Modified πN Superconvergence and the Static Model*

L. A. P. BALÁZS† AND JOHN M. CORNWALL†
Department of Physics, University of California, Los Angeles, California
(Received 27 March 1967)

Recently, superconvergence relations have been proposed for certain scattering amplitudes with spin at fixed momentum transfer. Unfortunately, the kinematics and high-energy behavior of πN scattering do not permit any such relations for this process. If, however, one assumes Regge-pole dominance at high energy, one can write superconvergence relations for certain amplitudes from which the leading Regge-pole contributions have been subtracted out. Assuming resonance dominance in the low-energy region, one has a sum rule relating these resonances to the Regge parameters. Comparison with experimental data in the forward direction shows that the Regge contribution is small and that the dominant resonances are the N and N^*; the sum rule is then approximately satisfied. Completely ignoring the Regge poles and making certain additional assumptions leads to a relativistic version of the static model, which gives a relation, for instance, between the N and N^* coupling constants. This relation is quite well satisfied. In addition one may derive the Adler PCAC (partially conserved axial-vector current) self-consistency condition. These results are shown to be related to the vanishing of the equal-time commutator of the pion currents.

I. INTRODUCTION

ONE of the most successful theories in strong-interaction physics has been the static model of P-wave πN scattering.[1] If one uses the N/D method with linear D,[2] one is led to the famous Chew-Low relation[3] $\gamma_{1/2,1/2}=2\gamma_{3/2,3/2}$ between the residue of the nucleon pole, $\gamma_{1/2,1/2}$, and the residue of the N^* pole, $\gamma_{3/2,3/2}$. Unfortunately, this result is usually destroyed if one does a more detailed calculation, which rarely leads to a linear D function. Another way of getting the Chew-Low result is to use partial-wave superconvergence,[4] assuming N and N^* dominance (see Appendix A). However, this uses the properties of the static model at infinite energies, the very region where it is least likely to be correct. Of course, one can always write a superconvergence relation for the relativistic partial-wave amplitude, but it is not clear how useful such a relation will be, since the left-hand cut is not very well known except near the physical threshold.[5]

In the present paper we have tried to avoid some of these difficulties by looking at amplitudes at fixed momentum transfer t, rather than at fixed angular momentum. Superconvergence relations for such amplitudes have been shown by de Alfaro, Fubini, Furlan, and Rossetti to hold for certain higher-spin amplitudes.[6] They have even been suggested for pseudoscalar meson-baryon scattering in the t-channel $I=2$ state.[7] Unfortunately, it can be shown that no fixed-t superconvergence rule can hold for the πN amplitude. We can, however, define a modified amplitude from which the leading Regge-pole terms have been subtracted out. If we assume Regge-pole dominance at high energies, this new amplitude will fall sufficiently fast (i.e., faster than s^{-1}) to be superconvergent.[8]

In Sec. II the modified superconvergence relation, which expresses an integral over the absorptive part of a πN amplitude in terms of Regge parameters, is derived. In Sec. III we look at this relation for vanishing momentum transfer by expressing the absorptive part in terms of low-energy resonance parameters. It is found that the N and N^* dominate all other resonances. For the Regge parameters we use the results of the recent analysis by Chiu, Phillips, and Rarita,[9] who found two sets of parameters which can fit the high-energy πN data. Their first set leads to a large Regge term, and it is not possible to satisfy our relation. The second set gives a comparatively small Regge contribution. The N and N^* are then the dominant terms and are related essentially through the usual static-model relation. Since this is in fact correct experimentally, our relation is thus satisfied approximately.

In Sec. IV, we make the approximation of dropping t-channel Regge contributions completely, since the analysis of Sec. III shows them to be small. This approximation amounts to saying that one of the πN amplitudes is in fact superconvergent. The resulting superconvergence relation is examined for nonvanishing t, and is shown to lead to a relativistic version of the static model. An infinite number of resonant states is needed to satisfy our relation exactly, but for low-lying resonances the relations virtually decouple into a set of statements, each concerning only a finite number of resonances. (This decoupling is exact, in the limit of complete mass degeneracy.) One set of states

* Work supported in part by the U. S. National Science Foundation.
† Alfred P. Sloan Foundation Fellow.
[1] G. F. Chew and F. E. Low, Phys. Rev. **101**, 1570 (1956).
[2] G. F. Chew, Phys. Rev. Letters **9**, 233 (1962).
[3] F. E. Low, Phys. Rev. Letters **9**, 277 (1962).
[4] This was first suggested by Y. Nambu. For a recent application, see, for example, M. Goldberg, Phys. Letters **24B**, 71 (1967).
[5] If, however, we keep only the short cuts from N and N^* exchange, we are led to essentially the same result as in the static theory.
[6] V. De Alfaro, S. Fubini, G. Furlan, and C. Rossetti, Phys. Letters **21**, 5 (1966).
[7] P. Babu, F. J. Gilman, and M. Suzuki, Phys. Letters **24B**, 65 (1967).
[8] After the completion of this work, we came to know that the idea of writing superconvergence relations for such modified amplitudes has also been proposed by A. A. Logunov, L. D. Soloviev, and A. N. Tavkhelidze [Phys. Letters **24B**, 181 (1967)]. These authors did not apply it to the B^+ amplitude, however.
[9] C. B. Chiu, R. G. N. Phillips, and W. Rarita, Phys. Rev. **153**, 1485 (1967).

which might be used to satisfy these relations consists of the N, $N^*(1238)$, and their Regge recurrences. Under certain circumstances, it is possible to satisfy the superconvergence relation by assuming only that the resonances corresponding to a single value of the orbital angular momentum are mass degenerate.

We can also derive the Adler partially conserved axial-vector current (PCAC) self-consistency condition on πN scattering,[10] either by using static-model kinematics or by assuming near-degenerate masses for the resonances which saturate the superconvergence relation.

We show that a theoretical basis for superconvergence is the statement that the equal-time commutator of two pion currents vanishes. This helps to understand why superconvergence is closely related to static-model results, in light of the work of Cook, Goebel, and Sakita, who got P-wave static-model results from the requirement that the dipole moments of the pion currents formed a commutative Lie algebra.[11] Because such algebras are noncompact, one can also understand why an infinite number of states is needed to satisfy the superconvergence relation.

A possible theoretical link between the Adler PCAC relation and superconvergence is a model of exact (or nearly exact) $SU(3)\times SU(3)$ with vector and axial-vector Yang-Mills mesons. For exact symmetry, we follow Nambu[12] and couple zero-mass pions to the axial-vector current in such a way that the axial-vector current is conserved. In this model, Adler's relation follows at once, but this model has the additional property of a superconvergent πN amplitude.

