fitting the data at larger $-t$. The results[18] are shown in Fig. 4 along with some representative data points.

Observation of rapid t variation in the differential cross section for $\bar{p}p$ charge exchange with $-t<0.02$ and/or verification of the energy dependence of the rapidly varying contribution for $-t<0.02$ in np charge exchange would be important confirmation of the presence of long-range OPE in nucleon-nucleon collisions. Note that we have assumed in the foregoing that the pion is "elementary"; i.e., the amplitudes f and f_4 decrease with increasing energy like (1). If the pion is, on the other hand, "Reggeized,"[19] these amplitudes would have a slightly different energy dependence. This effect is expected to be small.[19]

It will be interesting to learn more about the energy dependence of the structure of these charge-exchange peaks. Assuming our model to be correct, we expect the magnitudes of the parameters K, K_f, and K_{af} to decrease at high energies like, e.g., some power of energy.[20] We do not know whether they obey the same or different power laws.

We would like to take this opportunity to thank C. N. Yang and R. P. Feynman for stimulating and informative discussions, and M. E. Parkinson for help with computations.

[18] For more accurate estimations, the $\bar{p}p$ elastic scattering data should be used.

[19] G. F. Chew and S. C. Frautschi, Phys. Rev. Letters **8**, 41 (1962).

[20] The phases of these parameters may also be measured (see, e.g., Refs. 14 and 16). A. A. Lozunov, N. van Hieu, and I. T. Todorov [Ann. Phys. (N.Y.) **31**, 203 (1965)] showed that in the asymptotic region ($s\to\infty$ for fixed t), analyticity and crossing symmetry of scattering amplitudes relate their phases to their energy dependence. It would be interesting to compare such relations with experimental data. M. LeBellac [Nuovo Cimento, **42**, 443 (1966); CERN report (unpublished)] discusses such a comparison for πp scattering.

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Pion-Pion Scattering in a K-Matrix Model Incorporating Crossing Symmetry*

J. G. CORDES†

Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Cambridge, England

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A crossing-symmetric generalization of the K-matrix formalism is developed and used to construct a model for π-π scattering. The full amplitude satisfies elastic unitarity in the s channel and has the correct singularity structure at the elastic thresholds in the t and u channels. The two parameters present in the model are determined by requiring approximate satisfaction of the crossing relations in the neighborhood of the symmetry point. The resulting p-wave phase shift exhibits a resonance with mass about 800 MeV and width about 250 MeV. The f^0 is not reproduced. The s-wave scattering lengths in the $I=0$ and $I=2$ channels are $(a_0,a_2)=(-0.67,-0.30)$ and the effective Chew-Mandelstam coupling constant is found to be 0.18.

1. INTRODUCTION

IT has been known for many years that in potential theory the product $q\cot\delta$, considered as a function of energy, is regular at threshold and that consequently a power-series expansion in q^2 is valid there. This fact has frequently enabled useful parametrizations of scattering data in the form of the scattering-length and effective-range approximations.[1]

It is also well known that this result may be thought of from a somewhat different point of view, in which use is made of the K-matrix formalism. The characteristic feature of the K matrix, defined by

$$S=\frac{1+\frac{1}{2}iK}{1-\frac{1}{2}iK}, \tag{1.1}$$

lies in the fact that its Hermiticity is a necessary and sufficient condition for the S matrix to be unitary. To obtain further properties of K the scattering process under consideration should be specified in some detail. Thus we consider the elastic scattering of two identical particles of unit mass, with charge and spin both zero; the usual Mandelstam variables are defined as shown in Fig. 1. Taking two-particle matrix elements of K in the

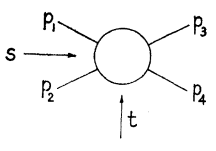

FIG. 1. Elastic pion-pion scattering.

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† Now at the Department of Physics, University of Toronto, Toronto, Canada.

[1] R. H. Dalitz, *Strange Particles and Strong Interactions* (Oxford University Press, New York, 1962).

energy region below the first inelastic threshold gives, after factoring out the energy-momentum-conserving δ functions, an analytic function $K(s,t)$. This K amplitude is real in the elastic-scattering region of the s channel as a result of the unitarity requirement that the matrix K be Hermitian. Since the K amplitude is also real in the unphysical energy region below threshold it follows that $K(s,t)$ must be regular at the elastic threshold $s=4$. (The details of this argument may be found in several places in the literature.[1-3]) The normal threshold singularity in the scattering amplitude may thus be said to have been "extracted."

The partial-wave K amplitudes, $k_l(s)$, will evidently also be regular at $s=4$ and consequently the normal threshold singularity in the scattering partial waves $a_l(s)$ may be explicitly displayed. Thus when a partial-wave decomposition of (1.1) is made one obtains

$$a_l(s)=\frac{k_l(s)}{1-\frac{1}{4}i\pi\rho(s)k_l(s)}, \tag{1.2a}$$

where

$$a_l(s)=\frac{4}{\pi}\frac{e^{i\delta_l(s)}\sin\delta_l(s)}{\rho(s)}, \tag{1.2b}$$

and

$$\rho(s)=\left(\frac{s-4}{s}\right)^{1/2}, \tag{1.2c}$$

explicitly revealing the square-root nature of the normal threshold branch point in $a_l(s)$.

The fact that $k_l(s)$ is regular at $s=4$ means that it is a much more slowly varying function than $\rho(s)$, at least for s very near threshold. In the hope that this situation persists for at least a short distance above threshold it is natural to try approximating $k_l(s)$ by a constant or a linear function of s. The usual scattering-length and effective-range approximations are in fact recovered by making these approximations on the function $k_l^{-1}(s)$, since by (1.2)

$$k_l^{-1}=\rho\cot\delta. \tag{1.3}$$

In this paper this extraction-of-singularities approach is extended to include the elastic thresholds in each of the three channels of a two-particle scattering amplitude, in the hope that such a crossing-symmetric calculation might prove more fruitful than the simple direct-channel analysis. The possibility that such an approach could form the basis of a calculational program within the framework of S-matrix theory was suggested by Landshoff and Olive.[4] A similar program in field theory has been discussed by Taylor.[5] Here the formalism will be applied to the case of π-π scattering, where previous calculations have indicated that the inclusion of only elastic unitarity may be sufficient to describe the low-energy behavior for a considerable distance above threshold. It is found that the complete π-π amplitude involves two parameters which may be determined by the requirement that crossing symmetry be satisfied in the neighborhood of the symmetry point.

In the next section the basic equations of the formalism are introduced and in Sec. 3 a power-series solution is developed. An approximate amplitude is obtained by selecting a particular (infinite) subset of terms of the expansion. In Sec. 4 the method is generalized to include isospin, in order to deal with π-π scattering. The numerical determination of the parameters is discussed in detail in Sec. 5. Finally, in Sec. 6 the results for the low partial-wave phase shifts in each isospin channel are presented and discussed.

2. THE BASIC EQUATIONS

When the connected part of the S matrix, iT, is introduced in the usual way ($S=1+iT$), (1.1) may be written in the form

$$T=K+\tfrac{1}{2}iTK, \tag{2.1}$$

or, in bubble notation, as shown in Fig. 2. This equation may be used to prove directly, with the assistance of the unitarity condition (shown diagrammatically in Fig. 3), that the K amplitude is lacking the branch point at $s=4$.

Our convention for the bubbles and for the integration over intermediate states may be illustrated by writing (2.1) out in full:

$$T(p_1,p_2,p_3,p_4)=K(p_1,p_2;p_3,p_4)$$
$$+\tfrac{1}{2}i\int d^4p_5d^4p_6\,T(p_1,p_2,p_5,p_6)\delta_+(p_5^2-1)\delta_+(p_6^2-1)$$
$$\times\delta^{(4)}(p_1+p_2-p_5-p_6)K(p_5,p_6;p_3,p_4).$$

The unitarity-type integral in this equation reduces, by using the δ functions, to an angular integration, so that the equation becomes

$$T(s,t)=K(s,t)+\tfrac{1}{16}i\rho(s)\int d\Omega\,T(s,t')K(s,t''),$$

where $t'=(p_1-p_5)^2$, $t''=(p_5-p_3)^2$, and the integration is over the angular orientation of the intermediate momentum $\mathbf{p}_5$ in the center-of-mass frame.

