ANNALS OF PHYSICS: **37**, 77–103 (1966)

Many-Channel Dynamics, Levinson's Theorem for Eigenamplitudes and One-Channel CDD Poles

D. ATKINSON, K. DIETZ, AND D. MORGAN*

CERN, Geneva, Switzerland

Multichannel dynamical amplitudes can have *CDD* poles when viewed as the solution of the appropriate one-channel inelastic problem. Manifestations of this phenomenon are investigated. The technique is to work via the eigenamplitudes, in terms of which the essential many-channel dynamics take a simple form; for them, individual Levinson's theorems are derived. On applying the method, it is found that, for a wide class of cases, the phenomenon always occurs in at least one channel and, in many cases, further analysis enables one to say in which one. A speculative application is made to SU_3 bootstraps.

I. INTRODUCTION

It has recently been pointed out (*1*) that the results of dynamical calculations in a single channel with prescribed inelasticity do not always agree with those of a related many-channel calculation. The purpose of this paper is to present an analysis of this phenomenon.[1]

A precise statement of the effect can be given as follows: suppose that t_{11} is the amplitude and η the inelasticity in channel one of a many-channel system. Then, if η and the left-hand discontinuity of t_{11} are used to calculate an amplitude, t, in a dynamic one-channel inelastic calculation, in general $t \neq t_{11}$. To get equality, one has to introduce a *CDD* pole.

Such considerations clearly have important bearing on the hypothesis that all particles of high energy physics are composite, for compositeness is now seen to be a relative concept. One has the choice of asking: Is such and such a particle a dynamical bound state or resonance in this or that set of channels? or, alternatively: Is it dynamic in any set of channels? The possibility opens that particle democracy, although formally existing, might in practice display oligarchic tendencies, and some particles be more dynamic than others. Every bootstrap hypothesis embodies a choice of channels in which the dynamical calculation is to be done.

* On leave of absence from the Rutherford High Energy Laboratory, Chilton, Berks., England.

[1] The present paper is an extension of our CERN preprint.

In the discussion to follow, a one-channel inelastic calculation will generally be taken to mean an application of the method of Frye and Warnock (*2*) (FW). This employs an N/D representation in which $\arg D = -\text{Re}\,\delta$. For brevity, the phrases failure of FW and success of FW will be used to designate the situations where respectively there are or are not one-channel *CDD* poles in t_{11}. Some brief comments will also be made on one-channel calculations using the method of Chew and Mandelstam (*3*) (CM). This is the appropriate method when the input information on the omitted channels is not η but $R = \sigma_{\text{tot}}/\sigma_{\text{el}}$. It should not be thought that the FW method receives more attention because it has more troubles: rather, it is because the troubles are easier to chart. It should be remarked that in this paper, as elsewhere (*1*), the underlying many-channel situation studied consists of n two-body channels.

Several analyses of the one-channel many-channel nonequivalence have appeared.[2] The present one is based on the use of eigenamplitudes and repeated application of Levinson's theorem as a criterion of dynamicality. A chain of argument is developed leading from dynamical many-channel solutions to dynamic eigenamplitudes and so to the question whether the physical one-channel amplitudes are in turn dynamical. The latter are expressible as linear sums of eigenamplitudes and their phase changes from threshold to infinite energy constitute the criterion. The problem is thus reduced to a study of the phase development of a vector sum of individual vectors with prescribed phase behavior. Sufficient conditions for success and failure are stated which essentially say that, if one of the individual vectors is always longer than the rest, then its phase behavior predominates. (This result, which is stated precisely in Section IV, will be termed the crank-shaft theorem.) The general conclusion is that failure results when there is a many-channel bound state or resonance and the physical channel under consideration is insufficiently strongly coupled to the resonating eigenamplitude. That is not to say that the phenomenon should be thought of as a weak coupling effect. In fact, the interchannel coupling can be large.

The arrangement of the rest of the paper is as follows. In Section II some simple examples are presented which show how failure can arise. These fall into two classes—weak coupling examples which furnish an immediate intuitive feeling for the phenomenon and, second, instances with coincident thresholds. These latter serve as an introduction both as to the general results and methods which follow. The next four Sections are devoted to establishing the general techniques. In Section III, the Levinson theorem for one-channel (FW) and (CM) amplitudes is discussed. In Section IV, the diagonalization of the many-channel S

[2] In addition to ref. *1* also preprints by J. B. Hartle and C. E. Jones, (Princeton University), P. Hertel (Heidelberg University), and Takeshi Kanki (Purdue University) have appeared during the preparation of the paper.

matrix is presented and in Section V a form of Levinson's theorem for the eigenamplitudes is derived. These results are then applied in Section VI to derive the crank-shaft theorem—a general criterion of the success or failure of FW calculations. Throughout this portion of the paper, the two-channel case is treated in generality and, whenever possible, results are proved for n channels with $n \geqq 2$. In particular, in contrast to the second set of examples of Section II, distinct thresholds are assumed. This allows in general a rich complexity in the relations between the eigenamplitudes and the physical amplitudes and, as a result, less detailed statements can be made than for the equal threshold case. In Section VII, it is shown that under a regime of moderate interchannel coupling (moderate in the sense of not too strong), the unequal threshold case is not qualitatively different from that of coincident thresholds and stronger statements can be made. The weak coupling example of Section II is then re-analyzed from the eigenamplitude viewpoint. The crank-shaft analysis for the CM method is sketched in Section VIII, and this concludes the general discussion. In Section IX, a speculative application of the present results to SU_3 symmetric bootstraps is outlined. In the final section, some comparisons are made with other work.

II. EXAMPLES OF THE FAILURE OF FW

In this section, certain specific examples are given in which an attempt to calculate a bound state by the FW method would fail. The reason for introducing these examples here is that they can be presented without the detailed formalism which will be developed in later Sections, and which will enable certain general criteria of failure to be studied.

The technique, in these examples, will be to suppose a two-channel system to be dynamically soluble by a matrix N/D method, in which specified bound states occur, but no CDD poles. The effective inelasticity, η, could then be calculated. A FW system could be set up, in which the left-hand discontinuity of t_{11}, the amplitude in channel one, is supplied, together with the inelasticity η. The question is whether the solution of the CDD pole free FW equations agrees, or not, with t_{11}. It can fail to do so if t_{11}, considered as a one-channel FW amplitude, has one or more CDD poles. The examples of failure will be cases in which t_{11} has no CDD pole in the two channel system, but has one such pole in the one-channel FW calculation.

In Section III, it will be shown that if certain conditions are satisfied by the discontinuity of t_{11} and by η, the FW solution satisfies Levinson's theorem, in the form

$$\mathrm{Re}\, \delta_{11}(\infty) = \pi(n_C - n_B) \tag{2.1}$$

where $\delta_{11}(s)$ is the phase shift, as a function of s, the energy squared, n_C is the number of CDD poles, and n_B the number of bound states. The convention

δ_{11} (threshold) $= 0$ is adopted. The criterion of a dynamical calculation is the absence of *CDD* poles, so that failure of such a calculation is equivalent to $n_C \neq 0$, that is, to nonobservance of the relation

$$\text{Re}\,\delta_{11}(\infty) = -\pi n_B \tag{2.2}$$

Accordingly, examples will be sought for which (2.2) is violated by the channel one amplitude.

As a first example, consider the case of weak interchannel coupling, in which there is a bound state in channel two. Then, writing

$$t = ND^{-1} \tag{2.3}$$

the channel one amplitude has the form

$$t_{11} = \frac{N_{11}D_{22} - N_{12}D_{21}}{D_{11}D_{22} - D_{12}D_{21}} \tag{2.4}$$

Since the channels are weakly coupled, one can write all the diagonal elements of N and D in the form

$$N_{11} = N_1 + O(\lambda^2) \quad \text{etc.} \tag{2.5}$$

where λ is some interchannel coupling parameter and the quantities N_1, D_1, etc., refer to the case of zero coupling. The off-diagonal elements will be of order λ. Then

$$t_{11} = N_1/D_1 + O(\lambda^2) \tag{2.6}$$

except in the neighborhood of zeros of D_1 and D_2. In the example, it is supposed that D_2 has a zero, corresponding to a bound state in channel two in the absence of interchannel coupling, but that D_1 has not. Let the position of this zero be $s = s_0$. Then both the numerator and denominator in (2.4) will have zeros at different values of s near s_0, when λ is small. If s_0 is below the threshold of channel one, the pole of t_{11} near s_0 will correspond to a bound state in channel one.