II. MODIFIED SUPERCONVERGENCE RELATION

Let the function $f(s)$ satisfy an unsubtracted dispersion relation

$$f(s)=\frac{1}{\pi}\int ds'\,\mathrm{Im}f(s')(s'-s)^{-1}. \qquad (1)$$

Suppose now that this function behaves for large s as s^a, with $a<-1$. Then it follows from Eq. (1) that for consistency we must have a "superconvergence" relation[6]

$$\int ds'\,\mathrm{Im}f(s')=0. \qquad (2)$$

If $a<2$ we can derive an additional relation

$$\int ds's'\,\mathrm{Im}f(s')=0\,, \quad \text{etc.} \qquad (3)$$

If we wish to apply this procedure to the invariant amplitude at fixed momentum transfer t, we must first know its asymptotic behavior. This is usually most easily obtained from a knowledge of the leading Regge poles, which give a behavior $s^{\alpha(t)-n}$, where $s=(\text{energy})^2$, n is an integer >0, and $\alpha(t)$ is the Regge trajectory. Thus if $\alpha(t)-n<-1$, we have superconvergence. Now in πN scattering we have two invariant amplitudes A and B, which give the T matrix through the relation

$$T=-A+\tfrac{1}{2}i\gamma\cdot(q_1+q_2)B\,, \qquad (4)$$

where q_1 and q_2 are the initial and final meson four-momenta, so that $t=(q_1-q_2)^2$. A Regge pole will give the behavior $A\propto s^\alpha$ and $B\propto s^{\alpha-1}$. Now the leading Regge trajectories are the P, P' in the $I=0$ and the ρ in the $I=1$ channel states. Since all of these have $\alpha>0$ (at least near $t=0$) we cannot have $\alpha-n<-1$, and so there is no superconvergence relation for the amplitude itself.

Suppose we now assume that the P,P', and ρ are the only trajectories with $\alpha(0)>0$. This is strongly suggested by high-energy data, which do not seem to require any other trajectories, and seem to imply that all other singularities are quite far to the left in the angular-momentum plane., Of course, there is always the specter of Regge cuts, which would certainly interfere with any superconvergence relations, but throughout this paper we shall ignore any possible cut contributions. If, therefore, we subtract out the explicit P, P', ρ contributions we will have

$$\begin{aligned} B-B_{\text{Regge}}&<\text{const}\,s^{-1} \quad \text{as} \quad s\to\infty\,,\\ A-A_{\text{Regge}}&<\text{const} \quad\quad \text{as} \quad s\to\infty\,. \end{aligned} \qquad (5)$$

We can therefore write a superconvergence relation for $B-B_{\text{Regge}}$. We still cannot write one for $A-A_{\text{Regge}}$ unless we make the much stronger assumption that there are no trajectories with $\alpha>-1$. We shall therefore not consider the A amplitude any further in the present paper.

Actually, of the two B amplitudes B^+ and B^- corresponding (except for constants) to the t-channel $I=0$ and $I=1$ states, only the B^+ amplitude gives a nontrivial superconvergence relation. This is because $\mathrm{Im}B^-(s',t)$ is antisymmetric with respect to the variable $(s'+\frac{1}{2}t-M^2-1)$, so that when we integrate it over all s' for fixed t we get zero identically. (This, of course, is also true of $\mathrm{Im}B^-(s't)$. On the other hand, $\mathrm{Im}B^+(s',t)$ is symmetric in this variable, so that Eq. (5) becomes

$$\pi g^2+\int_{M^2+1-t/2} ds'\,\mathrm{Im}[B^+-B^+{}_{\text{Regge}}]=0\,, \qquad (6)$$

where the first term is the contribution of the nucleon, with $g^2/4\pi=14.4$.

Unfortunately, the B^+ amplitude is not related in a simple way to cross sections. One way of testing Eq. (6) would be to make a partial-wave approximation $B^+\simeq B^+{}_{\text{PW}}$ for $s<s_0$ and the Regge approximation $B^+\simeq B^+{}_{\text{Regge}}$ for $s>s_0$, where s_0 is the approximate separation point between the low- and high-energy

[10] S. L. Adler, Phys. Rev. **137**, B1022 (1965).
[11] T. Cook, C. J. Goebel, and B. Sakita, Phys. Rev. Letters **15**, 35 (1965).
[12] Y. Nambu and G. Jona-Lasino, Phys. Rev. **122**, 345 (1961).

regions (s_0 is related to the strip width of Chew and Frautschi[13]). Then we would have

$$\pi g^2+\int^{s_0} ds'[\mathrm{Im}B^+{}_{\mathrm{PW}}-\mathrm{Im}B^+{}_{\mathrm{Regge}}]=0. \quad (7)$$

In evaluating this expression it is not too important which specific Regge form is used, provided that it is a good approximation for $s>s_0$. Thus one could use either the canonical form or the asymptotic form. The Regge contribution to Eq. (7) will be essentially the same provided these forms give approximately the same amplitude for $s>s_0$. This is discussed in more detail in Appendix C.

To evaluate $\mathrm{Im}B^+{}_{\mathrm{PW}}$, we could approximate it in terms of resonances. Actually Barger and Cline[14] have argued that the resonance and Regge regions may, in fact, overlap. They were able to explain a considerable amount of intermediate-energy data by a superposition of resonances and a Regge background. By using the asymptotic form for the latter they were able to reproduce the backward $\pi^- p$ elastic-differential cross section at lab momenta as low as 1.6 BeV/c ($s\simeq 200$). Thus a more useful approximation than the one leading to Eq. (7) might be set $B^+\simeq B^+{}_{\mathrm{Res}}$ for $s<s_0$, $B^+\simeq B^+{}_{\mathrm{Res}}+B^+{}_{\mathrm{Regge}}$ for $s_0<s<s_1$. Then s_1 would be some point just above the highest resonance, while s_0 is the lowest point up to which a Barger-Cline analysis is applicable (we could take $s_0\simeq 200$). With this approximation Eq. (6) becomes

$$\pi g^2+\int^{s_1} ds'\ \mathrm{Im}B^+{}_{\mathrm{Res}}-\int^{s_0} ds'\ \mathrm{Im}B^+{}_{\mathrm{Regge}}=0. \quad (8)$$

Since s_0 is fairly low, we might expect the Regge term to be comparatively small. We shall see in the following section that this is in fact the case.

