Pion-Pion Scattering and $K^{\pm} \to 3\pi$ Decay*

N. N. KHURI†
Institute for Advanced Study, Princeton, New Jersey

AND

S. B. TREIMAN‡
Palmer Physical Laboratory, Princeton University, Princeton, New Jersey

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The effects of final state pion-pion interactions on the spectrum of $K^{\pm} \to 3\pi$ decay is studied by dispersion relation methods. In the approximations adopted we are led to a set of linear integral equations for the amplitudes of the $K^{\pm} \to 3\pi$ decay. The kernels in these equations depend on the pion-pion S-wave scattering amplitudes. An approximate solution for these equations is obtained by iteration and the departures from a purely statistical spectrum for the decay are related to pion-pion S-wave scattering. The latter in turn is assumed to be well represented with a scattering length structure. The $K^{\pm} \to 3\pi$ spectrum then is parametrized by two quantities, the $T=0$ and $T=2$ pion-pion S-wave scattering lengths, a_0 and a_2. Such experimental results as presently exist indicate that $a_2 - a_0$ is positive and that roughly $a_2 - a_0 \approx 0.7$, in units of the pion Compton wavelength.

I. INTRODUCTION

It is a reasonable expectation,[1] based on centrifugal barrier considerations, that the amplitude for $K^{\pm} \to 3\pi$ decay should be essentially constant, the decay spectrum being governed therefore mainly by phase volume. In roughest approximation this expectation is borne out experimentally and implies that the pions come out predominantly in S waves in a state which is totally symmetric in isotopic spin space. Of the two such states, corresponding, respectively, to isotopic spin $T=1$ and $T=3$, it is the former which dominates as one deduces from the excess of τ-events ($K^+ \to \pi^- + \pi^+ + \pi^+$) over τ'-events ($K^+ \to \pi^+ + \pi^0 + \pi^0$). This is one of the well-known pieces of evidence for the conjectured $|\Delta T| = \frac{1}{2}$ selection rule for weak interactions (more precisely, it is evidence for $|\Delta T| < \frac{5}{2}$).

Despite these rough indications, a closer examination of the spectrum of the τ mode reveals a noticeable departure from the purely statistical.[2] Relative to the latter, the π^- distribution appears to be a growing function of energy; the π^+ distribution, a decreasing function. In fact the π^- variation from the lowest to the highest energies seems to be of order 50% and the π^+ variation about half of this. These are only rough estimates. The experimental errors are still quite large and it is not impossible that these effects are largely spurious. If confirmed, however, they would reflect either an "intrinsic" property of $K \to 3\pi$ decay or, what is our concern here, an effect of pion-pion "final state" interactions. Of course, this distinction between intrinsic and final state effects cannot be made precise in general terms. Nevertheless, in the dispersion treatment to be discussed here, we shall adopt a set of approximations which are well-defined, whether or not they are adequate, and the variations in the decay amplitude will be clearly related to pion-pion scattering effects.[3]

Our procedure is as follows. Concerning the weak interaction itself, we assume the validity of time-reversal invariance and of the $|\Delta T| = \frac{1}{2}$ selection rule. For charged K-meson decay, the latter implies that the final three-pion system has isotopic spin $T=1$. The decay process is then characterized by three amplitudes, corresponding to the three possible $T=1$ states. Next, on the basis of familiar heuristic arguments, we conjecture that these amplitudes satisfy certain dispersion representations. In the approximation where we retain only the lowest mass intermediate states, the absorptive parts in these representations involve products of the decay amplitudes themselves with the amplitudes for pion-pion scattering. Our dispersion relations are of the subtracted variety, a circumstance which one hopes will insure the rapid convergence of the dispersion integrals. We therefore suppose that only low-energy pion-pion scattering contributes significantly and we take into account S-wave effects only. A possible P-wave resonance at a relatively high-energy on our scale has been conjectured in another connection.[4] We are taking the view, however, that these and other high mass contributions m ayappreciably affect the "subtraction" constant, i.e. ,the absolute decay rate, but not the spectrum shape which concerns us here.

At this stage we have a set of coupled linear integral equations for the decay amplitudes, involving in their kernels the $T=0$ and $T=2$ S-wave pion-pion scattering

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‡ Alfred P. Sloan Foundation Fellow.
[1] R. Dalitz, Phys. Rev. **94**, 1046 (1954); see also E. Fabri, Nuovo cimento **11**, 479 (1954).
[2] S. McKenna, S. Natali, M. O'Connell, J. Tietge, and N. C. Varshneya, Nuovo cimento **10**, 763 (1958). For a compilation of earlier data see R. H. Dalitz, *Reports on Progress in Physics* (The Physical Society, London, 1957), Vol. 20, p. 163.
[3] A phenomenological analysis of pion-pion scattering effects on τ-decay has been carried out independently by B. S. Thomas and W. G. Holladay, Phys. Rev. **115**, 1329 (1959). They make use of the final state theorem of K. M. Watson, Phys. Rev. **88**, 1163 (1952); and assume only an attractive $\pi^+ - \pi^+$ force.
[4] W. R. Frazer and J. R. Fulco, Phys. Rev. Letters **2**, 365 (1959).