From (2.6) it can be seen that the phase of t_{11} will be close to that of $t_1 = N_1/D_1$ for small λ, and that for sufficiently small, but nonzero λ, the phase at infinity will be the same. However, there is no bound state in the uncoupled amplitude t_1, so that, by Levinson's theorem, its phase at infinity is zero. Hence

$$\text{Re}\,\delta_{11}(\infty) = 0 \tag{2.7}$$

but, since t_{11} has a bound state, the FW method would fail in channel one. The zero that was induced in the numerator of (2.4) is in fact a *CDD* pole of the one-channel amplitude. If s_0 is above the channel one threshold, t_{11} has a resonance, but no return of Re δ_{11} through $\pi/2$, and the FW method fails again. This example will be examined further in Section VII.

As a second example, consider the case in which the thresholds are coincident. It will not be necessary in this case to make a weak coupling assumption. The S matrix can be diagonalized in the physical region in terms of eigenphase-shifts by means of an energy dependent orthogonal similarity transformation. Thus, in particular,

$$\eta e^{2i\delta_{11}} \equiv S_{11} = \alpha^2 e^{2i\delta^{(1)}} + \beta^2 e^{2i\delta^{(2)}} \tag{2.8}$$

where $\alpha^2 + \beta^2 = 1$, and $\delta^{(i)}(s)$ are the eigenphase-shifts. The relation (2.8) holds in the physical region: but it can be continued into the complex s plane, a procedure that will be studied in detail in Section IV. A form of Levinson's theorem for the eigenamplitudes will be proved in Section V.

Suppose again that there is a bound state in eigenchannel two only. Thus, by Levinson's theorem

$$\begin{aligned} \delta^{(1)}(\infty) &= 0 \\ \delta^{(2)}(\infty) &= -\pi \end{aligned} \tag{2.9}$$

Clearly, S_{11} has a bound state, so that success or failure of the FW method depends on whether Re $\delta_{11}(\infty) = -\pi$ or Re $\delta_{11}(\infty) = 0$. From (2.8), it can be seen that $\alpha^2(s) < \frac{1}{2}$ for all physical s implies success and $\alpha^2(s) > \frac{1}{2}$ implies failure. This fact can be visualized by thinking of $\eta e^{2i\delta_{11}}$ as the sum of a long and a short vector.

This situation can be realized in a simple example by replacing the left-hand cut of each t_{ij} by a pole with residue Γ_{ij} and common position. Then the parameters of the diagonalization are

$$\frac{\alpha\beta}{\alpha^2 - \frac{1}{2}} = \frac{2\Gamma_{12}}{\Gamma_{22} - \Gamma_{11}} \tag{2.10}$$

and

$$\Gamma^{(1,2)} = \frac{\Gamma_{11} + \Gamma_{22}}{2} \mp \left[\left(\frac{\Gamma_{11} - \Gamma_{22}}{2}\right)^2 + \Gamma_{12}^2\right]^{1/2} \tag{2.11}$$

where $\Gamma^{(i)}$ are the residues of the corresponding poles of the eigenamplitudes. For coincident thresholds, the numbering of the eigenamplitudes is arbitrary and the convention $\Gamma^{(2)} > \Gamma^{(1)}$ will be adopted. Suppose that $\Gamma^{(2)}$ is large enough to produce a bound state, but that $\Gamma^{(1)}$ is not. Then a striking feature of (2.10) is that success or failure of FW for physical channel one (and the converse for channel two) depends simply on $\Gamma_{11} \gtrless \Gamma_{22}$. This condition does not require that the off-diagonal residue Γ_{12} be small. Hence, the example provides a contradiction for success given in BCS, namely, that Γ_{22} should not be strong enough to produce a bound state when $\Gamma_{12} = 0$. One can check that the resulting amplitudes do not have complex poles on the physical sheet.

Certain simple examples of the phenomenon of failure of an inelastic calculation have been adduced in this section in the two cases:

(a) distinct thresholds but weak coupling;

(b) coincident thresholds and possibly strong coupling.

The case of strong coupling with distinct thresholds is more complicated: and statements will be made in this more general situation in Section VI.

III. LEVINSON'S THEOREM WITH INELASTICITY

As has been noted in the Introduction, the method of calculating a one-channel inelastic scattering amplitude depends on the input information available. If one is provided with the inelasticity factor, given by

$$R = \frac{\sigma_{\text{tot}}}{\sigma_{\text{el}}} \tag{3.1}$$

then the relevant equations are those of Chew and Mandelstam (*3*). Here σ_{tot} is the total and σ_{el} the elastic cross section. If the effect of the other channels is represented by the inelasticity

$$\eta(s) = e^{-2\text{Im}\delta_l(s)} = \sqrt{1 - \frac{k^2\sigma_r(s)}{\pi(2l+1)}} \tag{3.2}$$

then one can use either the method of Frye and Warnock (*2*) or that of Froissart (*4*). Here $\delta_l(s)$ is the phase shift of the lth partial wave, which becomes complex above the lowest inelastic threshold, k is the cms momentum, and $\sigma_r(s)$ is the reaction cross section.

In this section, the methods of CM and FW will be considered in some detail. In fact, it will prove easier to state conditions for success or failure of a dynamical calculation in the FW system (Section VI) than for the CM system (Section VIII).

The problem to be considered is the set of conditions under which the dynamical calculation of an amplitude is possible, given the left-hand cut discontinuity and, in the one case $R(s)$, in the other $\eta(s)$. A calculation breaks down if a *CDD* pole occurs, for then the parameters of the pole are not fixed by the input information. The investigation therefore concerns the occurrence or absence of *CDD* poles. In this section, a form of Levinson's theorem that holds in the absence of *CDD* poles will be discussed, so that the remainder of the work is reduced to asking whether Levinson's theorem, in this form, is violated.

In the CM method, the partial wave amplitude $t(s)$ is written in the form

$$t(s) = N(s)/D(s) \tag{3.3}$$

in which $D(s)$ has the right-hand cut, and on which its phase is minus that of $t(s)$. Then $N(s)$ has only the left-hand cut. The CM equations lead to the following integral equation for $N(s)$:

$$N(s) = B(s) + \frac{1}{\pi}\int_{s_1}^{\infty} ds' \frac{B(s') - B(s)}{s' - s} \rho(s')R(s')N(s') \tag{3.4}$$

where

$$B(s) = \frac{1}{\pi}\int_{-\infty}^{0} \frac{\operatorname{Im} t(s')}{s' - s} ds' \tag{3.5}$$

here $\rho(s) = \sqrt{(s - s_1)/s}$ and s_1 is the threshold of the channel, supposed lower than any of the inelastic thresholds. If (3.4) has a solution, $D(s)$ is then calculable from

$$D(s) = 1 - \frac{1}{\pi}\int_{s_1}^{\infty} ds' \frac{\rho(s')R(s')N(s')}{s' - s} \tag{3.6}$$

Equations (3.4)-(3.6) have been written without *CDD* poles; and a necessary condition for them to be dynamical is that a unique solution exists.

In the FW method, on the other hand, a decomposition

$$t(s) = \bar{N}(s)/\bar{D}(s) \tag{3.7}$$

is made, in which $\bar{D}(s)$ carries the right-hand cut, as before, but on which its phase is $-\operatorname{Re}\delta_l(s)$. Then $\bar{N}(s)$ has both a right and a left-hand cut. FW derive the following integral equation for $\operatorname{Re}\bar{N}(s)$:

$$\frac{2\eta(s)}{1 + \eta(s)} \operatorname{Re}\bar{N}(s) = \bar{B}(s) + \frac{1}{\pi}\int_{s_1}^{\infty} ds' \frac{\bar{B}(s') - \bar{B}(s)}{s' - s} \rho(s') \frac{2\operatorname{Re}\bar{N}(s')}{1 + \eta(s')} \tag{3.8}$$

where

$$\bar{B}(s) = \frac{1}{\pi}\int_{-\infty}^{0} ds' \frac{\operatorname{Im} t(s')}{s' - s} + \frac{P}{\pi}\int_{s_1}^{\infty} ds' \frac{1 - \eta(s')}{2\rho(s')(s' - s)} \tag{3.9}$$

Then $\bar{D}(s)$ is given in terms of the solution of (3.8) by:

$$\bar{D}(s) = 1 - \frac{1}{\pi}\int_{s_1}^{\infty} ds' \frac{\rho(s')}{s' - s} \frac{2\operatorname{Re} N(s')}{1 + \eta(s')} \tag{3.10}$$

Again, for a dynamical system, the *CDD* pole free equations (3.8)-(3.10) must have a unique solution.

The form of Levinson's theorem that is relevant to a discussion of a dynamical CM system is

$$\varphi(\infty) - \varphi(s_1) = -\pi n_B \tag{3.11}$$

where $\varphi(s) = \arg t(s)$, and n_B is the number of zeros of $D(s)$ on the physical sheet. For a FW system, the Levinson theorem is

$$\mathrm{Re}\,\delta(\infty) - \delta(s_1) = -\pi\bar{n}_{\mathrm{B}} \tag{3.12}$$

where $\delta(s)$ is the phase shift, and $\bar{n}_{\mathrm{B}}$ is the number of zeros of $\bar{D}(s)$. The integer n_{B}, or $\bar{n}_{\mathrm{B}}$, is interpreted as the number of dynamical bound states in the CM, or FW methods, respectively.