III. SUM RULE AT $t\simeq 0$

To evaluate Eq. (7) let us first consider $B^+{}_{\mathrm{Regge}}$. The canonical form[15] is actually almost the same as the asymptotic form for $s>200$:

$$B^+{}_{\mathrm{Regge}}(s,t)=\sum_{P,P'}\frac{g(t)}{\sin\pi\alpha(t)}[P_{\alpha(t)}'(-z_t)-P_{\alpha(t)}'(z_t)], \quad (9)$$

where

$$2p_tq_tz_t=s+p_t^2+q_t^2,$$
$$p_t^2=\tfrac{1}{4}t-M^2,\quad q_t^2=\tfrac{1}{4}t-1.$$

Here $M=$ nucleon mass, with pion mass $=1$, and $g(t)$ is a function related to the residue of the pole (P and P') we are considering. If we use the asymptotic form

$$P_\alpha(z_t)\sim C(\alpha)z_t^{\alpha},$$

where

$$C(\alpha)=[2^\alpha\Gamma(\alpha+\tfrac{1}{2})][\pi^{1/2}\Gamma(\alpha+1)]^{-1},$$

we get an expression which can be compared with the parametric form given by Chiu, Phillips, and Rarita[9]:

$$B=-D_0\exp(D_1t)\alpha^2(\alpha+1)(E_L/E_0)^{\alpha-1}\times[(e^{-i\pi\alpha}-1)/\sin\pi\alpha], \quad (10)$$

where $2ME_L=s-M^2-1$, and E_0 is a scale factor corresponding to 1 BeV. Such a comparison at $t=0$ gives

$$g(0)=-D_0\alpha(\alpha+1)m_0^{1-\alpha}C(\alpha)^{-1}, \quad (11)$$

where $m_0=E_0$ (pion mass).

Let us return to a general value of t for the time being. From Eq. (9), for $s'>M^2+1-\frac{1}{2}t$,

$$\mathrm{Im}B^+{}_{\mathrm{Regge}}(s',t)=\sum_{P,P'}g(t)\left\{2p_tq_t\delta[s'+(p_t+q_t)^2]-P_\alpha{}'\left(\frac{s+p_t^2+q_t^2}{-2p_tq_t}\right)\Theta[s'+(p_t+q_t)^2]\right\}. \quad (12)$$

If we integrate over s', we find for the last term in Eq. (8)

$$\int^{s_0} ds'\ \mathrm{Im}B^+{}_{\mathrm{Regge}}(s',t)=2p_tq_t\sum_{P,P'}g(t)P_\alpha\left(\frac{s_0+p_t^2+q_t^2}{-2p_tq_t}\right). \quad (13)$$

If we use the asymptotic form for P_α, then at $t=0$, Eqs. (13) and (11) give

$$\int^{s_0} ds'\ \mathrm{Im}B^+{}_{\mathrm{Regge}}(s',0)=2Mm_0\sum_{P,P'}D_0\alpha(\alpha+1)\left(\frac{s_0-M^2-1}{2Mm_0}\right)^{\alpha}. \quad (14)$$

To evaluate $\mathrm{Im}B^+{}_{\mathrm{Res}}$ we can expand in terms of partial waves[16] and states of definite isotopic spin I:

$$B^+=\tfrac{1}{3}(B^{1/2}+2B^{3/2}), \quad (15)$$

where

$$B^I(s,0)=\sum_l\left(\frac{4\pi}{q^2}\right)\{(l+1)[E-M(l+1)]f_{l+}{}^I(W)+l(E+Ml)f_{l-}{}^I(W)\} \quad (16)$$

and $f_{l+}{}^I(W)$ is the partial-wave amplitude corresponding to total angular momentum $J=l+\frac{1}{2}$ and energy $W=\sqrt{s}$. If we have a resonance at $W=W_R$ with a partial width of $\Gamma_{l\pm}{}^I$ for decay into the πN channel, we will have a contribution

$$\mathrm{Im}f_{l\pm}{}^I(W)=\frac{\pi}{2q}\Gamma_{l\pm}{}^I\delta(W-W_R), \quad (17)$$

---

[13] G. F. Chew and S. C. Frautschi, Phys. Rev. **123**, 1478 (1961).
[14] V. Barger and D. Cline, Phys. Rev. Letters **16**, 913 (1966); V. Barger and M. Olsson, Phys. Rev. **151**, 1123 (1966).
[15] V. Singh, Phys. Rev. **129**, 1889 (1963).
[16] See, for instance, G. F. Chew, M. L. Goldberger, F. E. Low, and Y. Nambu, Phys. Rev. **106**, 1337 (1957).

TABLE I. Contributions to $(4\pi)^{-2}\int ds'\,\mathrm{Im}B^{+}{}_{\mathrm{Res}}$ from each bound state or resonance, using Eqs. (15)–(17). All masses and partial widths Γ are in MeV and are taken from the Particle Tables prepared for the XIII Conference on High-Energy Physics.[a]

Resonance	$J^P(l_{2I,2J})$	Γ(MeV)	$(4\pi)^{-2}\int ds'\,\mathrm{Im}B^+$
$N(940)$	$1/2^+(p_{11})$		14.4
$N_{1/2}'(1400)$	$1/2^+(p_{11})$	120	2.22
$N_{1/2}^*(1570)$	$1/2^-(s_{11})$	39	0.02
$N_{1/2}^*(1518)$	$3/2^-(d_{13})$	40	1.27
$N_{1/2}^*(1700)$	$1/2^-(s_{11})$	216	0.10
$N_{1/2}^*(1688)$	$5/2^-(d_{15})$	35	−0.55
$N_{1/2}^*(1688)$	$5/2^+(f_{15})$	73	2.58
$N_{1/2}^*(2190)$	$7/2^-$	60	1.27
$N_{1/2}^*(2650)$	$11/2^-$	21	0.52
$N_{3/2}^*(1236)$	$3/2^+(p_{33})$	120	−14.82
$N_{3/2}^*(1670)$	$1/2^-$	79	0.08
$N_{3/2}^*(1920)$	$7/2^+$	100	−3.51
$N_{3/2}^*(2420)$	$11/2^+$	27.5	−1.05
$N_{3/2}^*(2850)$	$15/2^+$	9	−0.40

[a] *Proceedings of the XIII International Conference on High-Energy Physics, Berkeley, California, 1966* (University of California Press, Berkeley, California, 1967). See also Ref. 17.

in the narrow-width approximation. If we combine Eqs. (15)–(17), we obtain an expression for $\int ds'\mathrm{Im}B^+$.

