0$ (Table VII).

(3) $m_1 = m_2 = m,\ m_3 = m_4 = m',\ m \neq m'$ ($\pi\pi \to N\bar{N}$)

With $m_1 = m_2 = m,\ m_3 = m_4 = m'$, Eq. (A-V.2) become

$$M_{\frac{1}{2}\frac{1}{2};00} = (p/m')\,A - k\cos\theta_s B,$$
$$M_{\frac{1}{2}-\frac{1}{2};00} = (ks^{1/2}/2m')\sin\theta_s B,$$

$$\hat{M}_{\frac{1}{2}\frac{1}{2};00} = M_{\frac{1}{2}\frac{1}{2};00}\left(\sum_i \lambda_i \text{ odd},\quad \eta = -1\right) \quad \text{is regular at} \quad s = 0,$$

$$\frac{1}{s^{1/2}}\hat{M}_{\frac{1}{2}-\frac{1}{2};00} = \frac{kB}{m'}\left(\sum_i \lambda_i \text{ even},\quad \eta = -1\right) \quad \text{is regular at} \quad s = 0 \qquad \text{(Table IX).}$$

(4) $m_1 = m_2 = m_3 = m_4 = m$ (nucleon–antinucleon elastic scattering)

Volkov and Gribov (*17*) have written the five independent helicity amplitudes as functions of the invariant amplitudes, $H_i (i = 1, 5)$ (Scalar, vector, tensor, axial vector, pseudoscalar):

$$M_{\frac{1}{2}\frac{1}{2};\frac{1}{2}\frac{1}{2}} = 4p^2H_1 - 4m^2\cos\theta_sH_2 - 4m^2\cos\theta_sH_3 - 4m^2H_4 - sH_5,$$
$$M_{\frac{1}{2}\frac{1}{2};-\frac{1}{2}-\frac{1}{2}} = -4p^2H_1 + 4m^2\cos\theta_sH_2 + 4\left(\frac{s}{4} + p^2\right)\cos\theta_sH_3 - 4m^2H_4 - sH_5,$$

$$M_{\frac{1}{2}-\frac{1}{2};\frac{1}{2}-\frac{1}{2}} = -s(1+\cos\theta_s)\,H_2 - 4m^2(1+\cos\theta_s)\,H_3 + 4p^2(1+\cos\theta_s)\,H_4,$$

$$M_{\frac{1}{2}-\frac{1}{2};-\frac{1}{2}\,\frac{1}{2}} = s(1-\cos\theta_s)\,H_2 + 4m^2(1-\cos\theta_s)\,H_3 + 4p^2(1-\cos\theta_s)\,H_4,$$

$$M_{\frac{1}{2}\,\frac{1}{2};\frac{1}{2}-\frac{1}{2}} = -2\sin\theta_s[s^{1/2}mH_2 + s^{1/2}mH_3].$$

One verifies directly that

$$p^2 M_{\frac{1}{2}\,\frac{1}{2};\frac{1}{2}\,\frac{1}{2}},\ p^2 M_{\frac{1}{2}\,\frac{1}{2};-\frac{1}{2}-\frac{1}{2}},\ \left[\cos\left(\frac{\theta_s}{2}\right)\right]^{-2} M_{\frac{1}{2}-\frac{1}{2};\frac{1}{2}-\frac{1}{2}},\ \left[\sin\left(\frac{\theta_s}{2}\right)\right]^{-2} M_{\frac{1}{2}-\frac{1}{2};-\frac{1}{2}\,\frac{1}{2}},$$

$$\text{and} \qquad (s^{1/2})^{-1}\sin\theta_s^{-1} M_{\frac{1}{2}\,\frac{1}{2};\frac{1}{2}-\frac{1}{2}}$$

have no kinematical singularities (Table VII).