For (3.11) and (3.12) to hold, certain conditions must be satisfied by the input quantities. The domains of validity of the two equations are not coextensive, although in many simple models both equations are satisfied (with $n_{\mathrm{B}} = \bar{n}_{\mathrm{B}}$). In a forthcoming publication (*5*), certain sufficient conditions will be given under which $R(s) \to \infty$ and $\eta(s) \to 0$ as $s \to \infty$ (*6*). In general, however, this involves introducing a subtraction into the N/D equations, and the consequent necessity of specifying a subtraction constant. In this paper it will be sufficient to restrict attention to the cases $R(s) \to 1$ and $1 - \eta(s) \to 0$, since these relations are satisfied by all the examples considered (in which only a finite number of channels contribute).

Sufficient conditions for the observance of (3.11) are:

$$\text{(a)} \qquad |B(s)| < Cs^{-\epsilon-1/2} \quad \text{for} \quad s_1 \leqq s < \infty, \quad \epsilon > 0 \tag{3.13}$$

$$\text{(b)} \qquad \lim_{s\to\infty} R(s) = 1$$

C is a constant. On the other hand, conditions under which (3.12) holds are:

$$\left.\begin{aligned} &(\bar{\mathrm{a}}) \qquad |\bar{B}(s)| < \bar{C}s^{-\bar{\epsilon}-1/2} \quad \text{for} \quad s_1 \leqq s < \infty, \quad \bar{\epsilon} > 0 \\ &(\bar{\mathrm{b}}) \qquad |1 - \eta(s)| < \bar{D}s^{-\bar{\eta}} \qquad\qquad\qquad\qquad \bar{\eta} > 0 \end{aligned}\right. \tag{3.14}$$

where $\bar{C}$, $\bar{D}$ are constants.

IV. DIAGONALIZATION OF THE S MATRIX

In this section the problem of diagonalizing a many-channel scattering matrix will be considered. Suppose that S is the scattering matrix which is already diagonal in all the conserved quantities (total angular momentum, isospin, etc.). Then the matrices T and t will be defined by

$$S = 1 + 2iT = 1 + 2i\rho^{1/2}t\rho^{1/2} \tag{4.1}$$

Here ρ is the phase space matrix

$$\rho_{ik} = \delta_{ik}\left(\frac{s - s_k}{s}\right)^{1/2} \tag{4.2}$$

where $\sqrt{s_k}$ is the threshold energy of the kth channel.

In this paper, only a finite number of two-body channels will be considered. Moreover, the formalism will be displayed for the simple case of two channels, although it is readily generalized to the n-channel case.

Suppose that there are two channels and two thresholds, $s = s_1$ and $s = s_2$, in general distinct. The matrix T can be diagonalized above the higher threshold, s_2, by a real, orthogonal, energy dependent matrix O (7). Thus one can write

$$T_D \equiv \begin{pmatrix} T^{(1)} & O \\ O & T^{(2)} \end{pmatrix} = OTO^{-1} = O\rho^{1/2} \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} \rho^{1/2} O^{-1} \tag{4.3}$$

for $s \geqq s_2$ where

$$O = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \quad \text{and} \quad \alpha^2 + \beta^2 = 1 \tag{4.4}$$

The quantities $T^{(i)}$ and the related $S^{(i)}(= 1 + 2iT^{(i)})$ have a key-role throughout the paper. It is of interest to display the physical matrix elements t_{ij} in terms of the diagonalized quantities $T^{(1)}$ and $T^{(2)}$. Equation (4.3) gives

$$\begin{aligned} \rho_1 t_{11} &= \alpha^2 T^{(1)} + \beta^2 T^{(2)} \\ \sqrt{\rho_1\rho_2}\, t_{12} &= \alpha\beta(T^{(1)} - T^{(2)}) \\ \rho_2 t_{22} &= \beta^2 T^{(1)} + \alpha^2 T^{(2)} \end{aligned} \tag{4.5}$$

The elements of the diagonalizing matrix O are given in terms of the t matrix elements by

$$\begin{aligned} \alpha^2(s) &= \tfrac{1}{2}(1 + [1 + K^2(s)]^{-1/2}) \\ \beta^2(s) &= \tfrac{1}{2}(1 - [1 + K^2(s)]^{-1/2}) \end{aligned} \tag{4.6}$$

where the function $K(s)$ satisfies

$$K(s) = \frac{2\sqrt{\rho_1\,\rho_2}\, t_{12}^{(s)}}{\rho_1\, t_{11}(s) - \rho_2\, t_{22}^{(s)}} \tag{4.7}$$

The eigenamplitudes $T^{(i)}(s)$ obey a very simple unitary condition above the higher threshold

$$\operatorname{Im} T^{(i)} = |\, T^{(i)} \,|^2 \qquad i = 1, 2, \quad s \geqq s_2 \tag{4.8}$$

so that one can write

$$T^{(i)} = e^{i\delta(i)} \sin \delta^{(i)} \qquad i = 1, 2, \tag{4.9}$$

where the $\delta^{(i)}(s)$, which will be termed "eigenphase-shifts," are real for $s_2 \leqq s < \infty$.

The analytic properties of the $T^{(i)}(s)$ are not always as simple as those of the $T_{ij}(s)$. For (4.5) gives

$$\begin{aligned} T^{(1)} &= \rho_1 t_{11} + \beta/\alpha\sqrt{\rho_1\,\rho_2}\, t_{12} \\ T^{(2)} &= \rho_2 t_{22} - \beta/\alpha\sqrt{\rho_1\,\rho_2}\, t_{12} \end{aligned} \tag{4.10}$$

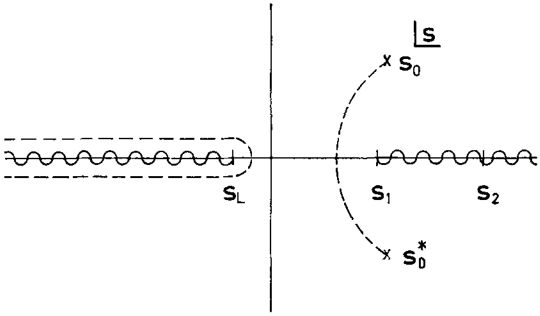

FIG. 1

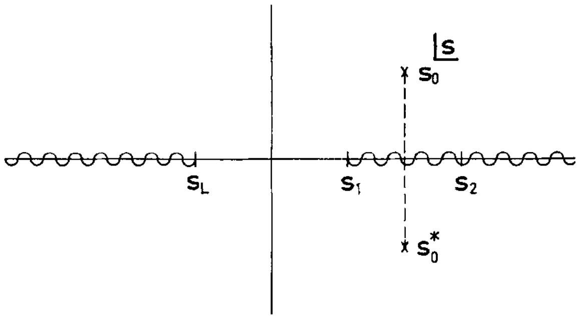

FIG. 2

FIGS. 1 AND 2. Analytic properties of the eigenamplitudes in the s plane; the dotted branch cuts arise from the diagonalization coefficients (Section IV).

so that the $T^{(i)}(s)$ can have, in addition to the "unitarity cut" $s_1 \leqq s < \infty$, any cuts arising from branch structure in the analytic continuations of $\alpha(s)$ and $\beta(s)$. A branch point occurs in $\alpha(s)$ and $\beta(s)$ whenever $K(s) = \pm i$.[3] To calculate the positions of all such branch points would require a detailed knowledge of the $t_{ij}(s)$. The cuts of $\alpha(s)$ and $\beta(s)$ in the s plane are taken to be the mapping of the cuts in the K plane defined in Fig. 3. Suppose that a complex branch point occurs at $s = s_0$ on the physical sheet. Then a branch cut must extend on some arc from s_0 to the conjugate branch point s_0^*. It is of importance to distinguish the case in which the "diagonalization cut" (i.e., a cut in $T^{(i)}(s)$ arising from $\alpha(s)$ and $\beta(s)$) does not cross the unitarity cut, $s_1 \leqq s < \infty$, from the cases in which such crossing occurs. If there is no cut crossing, the most general branch structure which can be induced, in the case that $K(s) = i$ at only one point $s = s_0$, is shown in Fig. 1. Beside the complex cut labelled (1), the real cut (2) is possible if, for instance, the branch point s_L is of the square root type (as it is

[3] Of course, $\alpha(s)$ and $\beta(s)$ have also all the branch points of the physical amplitudes (cf., (IV.6) and (IV.7)). For simplicity these are not emphasized here.