amplitudes. The scattering amplitudes themselves, in the absence of other indications, we parametrize in a scattering length representation; so that the whole problem is parametrized with two quantities, a_0 and a_2, the $T=0$ and $T=2$ scattering lengths. This scattering length structure accords reasonably well with the so-called S-wave dominant solution of the pion-pion problem obtained by Chew et al.[5]

We have not succeeded in finding a rigorous solution to our integral equation. Instead, in order to survey the situation we adopt the lowest order iterative solution, regarding the departures from a statistical decay spectrum as being small, a not unreasonable approach in view of the actual experimental indications. Our results reproduce the experimental situation well enough, with $a_2-a_0\approx 0.7$ (in units of the pion Compton wavelength).

II. KINEMATICS

Denote the three pions emerging from K-meson decay by the letters a, b, c; let k_a, k_b, k_c be their respective 4-momenta ($k_a^2=k_b^2=k_c^2=-\mu^2$, where μ is the pion mass); and let α, β, γ denote the respective charge states. Denote by K the 4-momentum of the K-meson ($K^2=-m^2$). Since we shall be considering only $K^\pm$ decay, and since we suppose the pions come out in $T=1$ states, we can as a matter of convenience pretend that the K meson has isotopic spin unity and that isotopic spin is conserved in the decay. This merely simplifies some of the writing and involves no error. Using this artifice, we denote by ρ the charge state of the K meson. The invariant matrix element M for $K\to 3\pi$ decay can then be written

$$M_{\rho;\alpha\beta\gamma}=A\delta_{\rho\alpha}\delta_{\beta\gamma}+B\delta_{\rho\beta}\delta_{\gamma\alpha}+C\delta_{\rho\gamma}\delta_{\alpha\beta}, \tag{1}$$

where the amplitudes A, B, C are functions of the scalar variables

$$\begin{aligned} s_a&=-(K-k_a)^2,\\ s_b&=-(K-k_b)^2,\\ s_c&=-(K-k_c)^2. \end{aligned} \tag{2}$$

Only two of these are independent, energy-momentum conservation implying

$$s_a+s_b+s_c=m^2+3\mu^2.$$

It is nevertheless convenient to display all three variables, so that we write $A=A(s_a,s_b,s_c)$ and similarly for B and C.

From the Bose statistics of the three pion system we have the symmetries

$$\begin{aligned} &(s_b\rightleftarrows s_c,\, s_a\rightleftarrows s_a): && (B\rightleftarrows C,\, A\rightleftarrows A),\\ &(s_a\rightleftarrows s_c,\, s_b\rightleftarrows s_b): && (A\rightleftarrows C,\, B\rightleftarrows B),\\ &(s_a\rightleftarrows s_b,\, s_c\rightleftarrows s_c): && (A\rightleftarrows B,\, C\rightleftarrows C). \end{aligned} \tag{3}$$

[5] G. F. Chew, S. Mandelstam, H. P. Noyes, UCRL-9001 (to be published).

Notice that $A=B=C$ at the symmetric point $s_0=s_a=s_b=s_c=\frac{1}{3}m^2+\mu^2$.

Consider now the τ-mode, $K^+\to\pi^++\pi^++\pi^-$. Let s_1 and s_2 refer to the positive pions, s_3 to the negative pion. The squared matrix element is then

$$|M_\tau|^2=|A(s_1,s_2;s_3)+B(s_1,s_2;s_3)|^2. \tag{4}$$

Similarly, for the τ' mode, $K^+\to\pi^0+\pi^0+\pi^+$, let s_1 and s_2 again refer to the like pions, s_3 to the unlike (π^+) pion. The squared element is

$$|M_{\tau'}|^2=|C(s_1,s_2;s_3)|^2. \tag{5}$$

For either mode, the decay spectrum is given by

$$d\omega=\frac{1}{(2\pi)^5}|M|^2\frac{d\mathbf{k}_1d\mathbf{k}_2d\mathbf{k}_3}{2K^0\cdot 2\omega_1\cdot 2\omega_2\cdot 2\omega_3}\delta(K-k_1-k_2-k_3), \tag{6}$$

where K^0, ω_1, ω_2, ω_3 are, respectively, the energies of the K meson and of the three pi-mesons. In the K-meson rest system the maximum pion kinetic energy is given by

$$\bar{T}=\frac{(m-\mu)^2-4\mu^2}{2m}\approx 50\text{ Mev}. \tag{7}$$

Let $t_i=T_i/\bar{T}$ be the kinetic energy of the ith pion, in units of $\bar{T}$. In a nonrelativistic treatment of the decay spectrum it is probably most convenient[1] to take as the two independent variables the quantities

$$\begin{aligned} x&=(2/\sqrt{3})(t_1-t_2),\\ y&=2t_3-1. \end{aligned} \tag{8}$$

Owing to energy-momentum conservation these variables satisfy

$$x^2+y^2\leq 1; \tag{9}$$

that is, $K\to 3\pi$ events must be within a unit circle in the x-y plane. Aside from a constant factor, the decay spectrum, in the nonrelativistic limit, is given by

$$d\omega(x,y)=|M|^2dxdy. \tag{10}$$

Notice that, in the case where the pions are emitted in S-waves in the totally symmetric isotopic spin state with $T=1$, we have $A=B=C=\text{constant}$, hence $|M_\tau|^2=4|M_{\tau'}|^2=\text{const}$. We are looking for departures from these conditions induced by final state pion-pion interactions.