II. Application and Test of the Crossing Matrix to πN Scattering

We want to write the crossing matrix between the amplitudes for the reactions

$$N_1 + \pi_2 \to N_3 + \pi_4, \qquad \text{channel } s,$$

$$\pi_4 + \pi_2 \to N_3 + \bar{N}_1, \qquad \text{channel } t.$$

In this case $\sigma(P) = 0$, and looking at Table XI, we get:

$$M^{(s)}_{\lambda_3 0;\lambda_1 0}(s,t,u) = e^{-i\pi\lambda_3} \sum_{\lambda_1'\lambda_3'} d^{1/2}(\chi_1)^{\lambda_1'}_{\lambda_1} d^{1/2}(\chi_3)^{\lambda_3'}_{\lambda_3} M^{(t)}_{\lambda_3'\lambda_1';00}(s,t,u),$$

$$\cos\chi_1 = -\frac{(s+m^2-\mu^2)\,t}{\mathscr{S}\mathscr{T}}, \qquad \sin\chi_1 = \frac{2m\Phi^{1/2}}{\mathscr{S}\mathscr{T}},$$

$$\cos\chi_3 = -\cos\chi_1, \qquad \sin\chi_3 = -\sin\chi_1,$$

with $\mathscr{S} = \mathscr{S}_{12} = \mathscr{S}_{34}$; $\mathscr{T} = \mathscr{T}_{31} = [t(t-4m^2)]^{1/2}$.

$\mathscr{T}$ is negative on the s-physical region, so that χ_1 is negative and thus $\chi_3 = \chi_1 + \pi$. Using parity conservation yields

$$M^{(s)}_{\frac{1}{2}0;\frac{1}{2}0} = M^{(s)}_{-\frac{1}{2}0;-\frac{1}{2}0}, \qquad M^{(s)}_{\frac{1}{2}0;-\frac{1}{2}0} = -M^{(s)}_{-\frac{1}{2}0;\frac{1}{2}0};$$

$$M^{(t)}_{\frac{1}{2}\,\frac{1}{2};00} = -M^{(t)}_{-\frac{1}{2}-\frac{1}{2};00}, \qquad M^{(t)}_{\frac{1}{2}-\frac{1}{2};00} = M^{(t)}_{-\frac{1}{2}\,\frac{1}{2};00}.$$

The crossing matrix then reads

$$M^{(s)}_{\frac{1}{2}0;\frac{1}{2}0} = i[\sin\chi_1 M^{(t)}_{\frac{1}{2}\,\frac{1}{2};00} - \cos\chi_1 M^{(t)}_{\frac{1}{2}-\frac{1}{2};00}],$$

$$M^{(s)}_{\frac{1}{2}0;-\frac{1}{2}0} = i[\cos\chi_1 M^{(t)}_{\frac{1}{2}\,\frac{1}{2};00} + \sin\chi_1 M^{(t)}_{\frac{1}{2}-\frac{1}{2};00}].$$

Check of the Crossing Relation

The standard way of writing the crossing matrix for πN scattering consists in eliminating the invariant functions A and B between the two equations which give $M^{(s)}_{\lambda_3 0;\lambda_1 0}$ and $M^{(t)}_{\lambda'_3\lambda'_1;00}$ (Section A-V.1). After some algebraic manipulations, we get

$$M^{(s)}_{\frac{1}{2}0;\frac{1}{2}0} = \frac{m}{p_t}\cos(\theta_s/2)\, M^{(t)}_{\frac{1}{2}\frac{1}{2};00} - \frac{E_s\sin(\theta_s/2)}{p_t} M^{(t)}_{\frac{1}{2}-\frac{1}{2};00},$$

$$M^{(s)}_{\frac{1}{2}0;-\frac{1}{2}0} = \frac{E_s}{p_t}\sin(\theta_s/2)\, M^{(t)}_{\frac{1}{2}\frac{1}{2};00} + \frac{m}{p_t}\cos(\theta_s/2)\, M^{(t)}_{\frac{1}{2}-\frac{1}{2};00}.$$