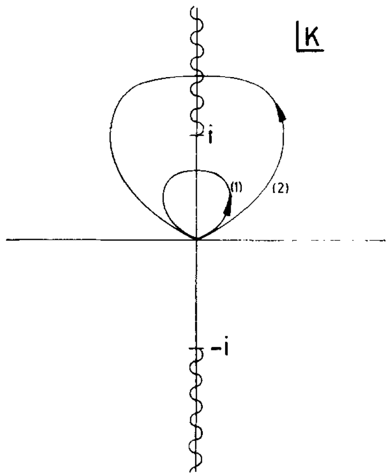

FIG. 3. Cuts in K plane; cf. Eq. (4.6). Curves (1) and (2) show two possible trajectories of the function $K(s)$ between the thresholds s_1 and s_2.

for two-particle exchange); but not if s_L is a logarithm branch point (as for one-particle exchange). Figure 2 shows an example of a diagonalization cut which crosses the unitarity cut between s_1 and s_2. Crossing can only occur above $s = s_2$ if $K(s) \to \infty$ at some point, for on $s_2 \leqq s < \infty$ α and β are real.

Another way of looking at the eigenamplitudes $T^{(1)}$ and $T^{(2)}$, as can be seen from Eqs. (4.10) and (4.6), is that they can be considered as different branches of one and the same analytic function.

It should be noted that due to the vanishing of the phase space factors ρ_i, the S matrix elements satisfy the equations $S_{ii} = 1$ and $S_{ij} = 0$ for $j \neq i$, at the ith threshold $s = s_i$. One of the eigenamplitudes therefore coincides with the ith physical amplitude at its threshold. In the case of no cut crossing, one is thus furnished with a natural labelling for the eigenamplitudes.

In this paper, in order to simplify the discussion, the case in which the extra diagonalization cuts do not intersect the unitarity cut will be treated exclusively. $\alpha^2(s)$ and $\beta^2(s)$, considered through (4.6) as functions of $K(s)$, can be defined on a K plane cut as shown in Fig. 3, with Re $(1 + K^2)^{-1/2}$ defined to be positive on sheet I, negative on sheet II. As s changes from s_1 to s_2, $K(s)$ undergoes a complex excursion, beginning and ending at $K = 0$. Paths (1) and (2) in Fig. 3 illustrate two possibilities. Path (1) corresponds to a case in which no diagonalization cut crosses the unitarity cut between s_1 and s_2. Then, from the definition of sheet I of $(1 + K^2)^{-1/2}$, it follows from (4.6) that

$$\text{Re}\ \alpha^2(s) > \text{Re}\ \beta^2(s)$$

for (4.11)

$$s_1 \leqq s \leqq s_2$$

If, in addition, $K(s)$ does not become infinite on $s_2 \leqq s \leqq \infty$, then

$$\alpha^2(s) > \beta^2(s) \quad \text{for} \quad s_2 \leqq s \leqq \infty \tag{4.12}$$

Path (2) in Fig. 3 illustrates one of the cases that have been excluded from the present discussion. For, in order that $\alpha(s)$ return to unity at $s = s_2$, it will suffer a discontinuity as the cut in the K plane is crossed. This will give rise to discontinuities in $T^{(1)}$ and $T^{(2)}$, through (4.10): that is, a diagonalization cut intersects the unitarity cut between $s = s_1$ and $s = s_2$.

The eigenamplitudes have a key role in the present analysis because it is to them that the essential dynmical features can be attributed. Thus, except for accidental degeneracies, each many channel bound state or resonance is reproduced in just one eigenamplitude. It is therefore evident that a form of Levinson's theorem for eigenamplitudes will provide a powerful tool for investigating many-channel dynamics.

V. LEVINSON'S THEOREM FOR EIGENAMPLITUDES

The next task is to investigate Levinson's theorem for the eigenphases. Suppose that the t_{ij} defined in (4.1) are calculated by a matrix ND^{-1} calculation, in which the equations are dynamical with nonsingular left-hand cuts. Precisely, it is assumed that all left-hand cut integrals $B_{ij}(s)$ satisfy

$$| B_{ij}(s) | < Cs^{-\epsilon-1/2} \quad \text{for} \quad s_1 \leqq s \leqq \infty; \quad \epsilon > 0 \tag{5.1}$$

(as in (3.13) and (3.14) for the one channel inelastic equations), and in the sense that there are no CDD poles. Then the determinant

$$\det D(s) \to 1, \qquad s \to \infty \tag{5.2}$$

Moreover, det $D(s)$ has, by assumption, no poles; and each zero is to be associated with a bound state.

From the definitions of the matrices N and D

$$t = ND^{-1} \tag{5.3}$$

one has the matrix relations

$$\begin{aligned} D = t^{-1}N &= \rho^{1/2}T^{-1}\rho^{1/2}N \\ &= \rho^{1/2}O^{-1}T_D^{-1}O\rho^{1/2}N \end{aligned} \tag{5.4}$$

using the notation of (4.1) and (4.3). Taking determinants of both sides

$$\det D = (\prod_i \rho_i)(\prod_i T^{(i)})^{-1} \det N \tag{5.5}$$

On the unitarity cut, det N is real, so that

$$\arg\,(\det D) = -\sum_i \delta^{(i)} \tag{5.6}$$

where

$$\delta^{(i)} = \arg\,(T^{(i)}/\rho_i) \tag{5.7}$$

and $\delta^{(i)}(s_1)$ is defined to be zero. In Eq. (5.7) the "natural" labeling (see Section IV) is assumed, whereby $T^{(i)}$ coincides at threshold with T_{ii}, and the dividing out of the factor ρ_i then serves to remove the kinematic zero of $T^{(i)}$. Since det D has no poles, and each zero corresponds to a bound state, (5.2) and (5.6) lead to

$$\sum_i n_{\mathrm{B}}^{(i)} = -\frac{1}{\pi}\sum_i \delta^{(i)}(\infty) \tag{5.8}$$

where $\sum n_{\mathrm{B}}^{(i)}$ means the total number of bound states in all channels (*8*).

By introducing further assumptions, it is possible to analyze the composite Levinson relation (5.8) into statements about each eigenphase-shift individually. First, it will be assumed that each zero of det D, which corresponds to a bound state, is simple. Furthermore, any such zero is plausibly associated, through (5.5), with a simple pole of just one diagonal element $T^{(i)}$. The possibility that several of the $T^{(i)}$ have poles or zeros, or that det N has a zero, can be shown to be unlikely. In the first place, a coincident zero of det D and det N implies, as in the one-channel case, that certain elements of N must satisfy homogeneous Fredholm quations. This almost never happens, since the spectrum of a Fredholm kernel is discrete. The possibility remains that more than one $T^{(i)}$ might have a pole (possibly multiple), while others have zeros, in such a way that the product in (5.5) gives a simple pole. In the two-channel case, such a contingency is ruled out by the assumption that the physical amplitudes T_{ij} have only simple poles: in the many-channel case a contradiction is not involved, but the possibility requires the satisfaction of detailed conditions; and it seems that its occurrence could only be a coincidence.

Thus it is natural to associate a pole of a particular $T^{(i)}$ with a bound state. Suppose that one writes

$$T^{(i)}/\rho_i = N^{(i)}/D^{(i)} \tag{5.9}$$

One channel N/D equations are to be written down, in which $D^{(i)}(s)$ has only the unitarity cut, while $N^{(i)}(s)$ carries the left-hand cut and any diagonalization cuts. The phase of $D^{(i)}(s)$ on its cut is defined to be $-\delta^{(i)}(s)$. Then, it follows from (5.1) that

$$D^{(i)}(s) \to 1 \qquad s \to \infty \tag{5.10}$$

so that

$$n_B^{(i)} - n_C^{(i)} = -\frac{1}{\pi}\,\delta^{(i)}(\infty) \tag{5.11}$$

Here $n_B^{(i)}$ is the number of zeros of $D^{(i)}(s)$ and $n_C^{(i)}$ is the number of *CDD* poles. Summing (5.11)

$$\sum_i n_B^{(i)} - \sum_i n_C^{(i)} = -\frac{1}{\pi}\sum_i \delta^{(i)}(\infty) \tag{5.12}$$

From (5.8) it is clear that one must have $n_C^{(i)} = 0$, so that (5.11) is replaced by

$$n_B^{(i)} = -\frac{1}{\pi}\,\delta^{(i)}(\infty) \tag{5.13}$$

which is the required form of Levinson's theorem for the eigenphases $\delta^{(i)}(s)$.

VI. THE CRANK-SHAFT THEOREM

In the preceding Sections III–V the tools have been assembled. They will now be employed for their predestined task, which was to state conditions under which a one-channel calculation would fail. When the thresholds are coincident, it is possible, in terms of a simple inequality on the diagonalization coefficients, to specify in which channel failure would occur; when the thresholds are distinct, one can in general assert only that failure would occur in channel one or channel two (or both).