One final remark needs to be made. We are assuming the validity of charge independence in the strong interactions. In computing the decay amplitudes we therefore neglect, for example, the difference in mass between charged and neutral pions. There is no reason to think that this is a bad approximation. But in any careful analysis of experimental data these mass differences should be taken into account in the phase volume effects. The question does not arise for the τ decay mode. It does arise, however, for the τ' mode, and therefore also in any careful comparison of the two modes. We are neglecting Coulomb effects which can presumably be handled by the standard penetration barrier corrections to the data.

III. DISPERSION REPRESENTATION

There exists as yet no entirely satisfactory formalism for treating decay processes in quantum field theory. However, in so far as we treat such processes only to lowest order in the weak interaction, there is little doubt as to the correct procedure.[6] The decay amplitude is presumably given by the matrix element of the weak interaction Hamiltonian density H, taken between the initial and final physical states determined by the strong interactions. For our process the matrix element in question is

$$\langle k_a k_b k_c \text{ out} | H(0) | K \rangle,$$

where the index "out" refers to the strong interactions only and $H(0)$ is a local operator. As a matter of minor convenience we shall prefer to work with the complex conjugate matrix element and it is this latter which we denote by M [Eq. (1)]. Nothing is lost thereby, since the decay spectrum is anyhow determined by the square of the absolute value of the matrix element. Since we are assuming invariance under time reversal, we can write

$$M = \langle K | H(0) | k_a k_b k_c \text{ out} \rangle. \tag{11}$$

The isotopic labels are momentarily suppressed.

Following the standard reduction procedures, we now reduce this to the form

$$M = \int dx_a dx_b\, e^{ik_a\cdot x_a + ik_b\cdot x_b} \times \langle K | A(j_\alpha(x_a), j_\beta(x_b); H(0)) | k_c \rangle, \tag{12}$$

where the symbol A denotes the advanced multiple commutator product[7]; and j_α and j_β are the currents for mesons of isotopic spin α and β, respectively. Even though the matrix element in (12) vanishes when either x_a or x_b lies outside the future light cone, we do not know how to proceed and obtain, even heuristically, a dispersion relation for M. To get around this difficulty we shall use the definition of the R and A products and write M as the sum of two terms. For the first of these terms we shall heuristically obtain a dispersion relation. The second term turns out to be equal to the absorptive part (multiplied by $-2i$) that appears in the dispersion relation for the first term. It can thus be absorbed into the integral to give a dispersion type representation for M. This trick not only enables us to obtain an integral representation for M, but as we shall see below also leads to considerable simplification of the problem of handling a three-body final state interaction.

It is an elementary matter to recast (12) into the form

$$M = \mathfrak{M} + \int dx_a dx_b\, e^{ik_a\cdot x_a + ik_b\cdot x_b} \times \langle K | [[j_\alpha(x_a), H(0)], j_\beta(x_b)]\theta(x_a) | k_c \rangle, \tag{13}$$

where

$$\mathfrak{M} = \int dx_a dx_b\, e^{ik_a\cdot x_a + ik_b\cdot x_b} \times \langle K | R(j_\alpha(x_a); j_\beta(x_b), H(0)) | k_c \rangle. \tag{14}$$

The symbol R denotes a retarded multiple commutator.[7] The identity (13) is an immediate consequence of the definition of the retarded and advanced products.

The matrix element $\mathfrak{M}$ has the same retarded operator product that one would obtain if he considers the amplitude for the scattering type reaction, $a+K\to b+c$. In fact, $\mathfrak{M}$ is a continuation to unphysical energy ($\omega_a<0$) of the amplitude for this reaction. Using translation invariance, we may write

$$\mathfrak{M} = \int dx dy\, e^{\frac{1}{2}i(k_a-k_b)\cdot x} \times \langle K | R(j_\alpha(x/2); j_\beta(-x/2), H(y)) | k_c \rangle. \tag{15}$$