The "plus bubble" in the unitarity condition (Fig. 3) is just the physical sheet amplitude T while the "minus bubble" represents the function obtained from T by

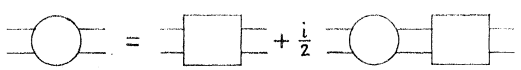

FIG. 2. The bubble notation version of the defining equation for the K matrix (2.1). The square blob in the figure corresponds to K.

[2] W. Zimmermann, Nuovo Cimento **21**, 249 (1961).
[3] R. J. Eden, P. V. Landshoff, D. I. Olive, and J. C. Polkinghorne, *The Analytic S Matrix* (Cambridge University Press, Cambridge, England, 1966).
[4] P. V. Landshoff and D. I. Olive, J. Math. Phys. **7**, 1464 (1966).
[5] J. G. Taylor, Nuovo Cimento Suppl. **1**, 857 (1963).

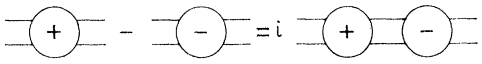

FIG. 3. The elastic-unitarity condition. The plus and minus bubbles are defined in the text.

continuing s counter-clockwise around the branch point at $s=4$.

If the details of the proof of the regularity of $K(s,t)$ at $s=4$ are examined (see, for example, Ref. 3, p. 231), it will be observed that the essential property of the phase-space factor in the unitarity-type integral in (2.1) is its discontinuity of $2\rho(s)$ across the elastic unitarity cut and that any other function with this property would serve as well.[6] There is, therefore, a large class of possible "K amplitudes" which may be generated by using different phase-space factors in the defining Eq. (2.1), the conventional K being obtained when simply $\rho(s)$ is used. This freedom in the choice of phase-space factor makes it possible to choose one without the awkward kinematic singularity at $s=0$ which is possessed by $\rho(s)$ itself. The simplest way of obtaining a function with the desired properties is by evaluating the integral

$$\frac{(s-s_0)}{i\pi}\int_4^\infty ds'\frac{\rho(s')}{(s'-s)(s'-s_0)}\equiv F(s,s_0)\,, \qquad (2.2)$$

where the subtraction is necessary to obtain a convergent integral.

It should be noted, however, that the use of this modified phase-space factor in defining a new $K(s,t)$ does not completely remove the singularity in K at $s=0$. There is still a singularity there due to a pinch of the integration contour in (2.1) by the normal threshold singularities in the momentum transfer variables in T and K. What is more, it seems likely that $K(s,t)$, no matter what is used for the phase-space factor, will have a natural boundary along the negative real s axis for the same reason that this probably occurs in the unphysical sheet amplitude.[7] We shall reserve until later the question of how these various singularities are to be taken into account in a practical calculation.

Returning to the modified phase-space factor F, the integral given in (2.2) is not the most general function satisfying our requirements, but since any others would require the introduction of further parameters we shall use (2.2). Evaluating the integral gives

$$\begin{aligned} &F(s,s_0)=G(s)-G(s_0);\\ &G(s)=\rho(s)+iR(s),\\ &R(s)=\frac{1}{\pi}\rho(s)\ln\frac{1+\rho(s)}{1-\rho(s)}, \end{aligned} \qquad (2.3)$$

where the principal branch of the logarithm is to be taken on the physical sheet [so that $R(s)\geq 0$ for $s\geq 4$]. Note that $G(s)$ is pure imaginary for $s\leq 4$; s_0 will therefore be restricted to be ≤ 4 so that the new K will still be real above threshold. The equation defining the modified K amplitude should properly be written

$$T=K'+\tfrac{1}{2}i\frac{F(s,s_0)}{\rho(s)}TK',$$

but we shall henceforth drop the prime on K' and also absorb the factor F/ρ into the convention for the intermediate lines whenever a unitarity-type integral occurs. The new relation between the partial waves [instead of (1.2)] will be

$$a_l(s)=\frac{k_l(s)}{1-\frac{1}{4}i\pi F(s,s_0)k_l(s)}. \qquad (2.4)$$

It is interesting to note that this modified K-matrix formalism bears a much closer resemblance to the N/D method than does the usual K matrix. For example, if a "scattering-length" approximation is made with the *modified* K matrix, the result is precisely the Chew-Mandelstam effective-range formula[8] which is usually derived by taking N to be a constant and using the determinantal approximation for D.

It was first pointed out by Zimmermann[2] that (2.1) could be extended to define a function which would have the elastic normal thresholds in each of the three channels removed. One simply defines a function $\lambda(s,t,u)$ by Fig. 4 and it is easy to see, by using the fact that $T(s,t,u)$ is a totally symmetric function of its three variables, that $\lambda(s,t,u)$ is regular at $s=4$, $t=4$, and $u=4$ and is also, of course, totally symmetric itself. Although Zimmermann was using the ordinary K matrix the situation remains unchanged when the modified phase-space factor is used. An equation like Fig. 4 has also been introduced and discussed by Taylor[5] within the context of field theory, but there the intermediate lines are off the mass shell.

We have already pointed out that $K(s,t)$, while regular at $s=4$, is singular at $s=0$ due to a pinch of the integration contour in the integral in (2.1). Conse-

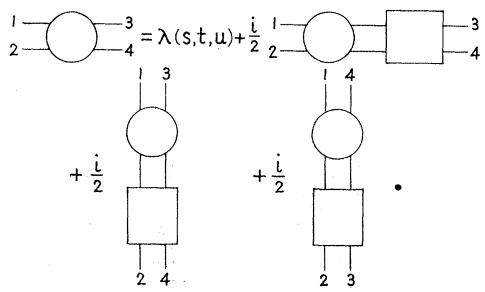

FIG. 4. Definition of the symmetrically reduced amplitude $\lambda(s,t,u)$.

[6] The author would like to thank Dr. P. V. Landshoff for this observation.

[7] P. G. Freund and R. Karplus, Nuovo Cimento **21**, 519 (1961).

[8] G. F. Chew and S. Mandelstam, Phys. Rev. **119**, 467 (1960).

FIG. 5. Definition of the function $L(s,t)$.

quently, by Fig. 4, $\lambda(s,t,u)$ must have branch points at $s=0$, $t=0$, and $u=0$, with just that structure required to make T regular at these points; λ and K are both finite at these singularities.

It is the equation in Fig. 4 which forms the basis of our approximate calculation of the scattering amplitude. It will be convenient to cast it into a somewhat different form, with the help of (2.1). Introducing the function $L(s,t)$ defined in Fig. 5, Figs. 2 and 4 may be written in the form

$$T(s,t)=K(s,t)+\tfrac{1}{2}iL(s,t)\,, \tag{2.5a}$$

$$T(s,t)=\lambda(s,t,u)+\tfrac{1}{2}i[L(s,t)+L(t,u)+L(u,s)]. \tag{2.5b}$$

Since $T(s,t,u)$ is a totally symmetric function of its three variables and $K(s,t,u)$ is symmetric in t and u, it follows that $L(s,t,u)$ has the same symmetry as K. Hence, by permuting the variables in (2.5a) twice, adding to (2.5a), and comparing with (2.5b), we find

$$T(s,t,u)=-\tfrac{1}{2}\lambda(s,t,u) +\tfrac{1}{2}[K(s,t)+K(t,u)+K(u,s)]. \tag{2.6}$$

Since it is difficult to take the singularities at $s=0$, $t=0$, and $u=0$ in λ and K explicitly into account and since those in λ must overlap those in each K in such a way as to annihilate one another when T is computed according to (2.6), we shall make the simplifying assumption that these singularities may be neglected, for the purposes of an approximate calculation of the amplitude, in both λ and K. The cross-channel normal thresholds in $K(s,t)$ (at $t=4$ and $u=4$) are of course retained and the scattering amplitude obtained from (2.6) will thus have the correct elastic branch points in each channel.