Now

$$\frac{m}{p_t}\times\cos(\theta_s/2) = \frac{2mt^{1/2}}{\mathscr{T}}\times\frac{i\Phi^{1/2}}{\mathscr{S}t^{1/2}} = i\sin\chi_1$$

and

$$\frac{E_s\sin(\theta_s/2)}{p_t} = \frac{(s+m^2-\mu^2)\,2t^{1/2}\,|\,t\,|^{1/2}\,s^{1/2}}{2s^{1/2}\mathscr{T}\mathscr{S}} = -i\,\frac{(s+m^2-\mu^2)\,t}{\mathscr{S}\mathscr{T}} = i\cos\chi_1,$$

so that our crossing matrix is correct in the πN case. In fact, such a comparison does not provide a test of the overall sign $\eta^{(s)}$ of the crossing matrix, since the relative sign between u and v spinors is arbitrary.

III. Tests of the Kinematical Constraints

(a) $\pi\pi \to N\overline{N}$

(i) $s = 4m_N^2$. With the help of invariant amplitudes we obtain (IV-1),

$$F^1_{++} + 2m_N pk\cos\theta_s F^2_{+-} = 0 \quad \text{at} \quad s = 4m_N^2 (4pk\cos\theta_s = t-u).$$

From (IV-12) the constraint reads

$$p^{s_3+s_4}T^s_{\tau_3\tau_4} \simeq p^{s_3+s_4+\tau_3+\tau_4} \text{ in a vicinity of } s = 4m_N^2$$

if

$$e^{i\theta_s} \simeq p^2. \qquad \text{(A-V.3)}$$

Hence the constraint is $pT^s_{++} \simeq p^2$; i.e., $p(M^s_{++} - iM^s_{+-}) \simeq p^2$, or in terms of F^1_{++}, F^2_{+-} defined by $pM^s_{++} = F^1_{++}$ and $pM^s_{+-} = F^2_{+-}s^{1/2}pk\sin\theta_s$,

$$F^1_{++} - is^{1/2}pk\sin\theta_s F^2_{+-} = 0 \quad \text{at} \quad s = 4m_N^2. \qquad \text{(A-V.4)}$$

But in this vicinity, $ipk \sin\theta_s \simeq -pk\cos\theta_s$ and $s^{1/2} = 2m_N$. So, (A-V.4) reads

$$F^1_{++} + 2m_N pk \cos\theta_s F^2_{+-} = 0 \qquad \text{at} \qquad s = 4m_N{}^2.$$

Remark. In a vicinity of $s = 4m_N{}^2$ where $e^{i\theta_s} \simeq p^{-2}$, the constraint would be written $pT^s_{--} \simeq p^2$, i.e., $F^1_{++} + is^{1/2}pk \sin\theta_s \simeq 0$ at $s = 4m_N{}^2$ (A-V.5) but now $ipk \sin\theta_s \simeq pk\cos\theta_s$ and (A-V.5) is identical to (A-V.3).

(ii) $s = 0$. Since Max $|\lambda' - \mu'| = 1$, we deduce from (A-IV.6) that there is no constraint at $s = 0$, which agrees with the result obtained with the help of invariant amplitudes.

(b) $N\pi \to N\pi$

(i) $\mathscr{S} = 0$. From (IV.14) we obtain

$$\mathscr{S}T^s_{\tau_3;\tau_1} = \mathscr{S}^{s_1+s_3+\tau_1+\tau_3} \qquad \text{if} \quad e^{i\theta_s} \text{ behaves like } \mathscr{S}^2.$$

The constraint reads

$$\mathscr{S}T^s_{++} \simeq \mathscr{S}^2; \qquad \text{i.e.} \quad \mathscr{S}M^s_{++} + i\mathscr{S}M^s_{+-} \simeq \mathscr{S}^2.$$

The regularization (Table VI) allows us to write

$$M^s_{++} = F^1_{++}\cos\left(\frac{\theta_s}{2}\right) \qquad M^s_{+-} = \frac{F^1_{+-}}{s^{1/2}}\sin\left(\frac{\theta_s}{2}\right),$$

where F^1_{++} and F^1_{+-} are the two R.H.A.