Equation (4.5) can be rewritten as follows

$$\begin{aligned} \eta e^{2i\delta_{11}} &= \alpha^2 e^{2i\delta(1)} + \beta^2 e^{2i\delta(2)} \\ \eta e^{2i\delta_{22}} &= \beta^2 e^{2i\delta(1)} + \alpha^2 e^{2i\delta(2)} \end{aligned} \tag{6.1}$$

According to Levinson's theorem, $\delta^{(1)}(\infty)$ and $\delta^{(2)}(\infty)$ are both zero modulo π. Thus (6.1) implies

$$\begin{aligned} \eta(\infty) &= 1 \\ \delta_{11}(\infty) &= 0 \bmod \pi \\ \delta_{22}(\infty) &= 0 \bmod \pi \end{aligned} \tag{6.2}$$

In general, for a finite number of channels, Levinson's theorem for the eigenamplitudes implies no inelasticity at $s = \infty$. This is an important consideration in other contexts.

The problem now is to relate the eigenphases $\delta^{(i)}$ to the physical phase shifts δ_{ii} at $s = \infty$. The first step in the solution is to observe that if there exists some $s_R \geqq s_2$ such that $\alpha^2(s) > \beta^2(s)$ for all $s_R \leqq s \leqq \infty$, then

$$\begin{aligned} \delta_{11}(\infty) &= \delta^{(1)}\infty + m\pi \\ \delta_{22}(\infty) &= \delta^{(2)}\infty - m'\pi \end{aligned} \tag{6.3}$$

where

$$m = \left[\frac{\delta_{11}(s_R) - \delta^{(1)}(s_R)}{\pi}\right]$$

and

$$m' = \left[\frac{\delta_{22}(s_R) - \delta^{(2)}(s_R)}{\pi}\right]$$

Here $[\gamma]$ means the nearest integer to γ. If, on the other hand, $\alpha^2(s) < \beta^2(s)$ for all $s_R \leqq s \leqq \infty$, (6.3) must be altered by interchanging $\delta^{(1)}$ and $\delta^{(2)}$. These simple results follow from a geometrical interpretation of (6.1) (an application of what will be termed the crank-shaft theorem (CST)).

In the simple case in which both thresholds are coincident ($s_1 = s_2$), and none of the diagonalization cuts of $T^{(1)}$ and $T^{(2)}$ intersect the unitarity cut $s_1 \leqq s \leqq \infty$, it has been shown that

$$\alpha^2(s) > \beta^2(s) \quad \text{for all} \quad s_1 \leqq s \leqq \infty \tag{6.4}$$

Furthermore, by definition

$$\delta_{11}(s_1) = \delta_{22}(s_1) = \delta^{(1)}(s_1) = \delta^{(2)}(s_1) = 0 \tag{6.5}$$

Thus, setting $s_R = s_1$ in (6.3), one has

$$\delta_{11}(\infty) = \delta^{(1)}(\infty)$$

and

$$\delta_{22}(\infty) = \delta^{(2)}(\infty) \tag{6.6}$$

Suppose that there is a dynamical bound state in eigenchannel two, but none in eigenchannel one. Then Levinson's theorem for the eigenphase shifts gives

$$\delta^{(1)}(\infty) = 0$$
$$\delta^{(2)}(\infty) = -\pi \tag{6.7}$$

so that (6.6) gives

$$\delta_{11}(\infty) = 0$$
$$\delta_{22}(\infty) = -\pi \tag{6.8}$$

However, from (6.1) it is clear that in general both physical channels have a bound state. Hence this bound state cannot be calculated dynamically in channel one, although it can be obtained from a *CDD* pole free FW calculation in channel two. Summarizing, the condition (6.4), or the equivalent assumption that no diagonalization cuts intersect the unitarity cut, is sufficient to ensure that a bound state arising from a pole in eigenchannel two cannot be calculated in physical channel one. Specific realizations of this phenomenon have already been given in Section II.

Next, the more complicated case in which the thresholds s_1 and s_2 are distinct will be considered, again with the assumption that no diagonalization cut intersect the cut $s_1 \leqq s \leqq \infty$. Some care is needed in the phase conventions. As in (5.7), one has

$$\delta^{(1)}(s_1) = \delta^{(2)}(s_1) = 0 \tag{6.9}$$

The physical phase shifts are defined to be zero at their respective thresholds

$$\begin{aligned} \delta_{11}(s_1) &= 0 \\ \delta_{22}(s_2) &= 0 \end{aligned} \tag{6.10}$$

The simple relations (6.6) no longer hold: however, it is possible to derive a single composite relation even when the thresholds are distinct (again in the case of no diagonalization cut intersection). Above the higher threshold, the unitarity condition is

$$S^{\dagger}S = 1 \tag{6.11}$$

which can be rewritten in terms of the matrix t that was introduced in (3.1) as

$$\frac{t - t^{\dagger}}{2i} = t^{\dagger}\rho t \tag{6.12}$$

For $s_1 \leqq s \leqq s_2$, on the other hand, (6.12) is modified by replacing $\rho_2 = \sqrt{(s - s_2)/s}$ by zero. This gives

$$\begin{aligned} \operatorname{Im} t_{11} &= \rho_1 |\, t_{11} \,|^2 \\ \operatorname{Im} t_{12} &= \rho_1 t_{11}^{*} t_{12} \qquad s_1 \leqq s \leqq s_2 \\ \operatorname{Im} t_{22} &= \rho_1 |\, t_{12} \,|^2 \end{aligned} \tag{6.13}$$

Moreover, (4.1) gives

$$T_{ij} = \sqrt{\rho_i \rho_j}\, t_{ij} \tag{6.14}$$

and for $s_1 \leqq s \leqq s_2$, $\rho_2 = i|\, \rho_2 \,|$. The first equation (6.13) implies, using

$$T_{11} = e^{i\delta_{11}} \sin \delta_{11} \tag{6.15}$$

that the physical channel one phase-shift δ_{11} is real below s_2. The second equation (6.13) shows that t_{12} has the phase δ_{11}. Hence one can write

$$T_{12} = \sqrt{i}\, Re^{i\delta_{11}} \tag{6.16}$$

where R is some real number. Finally, the third equation (6.13) gives

$$\operatorname{Re} T_{22} = -R^2 \tag{6.17}$$

The T matrix has the form

$$T = \begin{pmatrix} e^{i\delta_{11}} \sin \delta_{11} & \sqrt{i}Re^{i\delta_{11}} \\ \sqrt{i}Re^{i\delta_{11}} & -R^2(1 + i \cot \varphi) \end{pmatrix} \tag{6.18}$$

where φ is a real angle. From (4.3)

$$\det T = \det T_D \tag{6.19}$$

so that, combining (5.7) and (6.18), this gives

$$-iR^2 e^{i\delta_{11}} \frac{\sin (\delta_{11} + \varphi)}{\sin \varphi} = i \mid T^{(1)} T^{(2)} \mid e^{i\mathrm{Re}(\delta^{(1)} + \delta^{(2)})} \tag{6.20}$$

Equating phases on both sides of this equation (taking account of the conventions at $s = s_1$ given in (6.9) and (6.10)), one has unambiguously

$$\delta_{11} = \mathrm{Re}\ (\delta^{(1)} + \delta^{(2)}) \quad \text{for all} \quad s_1 \leqq s \leqq s_2 \tag{6.21}$$

Above the higher threshold, when (6.11) is in force, one has

$$\det (1 + 2iT) = \det (1 + 2iT_D) \tag{6.22}$$

i.e.,

$$e^{2i(\delta_{11} + \delta_{22})} = e^{2i(\delta^{(1)} + \delta^{(2)})}$$

which implies

$$\delta_{11} + \delta_{22} = \delta^{(1)} + \delta^{(2)} \qquad s \geqq s_2 \tag{6.23}$$

since this must agree with (6.21) at s_2, when $\delta_{22} = 0$.

Equation (6.23), taken at $s = \infty$, replaces the two equalities (6.6) that hold in the equal threshold case. If, as before, there is one dynamical bound state in eigenchannel two, but none in eigenchannel one, then

$$\delta^{(1)}(\infty) + \delta^{(2)}(\infty) = -\pi \tag{6.24}$$

Thus (6.23) implies that $\delta_{11}(\infty)$ and $\delta_{22}(\infty)$ cannot both equal $-\pi$, although each physical channel has a bound state. Evidently at least one channel must fail to be dynamical.

In the next section, the complications which can arise in the split threshold case will be discussed in detail. It will be shown how, for not too strong interchannel coupling, these complications do not arise and a stronger statement analogous to that for the equal threshold case will be obtained in place of Eq. (6.24).