Proceeding, then, on the assumption that the singularities at the origin in $\lambda(s,t,u)$ may be ignored, provided the K amplitude is suitably modified, the only remaining singularities in this function are those due to the inelastic thresholds at $s=16$, etc. In the hope that low-energy π-π scattering is dominated by elastic effects for a considerable distance above threshold, we shall not take the inelastic thresholds into account here. Thus, in effect, we are assuming that it is reasonable to take $\lambda(s,t,u)$ to be an entire function of its variables and furthermore, in the spirit of K-matrix calculations generally, we shall take λ to be a constant. Since $\lambda(s,t,u)$ is rigorously (i.e., before neglecting any singularities) real in the neighborhood of the symmetry point, and since we are approximating it by a singularity-free function, the constant λ will be taken to be real. We observe that in a Taylor-series expansion of $\lambda(s,t,u)$ about some point (the symmetry point, say), the first nonconstant terms involve the *second* derivatives of λ since the symmetry of λ requires any linear terms to combine into a constant (since $s+t+u=4$). This provides some support for the approximation of λ by a constant. The main test of the usefulness of this assumption, however, must lie in the physical significance of its consequences. Since the last three terms on the right-hand side in Fig. 4 correspond, in some sense, to the contributions to T from diagrams with a two-particle cut in at least one of the three channels, λ evidently represents the contribution of all other diagrams. Thus, taking a simple form of λ, such as a constant, effectively means that only the two-particle structure is being considered.[2,5] In this sense our basic equations roughly correspond to the strip-approximation equations.[5,9]

If one is interested in decay processes, for which the physical region is s, t, and $u\geq 4$ (considering s, t, and u as the subenergies in a three-body final state), the approximation of constant λ may be reasonable without even introducing a modified phase-space factor. This situation has been investigated by Taha.[10]

3. THE POWER-SERIES SOLUTION

Having thus decided to make λ a constant we may now consider Fig. 4 and the defining equation for the K matrix (2.1) as a pair of coupled integral equations for the scattering amplitude. The method we have adopted in order to obtain an approximate solution to these equations is to make a power-series expansion in the parameter λ and to sum as many terms (an infinite subset, as it turns out) in the resulting expansion as it appears feasible to handle. Once an approximate solution of the equations for $K(s,t)$ is obtained by this "perturbation" method we then have the option of determining T from K, either by (2.1), giving an amplitude satisfying elastic unitarity in the direct channel, or by (2.6), which clearly yields a crossing-symmetric amplitude. We regard it as more important to have unitarity satisfied in the elastic region and consequently (2.1) will be used. The extent to which the resulting amplitude satisfies crossing symmetry will naturally depend on the extent to which the approximations made in obtaining K are valid. The way in which the two parameters λ and s_0 (which are so far arbitrary) may then be used to minimize the violation of crossing will be discussed in more detail in Sec. 5.

The actual construction of the power-series solution will be facilitated by decoupling the two equations (2.1) and (2.5b) to give a single nonlinear integral equation for K only. This may be done by simply inserting (2.6) into (2.1), but in order to express the resulting equation in a convenient form we shall introduce an algebraic notation to denote the "composition" of two functions in a unitarity-type integral. Thus we define

$$K(s,t,u)\equiv K_1\,,$$
$$K(t,u,s)\equiv K_2\,,$$
$$K(u,s,t)\equiv K_3\,,$$

[9] G. F. Chew and S. C. Frautschi, Phys. Rev. **123**, 1478 (1961).
[10] M. O. Taha, Nuovo Cimento **42**, 201 (1966).

and also define $K_i \circ K_j$ to mean the composition of K_i and K_j, with the composition taking place in the s_j channel. (We treat s_1, s_2, and s_3 as interchangeable with s, t, and u.) This is illustrated in Fig. 6.

In this notation (2.1) reads

$$T_1 = K_1 + \tfrac{1}{2} i T_1 \circ K_1 ,$$

and (2.6) becomes

$$T_1 = -\tfrac{1}{2}\lambda + \tfrac{1}{2}\sum_{i=1}^{3} K_i .$$

Combining these equations gives

$$\begin{aligned} K_1 &= -\tfrac{1}{2}\lambda + \tfrac{1}{2}\textstyle\sum_i K_i - \tfrac{1}{2} i(-\tfrac{1}{2}\lambda + \tfrac{1}{2}\sum_i K_i) \circ K_1 \\ &= -\tfrac{1}{2}\lambda + \tfrac{1}{2}\textstyle\sum_i K_i + \tfrac{1}{8} i\pi F_1 \lambda K_1^{(0)} \\ &\qquad - \tfrac{1}{4} i \textstyle\sum_i K_i \circ K_1 , \end{aligned} \tag{3.1}$$

where $F_i \equiv F(s_i, s_0)$ and $K_i^{(l)}$ is the lth partial-wave projection, in the s_i channel, of K_i. We now insert the power-series expansion

$$K_i = \sum_{n=1}^{\infty} \lambda^n K_{i,n} \tag{3.2}$$

and separate out the coefficients of λ^n

$$\begin{aligned} K_{1,n} = &-\tfrac{1}{2}\delta_{n,1} + \tfrac{1}{2}\sum_i K_{i,n} + \tfrac{1}{8} i\pi F_1 K_{1,n-1}^{(0)} \\ &- \tfrac{1}{4} i \sum_{\substack{ipq \\ p+q=n}} K_{i,p} \circ K_{1,q} . \end{aligned} \tag{3.3}$$

It is now straightforward to solve this equation successively to any desired order. Thus one finds for the first three orders:

$$\begin{aligned} &K_{1,1} = 1 , \\ &K_{1,2} = \tfrac{1}{4} i\pi (F_2 + F_3) , \\ &K_{1,3} = -\tfrac{1}{16}\pi^2 [(F_2)^2 + (F_3)^2 + 4F_2 F_2^{(0)} + 4F_3 F_3^{(0)}] . \end{aligned} \tag{3.4}$$

In the expression for $K_{1,3}$, the function $F^{(0)}$ is the s-wave projection of $F(t, s_0)$, i.e.,

$$F^{(0)}(s, s_0) = \frac{1}{s-4} \int_{4-s}^{0} dt\, F(t, s_0) , \tag{3.5}$$

and the subscript on $F^{(0)}$ indicates as usual the variable on which $F^{(0)}$ depends. Since $G(t)$ has a branch point at $t=4$ the function $F^{(0)}(s)$ will have a branch point at $s=0$, an end-point singularity of the integral in (3.5), and consequently the third-order K matrix $K_{1,3}$ has branch points at $t=0$ and $u=0$, by (3.4).

It was to be expected that singularities of this type would arise in K, but now we see that when T is calculated by using (2.6) it will retain these undesirable

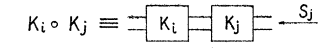

FIG. 6. A diagram illustrating the meaning of the "composition" of two amplitudes as used in the text.

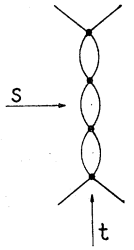

FIG. 7. A typical cross-channel chain diagram. The internal lines are on the mass shell but the use of the modified phase-space factor makes the amplitude for the graph the same as if the lines had been off shell.

singularities. In an exact treatment the function $\lambda(s,t,u)$ would have been singular in just such a way as to preserve the known cut-plane analyticity of T, as we have previously discussed. In accordance with the program outlined in the previous section we now separate the power-series expansion for K into two parts, retaining only that part with the desirable analytic structure; the other part we imagine as absorbed into λ. This division is, of course, somewhat arbitrary, but a natural separation does arise from the power-series solution, in which all those terms which are free of the singularities at $t=0$ and $u=0$ consist of the chain diagrams (Fig. 7) and are readily summed. These chains evidently form a geometric progression, with ratio $\frac{1}{4} i\pi\lambda F$, and thus our approximate expansion for K is simply

$$\begin{aligned} K(s,t,u) = &\lambda[1 - \tfrac{1}{4} i\pi\lambda F(t)]^{-1} \\ &+ \lambda[1 - \tfrac{1}{4} i\pi\lambda F(u)]^{-1} - \lambda . \end{aligned} \tag{3.6}$$

It is this approximate solution for K (appropriately generalized to include isospin) on which we base our π-π scattering model. In the next section the formulas obtained here are extended to describe particles with isospin 1.