The retarded product here vanishes when x is outside the future light cone. This is the standard circumstance which affords a heuristic basis for the conjecture of dispersion relations—for the time-like component of the vector $(k_a-k_b)/2$, in the "brick wall" system $\mathbf{K}+\mathbf{k}_c=0$. Generalizing to an arbitrary reference frame, we are lead to conjecture analyticity in the variable $s_a=-(K-k_a)^2$, for fixed $s_c=-(K-k_c)^2$. Leaving aside for the moment the question of subtractions, and restoring the isotopic spin labels in order to bring out the symmetries implied by (3), we have

$$\mathfrak{M}_{\rho;\alpha\beta\gamma} = \frac{1}{\pi}\int_{4\mu^2}^{\infty} \frac{\phi_{\rho;\alpha\beta\gamma}(s_a', s_c)}{s_a'-s_a-i\epsilon} ds_a' + \frac{1}{\pi}\int_{4\mu^2}^{\infty} \frac{\phi_{\rho;\beta\alpha\gamma}(s_b', s_c)}{s_b'-s_b+i\epsilon} ds_b' + G_{\rho;\alpha\beta\gamma}(s_c), \tag{16}$$

where $G(s_c)$ is at this stage an arbitrary function of the variable s_c. Notice the reversal of order of the labels α and β in the second term, a consequence of the symmetries (3). The dispersion relation (16) can be considered to be obtained from that for the scattering like reaction $a+K\to b+c$ by analytic continuation in the variable s_c from negative to positive values.

The absorptive parts ϕ are obtained in the standard way by taking the difference between $R(j_\alpha(x/2); j_\beta(-x/2), H(y))$ and $R(j_\beta(-x/2); j_\alpha(x/2), H(y))$. Ex-

[6] A detailed discussion of this point has been given by K. Symanzik, see the mimeographed notes on a seminar given during the course on Elementary particles at Oberwolfach, Black Forest, Germany, 1958 (unpublished).

[7] For a definition of the R products see H. Lehmann, K. Symanzik, and W. Zimmermann, Nuovo cimento **6**, 319 (1957); also K. Nishijima, Progr. Theor. Phys. (Kyoto) **17**, 765 (1957). The advanced or A products are obtained from the R products by changing the sign of the arguments of the θ-functions and putting $-i$ for each i.

panding in a sum over intermediate states $|n\rangle$, we have

$$\phi_{\rho;\alpha\beta\gamma}(s_a,s_c)$$

$$=-\frac{(2\pi)^4}{2}\sum_n\left\{\int dx\, e^{ik_a\cdot x}\langle K|R(j_\alpha(x);H(0))|n\rangle\right.$$

$$\times\langle n|j_\beta(0)|k_c\rangle+\langle K|j_\alpha(0)|n\rangle\int dx\, e^{ik_b\cdot x}$$

$$\left.\times\langle n|R(j_\beta(x);H(0))|k_c\rangle\right\}\delta(n-k_b-k_c). \quad (17)$$

The crucial remark now is that the function which appears on the right-hand side of (13), when expressed as an expansion over the set of states $|n\rangle$, is just precisely equal to $-2i\phi_{\rho;\alpha\beta\gamma}(s_a,s_c)$.[8] Thus we have

$$M_{\rho;\alpha\beta\gamma}=\mathfrak{M}_{\rho;\alpha\beta\gamma}-2i\phi_{\rho;\alpha\beta\gamma}. \quad (18)$$

Substituting (16) into (18), we obtain the following representation for M

$$M_{\rho;\alpha\beta\gamma}=\frac{1}{\pi}\int_{4\mu^2}^{\infty}\frac{\phi_{\rho;\alpha\beta\gamma}(s_a',s_c)}{s_a'-s_a+i\epsilon}ds_a'$$

$$+\frac{1}{\pi}\int_{4\mu^2}^{\infty}\frac{\phi_{\rho;\beta\alpha\gamma}(s_b',s_c)}{s_b'-s_b+i\epsilon}ds_b'+G_{\rho;\alpha\beta\gamma}(s_c). \quad (19)$$

Notice that the only difference between (19) and the dispersion relation (16) is the sign of the $i\epsilon$ in the first term.

The intermediate state with lowest mass which contributes to the first term on the right side of (17) is the two pion state, with threshold at $4\mu^2$. The lowest mass state which contributes to the second term is the $K-\pi$ state, with threshold at $(m+\mu)^2$. It will be our approximation to retain contributions only from the lowest mass states. Thus we neglect entirely the second term in (17); and in the first term we insert only the two pion contributions (the next state would be that of four pions, with threshold at $16\mu^2$). The physical values of the variables s_a, s_b, and s_c all lie in the range $4\mu^2<s<(m-\mu)^2$; and thus the maximum physical value of any s is much below the thresholds $16\mu^2$ and $(m+\mu)^2$. The neglect of higher mass contributions is even less serious because we shall ultimately adopt a subtracted form of the dispersion representation. The higher mass states can be regarded as contributing mostly to the subtraction constant, which we in any case do not attempt to compute. We are only neglecting their contributions to the *variations* of M with pion energies, at low energies. In other words we are attempting to deal with the shape, not the absolute level, of the decay spectrum.