VII. SPECIFIC EXAMPLES FOR DISTINCT THRESHOLDS

An apparatus has been produced in the foregoing sections to extend some of the results of the equal threshold case to the more general possibility $s_1 \neq s_2$.

In Section VI, it was found that the composite relation (6.23) replaces the two equations (6.6) of the degenerate system. A broad class of examples will now be exhibited for which (6.24) breaks up into the two equalities (6.6), even when the thresholds are distinct. To do this, it is necessary to study in detail the phases between s_1 and s_2.

It is convenient to rewrite the first equation (6.1) in the form

$$S_{11} = |\alpha^2 S^{(1)}| e^{2i(\delta^{(1)}+\varphi^{(1)})} + |\beta^2 S^{(2)}| e^{2i(\delta^{(2)}+\varphi^{(2)})} \tag{7.1}$$

where

$$\varphi^{(1,2)} = \frac{1}{2} \arg \begin{pmatrix} \alpha^2 & S^{(1)} \\ \beta^2 & S^{(2)} \end{pmatrix} - \delta^{(1,2)} \tag{7.2}$$

and $\varphi^{(1,2)} \equiv 0$ for $s \geqq s_2$. The φ's that have been introduced represent two effects: firstly phase changes of α^2 and β^2 between s_1 and s_2 (a phenomenon that will be called "twisting"), and, secondly possible deviations of arg $(S^{(i)})$ from $2\delta^{(i)}$, (which will be called "winding"). Note that these latter can only develop when $|S^{(i)}| \neq 1$ (see Eqs. (4.1) and (5.7)).

It is of interest to consider the case in which $\varphi^{(i)}(s_1) = \varphi^{(i)}(s_2)$, when there is no over-all twisting or winding. If $|\alpha^2 S^{(1)}| > |\beta^2 S^{(2)}|$ for all $s_1 \leqq s \leqq \infty$, then (7.1) can be treated by the methods used in Section VI for the equal threshold case. In a similar way it can be concluded that a bound state in eigenchannel two fails to be dynamical in the physical channel one. An equivalent statement regarding channel two (mutatis mutandis) cannot be made, since $\beta^2 = 0$ at $s = s_2$ and the condition $|\alpha^2 S^{(1)}| < |\beta^2 S^{(2)}|$ for all $s_1 \leqq s \leqq \infty$ cannot be satisfied.

Since a specific statement can be made when there is no twisting or winding in channel one, i.e.,

$$\varphi^{(1)}(s_1) = \varphi^{(1)}(s_2) \tag{7.3}$$

it is of interest to examine when this occurs. The existence of twisting (phase changes of α^2) is directly related to the locations of the diagonalization cuts. The function $\alpha^2(s)$ is defined in (4.6), in terms of $K(s)$, which is expressed in terms of the t matrix elements in (4.7). Because of the phase space factors in (4.7), $K(s)$ vanishes at the thresholds $s = s_1$ and $s = s_2$; it describes some complex trajectory between s_1 and s_2 (see Fig. 3). It is obvious that α^2 suffers no over-all phase change, i.e.,

$$\arg \alpha^2(s_1) = \arg \alpha^2(s_2) \tag{7.4}$$

provided the path of $K(s)$ does not cross one of the cuts (as in path (2) in Fig. 3). However, this eventuality has been excluded by our general assumption that no cut crossing is to occur (see the discussion at the end of Section IV). Note that a sufficient condition for this is

$$| K(s) | < 1 \qquad s_1 \leqq s \leqq s_2 \tag{7.5}$$

which can be written directly in terms of t matrix elements, thus

$$2| \sqrt{\rho_1\rho_2}\, t_{12} | < | \rho_1 t_{11} - i | \rho_2 | t_{22}, \qquad s_1 \leqq s \leqq s_2 \tag{7.6}$$

The requirement is thus that the interchannel coupling be not too strong.

Next, it will be shown that there is an appreciable class of amplitudes for which winding does not occur, that is, for which

$$\arg S^{(1)} = \delta^{(1)} \quad \text{for all} \quad s_1 \leqq s \leqq s_2 \tag{7.7}$$

At $s = s_1$, $S^{(1)} = 1$, and at $s = s_2$, $S^{(1)} = e^{2i\delta_{11}(s_2)}$, so that the trajectory of $S^{(1)}$ begins at s_1 and ends at s_2 on the unit circle. Four possibilities are shown in Fig. 4, of which only II and III exhibit winding. In fact, the criterion for no winding lies in the possibility of deforming the trajectory topologically on to the unit circle, without passing over the origin in the course of the deformation.

It can be shown that winding will not occur for weak coupling. For, referring to Fig. 4, winding can only occur if the trajectory of $S^{(1)}$, in its excursion from s_1 to s_2, intersects the positive real axis in the $S^{(1)}$ plane. For the quantity $[T^{(1)}]^{-1}$, introduced through Eqs. (4.1) and (4.3), the corresponding trajectory must not intersect the lines $-\infty < [\text{Im } T^{(1)}]^{-1} \leqq -2$ and $0 \leqq [\text{Im } T^{(1)}]^{-1} < \infty$. However, for zero coupling $[\text{Im } T^{(1)}]^{-1} = -1$ (see Fig. 5), so that for winding to occur the coupling must deform the trajectory $[\text{Im } T^{(1)}]^{-1} = -1$ by at least one unit. This is plausibly associated with at least moderate coupling.

It is of interest to re-analyze the example of Section II in which the decoupled

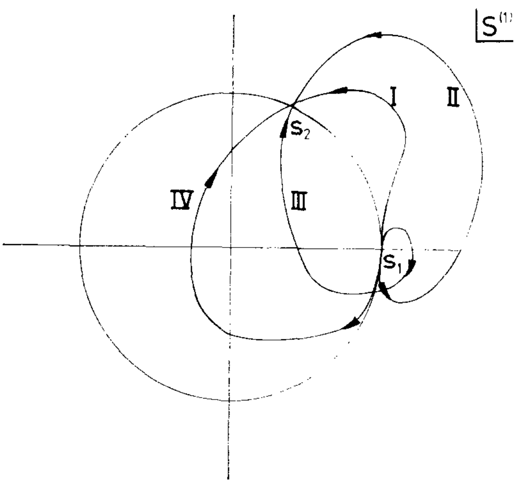

FIG. 4. Several instances of "winding" (Section VII)

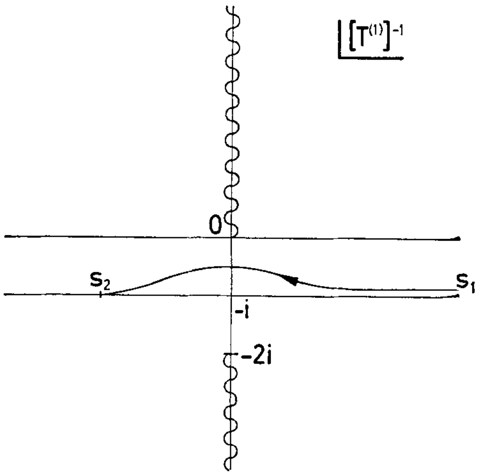

FIG. 5. Complex excursions of the inverse eigenamplitude $[T^{(1)}]^{-1}$ in the case of no "twisting" (Section VII).

channel two had a bound state between s_1 and s_2. Suppose that the t matrix be written

$$t = \begin{bmatrix} R_{11} - i\rho_1 & R_{12} \\ R_{12} & R_{22} - i\rho_2 \end{bmatrix}^{-1} \tag{7.8}$$

where the R's are the inverse K matrix elements. The weak coupling assumption of Section II implies that R_{12} is small, while the bound state in channel two requires $R_{22} - i\rho_2$ to have a zero between s_1 and s_2. From (4.1) it follows that

$$T = \begin{bmatrix} \cot \delta_1 - 2 & R_{12}/\sqrt{\rho_1\rho_2} \\ R_{12}/\sqrt{\rho_1\rho_2} & -\alpha(s - s_0)/\rho_2 \end{bmatrix}^{-1} \tag{7.9}$$

where $R_{11} = \rho_1 \cot \delta_1$ and the zero of $R_{22} - i\rho_2$ is supposed to occur at $s = s_0$. The coefficient α must be positive in order that the pole of S correspond to a true bound state, and not a ghost. From (7.9)

$$\det T = \frac{-\rho_2}{\alpha(\cot s_1 - i)(s - s_R)} \tag{7.10}$$

where

$$s_R = s_0 - \frac{R_{12}^2\, e^{i\delta_1} \sin \delta_1}{\alpha\rho_1}$$

so that Im $s_R < 0$.