4. INSERTION OF ISOSPIN INTO THE BASIC EQUATIONS

We define a K amplitude for each isospin channel in the following way:

$$T_I = K_I + \tfrac{1}{2} i T_I \circ K_I , \quad I = 0, 1, 2 . \tag{4.1}$$

Defining $L_I(s,t,u)$ as in Sec. 3, (4.1) becomes

$$T_I(s,t,u) = K_I(s,t,u) + \tfrac{1}{2} i L_I(s,t,u) . \tag{4.2}$$

We wish to find the analog in the isovector case of the crossing-symmetric two-particle structure equation (Fig. 4). This may be accomplished by defining the completely reduced amplitude $\lambda_I(s,t,u)$ as

$$\begin{aligned} \lambda_I(s,t,u) = &T_I(s,t,u) \\ &- \tfrac{1}{2} i[L_I(s,t,u) + L_I'(t,u,s) + L_I''(u,s,t)] . \end{aligned} \tag{4.3}$$

The new functions L_I', L_I'' will be chosen so that λ_I will be lacking the two-particle cuts in the t and u channels, respectively.

Since the scattering amplitude T_I satisfies the crossing relations

$$\begin{aligned} T_I(s,t,u) &= \beta_{IJ} T_J(t,s,u) \\ &= (-1)^J \beta_{IJ} T_J(t,u,s) , \end{aligned} \tag{4.4}$$

where β is the well-known π-π crossing matrix

$$\beta=\begin{bmatrix}1/3 & 1 & 5/3\\ 1/3 & 1/2 & -5/6\\ 1/3 & -1/2 & 1/6\end{bmatrix}, \tag{4.5}$$

it is clear that the t-channel cut in $\lambda_I(s,t,u)$ will be removed by the choice

$$L_I'(t,u,s)=(-1)^J\beta_{IJ}L_J(t,u,s), \tag{4.6}$$

for then

$$\begin{aligned}&\mathrm{disc}_t\lambda_I(s,t,u)\\ &=\mathrm{disc}_tT_I(s,t,u)-\tfrac{1}{2}i\,\mathrm{disc}_tL_I'(t,u,s)\\ &=[T_I(s,t+i\epsilon,u)-T_I(s,t-i\epsilon,u)]\\ &\quad-\tfrac{1}{2}i[L_I'(t+i\epsilon,u,s)-L_I'(t-i\epsilon,u,s)]\\ &=(-1)^J\beta_{IJ}[T_J(t+i\epsilon,u,s)-T_J(t-i\epsilon,u,s)]\\ &\quad-\tfrac{1}{2}i[L_I'(t+i\epsilon,u,s)-L_I'(t-i\epsilon,u,s)]\\ &=\tfrac{1}{2}i\{(-1)^J\beta_{IJ}[L_J(t+i\epsilon,u,s)-L_J(t-i\epsilon,u,s)]\\ &\quad-[L_I'(t+i\epsilon,u,s)-L_I'(t-i\epsilon,u,s)]\}\\ &=0.\end{aligned} \tag{4.7}$$

Similarly, L_I'' is chosen to satisfy

$$\begin{aligned}L_I''(u,s,t)&=(-1)^J\beta_{IJ}L_J'(u,s,t)\\ &=(-1)^{J+K}\beta_{IJ}\beta_{JK}L_K(u,s,t).\end{aligned} \tag{4.8}$$

Thus we have finally

$$\begin{aligned}\lambda_I(s,t,u)=T_I(s,t,u)&-\tfrac{1}{2}i[L_I(s,t,u)+(-1)^J\beta_{IJ}L_J(t,u,s)\\ &+(-1)^{J+K}\beta_{IJ}\beta_{JK}L_K(u,s,t)].\end{aligned} \tag{4.9}$$

It is straightforward to deduce from (4.9) that λ_I satisfies the same crossing relations as the scattering amplitude, i.e.,

$$\lambda_I(s,t,u)=(-1)^J\beta_{IJ}\lambda_J(t,u,s). \tag{4.10}$$

Now, by application of (4.2) and its crossed versions, we may write

$$\begin{aligned}T_I(s,t,u)=\lambda_I(s,t,u)&+T_I(s,t,u)-K_I(s,t,u)\\ &+(-1)^J\beta_{IJ}[T_J(t,u,s)-K_J(t,u,s)]\\ &+(-1)^I\beta_{IJ}[T_J(u,s,t)-K_J(u,s,t)],\end{aligned} \tag{4.11}$$

where use has also been made of the relation

$$(-1)^{J+K}\beta_{IJ}\beta_{JK}=(-1)^I\beta_{IK}. \tag{4.12}$$

With the aid of

$$\begin{aligned}(-1)^J\beta_{IJ}T_J(t,u,s)&=(-1)^I\beta_{IJ}T_J(u,s,t)\\ &=T_I(s,t,u),\end{aligned} \tag{4.13}$$

(4.11) becomes

$$\begin{aligned}T_I(s,t,u)=-\tfrac{1}{2}\lambda_I(s,t,u)&+\tfrac{1}{2}[K_I(s,t,u)\\ &+(-1)^J\beta_{IJ}K_J(t,u,s)+(-1)^I\beta_{IJ}K_J(u,s,t)],\end{aligned} \tag{4.14}$$

analogous to (3.3).

As a preliminary to developing the "perturbation theory" formalism we note that the requirement that $\lambda_I(s,t,u)$ satisfy (4.10) reduces the number of independent parameters, when λ_I is chosen to be constant (for each I), from three to one. This follows because (4.10), for λ_I independent of s, t, and u, requires the three-vector λ_I to be an eigenvector of the matrix $(-1)^J\beta_{IJ}$ with eigenvalue unity. This eigenvector is proportional to

$$\begin{bmatrix}5\\0\\2\end{bmatrix}\equiv(\hat{\lambda}_I); \tag{4.15}$$

thus we choose

$$\lambda_I=\lambda\hat{\lambda}_I. \tag{4.16}$$

It is interesting to note that if the subtraction point s_0 in the phase-space factor F is chosen to be $\frac{4}{3}$ it then follows that

$$T_I(\tfrac{4}{3},\tfrac{4}{3},\tfrac{4}{3})=K_I(\tfrac{4}{3},\tfrac{4}{3},\tfrac{4}{3})=\lambda_I, \tag{4.17}$$

so that in this case our λ would be the conventional one, frequently called the "renormalized coupling constant" (because of differences in normalization conventions this λ is in fact equal to $-4/\pi$ times the Chew-Mandelstam λ). However, we do not wish to fix s_0 arbitrarily, but prefer to use it, together with λ, to minimize the violation of crossing by our approximate amplitude, as described in the next section. Thus in general (4.17) will not be satisfied.

As a further preliminary to the extension of the perturbation theory developed in the previous section we recast the equations in matrix form. To this end we define

$$\begin{aligned}&K_{Ij}\equiv K_I(s_j,s_k,s_l),\quad I=0,1,2;\\ &\qquad j,k,l \text{ a cyclic permutation of } 1,2,3.\end{aligned} \tag{4.18}$$