We note here that in (17) no intermediate states of three pions appear. One might have thought that in studying a problem with a three pion final state, it would be necessary to deal with matrix elements of the form $\langle 3\pi|3\pi\rangle$. However, in resorting to the splitting given in (13) we have avoided this difficulty and have to deal only with $\langle 2\pi|2\pi\rangle$ and $\langle 4\pi|2\pi\rangle$ matrix elements. The latter one can presumably neglect since they start contributing to ϕ at a much larger threshold.

The absorptive part then involves a product of the decay amplitude M itself with the amplitude for pion-pion scattering. The sum over intermediate states is equivalent to an integration over the direction of the intermediate mesons, in the reference frame where $\mathbf{k}_b+\mathbf{k}_c=0$. To effect this integration, it is convenient to regard $M(k_a,k_b,k_c)$ as a function of s_a and of the angle θ_{bc} between $\mathbf{k}_a$ and $\mathbf{k}_b=-\mathbf{k}_c=\mathbf{k}$ in this reference frame. Thus k is the wave number of the scattering mesons in their center of momentum frame, and

$$s_a=4(k^2+\mu^2). \quad (20)$$

For the same kinds of reasons discussed above in relation to neglecting the higher mass states, we shall now suppose that only low-energy pion-pion scattering contributes significantly to the *shape* of the decay spectrum. We therefore take into account only S-wave scattering. As we have mentioned earlier the maximum physical value of s_a is $(m-\mu)^2\approx 6.25\mu^2$, on the other hand the P-wave resonance conjectured by Frazer and Fulco[4] is located at some value of $s_a>10\mu^2$.

Reflecting the isotropy which the assumption of pure S-wave scattering implies, we then find that the absorptive part ϕ depends only on s_a and is independent of s_c. On carrying out the integrations, we have

$$\phi_{\rho;\alpha\beta\gamma}(s_a)$$

$$=\sum_{\beta'\gamma'}\left[\int\frac{d\Omega_{bc}}{4\pi}M_{\rho;\alpha\beta'\gamma'}(s_a,\cos\theta_{bc})\right]f_{\beta'\gamma';\beta\gamma}(s_a); \quad (21)$$

where f is related to the pion-pion scattering amplitude by

$$f_{\beta'\gamma';\beta\gamma}=\tfrac{1}{3}(f_0-f_2)\delta_{\beta'\gamma'}\delta_{\beta\gamma}+\tfrac{1}{2}f_2(\delta_{\beta'\beta}\delta_{\gamma'\gamma}+\delta_{\beta'\gamma}\delta_{\gamma'\beta}), \quad (22)$$

with

$$f_T=e^{i\delta_T}\sin\delta_T;\quad T=0,\,2. \quad (23)$$

Here δ_0 and δ_2 are, respectively, the $T=0$ and $T=2$ S-wave phase shifts for pion-pion scattering.

It is now evident how we must choose the as yet unspecified function $G(s_c)$ which appears in (19). The Bose statistics dictate that this must have the same form as the other two terms in the equation, with the isotopic labels appropriately chosen to reflect the symmetries (3). Namely, we have

$$M_{\rho;\alpha\beta\gamma}(s_a,s_b,s_c)$$

$$=\frac{1}{\pi}\int_{4\mu^2}^{\infty}ds_a'\frac{\phi_{\rho;\alpha\beta\gamma}(s_a')}{s_a'-s_a+i\epsilon}+\frac{1}{\pi}\int_{4\mu^2}^{\infty}ds_b'\frac{\phi_{\rho;\beta\alpha\gamma}(s_b')}{s_b'-s_b+i\epsilon}$$

$$+\frac{1}{\pi}\int_{4\mu^2}^{\infty}ds_c'\frac{\phi_{\rho;\gamma\beta\alpha}(s_c')}{s_c'-s_c+i\epsilon}. \quad (24)$$

[8] *Note added in proof.*—This statement is of course only true for $s_a<16\mu^2$. At this stage Eq. (18) is used only for physical M. In the unphysical region M is defined by continuation from (19).

We now make a subtraction in each term, at the symmetric point $s_a=s_b=s_c=s_0$. At this point $A=B=C=D_0$ where D_0 is a complex constant. We can now use (1), (21), and (22) to obtain from (24) three separate dispersion representations for each of the amplitudes A, B, and C. We obtain

$$\begin{aligned} A(s_a,s_b,s_c)&=D_0+U(s_a)+V(s_b)+V(s_c),\\ B(s_a,s_b,s_c)&=D_0+V(s_a)+U(s_b)+V(s_c),\\ C(s_a,s_b,s_c)&=D_0+V(s_a)+V(s_b)+U(s_c); \end{aligned} \tag{25}$$