The physical S matrix elements can be calculated from (7.9). In particular, one has the approximate relation

$$S_{11} \approx e^{2i\delta_1} \frac{s - s_R^*}{s - s_R} \tag{7.11}$$

from which it follows that there is a resonance at $s = \text{Re}\, s_R$ in the physical channel one, but that

$$\text{Re}\, \delta_{11}(\infty) = \text{Re}\, \delta_1(\infty) + \pi = \pi \tag{7.12}$$

In the notation of Eq. (6.3), one has $m = 1$ and confirms that the resonance is not dynamical in the single channel framework for channel one. The mechanism of this effect in terms of the behavior of eigenamplitudes can be traced through in detail. One finds that $S^{(1)} \approx e^{2i\delta_{11}}$, $S^{(2)} \approx S_{22} \approx \text{const.}/(s - s_R)$. The transformation parameters $K(s)$ and β (cf., Eqs. (4.4), (4.7)) always remain small; but, in the neighborhood of the resonance, $S^{(2)}$ becomes compensatingly large, so as to control the phase change of S_{11}. Note that both eigenamplitudes patently satisfy Levinson's theorem and winding does not occur. The general form of the physical and eigenamplitudes is displayed in Fig. 6.

To summarize, for distinct thresholds and with strong coupling the analysis can be complicated through the occurrence of phase changes of the diagonalization coefficients (twisting) and of the expression ($\frac{1}{2} \arg (S^{(i)}) - \delta^{(i)}$) (winding).

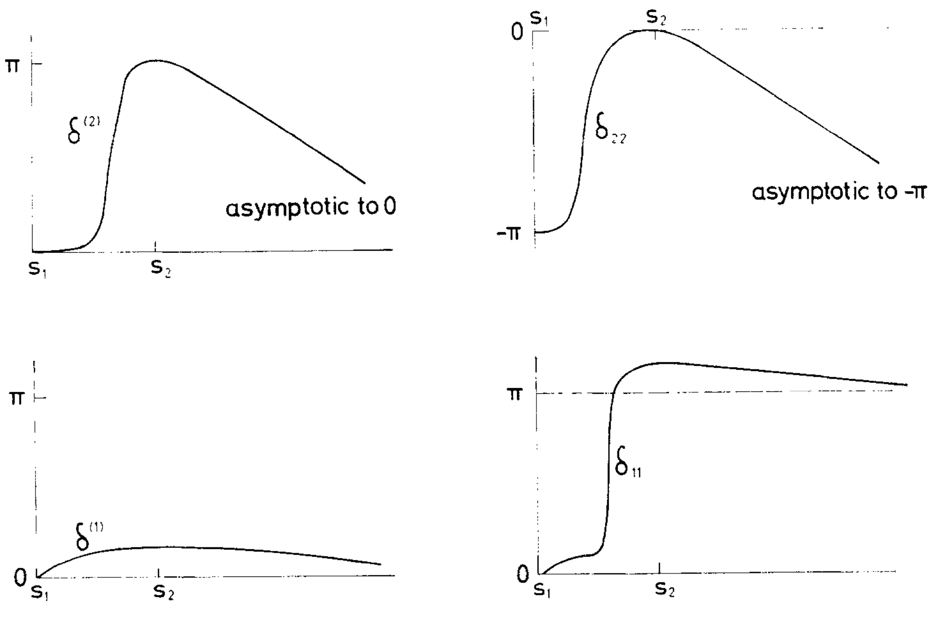

FIG. 6. Sketch of the behavior of the physical and eigenphases for a weak interchannel coupling example (Section VII).

Where neither effect is present, the simple sufficient condition involving $| \alpha^2 S^{(1)} | > | \beta^2 S^{(2)} |$ can be made.

Under the same assumption, the generalization to an n channel problem is immediate. If U is the transformation which diagonalizes the S matrix, and which is therefore unitary above the highest threshold, then

$$S_{11} = u_{11}^2 S^{(1)} + u_{21}^2 S^{(2)} + \cdots u_{n1}^2 S^{(n)} \tag{7.13}$$

in analogy with the two channel formulas. This equation can then be continued. If $S^{(1)}$ has a bound state, the resulting bound state in S_{11} will be dynamical if

$$| u_{11}^2 S^{(1)} | > | u_{21}^2 S^{(2)} | + \cdots + | u_{n1}^2 S^{(n)} | \tag{7.14}$$

on the whole of the right-hand cut. While the single channel dynamical calculation always succeeds for one of the channels in the two-channel case, it is possible in the many-channel case to construct examples in which the bound state is not calculable in any of the single channels.

VIII. NOTE ON THE CHEW-MANDELSTAM AND THE FROISSART METHODS

If $R = (\sigma_{\text{tot}}/\sigma_{\text{el}})$ is given instead of η, then the appropriate method of calculation is CM. Again there is the possibility of disagreement with the many-channel results; but the instances are not coextensive with those for the FW method. One can easily check that the weak coupling examples which have been considered go the same way for both methods. Squires (*1*) has given an example which fails for CM but which, it can be seen, would succeed for FW. One can also find examples which do the opposite and succeed for CM while failing for FW.

One would like to carry out for the CM method a parallel analysis to that which has been developed for the FW method. The essential difference comes at the point where one tries to develop the analogue of the crank-shaft theorem. One now has the relation

$$T_{11} \equiv | T_{11} | e^{2i\phi_{11}} = \alpha^2 \sin \delta^{(1)} e^{i\delta(1)} + \beta^2 \sin \delta^{(2)} e^{i\delta(2)} \tag{8.1}$$

and is led inexorably to requiring relations between $| \alpha^2 \sin \delta^{(1)} |$ and $| \beta^2 \sin \delta^{(2)} |$. The situation is thus more complicated than for FW and even in the case of energy independent α and β no decision is possible without detailed knowledge of $\delta^{(1)}$ and $\delta^{(2)}$.

In passing, it may be noted à propos of the FW method that Froissart has given an alternative calculational procedure when η is given. One defines

$$P(s) = \exp\left[-i \frac{\sqrt{s - s_1}}{\pi} \int_{s_1}^{\infty} \frac{\log \eta(s')\, ds'}{\sqrt{s' - s_1}(s' - s)} \right] \tag{8.2}$$

and then makes an elastic N/D decomposition of S/P (i.e., $S = P(1 + 2i\rho N/D)$). Thus, D has the phase $\delta - \beta/2$, where β is the phase of P on its cut. If η tends to a nonzero constant as $s \to \infty$, as would be expected for a finite number of channels, then $\beta(\infty) = 0$, and the previous FW classifications apply.

IX. AN APPLICATION TO SU_3 SYMMETRIC BOOTSTRAPS

The case of several channels with coincident thresholds was introduced into the present discussion because it furnishes a model which is easy to evaluate. One can also look for direct applications.[4] The most important realization of coincident thresholds in elementary particle physics occurs when there is some kind of internal symmetry. If the symmetry in question is only approximate, then in general the threshold coincidence will be approximate also; but one can ask whether useful "first order" statements can be made. In the case of charge independence (SU_2), the results are entirely academic. The symmetry breaking is small, so that to a good approximation one can and does work always in terms of the eigenstates and the questions treated in the present work are simply not asked. However, in the case of SU_3 symmetry, which is substantially broken in the sense that considerable phase changes occur between the displaced thresholds, there is more potential interest. The dynamics are now influenced by two diagonalizations—the one belonging to the symmetry and the other belonging to phase space. One can therefore speak of an actual diagonalization matrix U of the form discussed in Section IV and an "ideal" diagonalization matrix U_0 which would obtain if the symmetry were unbroken. The latter simply consists of Clebsch-Gordan coefficients (*9*). For the "ideal" case, it is easy to apply the rules of Section VI to the U_0 elements and deduce which single channel calculations would succeed and which fail. One may then conjecture that the resulting statements perhaps hold good for the actual situation of broken symmetry, particularly if the inequalities on the U_0 are strongly satisfied.[5]

Before proceeding to concrete instances, a few detailed points should be made. Firstly, the one-channel *CDD* poles which have been discussed only occur for resonances and bound states lying in energy below the highest threshold. This introduces what might appear to be an awkward distinction for a discussion of broken symmetries, although it arises naturally from the terms of the problem studied. In fact, it is not a difficulty if one categorizes situations according to actual physical masses. For the examples to be discussed, the resonance does lie below the highest threshold; and for the case of unbroken symmetry the resonances become bound states if masses are taken from the mass formula. Another related point is that in order to make useful comments on bootstrap calculations one really wants to discuss what happens when certain channels are omitted altogether. Now, if a channel has its threshold high in energy above

[4] This question arose from a discussion with V. L. Teplitz.

[5] This will be referred to as the "crank-shaft analysis survives symmetry breaking" conjecture or CASSB.

a multichannel resonance, it may very well be reasonable to omit it. Conversely, to omit a channel may be viewed as being equivalent to pretending that its threshold is very high.