In the neighborhood of $\mathscr{S} = 0$,

$$\mathscr{S}\cos\left(\frac{\theta_s}{2}\right) \simeq -i\mathscr{S}\sin\left(\frac{\theta_s}{2}\right).$$

Thus the constraint reads

$$s^{1/2}F^1_{++} - F^1_{+-} \simeq \mathscr{S}^2. \qquad \text{(A-V.6)}$$

Now we can express F^1_{++} and F^1_{+-} as functions of A and B,

$$F^1_{++} = \hat{M}^s_{++} = A + \frac{B(s - m_3{}^2 - m_4{}^2)}{2m_3},$$

$$F^1_{+-} = s^{1/2}\hat{M}^s_{+-} = \frac{A}{2m_3}(s + m_3{}^2 - m_4{}^2) + \frac{B}{2}(s + m_4{}^2 - m_3{}^2)$$

and check easily that (A-V.6) holds both at $s^{1/2} = m_3 + m_4$ and $s^{1/2} = m_3 - m_4$.

Remark. The constraint is nothing but $\hat{M}^s_{++} = \hat{M}^s_{+-}$ both at threshold and at pseudo-threshold, as noticed by Jones [16].

(c) $\bar{N}N \to N\bar{N}$

(i) $s = 4m^2$. From (IV-14),

$$p^2 T^s_{\tau_3\tau_4;\tau_1\tau_2} \simeq p^{2+\tau_1+\tau_2+\tau_3+\tau_4}$$

if $e^{i\theta_s}$ behaves like p^2, (where $p^2 = s - 4m^2$); i.e.,

$$p^2 T^s_{++;++} \simeq p^4, \quad p^2 T^s_{++;--} \simeq p^2, \quad p^2 T^s_{+-;+-} \simeq p^2, \quad \text{and} \quad p^2 T^s_{+-;-+} \simeq p^2;$$

(CPT conservation imply $T_{++;--} = T_{--;++}$, $T_{-+;-+} = T_{+-;+-}$, $T_{+-;-+} = T_{-+;+-}$). Let us define

$$\varphi_1 = M^s_{++;++}, \quad \varphi_2 = M^s_{++;--}, \quad \varphi_3 = M^s_{+-;+-},$$
$$\varphi_4 = M^s_{+-;-+}, \quad \text{and} \quad \varphi_5 = M^s_{++;+-}.$$

The constraints then read

$$p^2(\varphi_1 - \varphi_2) + p^2(\varphi_3 + \varphi_4) + 4ip^2\varphi_5 \simeq p^4, \tag{A-V.7}$$

$$-p^2(\varphi_1 - \varphi_2) + p^2(\varphi_3 + \varphi_4) \simeq p^2, \tag{A-V.8}$$

$$p^2(\varphi_1 + \varphi_2) + p^2(\varphi_3 - \varphi_4) \simeq p^2, \tag{A-V.9}$$

$$p^2(\varphi_1 + \varphi_2) - p^2(\varphi_3 - \varphi_4) \simeq p^2 \tag{A-V.10}$$

if $e^{i\theta_s} \simeq p^2$.

As already remarked (Section A-V.1),

$$p^2\varphi_1, \; p^2\varphi_2, \; \varphi_3/(1+z), \; \varphi_4/(1-z), \quad \text{and} \quad \varphi_5/s^{1/2}(1-z^2)^{1/2}$$

are kinematically regular (here, $z = \cos\theta_s$, hence $(1-z)/2 = -t/2p^2$, $1 + z = -u/2p^2$).