The various resonance parameters we have used were taken from Ref. 17. We have tabulated in Table I the contributions of the various particles to $\mathrm{Im}B^{+}{}_{\mathrm{Res}}$. We see in particular that the contribution of the $N^*(1236)$ almost cancels the contribution $\pi g^2=4\pi^2(14.4)$ of the N. This, of course, is just the usual conclusion of the static model, which assumes that the N and N^* are the only particles present, and that there are no t-channel contributions. It is trivial to show in this limiting case that Eq. (8) is equivalent to the Chew-Low result.

In our relativistic case, of course, we do have other contributions. From Table I, however, we see that the contribution of each of the other resonances is quite small compared with that of the N^*. If we add them all up, we get $4\pi^2(2.13)$, which is also not too large compared with the N^*.

The remaining relativistic term is the Regge contribution (14). Here, unfortunately, the D_0's are very poorly known. Chiu, Phillips, and Rarita give two possible solutions.[9] We have chosen their solution (b) because it has the property of allowing a secondary bump in the elastic differential cross section, similar to that observed. We then get

$$-\int^{s_0} ds'\,\mathrm{Im}B^{+}{}_{\mathrm{Regge}}(s',0)\approx 4\pi^2\times 2.6\,,$$

with $s_0\simeq(2M)^2\simeq 180$. Thus the total contribution of the left-hand side of Eq. (8) is $\simeq 4\pi^2(4.7)$. Since $\pi g^2\sim 4\pi^2(14.4)$, our equation is therefore satisfied to about 30%.

In view of the difficulty of determining the Regge-pole contribution to B^+ from experimental data, one might well wish to consider the analysis of this section from a different point of view, and regard the modified superconvergence relation as an additional constraint to be imposed on the Regge parameter determined from the experimental data. Needless to say, the parameters we had to use for some of the high-energy resonances are hardly known much better. Indeed, it is not very likely that we have even included all the resonances that occur in πN scattering. The static model, for instance, would suggest resonances in the p_{33}, d_{37}, and d_{35} states (see Appendix A). Their inclusion would certainly affect our results, although it is unlikely to change any of our qualitative conclusions.

[17] A. M. Rosenfeld *et al.*, Rev. Mod. Phys. **39**, 1 (1967).

IV. SUPERCONVERGENCE AND THE STATIC MODEL

In the last section, we found that the Regge-pole contribution was small compared to any single resonance. In this section, we shall simply drop the t-channel Regge-pole contribution altogether. The consequence is a sort of relativistic static model, which can be based on the hypothesis that the equal-time commutator of pion currents vanishes. The essential difference between superconvergence for fixed t (as we consider here) and superconvergence in the partial waves (fixed $\cos\theta$) is that we are evaluating the contribution of a given intermediate state in the same way as we would calculate the pole graphs in a relativistic field theory, using the appropriate Feynman propagator. Crossing symmetry becomes trivial to satisfy, and there is no need to project out partial waves with their attendant left-hand cuts.

To evaluate $\mathrm{Im}B^{+}{}_{\mathrm{Res}}$ we use the expansion

$$\mathrm{Im}B^I(s,t)=4\pi\sum_l\left\{\left[\frac{P_{l+1}'(z)}{E+M}-\frac{P_l'(z)}{E-M}\right]\times\mathrm{Im}f_{l+}-\left[\frac{P_{l-1}'(z)}{E+M}-\frac{P_l'(z)}{E-M}\right]\mathrm{Im}f_{l-}\right\},\quad(18)$$

where $z=1+t/2q^2$. If we insert Eq. (18) into Eq. (8) using Eqs. (15) and (17), we obtain

$$\pi g^2+4\pi^2\sum_{I,l}C_I\left\{\frac{W_R\Gamma_{l-}{}^I}{q_R(E_R-M)}P_l'(z_R)-\frac{W_R\Gamma_{l+}{}^I}{q_R(E_R-M)}P_l'(z_R)-\frac{W_R\Gamma_{(l+1)-}{}^I}{q_R(E_R+M)}P_l'(z_R)+\frac{W_R\Gamma_{(l-1)+}{}^I}{q_R(E_R+M)}P_l'(z_R)\right\}=0,\quad(19)$$

where

$$C_I=\tfrac{1}{3}\ \text{if}\ I=\tfrac{1}{2},\qquad =\tfrac{2}{3}\ \text{if}\ I=\tfrac{3}{2},$$

and the subscript R denotes that we are to evaluate at $W=W_R$. Of course, we are to set $\Gamma=0$ whenever there is no resonance in a given state.

Suppose we assume complete mass degeneracy. The z_R are then all equal, and Eq. (19) becomes an expansion in the P_l'. Since these form an orthonormal set, each coefficient of P_l' must vanish separately:

$$\pi g^2\delta_{l1}+4\pi^2\sum_I C_I\frac{W_R}{q_R}\left\{\frac{\Gamma_{l-}{}^I-\Gamma_{l+}{}^I}{E_R-M}-\frac{\Gamma_{(l+1)-}{}^I-\Gamma_{(l-1)+}{}^I}{E_R+M}\right\}=0. \quad (20)$$

Actually, the last term is usually quite small. In the static limit it is exactly zero. We therefore have the approximate equation

$$\pi g^2+4\pi^2\sum_I C_I\left\{\frac{W_R\Gamma_{l-}{}^I}{q_R(E_R-M)}-\frac{W_R\Gamma_{l+}{}^I}{q_R(E_R-M)}\right\}=0. \quad (21)$$

It is interesting to note that, within this approximation, Eq. (21) satisfies Eq. (19) even if we only assume mass degeneracy for the particles in a given orbital angular-momentum state l.

If we keep only the N and N^* for $l=1$, Eq. (21) is almost exactly satisfied experimentally if we take for W_R the mass of the N^*. In the static limit it is just the usual Chew-Low result $\gamma_{1/2,1/2}=2\gamma_{3/2,3/2}$. For $l=2$, we evaluated the various contributions in Eq. (21) by assuming that each reduced width in the degenerate case is approximately the same as it is for the corresponding physical resonance. Since $E-m\propto q^2$ at threshold this means that we must set

$$G_{l\pm}{}^I=\left[\frac{W_R\Gamma_{l\pm}{}^I}{q_R{}^{2l-1}(E_R-M)}\right]_{\text{degenerate}}\approx\left[\frac{W_R\Gamma_{l\pm}{}^I}{q_R{}^{2l-1}(E_R-M)}\right]_{\text{physical}}. \quad (22)$$

It is then found that the d_{13} and d_{15} resonances alone satisfy Eq. (21) very poorly. This is not very surprising, since the static model, for instance, suggests the existence of d_{33} and d_{35} resonances (see Appendix A). Indeed, it is simple to check that if we use static kinematics Eq. (21) is exactly satisfied by the solution of the d-wave static model.