where

$$\begin{aligned} U(s)&=\frac{(s-s_0)}{\pi}\int_{4\mu^2}^{\infty}ds' \\ &\times\frac{\tilde{A}(s')f_0(s')+\frac{1}{3}[\tilde{B}(s')+\tilde{C}(s')][f_0(s')-f_2(s')]}{(s'-s_0+i\epsilon)(s'-s+i\epsilon)}, \\ V(s)&=\frac{(s-s_0)}{\pi}\int_{4\mu^2}^{\infty}ds'\frac{\frac{1}{2}[\tilde{B}(s')+\tilde{C}(s')]f_2(s')}{(s'-s_0+i\epsilon)(s'-s+i\epsilon)}, \end{aligned} \tag{26}$$

and

$$\begin{aligned} \tilde{A}(s_a)&=\int\frac{d\Omega_{bc}}{4\pi}A(s_a,\cos\theta_{bc}),\\ \tilde{B}(s_a)&=\int\frac{d\Omega_{bc}}{4\pi}B(s_a,\cos\theta_{bc}),\\ \tilde{C}(s_a)&=\int\frac{d\Omega_{bc}}{4\pi}C(s_a,\cos\theta_{bc}). \end{aligned} \tag{27}$$

In (27) as in (21) the angles refer to those in the reference frame where $\mathbf{k}_b+\mathbf{k}_c=0$. We note here that because of the symmetries (3), $\tilde{A}$ obtained from the angular integration in the frame $\mathbf{k}_b+\mathbf{k}_c=0$ is identical with the $\tilde{B}$ obtained from the corresponding angular integration in the frame $\mathbf{k}_a+\mathbf{k}_c=0$. This is why only three functions $\tilde{A}$, $\tilde{B}$, and $\tilde{C}$ appear in (25) and we can define them all in the same reference frame as in (27).

It is evident now that we can, by integrating the equations (25) over $\cos\theta_{bc}$, obtain a set of coupled linear integral equations for the functions $\tilde{A}(s)$, $\tilde{B}(s)$, and $\tilde{C}(s)$. These functions are themselves analytic in the s plane, except for cuts along the real axis. We have seen no way to obtain a rigorous solution of this system of equations. Should the situation warrant it, one could resort to numerical methods. For the present, in order to obtain some indication of the effects involved we shall treat the pion-pion effects as a small perturbation and adopt a lowest order iterative solution of (25); i.e., we set $A=B=C=D_0$ in (27) and (26). In the physical region, in view of the actual experimental indications, this is a good trial solution since the departures from a statistical decay spectrum are small.

We are not concerned with the magnitude of D_0, and we can factor it out from the iterated solution of (25). From (4), (5), and (25) we can now write the squared matrix elements for the τ and τ' modes of K-meson decay. We have, aside from an over-all factor common to both,

$$\begin{aligned} |M_\tau|^2=4|1+\tfrac{1}{2}[U(s_1)+V(s_1)]& \\ +\tfrac{1}{2}[U(s_2)+V(s_2)]+V(s_3)|^2,& \end{aligned} \tag{28}$$

$$|M_{\tau'}|^2=|1+V(s_1)+V(s_2)+U(s_3)|^2; \tag{29}$$

where $U(s)$ and $V(s)$ are given by (26) with unity substituted for $\tilde{A}$, $\tilde{B}$, and $\tilde{C}$. In each of the above expressions, s_1 and s_2 refer to the like mesons, s_3 to the unlike meson; and in the K-meson rest system

$$s_i=(m-\mu)^2-2mT_i, \tag{30}$$

where T_i is the kinetic energy of the ith meson.

IV. SCATTERING LENGTH PARAMETRIZATION

In the lowest order iterative solution the basic amplitudes U and V have been reduced to well-defined integrals over the pion-pion scattering functions f_0 and f_2. These are, of course, not yet known experimentally; it was in fact in order to get some indication about them that we have undertaken the present analysis. We cannot expect to make much progress, however, unless we can parametrize these functions in a relatively simple way; either this, or else try one at a time specific and perhaps more complicated expressions which may be suggested from other sources. At present it seems most reasonable to adopt a scattering length characterization of the phase shifts; or rather, a relativistic generalization thereof. We take

$$(k/\omega)\cot\delta_T=1/a_T, \tag{31}$$

where k and ω are the center-of-mass momentum and energy, and a_T is the dimensionless scattering length (roughly, the conventional scattering length in units of the pion Compton wavelength). This structure accords reasonably well, up to moderate energies, with the so-called S-wave dominant solutions which Chew et al.[5] have obtained for the pion-pion scattering integral equations. With this form we then have

$$f_T=e^{i\delta_T}\sin\delta_T=ka_T/(\omega-ika_T), \tag{32}$$

and the integrals (26) are readily carried out.