Check. Following Gribov and Volkov (*17*), the five invariant amplitudes can be expressed as functions of the five independent helicity amplitudes:

$$H_1 = \frac{1}{8p^4}\left[p^2(\varphi_1 - \varphi_2) + p^2 z\left(\frac{\varphi_3}{1+z} - \frac{\varphi_4}{1-z}\right) - \frac{p^2 z}{m}(s + 4m^2)\frac{\varphi_5}{s^{1/2}\sin\theta_s}\right],$$

$$H_2 = -\frac{1}{8p^2}\left[\frac{\varphi_3}{1+z} - \frac{\varphi_4}{1-z} - \frac{4m}{s^{1/2}}\frac{\varphi_5}{\sin\theta_s}\right],$$

$$H_3 = -\frac{1}{8p^2}\left[\frac{\varphi_3}{1+z} - \frac{\varphi_4}{1-z} - \frac{s}{m}\frac{\varphi_5}{s^{1/2}\sin\theta_s}\right],$$

$$H_4 = -\frac{1}{8p^2}\left[\frac{\varphi_3}{1+z} + \frac{\varphi_4}{1-z}\right],$$

$$H_5 = -\frac{1}{2sp^2}\left[p^2(\varphi_1+\varphi_2) - p^2 z\left(\frac{\varphi_3}{1+z} - \frac{\varphi_4}{1-z}\right)\right.$$
$$\left. + m^2\left(\frac{\varphi_3}{1+z} + \frac{\varphi_4}{1-z}\right) + \frac{zp^2S^n}{m}\frac{\varphi_5}{s^{1/2}\sin\theta_s}\right].$$

We first notice that

$$\frac{\varphi_3}{1+z} \pm \frac{\varphi_4}{1-z} = -2p^2\left(\frac{\varphi_3}{u} \pm \frac{\varphi_4}{t}\right) = -2p^2\left[\left(\frac{\varphi_3 \mp \varphi_4}{u}\right) + \varphi_3 O(p^2)\right].$$

If we suppose that H_4 is finite at $p^2 = 0$, we deduce

$$p^2(\varphi_3 - \varphi_4) \simeq p^2. \qquad \text{(A-V.11)}$$

Furthermore,

$$p^2 z = -\frac{u+2p^2}{2} = \frac{t+2p^2}{2}.$$

Hence

$$p^2 z\left(\frac{\varphi_3}{1+z} - \frac{\varphi_4}{1-z}\right) = p^2(\varphi_3+\varphi_4) + 2p^2\left(\frac{\varphi_3}{u} + \frac{\varphi_4}{t}\right)p^2;$$

on the other hand,

$$p^2 \sin\theta_s \simeq +ip^2\cos\theta_s.$$

Hence

$$-p^2 z\frac{(s+4m^2)\,\varphi_5}{ms^{1/2}\sin\theta_s} = 4ip^2\varphi_5 + p^2\varphi_5 O(p^4).$$

H_1 now becomes

$$H_1 = \frac{1}{8p^4}\left[p^2(\varphi_1-\varphi_2) + p^2(\varphi_3+\varphi_4) + 4ip^2\varphi_5 + 2p^2\left(\frac{\varphi_3}{u}+\frac{\varphi_4}{t}\right)\right] + \frac{ip^2\varphi_5 O(p^4)}{8p^4}.$$

In order to avoid a pole in H_1, at $p^2 = 0$ we must impose

$$p^2(\varphi_1-\varphi_2) + p^2(\varphi_3+\varphi_4) + 4ip^2\varphi_5 + 2p^2\left(\frac{\varphi_3}{u}+\frac{\varphi_4}{t}\right)p^2 \simeq p^4.$$

With the help of (A-V.11) we get

$$p^2(\varphi_1 - \varphi_2) + p^2(\varphi_3 + \varphi_4) + 4ip^2\varphi_5 \simeq p^4. \quad \text{(A-V.12)}$$

On the other hand, the constraint

$$p^2(\varphi_3 + \varphi_4) + 2ip^2\varphi_5 \simeq p^2 \quad \text{(A-V.13)}$$

is necessary to cancel poles both in H_2 and H_3.