The highest orbital angular momentum we have checked is $l=3$. Here we have the two first Regge recurrences of the N and N^*, the $N^*(1688)$ with $J^P=\frac{5}{2}^+$, and the $N^*(1920)$ $(\frac{7}{2}^+)$, which are also approximately consistent with the static model.[16,18] If we use the experimental values for the partial widths of these resonances, we get $(G_{3-}{}^{1/2}/G_{3+}{}^{3/2})\simeq 3$. Equation (21) for $l=3$ predicts $(G_{3-}{}^{1/2}/G_{3+}{}^{3/2})=2$, which is certainly consistent with the experimental value in view of the experimental and theoretical uncertainties.

The exact relation (19) without mass degeneracy cannot be satisfied for any finite number of resonances. In the limit of mass degeneracy, one possible infinite sequence of states would be the N, N^*, and their Regge recurrences, that is, baryons with $I=\frac{1}{2}$, $J=\frac{1}{2},\frac{5}{2},\frac{9}{2},\cdots$, and $I=\frac{3}{2}$, $J=\frac{3}{2},\frac{7}{2},11/2,\cdots$,. In fact, the values given in Table I seem to indicate that these are in any case the dominant resonances in the πN problem. In the mass-degenerate limit, (20) is satisfied when the widths of all the $I=\frac{3}{2}$ resonances are equal, and equal to twice the widths of all the $I=\frac{1}{2}$ states. If mass degeneracy is not assumed, and the exact relation (19) is confronted with experiment using only the four lowest states [N, N^*, $N^*(1688)$, $N^*(1920)$], the N and N^* contributions cancel to a very high degree of accuracy. We shall, as in (21), ignore the terms in $(E_R+M)^{-1}$. The contributions of the $N^*(1688)$ and $N^*(1920)$ are ambiguous, since there are actually three relations to be satisfied, corresponding to the coefficients of t^0, t, and t^2, and these relations would be the same only if these two resonances had the same mass. It turns out that the coefficient of t in (19) very nearly vanishes when we insert experimental paritial widths and masses, but the relations for the constant coefficient and the t^2 coefficient are satisfied only to about 20–30% of the sum of the absolute values of the terms in (19). This is, of course, of the same order as the contribution of t-channel Regge poles, which we are ignoring in this section, so that the agreement may be considered satisfactory.

The superconvergence relations of this section have another interesting consequence: They give (approximately) the Adler self-consistency condition in the A^+ amplitude.[10] This condition says that

$$A^+(\nu=\nu_B=0,q_1{}^2=0)=g^2/M, \quad (23)$$

where

$$\nu=(s-u)/4M,\ u=2(M^2+1)-s-t,\ \nu_B=(-\mu^2+t)/4M,$$

and q_1 is the momentum of one of the pions. Recall that Adler derived (23) from the PCAC hypothesis, which has not yet been used in the present work. We shall interpret the condition (23) as holding at the point $s=u=M^2$, $t=0$, with the pion mass μ set to zero everywhere. (This is slightly different from Adler's original version, which had only one pion off the mass shell.) To see how it arises, we insert the expansion

$$\mathrm{Im}A^I(s,t)=4\pi\sum\left\{\left[\frac{W+M}{E+M}P_{l+1}'(z)+\frac{W-M}{E-M}P_{l-1}'(z)\right]\mathrm{Im}f_{l+}{}^I-\left[\frac{W+M}{E+M}P_{l-1}'(z)+\frac{W-M}{E-M}P_l'(z)\right]\mathrm{Im}f_{l-}{}^I\right\} \quad (24)$$

into the dispersion relation for $A^+=\frac{1}{3}(A^{1/2}+2A^{3/2})$:

$$A^+=\frac{1}{\pi}\int ds'\,\mathrm{Im}A^+(s',t)\left[\frac{1}{s'-s}+\frac{1}{s'-u}\right]. \quad (25)$$

[18] L. A. P. Balázs, V. Singh, and B. M. Udgaonkar, Phys. Rev. **139**, B1313 (1965).

If we make the resonance approximation (17), we obtain

$$A^+=4\pi\sum_{I,l}\frac{C_I}{q_R}\left(\frac{W_R+M}{E_R+M}\right)W_R[\Gamma_{l+}{}^I P_{l+1}'(z_R)-\Gamma_{l-}{}^I P_{l-1}'(z_R)]\left(\frac{1}{M_R{}^2-s}+\frac{1}{M_R{}^2-u}\right)$$
$$-4\pi\sum_{I,l}\frac{C_I}{q_R}\left(\frac{W_R-M}{E_R-M}\right)W_R(\Gamma_{l-}{}^I-\Gamma_{l+}{}^I)P_l'(z_R)\left(\frac{1}{M_R{}^2-s}+\frac{1}{M_R{}^2-u}\right). \quad (26)$$

In (23), the nucleon pole is not included, of course. We evaluate (23) at the point $t=0$ (or $z_R=1$), $s=u=M^2$, to get

$$A^+(s=u=M^2,\,t=0)=\frac{16\pi M_{33}\Gamma_{33}}{3q_{33}(M_{33}+M)(E_{33}-M)}+8\pi\sum_{l,I;\,l\geq 2}\frac{C_IW_R}{q_R}P_l'(1)$$
$$\times\left[\frac{\Gamma_{(l-1)+}{}^I-\Gamma_{(l+1)-}{}^I}{(E_R+M)(W_R-M)}-\frac{\Gamma_{l-}{}^I-\Gamma_{l+}{}^I}{(E_R-M)(W_R+M)}\right], \quad (27)$$

where we have explicitly isolated the contribution of the 33 resonance. The terms in brackets are small; indeed, in the limit of complete mass degeneracy, they would vanish by (19). Even without complete mass degeneracy, we argue as follows: The first term in brackets is less than the second by a factor $(W_R-M)(W_R+M)^{-1}$ [remembering that $E_R\pm M=(W_R\pm M)^2/2W_R$ when $\mu^2=0$]. If this term is ignored, then the second term will vanish by (21), as long as masses corresponding to the same value are degenerate. Then A^+ is dominated by the 33 term. But our previous superconvergence relation (21) says that the 33 term in (27) is related to the nucleon-pole parameters. Finally, we get

$$A^+(s=u=M^2,t=\mu^2=0)\simeq 2g^2/(M_{33}+M)\,, \quad (28)$$

which, of course, would be the Adler relation in the limit of mass degeneracy.