K is now to be thought of as a column vector with the nine components $(K_{01},K_{02},K_{03},K_{11},\cdots)$. With a corresponding notation for T (4.14) becomes

$$T_{I1}=-\tfrac{1}{2}\lambda\hat{\lambda}_{I1}+\tfrac{1}{2}\sum_{K,l}m_{I1;Kl}K_{Kl}, \tag{4.19}$$

where m is the 9×9 matrix

$$m=\tfrac{1}{6}\begin{bmatrix}6&2&2&0&-6&6&0&10&10\\ 2&6&2&6&0&-6&10&0&10\\ 2&2&6&-6&6&0&10&10&0\\ 0&2&-2&6&-3&-3&0&-5&5\\ -2&0&2&-3&6&-3&5&0&-5\\ 2&-2&0&-3&-3&6&-5&5&0\\ 0&2&2&0&3&-3&6&1&1\\ 2&0&2&-3&0&3&1&6&1\\ 2&2&0&3&-3&0&1&1&6\end{bmatrix}, \tag{4.20}$$

and

$$\hat{\lambda}=(5,5,5,0,0,0,2,2,2). \tag{4.21}$$

As in Sec. 3, the unitarity integrals will be denoted by $T_{Ij}\circ K_{Kl}$; with the composition taking place in the s_l channel. Thus, for example, the "unitarity" equations (4.1) are, in this notation,

$$T_{I1}=K_{I1}+\tfrac{1}{2}iT_{I1}\circ K_{I1}. \tag{4.22}$$

One of the advantages of this formulation is that an equation originally written down in one channel, such as (4.19), may immediately be extended to other channels. Thus (4.19) generalizes to

$$T_{Ij}=-\tfrac{1}{2}\lambda\hat{\lambda}_{Ij}+\tfrac{1}{2}\sum_{Kl}m_{Ij;Kl}K_{Kl}, \tag{4.23}$$

and similarly (4.22) becomes

$$T_{Ij}=K_{Ij}+\tfrac{1}{2}iT_{Ij}\circ K_{Ij} \tag{4.24}$$

(the summation convention is not used). Inserting (4.23) into (4.24) now gives the desired nonlinear integral equation involving only the K amplitude:

$$-\tfrac{1}{2}\lambda\hat{\lambda}_{Ij}+\tfrac{1}{2}\sum_{Kl}m_{Ij;Kl}K_{Kl}=K_{Ij}-\tfrac{1}{4}i\lambda\hat{\lambda}_{Ij}\circ K_{Ij}+\tfrac{1}{4}i\sum_{Kl}m_{Ij;Kl}K_{Kl}\circ K_{Ij}. \tag{4.25}$$

Assuming that a power-series expansion in λ exists,

$$K_{Ij}=\sum_{n=1}^{\infty}\lambda^n K_{Ij}^{(n)}, \tag{4.26}$$

we insert it into (4.25) and extract the coefficients of λ and λ^n $(n\geq 2)$:

$$-\tfrac{1}{2}\hat{\lambda}_{Ij}+\tfrac{1}{2}\sum_{Kl}m_{Ij;Kl}K_{Kl}^{(1)}=K_{Ij}^{(1)}, \tag{4.27a}$$

$$\tfrac{1}{2}\sum_{Kl}m_{Ij;Kl}K_{Kl}^{(n)}=K_{Ij}^{(n)}-\tfrac{1}{4}i\hat{\lambda}_{Ij}\circ K_{Ij}^{(n-1)}+\tfrac{1}{4}i\sum_{Kl,\,p+q=n}m_{Ij;Kl}K_{Kl}^{(p)}K_{Ij}^{(q)}. \tag{4.27b}$$

These equations (4.27) may be solved successively to obtain $K_{Ij}^{(n)}$ for any finite n. To this end we define the matrix

$$b=12(\tfrac{1}{2}m-I), \tag{4.28}$$

where I is the 9×9 unit matrix. Then, by (4.27) and (4.28),

$$\sum_{Kl}b_{Ij;Kl}K_{Kl}^{(1)}=6\hat{\lambda}_{Ij}, \tag{4.29a}$$

$$\sum_{Kl}b_{Ij;Kl}K_{Kl}^{(n)}=-3if_{Ij}^{(n-1)}+3i\sum_{Kl}m_{Ij;Kl}g_{Kl;Ij}^{(n)}, \tag{4.29b}$$

where we have defined

$$f_{Ij}^{(n)}=\hat{\lambda}_{Ij}\circ K_{Ij}^{(n)}, \tag{4.30a}$$

$$g_{Kl;Ij}^{(n)}=\sum_{p+q=n}K_{Kl}^{(p)}\circ K_{Ij}^{(q)}. \tag{4.30b}$$

To solve Eqs. (4.29) we have only to invert the matrix b, obtaining

$$K_{Ij}^{(1)}=6\sum_{Kl}d_{Ij;Kl}\hat{\lambda}_{Kl}, \tag{4.31a}$$

$$K_{Ij}^{(n)}=-3i\sum_{Kl}d_{Ij;Kl}f_{Kl}^{(n-1)}+3i\sum_{KlRs}d_{Ij;Kl}m_{Kl;Rs}g_{Rs;Kl}^{(n)}, \tag{4.31b}$$

where

$$d=b^{-1}=\frac{1}{72}\begin{pmatrix}0&2&2&0&-6&6&0&10&10\\2&0&2&6&0&-6&10&0&10\\2&2&0&-6&6&0&10&10&0\\0&2&-2&0&-3&-3&0&-5&5\\-2&0&2&-3&0&-3&5&0&-5\\2&-2&0&-3&-3&0&-5&5&0\\0&2&2&0&3&-3&0&1&1\\2&0&2&-3&0&3&1&0&1\\2&2&0&3&-3&0&1&1&0\end{pmatrix}. \tag{4.32}$$

It follows immediately from (4.31a) that the first-order term is given by

$$K_{Ij}^{(1)}=\hat{\lambda}_{Ij}, \tag{4.33}$$

and it is a straightforward but lengthy task to use this in (4.30) and (4.31b) to obtain $K_{Ij}^{(2)}$ and the higher-order terms successively. We shall quote only the results for the first three orders here. In conventional notation one obtains, in second order,

$$K_0^{(2)}(s,t,u)=\frac{15\pi i}{4}[F(t,t_0)+F(u,u_0)], \tag{4.34a}$$

$$K_1^{(2)}(s,t,u)=\frac{5\pi i}{4}[F(t,t_0)-F(u,u_0)], \tag{4.34b}$$

$$K_2^{(2)}(s,t,u)=\frac{9\pi i}{4}[F(t,t_0)+F(u,u_0)]; \tag{4.34c}$$

and in third order,

$$K_0^{(3)}(s,t,u)=-\frac{55\pi^2}{16}\{F(t,t_0)[F(t,t_0)+4F^{(0)}(t,t_0)]+(t\leftrightarrow u)\}, \tag{4.35a}$$

$$K_1^{(3)}(s,t,u)=-\frac{35\pi^2}{16}\left\{\left[F(t,t_0)+\frac{8}{7}F^{(0)}(t,t_0)\right]F(t,t_0)-(t\leftrightarrow u)\right\}, \tag{4.35b}$$

$$K_2^{(3)}(s,t,u)=-\frac{43\pi^2}{16}\left\{F(t,t_0)\left[F(t,t_0)+\frac{112}{43}F^{(0)}(t,t_0)\right]+(t\leftrightarrow u)\right\}. \tag{4.35c}$$

The function $F^{(0)}$ was defined in (3.9), and $t_0=u_0=s_0$.

In order to complete our program for this section, as outlined at the end of Sec. 3, we have now to sum the terms in the subset of the perturbation expansion which consist of the cross-channel chain diagrams (see Fig. 7). This is not quite as easy here as it was in the isoscalar case since what was previously a simple geometric progression now splits into two such progressions. To show this and to identify the coefficients, we try writing the nth-order K amplitude in the chain approximation in the form

$$K_I^{(n)}(s,t,u)=h_I^{(n)}[(F(t))^{n-1}+(-1)^I(F(u))^{n-1}], \quad n\geq 2, \tag{4.36}$$

where the $h_I^{(n)}$ are constants to be determined and the dependence of F on the subtraction point has been temporarily left out.