For H_5 the constraint reads

$$p^2(\varphi_1 + \varphi_2) - [p^2(\varphi_3 + \varphi_4) + 2ip^2\varphi_5] \simeq p^2$$

and with (A-V.13) we deduce

$$p^2(\varphi_1 + \varphi_2) \simeq p^2. \quad \text{(A-V.14)}$$

Eqs. (A-V.11)–(A-V.14) are obviously compatible with Eqs. (A-V.7)–(A-V.10).

(ii) *Constraint at* $s = 0$. From (A-IV.6) we deduce that there is a constraint if $\frac{1}{2}(|\lambda' - \mu'|)$ is an integer. The maximum value of $|\lambda' - \mu'|$ is 2 and is reached when $\lambda' = \lambda_4' - \lambda_2' = \pm 1$ and $\mu' = \lambda_3' - \lambda_1' = \mp 1$; i.e., for instance when $\lambda_1' = \lambda_4' = -\frac{1}{2}$ and $\lambda_3' = \lambda_2' = +\frac{1}{2}$.

The constraint reads

$$\sum d^{1/2}(-\chi_1)^{\lambda_1}_{-1/2}\, d^{1/2}(\chi_2)^{\lambda_2}_{+1/2}\, d^{1/2}(\chi_3)^{\lambda_3}_{+1/2}\, d^{1/2}(-\chi_4)^{\lambda_4}_{-1/2}\, M^{s}_{\lambda_3\lambda_4;\lambda_1\lambda_2} \simeq s,$$

where

$$\cos\chi_1 = -\cos\chi_2 = -\cos\chi_3 = \cos\chi_4 = \cos\chi = -\frac{s^{1/2}t^{1/2}}{[(s-4m^2)(t-4m^2)]^{1/2}}$$

and

$$\sin\chi_1 = \sin\chi_2 = -\sin\chi_3 = -\sin\chi_4 = \sin\chi = \frac{2m(4m^2 - s - t)^{1/2}}{[(s-4m^2)(t-4m^2)]^{1/2}},$$

so that $\chi_1 = \chi$, $\chi_2 = \pi - \chi$, $\chi_3 = \chi - \pi$, $\chi_4 = -\chi$. We now obtain

$$(2 - \sin^2\chi)\,\varphi_1 + \sin^2\chi\varphi_2 - \sin^2\chi\varphi_3 - \sin^2\chi\varphi_4 \simeq s.$$

Noticing that at $s = 0$, $\cos^2\chi = 0$, and $\sin^2\chi = +1$, the constraint takes up the following form:

$$\varphi_1 + \varphi_2 - \varphi_3 - \varphi_4 = 0 \quad \text{at} \quad s = 0. \quad \text{(A-V.15)}$$

Check:

$$H_5 = -\frac{1}{2s}\left\{\varphi_1 + \varphi_2 + \frac{\varphi_3}{1+z}\left(-z + \frac{m^2}{p^2}\right) + \frac{\varphi_4}{1-z}\left(z + \frac{m^2}{p^2}\right)\right\}$$

$$+ \text{ finite term at } s = 0.$$

Noticing that

$$\frac{\varphi_3}{p^2(1+z)}(-p^2 z + m^2) = -\varphi_3\left(1 + \frac{s}{2u}\right),$$

$$\frac{\varphi_4}{p^2(1-z)}(p^2 z + m^2) = -\varphi_4\left(1 + \frac{s}{2t}\right),$$

we must have the same condition (A-V.15): $\varphi_1 + \varphi_2 - \varphi_3 - \varphi_4 = 0$ at $s = 0$ in order to avoid a pole in H_5 at $s = 0$.

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