Actually, if we are willing to make the static approximation for some of our kinematic factors, we can easily derive the Adler result without making resonance approximations, degenerate or otherwise. Comparing Eqs. (18) and (24) directly, we see that

$$\mathrm{Im}A^I(s,t)\approx-(W-M)\,\mathrm{Im}B^I(s,t) \quad (29)$$

if we drop the terms which would vanish in the static limit. From Eq. (25) we thus have

$$A^+(s=u,t=0)\approx\frac{1}{\pi}\int ds'\frac{2(W'-M)}{W'^2-M^2}\,\mathrm{Im}B^+(s',0)$$
$$\approx\frac{-1}{M\pi}\int ds'\,\mathrm{Im}B^+(s',0)\,, \quad (30)$$

where we have made the static approximation $W'\approx M\gg 1$. Using Eq. (7) we are immediately led to Eq. (23), if we ignore the Regge term, which represents high-energy effects, and which, as we have seen, is small.

V. THEORETICAL SPECULATIONS

The original approach to superconvergence[19] was via current-commutation relations. We might ask if there is a corresponding commutation relation for pion currents which reproduces our modified superconvergence relations, in particular the relativistic static model of Sec. IV. Such a commutation relation between operators would furnish a global characterization of modified superconvergence rules, to replace the large set of individual rules which hold for invariant amplitudes for a specific process (e.g., $\pi N\to\pi N^*$). This kind of program has already been carried out for vector currents by Bardakci and Segrè.[20]

It is a general rule that the behavior of a scattering amplitude at large values of the invariants depends on some sort of equal-time commutator. This concept has been explored in detail by Bjorken,[21] and Cornwall[22] has shown how the same analysis can be carried through using a modified form of the Jost-Lehmann-Dyson representation. A similar analysis for πN nonforward scattering has been made by Domokos and Karplus,[23] and we refer the reader to that paper for details. Let us write a commutator between two nucleons in the Breit frame $\mathbf{p}=-\mathbf{p}'$:

$$\langle p'|[J^\alpha(\mathbf{x},0),J^\beta(y,0)]|p\rangle=\tfrac{1}{2}\tau^\gamma\epsilon^{\alpha\beta\gamma}\delta(\mathbf{x}-\mathbf{y})C(t)+i\delta_{\alpha\beta}\partial_k\delta(\mathbf{x}-\mathbf{y})D_k(t)\,, \quad (31)$$
$$D_k(t)=-(i/M)\chi^\dagger(\boldsymbol{\sigma}\times\mathbf{p})_k\chi D(t)\,,$$

where α, β, and γ are isospin indices, J is the pion current, and $t=(p-p')^2$. Karplus and Domokos[23] relate C and D to the electric isovector and magnetic isoscalar form factors, respectively. Whatever C and D may be, they come from t-channel exchange processes, and we might then follow the principles of the Chew-Low static model by simply setting C and $D=0$, which amounts to ignoring t-channel exchange processes completely.

The relation of (31) to superconvergence is found from Eqs. (15) and (16) of Karplus and Domokos:

$$\nu\to\infty:\, B^+(\nu,t)\sim -iMD(t)/\nu$$
$$(1-t/4M^2)^{1/2}A^-(\nu,t)+\nu B^-(\nu,t)\sim -iC(t)/\nu\,. \quad (32)$$

[19] S. Fubini and G. Segrè, Nuovo Cimento **65**, 641 (1966).

[20] K. Bardakci and G. Segrè, Phys. Rev. **159**, 1263 (1967).

[21] J. D. Bjorken, Phys. Rev. **148**, 1467 (1966).

[22] J. M. Cornwall, Phys. Rev. Letters **16**, 1174 (1966).

[23] G. Domokos and R. Karplus, Phys. Rev. **153**, 1492 (1967).

Clearly, if C and D vanish, we get the superconvergence relations. We propose, as a relativistic generalization of the static model, that *pion currents commute on the light cone*:

$$\delta(x^2)[J^\alpha(x),J^\beta(0)]=0\,, \qquad (33)$$

which amount to setting C and D equal to zero. We get two superconvergence rules:

$$\frac{1}{\pi}\int d\nu\ \mathrm{Im}B^+(\nu,t)=0\,, \qquad (34a)$$

$$\frac{1}{\pi}\int d\nu\left[\left(1-\frac{t}{4M^2}\right)^{1/2}\mathrm{Im}A^-(\nu,t)+\nu\ \mathrm{Im}B^-(\nu,t)\right]=0\,. \qquad (34b)$$

The first we have already explored in some detail in Sec. IV. Although we have not yet carried out a detailed analysis of the second rule for $t\neq 0$, it is easy to study (34b) for $t=0$, where the integrand is simply related to total cross sections. Preliminary calculations seem to indicate that it is in fact satisfied (see also Ref. 8). It is very instructive to consider it in the approximation saving only the N and N^*. The result is the condition

$$M(E-M)g^2=M^*(E^*-M)\times[(8\pi/3)M^*\Gamma^*/q^*(E^*-M)]\,, \qquad (35)$$

where E and E^* are to be evaluated at $S=M^2$, $S=M^{*2}$, respectively. We have factored out $M(E-M)$ for each pole for the following reasons: If $M(E-M)$ is removed from the left side of (35), and $M^*(E^*-M)$ from the right, the resulting equation is just the condition arising from (34a). Thus, if N and N^* were mass degenerate, (34b) would be redundant. As it is, if one puts physical values into (35), the two terms have opposite signs. But both terms contribute only a very small amount to the full integral (34b), and there is every reason to believe that higher resonances will correct the situation. They contribute a tiny amount because of the factor $E-M$, which is of the order $\mu^2/2M$. At this stage, therefore, it is eminently plausible to accept (33) as a relativistic statement of the static model. Of course, (33) is only an approximation, since the functions C and D of (31) are certainly not zero. But we might use, e.g., the model of Karplus and Domokos to evaluate the corrections to the static model which come from t-channel vector-meson exchange, using Eq. (32).

Equation (33) is relevant to the static model as expressed in the strong-coupling theory, as Cook, Goebel, and Sakita have pointed out.[11] These authors consider a commutative algebra formed from the dipole moments of the pion current, and show that a (necessarily infinite) number of resonances, beginning with the N and N^*, form a representation of the algebra. Most of these resonances have $I\geq\frac{5}{2}$, and therefore do not appear in πN scattering. The present authors are currently exploring the consequences of Eq. (33) when sandwiched between higher resonances, and we hope to report on this work in the near future.