Now application of (4.19) yields the nth-order scattering amplitudes

$$\begin{aligned}T_0^{(n)}(s,t,u)&=\tfrac{1}{6}\{(2h_0^{(n)}+6h_1^{(n)}+10h_2^{(n)})(F(s))^{n-1}\\&\quad+(4h_0^{(n)}-3h_1^{(n)}+5h_2^{(n)})\\&\quad\times[(F(t))^{n-1}+(F(u))^{n-1}]\},\\T_1^{(n)}(s,t,u)&=\tfrac{1}{12}(-2h_0^{(n)}+9h_1^{(n)}+5h_2^{(n)})\\&\quad\times[(F(t))^{n-1}-(F(u))^{n-1}],\\T_2^{(n)}(s,t,u)&=\tfrac{1}{12}\{(4h_0^{(n)}-6h_1^{(n)}+2h_2^{(n)})(F(s))^{n-1}\\&\quad+(2h_0^{(n)}+3h_1^{(n)}+7h_2^{(n)})\\&\quad\times[(F(t))^{n-1}+(F(u))^{n-1}]\}.\end{aligned} \tag{4.37}$$

Noticing that the coefficient of $(F(t))^{n-1}$ must be the same in $K_I^{(n)}$ as in $T_I^{(n)}$ gives one relation between the $h_I^{(n)}$:

$$2h_0^{(n)}+3h_1^{(n)}-5h_2^{(n)}=0, \tag{4.38a}$$

while the observation that the coefficient of $(F(s))^{n-1}$ in $T_I^{(n)}$ must be

$$\hat{\lambda}_I(\tfrac{1}{4}i\pi\hat{\lambda}_I)^{n-1}$$

yields two more relations

$$\tfrac{1}{3}(h_0^{(n)}+3h_1^{(n)}+5h_2^{(n)})=5(i5\pi/4)^{n-1}, \tag{4.38b}$$

$$\tfrac{1}{6}(2h_0^{(n)}-3h_1^{(n)}+h_2^{(n)})=2(i\pi/2)^{n-1}. \tag{4.38c}$$

These three equations (4.38) have as their solution

$$h_0^{(n)}=(5/3)[(i5\pi/4)^{n-1}+2(i\pi/2)^{n-1}], \tag{4.39a}$$

$$h_1^{(n)}=(5/3)[(i5\pi/4)^{n-1}-(i\pi/2)^{n-1}], \tag{4.39b}$$

$$h_2^{(n)}=\tfrac{1}{3}[5(i5\pi/4)^{n-1}+(i\pi/2)^{n-1}]. \tag{4.39c}$$

The chains in each isospin channel are now simple to sum:

$$K_0(s,t,u)=(5/3)\lambda[\alpha^{-1}(t)+2\beta^{-1}(t)]+(t\leftrightarrow u)-5\lambda, \tag{4.40a}$$

$$K_1(s,t,u)=(5/3)\lambda[\alpha^{-1}(t)-\beta^{-1}(t)]-(t\leftrightarrow u), \tag{4.40b}$$

$$K_2(s,t,u)=\tfrac{1}{3}\lambda[5\alpha^{-1}(t)+\beta^{-1}(t)]+(t\leftrightarrow u)-2\lambda, \tag{4.40c}$$

where

$$\alpha(t)=1-(i5\pi\lambda/4)F(t,t_0), \tag{4.41a}$$

and

$$\beta(t)=1-\tfrac{1}{2}i\pi\lambda F(t,t_0). \tag{4.41b}$$

5. DETERMINATION OF THE PARAMETERS

We wish to use the expressions (4.40), obtained in the chain approximation, for the K amplitude and to obtain from it the scattering partial waves according to (2.4). Since the scattering amplitude obtained in this way will not, of course, satisfy exact crossing symmetry (since we have had to make approximations in solving the original equations) it is reasonable to attempt to determine the two parameters (s_0 and λ) in such a way as to minimize the violation of crossing symmetry. One might imagine doing this in either of two ways. One possibility would be to work entirely within the framework of the model defined by Eqs. (4.40) and (2.4), making no further reference to the two-particle cut structure equations (4.9). In this case one would simply try to minimize the violation of the crossing relations (4.4) (or of the partial-wave approximation to these equations). An alternative approach would be to attempt to choose the parameters in the approximate solution (4.40) in such a way as to make the original equations of the model as well satisfied as possible; that is, to match the amplitude calculated from (4.40) and (2.4) with that obtained by using (4.40) and (4.14).

To make the best use of either of these methods would involve minimizing some "crossing function," computed as an average over some region of the s-t plane, with respect to the parameters. Such a procedure has been described by Blankenbecler *et al.*[11] and, of course, requires the use of a computer. The results of such a calculation cannot be expected to be entirely unambiguous, however.[12] It is quite likely that more than one minimum would be obtained, and a choice would then have to be made on the basis of the relative sharpness and/or deepness of the various minima. Even then it might prove necessary to make use of some of the known experimental features of the π-π system, such as the position or at least the existence of the ρ meson, since we can hardly expect that inclusion of the elastic-scattering channels only is sufficient to completely determine matters. Indeed, it has generally been found necessary in previous calculations to include something of inelastic effects if the ρ was to come out of the calculation at all. While inelastic effects have not been explicitly taken into account in our model it is nevertheless the case that the $k_I^{(l)}(s)$ become complex at sufficiently high energies, thereby violating the elastic unitarity condition on the $a_I^{(l)}(s)$. This will be discussed in more detail in the next section. To return to the object of the present discussion, since there are

[11] R. Blankenbecler, J. J. Brehm, L. F. Cook, and R. E. Kreps, Phys. Rev. **133**, B1526 (1964).
[12] R. E. Kreps, Phys. Rev. **141**, 1380 (1966).

only two parameters in the present model it is possible to form a rough idea of its capabilities without resorting to a machine calculation. It is this preliminary investigation which is described in the present paper.

Since the description of resonances will evidently form a vital part of the discussion we must first develop the resonance position and width formulas in the modified K-matrix formalism. It follows from (1.2c) and (2.4) that the phase shifts are given by

$$\delta_I{}^{(l)}(s)=\tan^{-1}\left(\frac{\rho(s)}{(4/\pi)[k_I{}^{(l)}(s)]^{-1}+R(s)-\mu}\right), \quad (5.1)$$

where $\mu=-iG(s_0)$ and $R(s)$ is defined in (2.3). Thus the phase shift will be $\frac{1}{2}\pi(\pm n\pi)$ at values s_r of s such that

$$k_I{}^{(l)}(s_r)=(4/\pi)[\mu-R(s_r)]^{-1}. \quad (5.2)$$

The condition that this be a true resonance (phase shift *increasing* through $\frac{1}{2}\pi$) may be derived by writing the amplitude in the neighborhood of s_r in the form

$$a_I{}^{(l)}(s)=\frac{4}{\pi}\frac{1}{\rho}\frac{\Delta}{s_r-s-i\Delta},$$
$$\Delta=\frac{k_I{}^{(l)}\rho}{(\mu-R)k_I{}^{(l)\prime}-kR'}=\frac{\rho}{\frac{1}{4}\pi(\mu-R)^2k_I{}^{(l)\prime}-R'}, \quad (5.3)$$

where the primes denote differentiation with respect to s and all functions in (5.3) are evaluated at s_r. The second equality in (5.3) is a result of using the resonance condition (5.2). Thus, for $\Delta>0$, there will be a true resonance mass $\sqrt{s_r}$ with width

$$\Gamma=\Delta/\sqrt{s_r},$$

in units of the pion mass. Since $R'(s)>0$ for all s above threshold we see by (5.3) that a necessary condition for resonance is

$$k_I{}^{(l)\prime}(s_r)>0.$$

We may now inquire whether a p-wave resonance is possible for any choice of the parameters, given the $I=1$ K amplitude (4.40b). It is convenient to use

$$\alpha_0=\alpha(0)=1+\frac{5\pi\lambda}{4}\left(\frac{2}{\pi}-\mu\right)$$

as a parameter instead of s_0. Now inspection reveals that the only region of the (α_0,λ) plane which is capable of describing a p-wave resonance (except for very large values of λ, say $|\lambda|>1$, which we reject as inconsistent with our neglect of all but the chain diagrams) is $(\alpha_0\lesssim-0.1,\ \lambda>0)$. The position of the resonance will be (very roughly) about that of the ρ (750 MeV) for points near the line $\alpha_0+2\lambda=0.05$. The parameters giving the experimental mass and generally accepted width (120 MeV) are $(\alpha_0,\lambda)=(-0.61, 0.27)$. As is customary the width is found to be much more sensitive to variations of the parameters than is the position.

The actual procedure we have used for approximately locating a point of minimum crossing violation will now be described. The parameter search was begun in the neighborhood of the ρ; the possibility of relaxing this is discussed later. In keeping with what we have said earlier about the aims of the present work the crossing conditions will only be applied at a single point in the s-t plane, the symmetry point $(\frac{4}{3},\frac{4}{3})$, although a few remarks will be made about the qualitative behavior away from that point.