For practical purposes it is most convenient to take as variables not the invariant quantities s_i, but rather the kinetic energies of the pi mesons in the K-meson rest frame. Measured in units of $\bar{T}$, the maximum kinetic energy, these are denoted by t_i, and we have

$$t_1+t_2+t_3=(m-3\mu)/\bar{T}=3t_0;$$

where t_0 corresponds to the symmetric point kinetic energy, $t_0\approx\frac{1}{2}$. Notice that the kinetic energy t of one of the pions, and the momentum k (measured in units of the pion mass) of each of the other two in their mutual center of momentum system are related by

$$k^2=\rho^2(1-t);\quad \rho^2=m\bar{T}/2\mu^2\approx 0.64. \tag{33}$$

With t and k, likewise t_0 and k_0, so related, our basic integral is

$$I_T(t)=\frac{2}{\pi}(k^2-k_0^2)\times\int_0^\infty \frac{f_T(k')}{(k'^2-k^2+i\epsilon)(k'^2-k_0^2+i\epsilon)}k'dk'. \quad (34)$$

This can be readily evaluated. However, inasmuch as we are presently treating the pion-pion effects as a small perturbation, we are in effect computing to lowest order in a_T. For consistency we must therefore take

$$f_T \simeq ka_T/\omega. \quad (32')$$

This quantity is bounded in magnitude by a_T, assumed small. This is in fact why we adopted the relativistic form (31) rather than the conventional expression ($\omega\to 1$), where $|f_T|\to 1$ as $k\to\infty$. This latter behavior would be inconsistent with our perturbative approach. We now find

$$I_T(t)=J_T(t)-J_T(t_0),$$

with

$$J_T(t)=(ka_T/\omega)\left\{-i+\frac{2}{\pi}\ln(\omega-k)\right\}. \quad (35)$$

The basic amplitudes U and V, in terms of which the decay matrix elements are formed [see (28) and (29)], are given by

$$\begin{aligned} U(t)&=\tfrac{1}{3}[5I_0(t)-2I_2(t)],\\ V(t)&=I_2(t). \end{aligned} \quad (36)$$

To lowest order we can drop the square of these quantities in evaluating (28) and (29), so that only the real part of $J_T(t)$ comes into play. For this, in good enough approximation we can set $\ln(\omega-k)\approx -k$, hence

$$\text{Real } J_T(t)\to -(2/\pi)(k^2/\omega)a_T. \quad (35')$$

We now have

$$\tfrac{1}{4}|M_\tau(t_1,t_2,t_3)|^2\simeq 1-(2/\pi)a_2\rho^2\{2h(t_3)+\tfrac{1}{3}h(t_1)+\tfrac{1}{3}h(t_2)\} -(10/3\pi)a_0\rho^2\{h(t_1)+h(t_2)\}; \quad (37)$$

$$|M_{\tau'}(t_1,t_2,t_3)|^2\simeq 1-(2/\pi)a_2\rho^2\{2h(t_1)+2h(t_2)-\tfrac{4}{3}h(t_3)\} -(20/3\pi)a_0\rho^2h(t_3); \quad (38)$$

with

$$h(t)=\frac{1-t}{[1+\rho^2(1-t)]^{\frac{1}{2}}}-\frac{1-t_0}{[1+\rho^2(1-t_0)]^{\frac{1}{2}}};\quad t_0\approx\tfrac{1}{2}. \quad (39)$$

For rough purposes it is a good enough approximation, finally, to set $t=t_0$ in the denominator of (39); this involves an error at most 15%—at the end of the spectrum. We then find

$$\begin{aligned}\tfrac{1}{4}|M_\tau|^2&\simeq 1+\frac{5}{3\pi}\frac{\rho^2}{(1+\frac{1}{2}\rho^2)^{\frac{1}{2}}}(a_2-a_0)(2t_3-1);\\ |M_{\tau'}|^2&\simeq 1+\frac{10}{3\pi}\frac{\rho^2}{(1+\frac{1}{2}\rho^2)^{\frac{1}{2}}}(a_0-a_2)(2t_3-1).\end{aligned} \quad (40)$$

One sees that to lowest order in the scattering lengths the correction terms both for τ and τ' decay depend, essentially linearly, only on the energy of the unlike particle. In each case the coefficient is determined by the difference of the scattering lengths, and the τ' coefficient has twice the magnitude as that for τ, but with the opposite sign.

Gell-Mann and Rosenfeld[9] showed that a good fit to the τ-decay data, obtained from a plot compiled by Dalitz,[2] is obtained with

$$|M_\tau|^2\sim 1+(2/10)(2t_3-1). \quad (41)$$

Comparing this result with (40) we have

$$\frac{5}{3\pi}\frac{\rho^2}{(1+\frac{1}{2}\rho^2)^{\frac{1}{2}}}(a_2-a_0)\approx 0.2,$$

hence

$$a_2-a_0\approx 0.7. \quad (42)$$

If this is a difference of much larger numbers then, of course, our perturbation approach is not valid. However, if the scattering lengths are comparable to unity then our first order approximation may not be misleading. If one iterates twice and looks at the terms of order a^2, one finds that for scattering lengths of order unity the second order terms are at the worst points only 25% of the first order terms.