The Adler constraint on A^+ depends only on PCAC and nothing else, yet we were able to derive it using only superconvergence, and not PCAC. One possible link between PCAC and superconvergence is a hypothetical scheme to enforce nearly exact chiral $SU(2)\times SU(2)$ symmetry with the aid of V and A Yang-Mills mesons, plus pions and a sigma meson. It is easiest to begin by thinking of a world where $SU(2)\times SU(2)$ is exact, and the pions have zero mass, as proposed by Nambu.[12] PCAC can then be gotten by giving the pions their physical mass. One begins by requiring that the amplitude $T_{\mu\nu}(k^1,k^2,p^1,p^2)$ describing the elastic scattering of a baryon (momentum p^1) and an axial-vector Yang-Mills meson (momentum k^1) obey $k_\mu^1T^{\mu\nu}=k_\nu^2T^{\mu\nu}=0$, in the limit of exact chiral symmetry. Since pions couple to axial-vector mesons, part of this constraint involves πN scattering. If we describe πN scattering in terms of resonances plus Yang-Mills quanta exchanged in the t channel, the resulting conditions appear to be modified superconvergence conditions. Details of this approach will be published later.

VI. CONCLUSION

We have seen that it is possible to write a modified superconvergence relation for the B^+ amplitude, which relates the low-energy resonances to high-energy Regge parameters. The later term is small and it is found that the relations between the resonances agree more or less with the predictions of the static model. In a sense, then, our approach can be thought of as a relativistic justification for the success of that model.

Of course, if we are willing to assume that there are no residual singularities in the angular-momentum plane with $\alpha\geq -1$ when we extract the P,P' and ρ contributions, we can write additional relations. We have a superconvergence relation for $(A^- - A^-_{\mathrm{Regge}})$ as well as one for $\nu(B^- - B^-_{\mathrm{Regge}})$. Actually, the sum of these two is particularly convenient at $t=0$, since the imaginary part is simply related to the total cross section through the optical theorem. So far, only preliminary calculations have been made, but these seem to indicate that this relation is in fact satisfied.

In addition to superconvergence relations for amplitudes it may be useful to look at relations for derivatives of amplitudes, with respect to both t and s. The former can, of course, be obtained trivially from the relations for the amplitude at any t and have the effect of eliminating the low partial waves. The latter have to be obtained by first differentiating both sides of a dispersion relation with respect to s and then integrating by parts (see Appendix B). This leads to a dispersion relation for the derivative of the amplitude. From this we can get a superconvergence relation. The advantage of dealing with derivatives of amplitudes is that they have better convergence properties, with a behavior at infinity which is down by a factor of s^{-1}. One disadvantage may be that they probably depend more on some of the detailed structure of amplitudes. In the case of resonances, for instance, they would presumably depend on the shape as well as the width of a resonance.

ACKNOWLEDGMENTS

We would like to acknowledge our debt to Professor S. Fubini and Dr. F. Segrè, whose lectures on superconvergence and current-commutation relations inspired us to carry out this work. In addition we would like to thank our colleagues at UCLA, especially Professor S. Patil, for stimulating conversations.

APPENDIX A

We discuss here briefly the πN static model in the P and D waves. The partial-wave amplitude

$$g_{IJ}(\omega)=\frac{e^{i\delta}\sin\delta}{q^{2l+1}} \tag{A1}$$

satisfies the crossing relation

$$g_{IJ}(-\omega')=\sum_{I'J'}\alpha_{II'}\beta_{JJ'}g_{I'J'}(\omega'), \tag{A2}$$

where δ=phase shift, ω=pion energy, $q^2=\omega^2-1$, and α and β are the isotopic spin-crossing[18] matrices which are given by

$$\alpha=\begin{pmatrix}-\frac{1}{3} & \frac{4}{3}\\ \frac{2}{3} & \frac{1}{3}\end{pmatrix} \tag{A3}$$

for $I=\frac{1}{2}, \frac{3}{2}$ and by

$$\beta=\frac{1}{2l+1}\begin{pmatrix}-1 & 2l+2\\ 2l & 1\end{pmatrix} \tag{A4}$$

for $J=l-\frac{1}{2}$, $l+\frac{1}{2}$. We can write a dispersion relation for g

$$g_{IJ}(\omega)=\frac{1}{\pi}\int_1^\infty d\omega'\frac{\text{Im}g_{IJ}(\omega')}{\omega'-\omega}+\frac{1}{\pi}\int_1^\infty d\omega'\frac{\text{Im}g_{IJ}(-\omega')}{\omega'+\omega}. \tag{A5}$$

Since unitarity demands that $g_{IJ}(\omega)$ be bounded by ω^{-2l-1}, as can be seen from Eq. (A1), we must have a superconvergence relation for $l\geq 1$

$$\int_1^\infty d\omega'[\text{Im}g_{IJ}(\omega')-\text{Im}g_{IJ}(-\omega')]=0. \tag{A6}$$

We next make the usual assumption that the integral is dominated by resonances and bound states. If we make the usual narrow-width approximation

$$\text{Im}g_{IJ}(\omega)=\pi\gamma_{IJ}\delta(\omega-\omega_{IJ}), \tag{A7}$$

where ω_{IJ} is the position of the particle and ω_{IJ} its reduced width, then Eqs. (A2) and (A6) give

$$\gamma_{IJ}-\sum_{I',J'}\alpha_{II'}\beta_{JJ'}\gamma_{I'J'}=0. \tag{A8}$$

In the P wave, if we keep only the N and N^*, these are four redundant equations giving the usual result $\gamma_{1/2,1/2}=2\gamma_{3/2,3/2}$. This is satisfied quite well by experiment.

If we have two particles in a given state (I,J), we again have Eq. (A8) but with $\gamma_{IJ}\to\gamma_{IJ}+\gamma_{IJ}'$, where γ_{IJ}' is the reduced width for the second particle. Suppose, for instance, that we include the $N_{1/2}'(1400)$ particle (the Roper resonance) along with the nucleon in the $(\frac{1}{2},\frac{1}{2})$ state. Equation (A8) is then no longer satisfied by the experimental widths. The only way to rectify this is to add another particle in the $(\frac{3}{2},\frac{3}{2})$ state with $\gamma_{3/2,3/2}'=\frac{1}{2}\gamma_{1/2,1/2}'$ (it is straightforward to show that an extra particle in any other state will only worsen the situation). This particle is presumably more massive then the N^*. Its inclusion would improve the sum rule (14), since we would have to add a contribution which cancels that of the $N_{1/2}'(1400)$.