One relation between the parameters will be obtained by simply requiring the amplitude obtained from (2.4) to satisfy crossing at the symmetry point. At this point (4.4) requires $T_0=2.5T_2$, which, when expanded into partial waves, becomes

$$a_0{}^{(0)}(\tfrac{4}{3})-2.5a_0{}^{(2)}(\tfrac{4}{3})+\cdots$$
$$=2.5[a_2{}^{(0)}(\tfrac{4}{3})-2.5a_2{}^{(2)}(\tfrac{4}{3})+\cdots]. \quad (5.4)$$

This equation defines a curve in the resonance region of the parameter space, part of which is shown in Fig. 8.

In solving (5.4) to obtain this curve the partial-wave expansion was terminated at the d waves, their contribution being only about 5% that of the s waves. The position of the ρ is indicated by the heavy dot in Fig. 8; its distance from the $T_0=\frac{5}{2}T_2$ curve only appears large because only a small section of the (α_0,λ) plane is shown in the figure. If the experimental ρ mass is used to eliminate the remaining degree of freedom, by varying along the curve in Fig. 8 until a resonance mass of 750 MeV is "predicted" [by using (5.2)], the parameters found are $(-0.54, 0.24)$, with a resulting predicted width of about 180 MeV. This is close to one recent experimental determination[13] (170 MeV) but is above the generally accepted range of 100–140 MeV.

It was our original intention to use the derivative crossing conditions on the partial amplitudes, as derived by Chew and Mandelstam,[14] to make the final choice of parameters. However, the derivative conditions were found to be so sensitive to higher partial waves that this method was rejected as unsuitable for hand calculation. It should be pointed out here that not only are there convergence difficulties but also the partial-wave

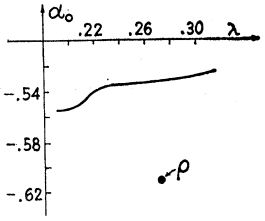

FIG. 8. Curve on which $T_0(\frac{4}{3},\frac{4}{3})=\frac{5}{2}T_2(\frac{4}{3},\frac{4}{3})$. The ρ parameters are $(-0.61,0.27)$; the apparent distance from this point (heavy dot in the figure) to the curve is exaggerated by the large scale of this figure.

[13] G. Wolf, Phys. Letters **19**, 328 (1965).
[14] G. F. Chew and S. Mandelstam, Nuovo Cimento **19**, 752 (1961).

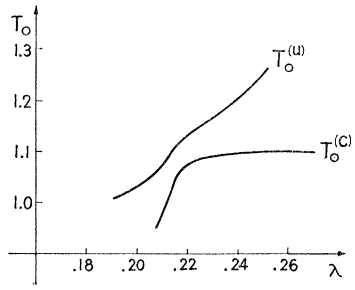

FIG. 9. A plot showing how the discrepancy $T_0^{(u)}-T_0^{(c)}$, considered as a function of λ, is minimized for $\lambda=0.22$.

projection integrals have to be evaluated numerically and the errors in this increase with l because of the increased oscillatory behavior of the integrands. This does not mean that the derivative conditions were badly violated but rather that, since the deviations were in fact rather small, it was not possible to select a particular pair of parameters by this method with any degree of confidence. A similar situation prevailed when we attempted to use crossing relations at points away from the symmetry point. This approach would presumably be feasible on a computer but for the purposes of the present investigation we can make use of the alternative method, mentioned at the beginning of this section, in which (4.14) is employed. Since we wish to place more weight on the "internal" condition (5.4) than on the second condition the following procedure is adopted: We search along the curve of Fig. 8 for a minimum deviation of $T_0(\frac{4}{3},\frac{4}{3},\frac{4}{3})$, as calculated in (5.4), with the $T_0(\frac{4}{3},\frac{4}{3},\frac{4}{3})$ obtained from (4.14). The latter amplitude is given by

$$T_0^{(c)}(\tfrac{4}{3},\tfrac{4}{3},\tfrac{4}{3})=-(5/2)\lambda+(5/6)K_0(\tfrac{4}{3},\tfrac{4}{3},\tfrac{4}{3})+(5/3)K_2(\tfrac{4}{3},\tfrac{4}{3},\tfrac{4}{3})$$
$$=-10\lambda+(25/3)\lambda\alpha^{-1}(\tfrac{4}{3})+(20/3)\lambda\beta^{-1}(\tfrac{4}{3}), \quad (5.5)$$

where the superscript c reminds us that this amplitude has been obtained from the crossing-symmetric equation (4.14). A plot of $T_0^{(c)}(\frac{4}{3},\frac{4}{3},\frac{4}{3})$, as (α_0,λ) are varied along the curve in Fig. 8, is shown in Fig. 9 together with $T_0^{(u)}$ (u for unitary), calculated as in (5.4).

The discrepancy between these two functions will evidently be minimized in the neighborhood of $(\alpha_0,\lambda)=(-0.54, 0.22)$. Note that the same result would have been obtained by using the $I=2$ amplitude T_2, since any amplitude computed from (4.14) automatically satisfies crossing. These parameters $(-0.54, 0.22)$ imply a p-wave resonance at

$$s=33m^2=(5.75m)^2=(805\text{ MeV})^2$$

with width $\Gamma\approx 1.8m_\pi=252$ MeV. The general features of the phase-shift behavior in this and other channels are described in the next section.

The first question which now comes to mind is what other minima, if any, of the second crossing condition occur on the curve of Fig. 8. This can only be answered decisively by an extensive machine search; all we can say here is that no other points in the resonance region of the (α_0,λ) plane were noticed for which both crossing conditions were well satisfied. In any case, the above minimum is certainly the *nearest* to the experimental position.

The other main question of interest is that of the magnitude of the shift in our approximately determined parameters which will undoubtedly result when crossing dissatisfaction is minimized over a region of the s-t plane rather than at a single point. The behavior of the slopes of the partial amplitudes (as mentioned earlier) certainly provides reason for hoping that crossing is reasonably well satisfied over at least a small region around the symmetry point, but again this question can only be properly decided by a machine calculation.

6. RESULTS AND DISCUSSION

In this section we shall first briefly describe the main features of the low partial-wave phase shifts (in each isospin channel) resulting from the choice of parameters $(\alpha_0,\lambda)=(-0.54, 0.22)$, as determined in the previous section. In conclusion a comparison with the known experimental features of the π-π system will be made and the relation with the results of some other theoretical models is discussed.

The s-wave scattering lengths in the $I=0$ and $I=2$ channels, a_0 and a_2, may be readily found by using (2.4). One has

$$a_I^{-1}=(4/\pi)[a_I^{(0)}(4)]^{-1}=(4/\pi)[k_I^{(0)}(4)]^{-1}-\mu, \quad I=0,2 \quad (6.1)$$

where the factor $4/\pi$ in the first equality is necessary to conform to the conventional normalization. With $(\alpha_0,\lambda)=(-0.54, 0.22)$, (6.1) gives

$$a_0=-0.67, \quad a_2=-0.30.$$

Scattering lengths for the higher partial waves are also frequently defined by

$$a_I^{(l)}\equiv\frac{\pi}{4}4^l\lim_{s\to 4}\frac{a_I^{(l)}(s)}{(s-4)^l}. \quad (6.2)$$

These will, of course, only be finite in a model in which the partial waves have the correct threshold behavior. That such is the case in our model follows automatically from the fact that the full K matrix (4.40) has the correct singularity structure at the cross channel (t and u) normal thresholds. Evaluating (6.2) gives, for $l=1$ and 2,

$$a_1^{(1)}=0.013, \quad a_0^{(2)}=0.012, \quad a_2^{(2)}=0.002.$$

The low-energy behavior of the $I=0$ s-wave phase shift $\delta_0^{(0)}(s)$ is illustrated in Fig. 10.