A further point is that charge independence (SU_2) is such a good symmetry that one wants in practice in the discussion of broken SU_3 to refer to SU_2 eigenstates (precisely, eigenstates of I, I_3, and Y) rather than the physical states. This means that the elements of U_0 will not actually comprise Clebsch-Gordan coefficients but isoscalar factors (see ref. (*9*), Eqs. (10.5) and (10.6)). In principle, an extension of the methods of the present paper is implied, in that one now considers the reduction of an n channel problem to one with m channels ($n > m \geqq 1$). In general, this would be a complicated process to analyze; in this special case it is trivial.

We now turn to some examples. Consider first the ρ resonance and assume, as is conventional, that this can be derived from a many channel dynamical calculation with channels $\pi\pi$, $K\bar{K}$, and $\eta\eta$. (This may very well not be true.) The corresponding "conventional" assumption will be made in the subsequent examples. Under unbroken octet symmetry, one can write

$$|\rho\rangle = \sqrt{2/3}\,|\pi\pi\rangle + \sqrt{1/3}\,|K\bar{K}\rangle \tag{9.1}$$

or, conversely, for the $J^P = 1^-$ state

$$|\pi\pi\rangle = \sqrt{2/3}\,|8_2\rangle + \sqrt{1/6}\,|10\rangle - \sqrt{1/6}\,|\overline{10}\rangle \tag{9.2}$$

Note that in Eqs. (9.1) and (9.2) use has been made for the first time of the diagonalization matrix operating on state vectors ($|\xi'\rangle = U_0\,|\xi\rangle$) rather than as hitherto on the scattering matrix ($S' = U_0^{-1}SU_0$). A certain conciseness is thereby achieved. It is now simple to apply the crank-shaft rules ($2/3 > 1/3$) and deduce that a single channel dynamical calculation in the $\pi\pi$ system with the correct prescribed inelasticity should yield the resonance.[6] This is a logical deduction and also useless. But now by the CASSB conjecture, the statement can be read as referring to the physical situation. In this case, since the $K\bar{K}$ threshold lies rather high, it is plausible that the $K\bar{K}$ channel can be omitted altogether.

Generally speaking, interesting cases are to be sought where the many-channel aspect is fairly complicated. For another example, consider the Y_1^*(1385 MeV), assumed to be a member of a $J^P = 3/2^+$ decuplet. In exact SU_3,[7] one has

[6] Concerning another aspect of the ρ, E. Abers has recently made a numerical calculation of the two channel problem $\pi\pi$ and $\pi\omega$. Using the Zachariasen-Zemach method, he finds that the ρ is a dynamical bound state only in the $\pi\pi$ channel. (Private communication.)

[7] The formalism is here implicitly extended to the case of two-body channels with unequal mass.

$$| Y_1^* \rangle = -\frac{1}{\sqrt{6}} | \bar{K}N \rangle + \frac{1}{\sqrt{6}} | K\Xi \rangle + \frac{1}{\sqrt{6}} | \pi\Sigma \rangle + \frac{1}{2} | \pi\Lambda \rangle - \frac{1}{2} | \Sigma\eta \rangle \tag{9.3}$$

$$(1434) \qquad (1812) \qquad (1331) \qquad (1253) \qquad (1742)$$

The figures in brackets under the decay products are real thresholds in MeV. In this case, one would guess that a calculation with $K\Xi$ and $\Sigma\eta$ omitted would probably be successful but to have the three channels $\pi\Lambda$, $\pi\Sigma$, and $\bar{K}N$ was probably an irreducible minimum.

In conclusion, it should be remarked that the whole question of the consistency of broken SU_3 with conventional bootstraps is as yet extremely ill understood. What has been offered here is a very simple rule for making prima facie judgments on the correctness of particular dynamical models when the general notion of SU_3 bootstraps is assumed. The resulting statements should, of course, only be interpreted as a guide. For neither of the examples treated were they very surprising. One has simply supplied a slightly stronger albeit still tenuous rationale for making them.

X. CONCLUSIONS

The question has been asked: When do one channel dynamical calculations with prescribed inelasticity reproduce the results of many-channel dynamics? The technique employed was to work via the eigenamplitudes. To this end, the diagonalization procedure was analyzed in detail and individual Levinson's theorems derived for the separate eigenamplitudes. A study of the relation of the physical to the eigenamplitudes yielded the crank-shaft theorem (Section VI). This led, for all the cases where the eigenamplitudes are analytic in a neighborhood of the unitarity cut (Section IV), to the following general result: a multichannel dynamical resonance (or bound state) will appear as a *CDD* pole in the single channel inelastic amplitude for at least one of the channels (Section VI). In the case of equal thresholds (Section VI), it is possible by testing a simple inequality on the diagonalization coefficients to say in which channel failure will occur and in which success. After an analysis of "twisting" and "winding" (Section VII), an analogous criterion for an important subset of cases with distinct thresholds could also be stated.

In Section IX, these results were tentatively applied to the physically interesting question of bootstraps in broken SU_3. The idea, admittedly speculative, was to employ the above rules with exact SU_3 diagonalization coefficients to yield prognostications on the actual broken SU_3 situation. The result in the examples considered was that the usual notion of taking only the lowest lying thresholds was upheld by our criterion.

It is of interest to compare the present approach with that of Bander, Coulter, and Shaw (*1*). These authors study the amplitude t_{11} as a function of the coupling constants Γ_{ij} in the underlying two channel situation. They remark that, for increasing Γ_{22}, the onset of failure coincides with the emergence of a zero of S_{11} from an unphysical sheet through the inelastic cut. In terms of the present formalism, $S_{11} = \alpha^2 S^{(1)} + \beta^2 S^{(2)}$ with the $S^{(i)}$ unimodular and α^2 and β^2 real. Thus, $S_{11} = 0$ implies $S^{(1)} = -S^{(2)}$ and $\alpha^2 = \beta^2$. This can be readily understood on the present "crankshaft" picture, if one considers a continuous transition from a success to a failure regime. The curve described by S_{11} in the complex plane will sweep through a series of configurations, first circumscribing $S_{11} = 0$, then, at the transition, passing through it, and finally cutting the real axis to the right of the origin. A *CDD* pole has then emerged.

The whole subject of the present paper has been how one channel *CDD* poles can arise in the framework of multichannel partial wave dispersion relations. A different facet of the same physical principle has been exposed by Mandelstam (*10*).[8] The problem which he considers has many channels through spin orbit coupling, and the question studied is what happens when a continuation is made in total angular momentum j from high values down to the value j_0 where one of the orbital channels becomes "nonsensical". The quantities compared are (a) the result from the continuation and (b) the "physical" amplitude which results from a calculation at the value $j = j_0$ with the nonsense channels omitted. It is concluded that (a) and (b) differ by *CDD* poles.

Clearly there remain tasks for the future, in particular a realistic treatment of high energies, with the inclusion of an infinite number of channels.

As a final word, it is worth reiterating that the present topic highlights a problem which faces the proponents of universal bootstraps. It may be that all the particles of high energy physics are compound dynamical states; but, where, from among the infinity of possibly inequivalent channels to which a particle is coupled, is the dynamics to be done?

Acknowledgments

The authors thank Dr. V. L. Teplitz for the useful comments referred to in the text. One of us (D. M.) would like to thank Professor L. Van Hove for his hospitality at CERN.

Received: August 23, 1965

REFERENCES

1. M. Bander, P. W. Coulter, and G. L. Shaw, *Phys. Rev. Letters* **14**, 270 (1965), (hereinafter referred to as BCS); D. Atkinson, K. Dietz, and D. Morgan, CERN preprint TH. 531; see also E. J. Squires, *Nuovo Cimento* **34**, 1751 (1964); and H. Munczek, *Phys. Letters* **13**, 92 (1964).

[8] The authors thank V. L. Teplitz for pointing out this parallel.

2. G. Frye and R. L. Warnock, *Phys. Rev.* **130,** 478 (1963).
3. G. F. Chew and S. Mandelstam, *Phys. Rev.* **119,** 467 (1960).
4. M. Froissart, *Nuovo Cimento* **22,** 191 (1961).
5. D. Atkinson and D. Morgan, Levinson Theorem for S-matrix theory with inelasticity. *Nuovo Cimento*, in press.
6. R. L. Warnock, *Phys. Rev.* **131,** 1320 (1963).
7. R. C. Hwa and D. Feldman, *Ann. Phys. (N.Y.)* **21,** 453 (1963).
8. M. T. Vaughn, R. Aaron, and R. D. Amado, *Phys. Rev.* **124,** 1258 (1961).
9. J. J. de Swart, *Rev. Mod. Phys.* **35,** 916 (1963).
10. S. Mandelstam, UCRL-11686.