Turning now to the D wave, we find that it is impossible to satisfy Eq. (A8) for all possible I and J by keeping only d_{13} and d_{15} resonances. We have to have additional resonances in the d_{33} and d_{35} states. Indeed, Eq. (A8) reduces to the two equations

$$\gamma_{3/2,3/2}=\tfrac{1}{10}(\gamma_{1/2,3/2}+9\gamma_{1/2,5/2}) \tag{A9}$$

and

$$\gamma_{3/2,5/2}=\tfrac{1}{5}(3\gamma_{1/2,3/2}+2\gamma_{1/2,5/2}). \tag{A10}$$

A d_{35} resonance would improve Eq. (14) while a d_{33} particle would worsen it. Equations (A9) and (A10), together with the experimental widths of the d_{13} and d_{15} resonances, suggest that the d_{35} would probably be more important than the d_{33} contribution.

APPENDIX B

We show here how one can set up superconvergence relations for derivatives of amplitudes. Suppose one has a simple dispersion relation for $f(s)$

$$f(s =\frac{1}{\pi}\int_{\xi_0}^\infty ds'\frac{\text{Im}f(s')}{s'-s}. \tag{B1}$$

If we differentiate both sides with respect to s, we have

$$\begin{aligned}f'(s)&=\frac{1}{\pi}\int_{\xi_0}^\infty ds'\,\text{Im}f(s')\frac{\partial}{\partial s}\left(\frac{1}{s'-s}\right)\\ &=-\frac{1}{\pi}\int_{\xi_0}^\infty ds'\,\text{Im}f(s')\frac{\partial}{\partial s'}\left(\frac{1}{s'-s}\right).\end{aligned} \tag{B2}$$

Integrating by parts, we obtain

$$f'(s)=-\frac{1}{\pi}\frac{\mathrm{Im}f(\xi_0)}{\xi_0-s}+\frac{1}{\pi}\int_{\xi_0}^{\infty}ds'\frac{\mathrm{Im}f'(s')}{s'-s}. \tag{B3}$$

If $f'(s)$ decreases faster than s^{-1} at infinity we have the superconvergence relation

$$\mathrm{Im}f(\xi_0)+\int_{\xi_0}^{\infty}ds'\,\mathrm{Im}f'(s')=0. \tag{B4}$$

The above derivation, of course, is valid only if $\mathrm{Im}f(\xi_0)$ is finite. If it is not, we could still get a relation by taking some point $s=\xi_1$ above $s=\xi_0$. Writing Eq. (B2) as

$$f'(s)=-\frac{1}{\pi}\left(\int_{\xi_0}^{\xi_1}+\int_{\xi_1}^{\infty}\right)ds'\,\mathrm{Im}f(s')\frac{\partial}{\partial s'}\left(\frac{1}{s'-s}\right) \tag{B5}$$

and integrating only the second integral by parts, we have

$$f'(s)=\frac{1}{\pi}\int_{\xi_0}^{\xi_1}ds'\frac{\mathrm{Im}f(s')}{(s'-s)^2}+\frac{1}{\pi}\frac{\mathrm{Im}f(\xi_1)}{\xi_1-s}+\frac{1}{\pi}\int_{\xi_1}^{\infty}ds'\frac{\mathrm{Im}f'(s')}{s'-s}. \tag{B6}$$

If, again, $f'(s)$ decreases faster than s^{-1} at infinity, we must have

$$\mathrm{Im}f(\xi_1)+\int_{\xi_1}^{\infty}ds'\,\mathrm{Im}f'(s')=0. \tag{B7}$$

APPENDIX C

We will look at the problem of writing modified superconvergence relations from a slightly different point of view here. The results are identical with the ones given in Secs. II and III, however.

Suppose we again consider the B^+ amplitude and look at some large negative value of t. Here we expect to find $\alpha<0$, so that $B<(\text{const})s^{-1}$. We can then write a normal superconvergence relation for B^+:

$$\pi g^2+\int_{M^2+1-t/2}^{\infty}ds'\,\mathrm{Im}B^+(s',t)=0. \tag{C1}$$

Let us now break up the integral into a low-energy partial-wave region and high-energy Regge region separated by $s=s_0$. We then have

$$\pi g^2+\int_{M^2+1-t/2}^{s_0}ds'\,\mathrm{Im}B^+{}_{\mathrm{PW}}+\int_{s_0}^{\infty}ds'\,\mathrm{Im}B^+{}_{\mathrm{Regge}}=0. \tag{C2}$$

If we use the asymptotic form for $s>s_0$

$$\mathrm{Im}B^+{}_{\mathrm{Regge}}(s,t)=-\sum_{P,P'}g(t)\alpha C(\alpha)\left(\frac{s+p_t^2+q_t^2}{-2p_tq_t}\right)^{\alpha-1}, \tag{C3}$$

we can explicitly evaluate the second integral in Eq. (C2). Eq. (C2) then reduces to

$$\pi g^2+\int_{M^2+1-t/2}^{s_0}ds'\,\mathrm{Im}B^+{}_{\mathrm{PW}}-2p_tq_t\times\sum_{P,P'}g(t)C(\alpha)\left(\frac{s+p_t^2+q_t^2}{-2p_tq_t}\right)^{\alpha}=0. \tag{C4}$$

We can now continue this expression to $t=0$. Using Eq. (11) we then have exactly the result given by Eqs. (7) and (14).

The reason for the equivalence of the above results follows from the fact that $\mathrm{Im}B^+{}_{\mathrm{Regge}}$ itself satisfies a superconvergence relation

$$\int_{M^2+1-t/2}^{\infty}ds'\,\mathrm{Im}B^+{}_{\mathrm{Regge}}(s',t)=0 \tag{C5}$$

for large negative t, where it converges. This can be easily shown to be true from Eq. (12). We therefore have

$$\int_{s_0}^{\infty}ds'\,\mathrm{Im}B^+{}_{\mathrm{Regge}}=-\int_{M^2+1-t/2}^{s_0}ds'\,\mathrm{Im}B^+{}_{\mathrm{Regge}}, \tag{C6}$$

which explains the equivalence of Eqs. (C2) and (7). It also shows why it does not matter which Regge form one uses provided that it is a good approximation for $s>s_0$. Finally, it explains why Eqs. (C1) and (6) are consistent with each other for large negative t. All we have to do is subtract Eq. (C5) from Eq. (C1) to get Eq. (6). It is only for small t that there is a difference, since Eq. (6) can be continued into this region, whereas Eq. (C1) cannot.