We have not continued the plot to high energies because of the complication resulting from the presence of a pole in our approximate K-matrix elements (4.40). The function $\alpha(t)$ has a zero at about $t=-45$ resulting

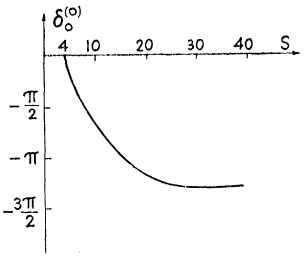

FIG. 10. The $I=0, l=0$ phase shift $\delta_0^{(0)}$.

in a branch point in all the $k_I^{(l)}(s)$ at about $s=50$ (1000 MeV). Since the large residue of the pole in K_I results in the imaginary part of $k_I^{(l)}$ dominating the real part (until very high energies are reached) the violation of elastic unitarity in fact remains small well beyond this additional branch point. It will be seen that no resonances occur in this channel (at least in the region of real phase shifts), although there will be a broad bump in the cross section as the phase shift decreases through $-\frac{1}{2}\pi$ at 440 MeV. This bump may perhaps be connected with the ABC anomaly[15] and/or the suggested scalar meson σ,[16] which, if it exists, is expected to have a fairly large width.[17]

Turning now to the d wave in this isospin channel, the phase shift $\delta_0^{(2)}$ remains between 0 and $\pi/2$ below the branch point discussed above and so the f^0 (at 1250 MeV), which after the ρ is probably the best established feature of the π-π system, is not found. Above the branch point the phase shift $\delta_0^{(2)}$ becomes quite large (possibly even reaching $\pi/2$) at times, but the results of the model cannot be regarded as significant in this region. In the $I=2$ channel, the s wave remains between 0 and $-\pi/2$ while the d wave, after starting out positive, crosses the axis but stays small in magnitude.

Finally we note the curious feature that, as well as the p-wave resonance, the model also predicts resonances in each of the higher odd partial waves with masses successively increasing but limited above by the position of the pole in K (about 1000 MeV). It seems unlikely that these higher resonances should be taken very seriously since they only arise as a result of the pole in K_1 and this pole cannot be present in the "true" K_1. However, even if this pole is smoothed out by something not included in the present model one might possibly expect to see the spin-3 resonance (assuming of course that our K does in fact provide a reasonable description of the "origins" of the ρ).

It is notoriously difficult to compare the predictions of a π-π scattering model with experimental results since the latter, with the exception of the two established resonances, are in a continual state of flux. It is true that the majority of experimental (and a few theoretical) papers over the past few years have favored positive s-wave scattering lengths, but the evidence is as yet far from conclusive. As an indication of the probable magnitude of a_0 we may quote one of the most recent analyses[17] in which the data were found to be consistent with $a_0=\pm 1.1$. Several groups[16] have found evidence for a scalar ($I=l=0$) resonance, the ϵ^0, at about the mass of the ρ, although it has not shown up in all experiments.[18] There has also been some recent support[17] for the other scalar resonance, σ. If either of these scalar resonances exist the phase shift $\delta_0^{(0)}$ would have to be increasing through $\frac{1}{2}\pi$ at the resonance mass, in contradiction to the behavior we find. The lack of a resonance-generating centrifugal barrier makes it very difficult for a π-π model to generate s-wave resonances unless other channels in which the particle could occur as a bound state are included. On the other hand, if the bumps in the cross section turn out to be explicable on some other nonresonant basis (such as $\delta_0^{(0)}$ *decreasing* through $\frac{1}{2}\pi$), then Cook's conjecture[19] may be tenable. This is a proposal that $\delta_0^{(0)}(s)$ may decrease from 2π at threshold to 0 at infinity. On passing through $\frac{3}{2}\pi$ and $\frac{1}{2}\pi$ broad bumps in the cross section would be produced which could conceivably correspond to those observed (the σ and ϵ^0). The low-energy behavior of our $\delta_0^{(0)}$ is consistent with this explanation of the bump at the σ mass but not for the one at the ρ mass.

The simplest way of comparing our results with those of most other calculations[20–23] is by determining an effective Chew-Mandelstam[14] coupling constant defined by

$$-\frac{4}{\pi}\lambda_{\text{c.m.}}=\tfrac{1}{5}T_0(\tfrac{4}{3},\tfrac{4}{3},\tfrac{4}{3})=\tfrac{1}{2}T_2(\tfrac{4}{3},\tfrac{4}{3},\tfrac{4}{3}).$$

From Fig. 9 it may be seen that in the neighborhood of $\lambda=0.22$ the existence of the second parameter s_0 has the approximate effect of simply reversing the sign of λ. Then, when account has been taken of the normalization factor $\frac{1}{4}\pi$, we find

$$\lambda_{\text{c.m.}}=18.$$

It seems to be a general feature of all these models that they agree on low-energy behavior for a given value of $\lambda_{\text{c.m.}}$. Thus, for example, the scattering lengths obtained by Saperstein and Uretsky[21] (using fourth-order perturbation theory for the discontinuity on the left-hand

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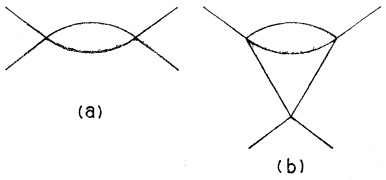

FIG. 11. (a) The "bubble" graph. The combination of on-shell internal lines and modified phase-space factor leads to the usual Feynman amplitude for graphs of this form. (b) A non-chain-type graph, which is *not* converted to a Feynman diagram by using the modified phase-space factor. This graph occurs in the third order of the power-series expansion in λ of $K(s,t)$ and possesses a singularity at $t=0$.

cut) for $\lambda_{c.m.}=0.18$ are

$$a_0=-0.63, \quad a_2=-0.32, \quad a_1^{(1)}=0.08,$$

the first two of which are quite similar to ours. They also find, for this value of $\lambda_{c.m.}$, a p-wave resonance about five times broader than the ρ. This resonance occurs at the ρ mass by taking $\lambda_{c.m.}=\pm 0.2$ in their model. Alexanian and Wellner[22] found that to reproduce the experimental ρ required $\lambda_{c.m.}=0.24$. Their scattering lengths are

$$a_0=-0.78, \quad a_2=-0.44, \quad a_1^{(1)}=0.03,$$

and the resonance width was determined by them to be 100 MeV. These authors were using a pure $\lambda\phi^4$ model for the coupling but with an improved convergence scheme.

On the other hand it has generally been found in calculations based on dispersion relations that a negative $\lambda_{c.m.}$ is necessary to obtain a p-wave resonance. Thus, for example, Moffat *et al.*,[23] whose calculations are based on dispersion relations for the inverse partial-wave amplitudes, found a solution with a p-wave resonance at 700 MeV for $\lambda_{c.m.}=-0.1$.

The only other calculation known to the author in which *all* parameters are determined (also by crossing requirements) is that of Blankenbecler *et al.*[11] The N/D formulation of the partial-wave dispersion relations was used in a form which guaranteed that the full partial-wave sum exhibited the correct singularity structure in the crossed channels; inelastic effects due to the π-ω and ρ-ρ channels were also included in an approximate way. They found a p-wave resonance at 810 MeV with width 350 MeV. Their scattering lengths also turned out to be negative,

$$a_0=-1.72, \quad a_2=-1.85.$$

The two (related) features of partial-wave dispersion-relations calculations (see, for example, Kang[20] and references contained therein) which have caused most difficulty are the problems of forcing the correct threshold behavior in the individual partial waves and also, of course, the convergence difficulties encountered when dealing with integrals over an infinite energy range. These difficulties are both absent in the present work, where the unitarity-like integrals are over finite energy intervals.

While this work has been based on a purely S-matrix theory approach to the strong interaction problem the results obtained by the method of solution adopted here have a certain similarity to the $\lambda\phi^4$ model for π-π interactions; in fact the two coincide in first- and second-order perturbation theory (for $s_0=4$). This is essentially due to the use of the modified "phase-space" factor F which effectively converts the simple bubble graph [Fig. 11(a)] with on-mass-shell intermediate lines to one with off-mass-shell intermediate lines. This relationship does not follow for the terms occurring in higher order which are not simply composed of a chain of bubbles, as in Fig. 11(b). It is the graph in Fig. 11(b), incidentally, which contributes the function $F^{(0)}$ containing the unwanted singularity at $t=0$, as discussed in Sec. 3.

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