Let us introduce the functions W_τ^-, W_τ^+, and $W_{\tau'}^+$, which describe, respectively, the uncorrelated energy distributions of the π^- and π^+ mesons in τ decay and of the π^+ meson in the τ' decay. In order to bring out most clearly the effects under consideration we measure these relative to the purely statistical distributions. We then find

$$W_\tau^-(t)=1+\frac{5}{3\pi}\frac{\rho^2}{(1+\frac{1}{2}\rho^2)^{\frac{1}{2}}}(a_2-a_0)(2t-1), \quad (43)$$

$$W_\tau^+(t)=1-\frac{5}{6\pi}\frac{\rho^2}{(1+\frac{1}{2}\rho^2)^{\frac{1}{2}}}(a_2-a_0)(2t-1), \quad (44)$$

$$W_{\tau'}^+(t)=1-\frac{10}{3\pi}\frac{\rho^2}{(1+\frac{1}{2}\rho^2)^{\frac{1}{2}}}(a_2-a_0)(2t-1). \quad (45)$$

The three slopes stand in the ratio $\tau(-):\tau(+):\tau'(+)=1:-\frac{1}{2}:-2$.[10] On the experimental side it

[9] M. Gell-Mann and A. H. Rosenfeld, *Annual Review of Nuclear Science* (Annual Reviews, Inc., Palo, Alto, California, 1957), Vol. 7, p. 407.

[10] S. Weinberg, Phys. Rev. Letters 4, 87 (1960).

happens that the best least squares fit to the data on τ decay given by McKenna et al.[2] gives just about the ratio implied above, though the errors are large and this agreement must not be taken too seriously. The experimental indication is that the π^- spectrum (relative to statistical) is an increasing function of the energy the variation amounting to 50% from one end of the spectrum to the other. With $a_2-a_0\approx 0.8$, Eq. (43) would reproduce the least squares fit to the data of McKenna et al.

Most of the S-wave dominant solutions of the pion-pion integral equations obtained by Chew, Mandelstam, and Noyes[5] have the following general properties: (i) a_2 and a_0 have the same sign; (ii) the ratio a_0/a_2 is of the order $\frac{5}{2}$. If one accepts these properties, then our results will lead to the conclusion that both a_0 and a_2 are negative, hence a repulsive S-wave pion-pion interaction. The values $a_2\approx -0.3$ and $a_0\approx -1$ will give agreement with the data and correspond to $\lambda\approx 0.15$, where λ is the Chew-Mandelstam pion-pion coupling constant.

Finally, we point out that a very useful test of the results of this paper would be provided by looking at the π^+ spectrum in the τ'-decay mode.[10]

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Unstable Particles in a General Field Theory

J. GUNSON AND J. G. TAYLOR*
Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Cambridge, England
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The problem of unstable particles in quantum field theory is treated as one of the interpretation of complex singularities appearing in the analytic continuation of scattering amplitudes into unphysical sheets of their Lorentz invariant variables. Suitable continuations are shown to hold under certain restrictive assumptions in a general field theory, making use of unitarity and causality of the S matrix. The extra singularities appearing in the continuation are fixed isolated poles, in accordance with a conjecture of Peierls.

1. INTRODUCTION

THE problem of the definition of unstable particles in quantum field theory has received much attention recently.[1] The difficulty met with in framing such a definition lies in the fact that the asymptotic "in" and "out" states containing such particles do not exist and so the usual methods of field theory based on these asymptotic states do not apply. Two main methods of approach to this problem have appeared. One of these[2] is to define the mass and lifetime of an unstable particle in terms of the average mass and mass spread of a certain mass distribution appearing in the spectral representation of the propagator. A limitation is imposed on the high-energy behavior of the vertex function in order that the mass and lifetime defined in this manner exist. This limitation is not satisfied by the Lee model with a certain cutoff[1] and may not be satisfied in a realistic field theory. For this reason we follow here the suggestion of Peierls[3] that unstable particles may be associated with poles appearing in the unphysical sheets of the analytic continuation of the particle propagator in momentum space. The work of Lévy[1] and others shows that the Lee model provides a satisfactory demonstration of this suggestion. In this paper, we attempt to extend the methods developed by Lévy to a more general field theory. However, it is not yet possible to give a satisfactory definition of the local field to be associated with an unstable particle in a general field theory satisfying the usual axioms of causality, Lorentz invariance, and an asymptotic condition. In consequence, we consider mainly the interpretation of poles appearing in the scattering amplitudes and then show that the same poles should occur in the propagator if it exists. We do not consider here the decay properties of unstable particles as a function of time but only how they manifest themselves as resonances in scattering or production processes.

In Sec. 2, we discuss the continuation of the two-particle scattering amplitude into the first unphysical

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[1] M. Lévy, Nuovo cimento 13, 115 (1959). This paper contains further references.
[2] P. T. Matthews and A. Salam, *1958 Annual International Conference on High-Energy Physics, Cern*, edited by B. Ferretti (CERN Scientific Information Service, Geneva, 1958). Also Phys. Rev. 112, 283 (1958).
[3] R. E. Peierls, *Proceedings of the Glasgow Conference on Nuclear and Meson Physics* (Pergamon Press, New York, 1954), p. 296.